ON THE CONSTRUCTION OF FIELD THEORIES FOR HIGHER SPIN MASSLESS PARTICLES

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CONTENTS

I INTRODUCTION AND SUMMARY

The description of the most fundamental constituents of nature is centered around field theories of massless particles. The fundamental forces are thought to be mediated by such particles. The helicity, or spin, is one for those particles which account for the electromagnetic, the weak and the strong interactions, and the spin is two for the gravitational interactions. Forces due to particles of other helicities have not been observed.

In supergravity theories a key role is played by massless spin-3/2 particles, together with spin-2 particles. The main reason for the development of supergravity is not so much to explain new interactions, but to give a theoretically more consistent and elegant description.

In this thesis an attempt is presented to construct theories of interacting massless particles with higher spins. As massless particles are involved, these theories must have local symmetries, except for spin 0 and spin ½. Theories of free, i.e. non-interacting massless particles already exist. These free theories are invariant under commuting local symmetry transformations, which are generalisations of the familiar gauge transformation of the electromagnetic four potential. Free field theories are described in chapter II. They form the basis of our approach to the construction of theories with interactions.

Chapter III gives a formal description of the structrure of the kind of massless higher spin theories we are looking for. It is the structure of an "extended massless free field theory". Every existing theory of massless particles can be viewed as such a theory. Chapter IV deals with the algebraic structure of those theories. Here we coin the word "bracket structure", after Schouten, for an elegant type of structure which both underlies Yang-Mills theory and General Relativity. Other structures are shown to be possible as well. An example of such an alternative structure is formed by supergravity theories. Both chapter III and IV are amply illustrated with the examples of known standard theories.

The main reason why theories of higher massless spins are discussed from the extended massless theory point of view is the approach chosen in this thesis to search for new theories. Two basic approaches exist to construct new theories. Conventionally one tries to extend a globally invariant theory to a local one. But another way is to try to extend a locally invariant free theory to a locally invariant interacting one. Here we have chosen for the second approach. How a new theory may be obtained in this way can be found in chapter V. The algorithm is not ours, but we did apply it for the first time to find new interactions rather than reobtaining known ones.

Actually the algorithm only constructs the interaction in first order in the coupling constant. Our original hope was that in this way one would have enough information to make a direct guess for a complete theory. However this happens not to be the case.

By explicit construction it is established that consistent first order interaction lagrangians in general do exist. They give rise to non-commuting first order gauge transformations. These interactions are cubic in the fields. Either they contain three fields which carry all the same spin, or two of the fields carry the same spin and the third a different one. There is a lot of systematics in the properties of these first order lagrangians and their transformation rules and commutators. All this can be found in chapter VI. The explicit forms of the interactions are listed in Appendix C.

We do not repeat every time the construction of these lagrangians. Instead, in chapter VII, we give, as examples, how the construction algorithm can be applied to find first order spin-1 and spin-2 selfinteractions, and the interaction of gravitation with spin-1/2 and spin-3/2 particles. The first ones can be found in the literature, the second ones are new, although the results, of course, are just first order formulations of well known (super)-gravity theories. Furthermore it is shown how to extract the algebraic structure of the full theory from the first order theory. In particular it is shown that this is more difficult for theories of the supergravity type than for the selfcontained spin-2 selfinteraction theory.

The next chapter, chapter VIII, deals with spin-3 selfinteractions. A non-trivial selfinteraction is found. The coupling constant has the dimension of an inverse mass squared and the interaction is fully antisymmetric in the spin-3 field, like the Yang-Mills interaction. However there is no self contained spin-3 theory like General Relativity is a self contained spin-2 theory. Some possible ways out to find a suitable algebraic structure for a theory involving massless spin-3 fields are indicated.

Finally matter gauge couplings are studied in chapter IX. Here a gauge field is meant to be a massless field with spin equal or higher than one, with which a local invariance is associated. As an example the coupling of a massless spin-s boson to massless spin-0 fields is studied. Again first order interactions can be found, and again the algebraic structure is problematic for higher spins. Probably, if one tries to include one type of higher spin interaction, one is forced to include them all. An alternative notation instead of symmetric tensors is proposed to deal with the algebraic properties of such systems.

Summarizing, the situation for interactions of higher spin massless fields is as follows. Not only do free lagrangians exist for arbitrary spin, but for all higher spins consistent cubic first order interactions can be found. There is a lot of systematics in the properties of these first order interactions. However, although a complete theory has not yet been given, it is highly probable that such a complete theory has to describe either all massless spins higher than two or no such spins at all.

II MASSLESS FREE FIELD THEORIES

1. Introduction

Field theories for free massless particles exist for all integer and half integer spin. Massless particles correspond to irreducible unitary representations of the covering group of the Poincaré group [1]. The spin of a particle is determined by the representation of the covering group of the "little group" of its four-momentum. The little group is the subgroup of the Poincaré group which leaves the four-momentum of the particle invariant. For massless particles the four-momentum is lightlike and the little group is isomorphic with the Euclidean group E_2. Its generators are the rotation around the particle three-momentum axis and two others, called translations. The translation generators must be represented trivially in order to have a finite dimensional irreducible representation. Finite spins correspond to finite dimensional irreducible representations of the little group. Thus the translation like elements of the little group leave the spin of the irreducible representations of the Poincaré group unaffecrted. This is the source of "gauge invariance".

In a field theory of massless particles this gauge invariance can be assured in a trivial way, by using gauge invariant fields, like $F_{\mu\nu}$ in classical electromagnetism . Gauge invariance also can be assured in non trivial ways. Then the fundamental field is not gauge invariant, like A_μ in the quantum theory of electrodynamics, and the fields of the free field theories discussed below.

Lagrangian field theories describing free massless particles of arbitrary spin were obtained several years ago [2-5]. For bosons these theories are generalisations of the Maxwell Lagrangian for spin 1, and for fermions of the Rarita-Schwinger Lagrangian for spin 3/2. Later these theories will be taken as a starting point for the construction of theories of interacting massless particles.

Before we proceed to describe the massless field theories, we want to point out that this starting point may be questioned. One might argue that one should use other generalisations of the Maxwell equation. For instance, for spin 2, it is suggested that the quantity which is analoguous to A_μ is not $h_{\mu\nu}$ but rather the Christoffel symbol $\Gamma^\mu{}_{\alpha\beta}$, consisting of first order derivatives of $h_{\mu\nu}$ [6]. Proceeding in this way the field equation would contain more than two derivatives. This we consider undesirable and that is why we prefer the lagrangians presented here.

The same notation as in reference [5] is used here, except for that we use the Bjorken-Drell metric diag(+,−,−,−). This metric tensor will be denoted by $\eta_{\mu\nu}$.

2. Massless bosons

To describe a massless spin-s boson, a fully symmetric real tensor field $\phi_{\mu_1 \ldots \mu_s}(x)$ is used. For $s \geq 4$ it must have zero double trace:

(2.1) $$\phi^{\alpha\beta}{}_{\alpha\beta\mu_5 \ldots \mu_s}(x) = 0$$

Like the Maxwell equation, the field equation is a second order differential equation:

(2.2) $$W_{\mu_1 \ldots \mu_s}(x) = 0$$

where $$W_{\mu_1 \ldots \mu_s} = \partial^\lambda \partial_\lambda \phi_{\mu_1 \ldots \mu_s} - \Sigma^1_\mu \, \partial_{\mu_1} \partial^\rho \phi_{\rho\mu_2 \ldots \mu_s} + \Sigma^2_\mu \, \partial_{\mu_1} \partial_{\mu_2} \phi^\lambda{}_{\lambda\mu_3 \ldots \mu_s}$$

Here the notation Σ^1_μ denotes a symmetrized sum of s terms, and the symbol Σ^2_μ a symmetrized sum over all pairs μ_i, μ_j of $\frac{1}{2}s(s-1)$ terms. The left hand side of (2.2) has zero double trace, like $\phi_{\mu_1 \ldots \mu_s}$ itself. We refer to the literature [2,3,4] for the proof that the physical degrees of freedom of a field satisfying (2.2) are those of a massless particle of helicity $\pm s$.

The field equation (2.2) is invariant for the following gauge transformation

(2.3) $$\delta\phi_{\mu_1 \ldots \mu_s} = \Sigma^1_\mu \, \partial_{\mu_1} \xi_{\mu_2 \ldots \mu_s}$$

where the gauge parameters $\xi_{\mu_2 \ldots \mu_s}(x)$ are required to have vanishing trace.

(2.4) $$\xi^\lambda{}_{\lambda\mu_4 \ldots \mu_s}(x) = 0$$

Gauge transformations on ϕ commute. For a massless spin-0 field there is no gauge transformation. The double tracelessness of ϕ is preserved by gauge transformations.

The existence of this gauge transformation is related to the fact that the quantities $W_{\mu_1 \cdots \mu_s}$, which form the left hand side of the equation of motion (2.2), are not independent. To be specific, for arbitrary ϕ the following identity holds

$$(2.5) \qquad \partial^\lambda W_{\lambda\mu_2 \cdots \mu_s} - \tfrac{1}{2} \sum_\mu^1 \partial_{\mu_2} W^\lambda{}_{\lambda\mu_3 \cdots \mu_s} = 0$$

This identity we shall call, for lack of a better name <u>source constraint</u>. This name is chosen because (2.5) implies that any source $T_{\mu_1 \cdots \mu_s}(x)$ which would appear at the right hand side of (2.2) must satisfy (2.5) with W replaced by T. So, actually, the identity (2.5) <u>is</u> not a source constraint, but rather <u>implies</u> a source constraint.

The field equation (2.2) may be derived from an action principle. The following lagrangian

$$(2.6) \quad L_0 = (-1)^{s+1} \{ \tfrac{1}{2}\phi_{\mu_1 \cdots \mu_s} W^{\mu_1 \cdots \mu_s} - \tfrac{1}{8} s(s-1) \phi^\lambda{}_{\lambda\mu_3 \cdots \mu_s} W_\rho{}^{\rho\mu_3 \cdots \mu_s} \}$$

gives rise to a field equation equivalent to (2.2)

$$(2.7) \qquad L_{0,\phi_{\mu_1 \cdots \mu_s}} = W_{\mu_1 \cdots \mu_s} - \tfrac{1}{2} \sum_\mu^2 \eta_{\mu_1\mu_2} W^\rho{}_{\rho\mu_3 \cdots \mu_s} = 0$$

Here the symbol $L_{0,\phi}$ is used to denote the coefficient of $\delta\phi$ in the variation of the action due to a change in ϕ. In other words, $L_{0,\phi}$ stands for the left hand side of the equation of motion as derived from the lagrangian L_0 . The source constraint (2.5) for W, when expressed in terms of $L_{0,\phi}$, states that its traceless divergence vanishes identically:

$$(2.8) \qquad \partial^\rho (L_{0,\phi_{\rho\mu_2 \cdots \mu_s}} - \frac{1}{2(s-1)} \sum_\mu^2 \eta_{\mu_2\mu_3} L_{0,\phi^\lambda{}_{\lambda\rho\mu_4 \ldots \mu_s}}) = 0$$

The fact that the traceless divergence rather than the full divergence is involved, stems from the fact that the gauge parameters are traceless. Explicit forms of the lagrangians L_0, as they follow from (2.2) and (2.6) are given in Appendix B for spin 0, 1, 2 and 3.

3. Massless fermions

The description of a massless spin $(s+\frac{1}{2})$ fermion is very much like that of a spin-s boson. It is represented by a fully symmetric Majorana tensor-spinor field $\psi_{\mu_1\cdots\mu_s}(x)$, where the spinorial index is suppressed. Some properties of Majorana spinors can be found in Appendix A. For $(s+\frac{1}{2})>(7/2)$ the triple gamma contraction is required to vanish

$$(2.9)\qquad \gamma^\alpha\gamma^\beta\gamma^\delta\psi_{\alpha\beta\delta\mu_4\cdots\mu_s} = \gamma^\alpha\psi_{\alpha\ \ \rho\mu_4\cdots\mu_s}^{\ \ \rho} = 0$$

The field equation is a first order differential equation which reduces to the Rarita-Schwinger equation for $(s+\frac{1}{2})=3/2$:

$$(2.10)\qquad Q_{\mu_1\cdots\mu_s} = 0$$

where
$$Q_{\mu_1\cdots\mu_s} = \not{\partial}\psi_{\mu_1\cdots\mu_s} - \sum_\mu^1 \partial_{\mu_1}\gamma^\rho\psi_{\rho\mu_2\cdots\mu_s}$$

Note that the triple gamma contraction of Q vanishes.

In case $s>3/2$, the field equation (2.10) is invariant for the gauge transformation

$$(2.11)\qquad \delta\psi_{\mu_1\cdots\mu_s} = \sum_\mu^1 \partial_{\mu_1}\varepsilon_{\mu_2\cdots\mu_s}$$

Here the gauge parameter ε is required to have vanishing gamma trace:

$$(2.12)\qquad \gamma^\rho\varepsilon_{\rho\mu_3\cdots\mu_s} = 0$$

Such gauge transformations commute and they preserve the triple-gamma-tracelessness of ψ. Related to the gauge transformation is the following source constraint for the left hand side of the field equation

$$(2.13)\qquad \partial^\rho Q_{\rho\mu_2\cdots\mu_s} - \tfrac{1}{2}\not{\partial}\gamma^\lambda Q_{\lambda\mu_2\cdots\mu_s} - \tfrac{1}{2}\sum_\mu^1 \partial_{\mu_2}Q^\lambda_{\ \lambda\mu_3\cdots\mu_s} = 0$$

The lagrangian

$$(2.14)\qquad L_0 = i(-1)^s\{\tfrac{1}{2}\bar{\psi}_{\mu_1\cdots\mu_s}Q^{\mu_1\cdots\mu_s} - \tfrac{1}{4}s\,\bar{\psi}_{\mu_2\cdots\mu_s\rho}\gamma^\rho\gamma_\sigma Q^{\sigma\mu_2\cdots\mu_s}$$
$$- \frac{1}{8}s(s-1)\,\bar{\psi}^\rho_{\ \rho\mu_3\cdots\mu_s}Q^{\lambda\ \ \mu_3\cdots\mu_s}_{\ \lambda}\}$$

leads to a field equation equivalent to (2.10):

$$(2.15)\quad L_{0,\bar{\psi}_{\mu_1\cdots\mu_s}} = Q_{\mu_1\cdots\mu_s} - \tfrac{1}{2}\sum_{\mu}^{1}\gamma_{\mu_1}\gamma^{\rho}Q_{\rho\mu_2\cdots\mu_s} - \tfrac{1}{2}\sum_{\mu}^{2}\eta_{\mu_1\mu_2}Q^{\rho}{}_{\rho\mu_3\cdots\mu_s} = 0$$

When expressed in terms of $L_{0,\psi}$, the source constraint (2.13) states that its gamma-contractionless divergence vanishes:

$$(2.16)\qquad \partial^{\rho}\Big(L_{0,\bar{\psi}_{\rho\mu_2\cdots\mu_s}} - \frac{1}{2s}\sum_{\mu}^{1}\gamma_{\mu_2}\gamma^{\sigma}L_{0,\bar{\psi}_{\sigma\rho\mu_3\cdots\mu_s}} + \frac{1}{2s}\sum_{\mu}^{2}\eta_{\mu_2\mu_3}L_{0,\bar{\psi}^{\lambda}{}_{\lambda\rho\mu_4\cdots\mu_s}}\Big) = 0$$

Explicit forms of the free lagrangians for massless fields with spin 1/2, 3/2 and 5/2 fields can be found in Appendix B.

4. Field strengths

It is of interest that for all these spins there is a free field strength which generalizes the $F_{\mu\nu}$ of the spin-1 case [5]. The field strength is the simplest gauge invariant object linear in the field which does not vanish when the free field equations are satisfied. For spin-s bosons it is a linear combination of terms containing s derivatives in the symmetric s-tensor field:

$$(2.17)\qquad F_{\mu_1\cdots\mu_s;\rho_1\cdots\rho_s} = \sum_{j=0}^{s}(-1)^{j}\binom{s}{j}^{-1}\sum_{\rho}^{j}\sum_{\mu}^{j}\partial_{\mu_1}\cdots\partial_{\mu_j}\partial_{\rho_{j+1}}\cdots\partial_{\rho_s}\phi_{\rho_1\cdots\rho_j\mu_{j+1}\cdots\mu_s}$$

Here the symbol $\sum_{\mu}^{j}$ denotes a symmetrized sum over the $\binom{s}{j}$ permutations of the indices $\{\mu_j\}$, and $\sum_{\rho}^{j}$ similar for $\{\rho_j\}$. Even if the trace of the gauge parameters does not vanish, still the field strength is invariant for (2.3).

For spin-$(s+\frac{1}{2})$ fermions the expression for the field strength is obtained by replacing the s-tensor ϕ everywhere in (2.17) by the s-tensor-spinor ψ.

The field strength is fully symmetric in the indices $\{\mu_j\}$ and fully symmetric in the indices $\{\rho_j\}$. For the simultaneous interchange of both sets of indices, it is symmetric for s even, and antisymmetric for s odd. Furthermore the field strength enjoys a cyclicity property:

$$(2.18) \qquad \sum_{\mu}^{1} F_{\mu_1 \cdots \mu_s ; \mu_{s+1} \rho_1 \cdots \rho_{s-1}} = 0$$

For s=2, the field strength (2.17) is a linear combination of the linearized standard Riemann curvature and vice versa:

$$(2.19) \qquad F_{\mu_1\mu_2;\rho_1\rho_2} = \frac{1}{2}\left(R^0_{\rho_1\mu_1;\rho_2\mu_2} + R^0_{\rho_1\mu_2;\rho_2\mu_1} \right)$$

$$R^0_{\rho_1\mu_1;\rho_2\mu_2} = \frac{4}{3} F_{\rho_1\rho_2;\mu_1\mu_2} + \frac{2}{3} F_{\rho_1\mu_1;\rho_2\mu_2}$$

Here $R^0_{\rho_1\mu_1;\rho_2\mu_2}$ represents the linearized Riemann tensor.

III EXTENDED MASSLESS THEORIES

1. Introduction

Our aim is to construct theories of massless interacting higher spin particles which will be done in later chapters (V-IX). In this chapter, and in the next one, it will be stated in more detail what kind of theory we are after. The structure proposed here is realized in known theories of spin-1, spin-2 and spin-3/2 fields, as will be shown in the examples at the end of this chapter.

Of course those features will be stressed most which are important with respect to the construction of new theories. Also, our presentation must be valid for any spin, and as general as possible in order not to exclude interesting possibilities a priori. These requirements force our discussion to be less elegant than an exposition on e.g., gravitational interactions alone could have been.

In section 2 it is shown how interacting theories can be viewed as extensions of free theories. Some subtleties, due to the possibility to redefine the field and the parameters of the gauge transformation, are discussed in section 3. To illustrate these ideas, in section 4 it is shown how some well-known theories of massless particles can be viewed as extensions of free theories. That they can also be constructed when starting from a free theory will be shown in Chapter VII.

2. Extending a free lagrangian

In the previous chapter we described free lagrangians for massless spin-s particles. For $s>\frac{1}{2}$ these lagrangians were invariant for gauge transformations. The change of the field under such a gauge transformation may be written symbolically in the form

(3.1) $$\phi \rightarrow \phi' = \phi + \partial\xi$$

The gauge invariance of the free lagrangian is equivalent to the "free source constraint" (2.8;2.16), symbolically written as

(3.2) $$\partial L_{0,\phi} = 0$$

Remember that $L_{,\phi}$ denotes the left hand side of the equation of motion corresponding to a lagrangian L. In general $L_{,\phi}$ is not zero. It is zero only for fields which satisfy the equations of motion. However, (3.2) states that $\partial L_{0,\phi}$ vanishes always, whatever ϕ.

Let us assume that asymptotic regions exist, where the particles participating in a collision are free. There are reasons to doubt the validity of this assumption, but let us not worry about that. Then in the asymptotic region we may use free field theories. Such theories have to be gauge invariant, otherwise there is a conflict with Lorentz invariance. It is reasonable to assume that this asymptotic invariance induces a gauge invariance of the full lagrangian. The gauge invariance assures that the full lagrangian describes the same number of physical degrees of freedom as the free one. Let us remember that for spin 1 and for spin 2, the gauge transformations are quite different from those of the free theory. The main difference is that they no longer commute. Reasoning in this way, the proposed structure of theories for massless interacting particles is as follows.

Essentially only one more assumption is made about the structure of the lagrangian. It must be possible to write the full lagrangian as a polynomial in the fields and their derivatives. The same must apply to the source constraint of the full theory. If the lagrangian is viewed as a power series in terms of a coupling constant g,

$$L = L_0 + gL_1 + g^2L_2 + \dots \tag{3.3}$$

the lowest order lagrangian L_0 must be one of the free lagrangians of the preceding chapter, or a linear combination of them.

Thus, in the limit of the coupling constant going to zero, the full lagrangian reduces to the lagrangian of a free theory.

Unless otherwise stated, only the case where L_i is (i+2)-linear in the fields will be considered.

Then the number of derivatives involved is directly related to the dimension of the coupling constant g, as follows from the dimensional rules ($\hbar$=c=1):

$$[\text{lagrangian density}] = [\text{mass}]^4$$

(3.4)
$$[\text{derivative}] = [\text{mass}]$$
$$[\text{Bose field}] = [\text{mass}]$$
$$[\text{Fermion field}] = [\text{mass}]^{3/2}$$

For instance, for a coupling constant with dimension $-D$ (i.e. $[g]=[\text{mass}]^{-D}$) the number of derivatives in a first order lagrangian trilinear in bose fields is $(D+1)$, and in a first order lagrangian bilinear in fermion fields and linear in a bose field it is (D).

Let us write the source constraint of the full lagrangian as a power series in the coupling constant g:

(3.5)
$$B(\phi, L_{,\phi}) = 0$$

where
$$B(\phi, L_{,\phi}) = \partial L_{,\phi} + gB_1(\phi; L_{,\phi}) + g^2B_2(\phi,\phi; L_{,\phi}) + \ldots$$

Here $B(\phi, L_{,\phi}) = B\big(\phi(x), L_{,\phi}(x)\big)$ acts linearly on $L_{,\phi}$ and has a "polylinear" dependence on ϕ. B has the same tensorial type as the gauge parameter of the free theory. B may depend on space-time derivatives of ϕ and $L_{,\phi}$ as well, i.e.

(3.6)
$$B(\phi, L_{,\phi}) = b_{00}(\phi, L_{,\phi}) + b_{10}(\partial\phi, L_{,\phi}) + b_{01}(\phi, \partial L_{,\phi}) + \ldots$$

The B_i are i-linear in ϕ. The total number of space-time derivatives occurring in a particular B_i is fixed by the dimensions of the field and the left hand side of the field equation $L_{,\phi}$.

Since L can contain second and higher order space-time derivatives of the field, the Euler-Lagrange equations contain in general more terms than the standard two:

(3.7)
$$L_{,\phi} = \frac{\partial L}{\partial\phi} - \partial_\alpha \frac{\partial L}{\partial(\partial_\alpha\phi)} + \partial_\alpha\partial_\beta \frac{\partial L}{\partial(\partial_\alpha\partial_\beta\phi)} - \partial_\alpha\partial_\beta\partial_\gamma \frac{\partial L}{\partial(\partial_\alpha\partial_\beta\partial_\gamma\phi)} + \ldots$$

Substituting the right hand side of (3.3) in (3.5), terms can be collected order by order. To zeroth order in g we find back the source constraint of the free field theory

(3.8)
$$\partial L_{0,\phi} = 0$$

From the first order in g we find

$$(3.9) \qquad \partial L_{1,\phi} + B_1(\phi;\, L_{0,\phi}) = 0$$

As B_1 acts linearly on $L_{0,\phi}$, one has therefore

$$(3.10) \qquad \text{for } \phi \text{ such that } L_{0,\phi}=0 : \qquad \partial L_{1,\phi} = 0$$

The above equation lies at the basis of our construction algorithm for interacting theories. This will be discussed in chapter V.

A gauge transformation for the lagrangian (3.3) is implied by the source constraint (3.5). In order to demonstrate this, we contract $B(\phi,\, L_{,\phi})$ with a gauge parameter ξ. The resulting expression is a scalar density which vanishes for every ξ. After an integration over space-time one has

$$(3.11) \qquad \int d^4x\; \xi\; B(\phi,\, L_{,\phi}) = 0 \qquad \text{for all } \xi(x) \text{ and all } \phi(x)$$

Next, by means of partial integrations all space-time derivatives of B acting on $L_{,\phi}$ are shifted away to ξ and ϕ, the other argument of B. The result can be viewed as an new operator $T = T\big(\phi(x),\xi(x)\big)$ linear in the gauge parameter ξ, contracted with $L_{,\phi}$:

$$(3.12) \qquad \int d^4x\; T(\phi,\xi)\; L_{,\phi} = 0 \qquad \text{for all } \xi(x) \text{ and all } \phi(x)$$

On the other hand, if an action is required to be invariant for a transformation

$$\phi(x) \rightarrow \phi'(x)$$

one must have

$$(3.13) \qquad S[\phi(x)] = S[\phi'(x)]$$

If only invariance under an infinitesimal transformation is required, we must have

$$(3.14) \qquad S[\phi] = S[\phi'] + O\big((\phi'-\phi)^2\big)$$

Then it is sufficient that

(3.15) $$\int d^4x\, L_{,\phi}\, (\phi'-\phi) = 0$$

So (3.12) implies that L is invariant under infinitesimal gauge transformations of the form

(3.16) $$\phi \rightarrow \phi' = \phi + T(\phi,\xi)$$

$$= \phi + \partial\phi + g\, T_1(\phi;\xi) + g^2\, T_2(\phi,\phi;\xi) + \ldots$$

where T is expanded as a power series in g analoguous to the expansion of the source constraint in (3.5). Not only there is a one-to-one correspondence between the source constraint operator B and the infinitesimal gauge transformation T, but this correspondence holds order by order in g, as can be checked easily.

Here we want to add the following remark. For free theories there is no difference between the finite and the infinitesimal version of the gauge transformation. Yet in general finite and infinitesimal transformations are different concepts. When in the following the term gauge transformation is used, we will always mean infinitesimal gauge transformation.

Let us study the effect of two successive gauge transformations.

(3.17) $$\phi \rightarrow \phi_1 = \phi + T(\phi,\xi_1) \rightarrow \phi_{12} = \phi_1 + T(\phi_1,\xi_2)$$

Often a different notation will be employed:

(3.18) $$\phi \rightarrow \phi_1 = \phi + \delta_{\xi_1}\phi \rightarrow \phi_{12} = \phi_1 + \delta_{\xi_2}\phi_1$$

The commutator on the field ϕ will be defined as follows:

(3.19) $$[\delta_{\xi_2},\delta_{\xi_1}]\,\phi = \phi_{12} - \phi_{21}$$

For free theories, this commutator vanishes. In theories with interactions we shall study the transformation induced by the commutator:

(3.20) $$\phi \rightarrow \phi + [\delta_{\xi_2},\delta_{\xi_1}]\phi$$

In general $\phi_{12}-\phi_{21}$ is not of the form $T(\phi,\xi)$. Nevertheless, as shown next, the action is invariant for the transformation induced by the commutator.

Let us first express the commutator on the field in terms of T:

$$[\delta_{\xi_2},\delta_{\xi_1}]\phi = T(\phi+T(\phi,\xi_1),\xi_2)-T(\phi,\xi_2)+T(\phi,\xi_1)-T(\phi+T(\phi,\xi_2),\xi_1) \tag{3.21}$$

Let us define a special kind of directional derivative:

$$D_\phi(f(\phi),\psi(\phi)) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\{f(\phi+\varepsilon\psi(\phi))-f(\phi)\} \tag{3.22}$$

Because the gauge parameters are infinitesimal, this enables us to rewrite the commutator as follows:

$$[\delta_{\xi_2},\delta_{\xi_1}]\phi = D_\phi(T(\phi,\xi_2),T(\phi,\xi_1)) - D_\phi(T(\phi,\xi_1),T(\phi,\xi_2)) \tag{3.23}$$

Employing the same notation, the change of the action due to a change $\delta\phi$ in ϕ is given by

$$\delta S = \int d^4x\, D_\phi(L, \delta\phi) \tag{3.24}$$

In particular the change in the action induced by the commutator is

$$\delta S^{com} = \int d^4x\, D_\phi(L, D_\phi(T(\phi,\xi_2),T(\phi,\xi_1))) - (1\leftrightarrow 2) \tag{3.25}$$

On the other hand, we know the action is invariant for the transformations (3.17). This can be expressed by the fact that the following functional G

$$G[\phi(x),\xi(x)] = \int d^4x\, D_\phi(L, T(\phi,\xi)) \tag{3.26}$$

vanishes for every $\phi(x)$, $\xi(x)$.

Of course the functional derivative of G must vanish too. As a consequence we find

$$0 = \int d^4x\, D_\phi(D_\phi(L,T(\phi,\xi_2)), T(\phi,\xi_1)) \tag{3.27}$$

Now compare (3.25) and (3.27). Substituting the explicit definition (3.22) of

D, one sees that the antisymmetrical part in ξ_1 and ξ_2 of the r.h.s of (3.27) is identical to the r.h.s of (3.25). This completes the proof that the action is invariant for the commutator transformation.

3. Field redefinitions and gauge parameter redefinitions

Although for every interacting theory the lagrangian can be cast in the form (3.3), the reverse is not true. Some lagrangians, here called "fake interactions", do not represent an interaction but merely a free field theory. Later, when trying to construct new interactions, we must take care of not mistaking these fake interactions for real ones.

Fake interaction lagrangians are connected to the standard free lagrangians of chapter II by means of field redefinitions. Such a field redefinition has the generic form

$$(3.28) \qquad \phi \rightarrow \phi_r = R(\phi) = \phi + gR_2(\phi,\phi) + g^2R_3(\phi,\phi,\phi) + \ldots$$

Here the R_i are i-linear in ϕ. R is required to have an inverse

$$(3.29) \qquad \phi = R^{-1}(\phi_r) - \phi_r - gR_2(\phi_r,\phi_r) + \ldots$$

Substituting the series expansion of ϕ_r in the standard free lagrangian gives a fake interaction:

$$(3.30) \qquad L_0(\phi) \rightarrow L_0{}^r(\phi) = L_0(\phi_r(\phi))$$

The suffix "r" stands for redefinition.

The gauge transformation changes accordingly. If $\phi \rightarrow \phi + \partial\xi$ is an invariance of L_0 , then $L_0{}^r$ is invariant for

$$(3.31) \qquad \phi \rightarrow R^{-1}(R(\phi)+\partial\xi) = \phi + \partial\xi + T^r(\phi,\xi)$$

Two characteristics of fake interactions can be noted. First, the gauge transformations (3.31) commute, since under two successive transformations $\phi \rightarrow R^{-1}(R(\phi)+\partial\xi+\partial\eta)$. Second, for fields $\phi(x)$ which satisfy the free field equation, the first-order-in-g part of the lagrangian $L_0{}^r$ vanishes modulo total divergencies, as it equals $gR_2(\phi,\phi)\ L_{0,\phi}$.

The other topic we want to discuss in this section is the possibility of gauge parameter redefinitions. Let us make the following substitution for the gauge parameter

$$\xi \rightarrow \xi^r = \xi + g\Xi(\phi,\xi)$$

$$= \xi + g\Xi_1(\phi;\xi) + g^2\Xi_2(\phi,\phi;\xi) + \ldots \tag{3.32}$$

with, as usual, the Ξ_i i-linear in ϕ and linear in ξ. Then the transformation rule (3.16) will be changed to

$$\phi \rightarrow \phi' = \phi + T(\phi,\xi^r(\phi,\xi)) = \phi + T^{gr}(\phi,\xi)$$

$$= \phi + \partial\xi + gT_1{}^{gr}(\phi;\xi) + g^2T_2{}^{gr}(\phi,\phi;\xi) + \ldots \tag{3.33}$$

The suffic "gr" stands for gauge parameter redefinition.

It is important to be aware of the existence of field and gauge parameter redefinitions. As will be shown later on they give some freedom to modify a constructed $T(\phi,\xi)$ term in a gauge transformation. Both transformations, (3.16) and (3.33), leave the action invariant. Unlike field redefinitions, however, gauge parameter redefinitions do affect the commutator of the gauge transformations.

4. Examples of extended theories

To illustrate the ideas of the preceding sections we give some examples. The pure gauge Yang-Mills action, the pure gravitational Einstein-Hilbert action, the interaction of a spin-½ field with gravitation, and the supergravity action can be viewed as extended free theories. We start in this section from the known theories and separate the free lagrangian from the various terms in the power series expansion of the interaction. In Chapter VII it will be shown that the interaction terms can be actually be constructed by means of the methods of Chapter V.

The Yang-Mills lagrangian is given by [7,8,9]

$$L = -\frac{1}{4g^2} G_{\mu\nu}{}^{i} G^{\mu\nu i} \tag{3.34}$$

where

(3.35) $$G_{\mu\nu}^{\ i} = gF_{\mu\nu}^{\ i} - g^2 f^{ijk} A_\mu^{\ j} A_\nu^{\ k}, \qquad F_{\mu\nu}^{\ i} = \partial_\nu A_\mu^{\ i} - \partial_\mu A_\nu^{\ i}$$

The internal indices i,j,k run from 1 to some number N. Summation over repeated indices is implied. The f's have to be the structure constants of some Lie group. Clearly the lagrangian has the form of an extended theory, for we can write

(3.36) $$L = L_0 + gL_1 + g^2L_2$$

with L_0 a sum of free spin-1 lagrangians

(3.37) $$L_0 = -\tfrac{1}{4}F_{\mu\nu}^{\ i} F^{\mu\nu i}$$

and L_1 and L_2 given by

(3.38) $$L_1 = \tfrac{1}{2} f^{ijk} F^{\mu\nu i} A_\mu^{\ j} A_\nu^{\ k}$$
$$L_2 = -\tfrac{1}{4} f^{ijk} f^{ilm} A_\mu^{\ j} A_\nu^{\ k} A^{\mu l} A^{\nu m}$$

The infinitesimal form of the gauge transformation which leaves the lagrangian (3.34) invariant is given by

(3.39) $$A_\mu^{\ i} \rightarrow (A_\mu^{\ i})' = A_\mu^{\ i} + \partial_\mu \Lambda + g f^{ijk} A_\mu^{\ j} \Lambda^k$$

In the limit of vanishing g this reduces to the gauge transformation for a free spin-1 field. The Yang-Mills lagrangian is special in two ways. The series (3.36) is finite, and, not unrelated, the coupling constant g is dimensionless.

For the pure gravitational Einstein-Hilbert action [9,10] things are not as transparent as for the Yang-Mills case. The action is a functional of the metric field $g_{\mu\nu}(x)$

Before proceeding further, we must warn the reader. Contrary to the usual convention, indices are always raised and lowered with the (constant) Minkowski metric $\eta_{\mu\nu}$, and never with $g_{\mu\nu}(x)$. In particular a distinction must be made between $g_{\mu\nu}(x)$ and its inverse $\tilde{g}^{\mu\nu}(x)$. Using the above notation, the pure gravitational lagrangian is given by

$$(3.40) \qquad L = \frac{2}{\kappa^2}\sqrt{-g}\,R$$

where

$$(3.41) \qquad \sqrt{-g} = \left[\det(-g_{\mu\nu})\right]^{\frac{1}{2}}$$

$$(3.42) \qquad R = \tilde{g}^{\kappa\mu} R^{\nu}{}_{\kappa\nu\mu} \qquad\qquad \tilde{g}^{\mu}{}_{\lambda}(x)\, g_{\mu\rho}(x) = \eta_{\lambda\rho}$$

$$R^{\nu}{}_{\kappa\lambda\mu} = \partial_{\mu}\Gamma^{\nu}{}_{\kappa\lambda} - \partial_{\lambda}\Gamma^{\nu}{}_{\kappa\mu} + \Gamma^{\nu}{}_{\rho\mu}\Gamma^{\rho}{}_{\kappa\lambda} - \Gamma^{\nu}{}_{\rho\lambda}\Gamma^{\rho}{}_{\kappa\mu}$$

$$\Gamma^{\nu}{}_{\kappa\lambda} = \tfrac{1}{2}\,\tilde{g}^{\nu\mu}\,(\partial_{\lambda}g_{\mu\kappa}+\partial_{\kappa}g_{\mu\lambda}-\partial_{\mu}g_{\kappa\lambda})$$

The action is invariant for general coordinate transformations. This can be expressed by the invariance of the lagrangian (3.40) for the infinitesimal transformation

$$(3.43) \qquad g_{\mu\nu} \rightarrow g_{\mu\nu} + \kappa[(\partial_{\rho}g_{\mu\nu})\xi^{\rho} + g_{\rho\mu}\partial_{\nu}\xi^{\rho} + g_{\rho\nu}\partial_{\mu}\xi^{\rho}]$$

where the l.h.s and the r.h.s refer to the same space-time point x, and $\xi^{\rho}(x)$ is the gauge parameter field.

If we want to write (3.40) as the lagrangian of an extended free field theory, we must not use $g_{\mu\nu}$ as field variable. Instead we use $h_{\mu\nu}(x)$ which is defined by

$$(3.44) \qquad g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}(x)$$

From (3.44) power series expansions for $\sqrt{(-g)}$, $\tilde{g}$ and Γ are obtained easily:

$$\sqrt{-g} = \exp\left(\tfrac{1}{2}\mathrm{Tr}(\ln(g^{\mu}{}_{\mu}))\right) = 1 + \tfrac{1}{2}\kappa h^{\mu}{}_{\mu} - \tfrac{1}{4}\kappa^2 h^{\mu}{}_{\lambda}h^{\lambda}{}_{\mu} + \tfrac{1}{8}\kappa^2 h^{\lambda}{}_{\lambda}h^{\mu}{}_{\mu} + \kappa^3\ldots$$

$$(3.45) \qquad \tilde{g}^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu}{}_{\rho}h^{\rho\nu} - \kappa^3 h^{\mu}{}_{\rho}h^{\rho}{}_{\lambda}h^{\lambda\nu} + \kappa^4\ldots$$

$$\Gamma^{\nu}{}_{\kappa\lambda} = \tfrac{1}{2}(\partial_{\lambda}h_{\mu\kappa}+\partial_{\kappa}h_{\mu\lambda}-\partial_{\mu}h_{\kappa\lambda})(\eta^{\mu\nu}-\kappa h^{\mu\nu}+\kappa^2\ldots$$

With the help of these formulas a series expansion for the lagrangian can be given (total divergencies are dropped)

(3.46) $$L = L_0 + \kappa L_1 + \kappa^2 L_2 + \kappa^3 L_3 + \ldots$$

L_0 turns out to be the standard lagrangian of a massless spin-2 field as given in chaper II:

(3.47) $$L_0 = \tfrac{1}{2}(\partial_\rho h_{\mu\nu})^2 - \tfrac{1}{2}(\partial_\mu h^\lambda{}_\lambda)^2 - (\partial^\rho h^{\mu\nu})(\partial_\mu h_{\rho\nu}) + (\partial^\rho h_{\rho\mu})(\partial^\mu h^\nu{}_\nu)$$

Contrary to the Yang-Mills case the series expansion (3.46) for L contains an infinite number of terms. From (3.43) an infinitesimal gauge transformation for $h_{\mu\nu}(x)$ follows

(3.48) $$h_{\mu\nu} \to h_{\mu\nu}{}' = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + \kappa(\partial_\rho h_{\mu\nu}\xi^\rho + h_{\rho\mu}\partial_\nu\xi^\rho + h_{\rho\nu}\partial_\mu\xi^\rho)$$

Thus we see that the gauge transformation of the field $h_{\mu\nu}$ is an extension of a spin-2 free field gauge transformation. Thus the pure gravitational lagrangian can be written as an extension of a free massless spin-2 theory.

The gravitational interaction of half-integer spin fields cannot be described in terms of the metric $g_{\mu\nu}(x)$ alone. One is forced to introduce a <u>vierbein</u> $e^a{}_\mu(x)$, which is not symmetric in a and μ, and thus contains six (unphysical) degrees of freedom more than g. As indicated by the notation, we actually have two kind of indices, "world" indices, denoted by Greek characters, and "Lorentz" indices, denoted by Latin ones. How fundamental this distinction maybe, for our considerations -we merely want to point out that these theories can be written as extensions of free theories- this distinction is irrelevant.

For the coupling of a massless <u>spin-½ field to gravitation</u>[9,11], there are two physically distinct possibilities, "with torsion" or "without torsion". The mass of the spin-½ field is taken zero only for simplicity, a massive spin-½ field can be coupled in the same way.

In case there is no torsion, the lagrangian is given by

(3.49) $$L = \frac{2}{\kappa^2} e R + e \frac{i}{2} \bar{\psi} \gamma^i e^\mu{}_i D_\mu \psi$$

with

(3.50) $$e = \det(e^a{}_\mu)$$

(3.51) $$R = \tilde{e}^{\mu}_{a}\tilde{e}^{\nu}_{b}R^{ab}_{\mu\nu}(\omega) \qquad \tilde{e}^{\mu}_{a}e_{b\mu} \equiv \eta_{ab}$$

$$R^{ab}_{\mu\nu}(\omega) = \partial_{\mu}\omega_{\nu}^{ab} - \partial_{\nu}\omega_{\mu}^{ab} - \omega_{\mu}^{ac}\omega_{\nu c}^{\ \ b} + \omega_{\nu}^{ac}\omega_{\mu c}^{\ \ b}$$

(3.52) $$\omega_{\mu}^{ab} = \omega_{\mu}^{ab}(e)$$

(3.53) $$\omega_{\mu}^{ab}(e) = \tfrac{1}{2}(-\tilde{e}^{\nu a}(\partial_{\mu}e^{b}_{\ \nu}-\partial_{\nu}e^{b}_{\ \mu}) + \tilde{e}^{\nu b}(\partial_{\mu}e^{a}_{\ \nu}-\partial_{\nu}e^{a}_{\ \mu})$$
$$-(\tilde{e}^{\rho a}\tilde{e}^{\sigma b}-\tilde{e}^{\rho b}\tilde{e}^{\sigma a})(\partial_{\sigma}e_{c\rho})e^{c}_{\ \mu})$$

and

(3.54) $$D_{\mu}\psi = (\partial_{\mu} - \tfrac{1}{4}\omega_{\mu}^{ab}\sigma_{ab})\psi$$

The first term in the lagrangian is equivalent to the pure gravitational lagrangian (3.40) if one makes the identification

(3.55) $$e_{a\mu}(x)\, e_{a\nu}(x) = g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}(x)$$

(implying $\tilde{e}^{\mu}_{\ b}\tilde{e}^{\nu}_{\ b} = \tilde{g}^{\mu\nu}$ and $e_{\mu b}\tilde{e}^{\ b}_{\nu} = \eta_{\mu\nu}$)

Although the first term can be expressed as a function of $h_{\mu\nu}(x)$ alone, this is not true for the spin-½-gravitation interaction term. Here we really do need the vierbein.

In the torsion case, formulas (3.52)-(3.55) still apply with one major exception. Now we have

(3.56) $$\omega_{\mu}^{ab} = \omega_{\mu}^{ab}(e) + e_{\mu k}K^{kab}(\psi)$$

(3.57) $$K^{kab}(\psi) = -\frac{\kappa^2}{16}\, i\, \bar{\psi}\, (\gamma^{k}\sigma^{ab}-\gamma^{a}\sigma^{bk}+\gamma^{b}\sigma^{ak}-\eta^{ak}\gamma_{\ell}\sigma^{\ell b}+\eta^{bk}\gamma_{\ell}\sigma^{\ell a})\,\psi$$

So in case of torsion the first term of the lagrangian is not equivalent to the Einstein-Hilbert lagrangian (3.40). On substitution of the formulas for ω_{μ}^{ab} the difference between the two versions of the lagrangians turns out to be only of order κ^2 and quartic in the spin-½ field:

$$L(\omega(e,\psi)) - L(e(\omega)) = \tag{3.58}$$

$$= -\frac{e\kappa^2}{16}\left((\bar\psi\gamma^k\sigma^{ij}\psi)(\bar\psi\gamma_j\sigma_{ki}\psi) - \tfrac{1}{2}(\bar\psi\gamma^k\sigma_{ij}\psi)^2 - \tfrac{1}{2}(\bar\psi\gamma^i\sigma_{ij}\psi)^2\right)$$

When using vierbeins, there are two kinds of gauge parameters. Besides the vector parameter ξ_μ related to general coordinate transformations, there is the antisymmetric tensor parameter Ω^{ab}, related to local Lorentz transformations. The infinitesimal gauge transformations of the fields which belong to this system are

$$\delta e_{a\mu} = \kappa\left(\xi^\lambda(\partial_\lambda e_{a\mu}) + (\partial_\mu\xi^\lambda)e_{a\lambda} + \Omega_{ab}e^b{}_\mu\right) \tag{3.59}$$

$$\delta\psi = \kappa\,\xi^\lambda\,\partial_\lambda\psi + \tfrac{1}{2}\kappa\,\Omega^{ab}\,\sigma_{ab}\,\psi \tag{3.60}$$

Note that this is consistent with the case of pure gravitation, since (3.48) follows from (3.55) and (3.59). In order to exhibit the structure of an extended free theory we must change our variable $e^a{}_\mu$ again. This can be accomplished in many ways. Here we choose $h_{\mu\nu}(x)$ defined by (3.55) and the antisymmetric part of the vierbein (when viewed as a 4x4 matrix) as new variables

$$e_{a\mu}\,e_{a\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \tag{3.61}$$

$$(e_{a\mu} - e_{\mu a}) = \kappa\,v_{a\mu}$$

implying $e_{a\mu} = \eta_{a\mu} + \frac{1}{2}\kappa h_{a\mu} + \frac{1}{2}\kappa\,v_{a\mu} + O(\kappa^2)$. At this point the distinction between Lorentz and world indices is lost. Expressing the lagrangian (3.49) in terms of h,v and ψ, one finds a power series

$$L(h,v,\psi) = L_0 + \kappa L_1 + \kappa^2 L_2 + \kappa^3\ldots \tag{3.62}$$

with L_0 the sum of the same free spin-2 lagrangian (3.47) as before, and the free spin-$\frac{1}{2}$ lagrangian

$$L_0(\text{spin-}\tfrac{1}{2}) = \frac{1}{2}\bar\psi\gamma^\mu\partial_\mu\psi \tag{3.63}$$

The antisymmetric field $v_{a\mu}$ does not enter in L_0. The difference between the formulations with and without torsion appears only in order κ^2.

The gauge transformation (3.59) of $e_{a\mu}$ gives rise to transformation laws for $h_{\mu\nu}$ and $v_{a\mu}$. The transformation law for $h_{\mu\nu}$ remains the same (3.48) as for the Einstein-Hilbert lagrangian. In particular $h_{\mu\nu}$ does not transform under Ω^{ab}. The field $v_{a\mu}$ is pure gauge in the sense it can be adjusted at will by the Ω-dependent part of (3.59). Written in terms of v and h its transformation rule looks like

$$\delta v = \Omega + \kappa T_1(v,h;\Omega,\xi) \quad \text{rather than} \quad \delta v = \partial\Omega + \kappa T_1(v,h;\Omega,\xi) .$$

This occurence of a pure gauge field is a generalisation of the extended theory scheme discussed in the preceding sections. In the transformation rule (3.60) for ψ there is no inhomogeneous term present as its corresponding free lagrangian has no gauge invariance.

So we can conclude that the lagrangians of gravitationally interacting spin-½ fields can be written as extensions of the theories of massless spin-2 and spin-½ fields.

The last example we want to discuss in this chapter is the minimal <u>supersymmetric</u> extension of the pure gravitational lagrangian [12-14]. This lagrangian can be expressed in terms of a vierbein $e_{a\mu}$ and a Majorana vector spinor field ψ_μ as follows

$$\text{(3.64)} \qquad L = \frac{2}{\kappa^2} eR - \tfrac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma^5 \gamma_\nu D_\rho \psi_\sigma$$

Here e and R are given by (3.50) and (3.51). There is torsion present:

$$\text{(3.65)} \qquad \omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e) + e_{k\mu} K^{kab}(\psi_.)$$

$$K^{kab} = -i \frac{\kappa^2}{8} (\bar{\psi}^k \gamma^a \psi^b - \bar{\psi}^k \gamma^b \psi^a + \bar{\psi}^a \gamma^k \psi^b)$$

The covariant derivative is given by

$$\text{(3.66)} \qquad D_\rho \psi_\sigma = (\partial_\rho - \tfrac{1}{2} \omega_\rho{}^{ab} \sigma_{ab}) \psi_\sigma$$

Here no mass term can be included. Three kinds of infinitesimal gauge transformations leave this supersymmetric lagrangian invariant: general coordinate transformations, local Lorentz transformations and supersymmetry transformations. They are parametrized by a vector field ξ_μ , an antisymmetric tensor-field Ω^{ab} and a spinor field ε respectively.

The transformation of the vierbein due to ξ and Ω remains the same as (3.59), while its sypersymmetry transformation is given by

$$\delta e_{a\mu} = \frac{1}{8}\kappa^2 \bar{\varepsilon}\gamma_a \psi_\mu \tag{3.67}$$

The transformation rule for the vector-spinor ψ_μ under ξ and Ω is given by

$$\delta\psi_\mu = \xi^\lambda \partial_\lambda \psi_\mu + \kappa(\partial_\mu \xi^\lambda)\psi_\lambda + \tfrac{1}{2}\kappa\Omega^{ab}\sigma_{ab}\psi_\mu \tag{3.68}$$

and its supersymmetry transformation by

$$\delta\psi_\mu = D_\mu \varepsilon \tag{3.69}$$

Now we express the vierbein in the fields h and v as done in (3.61). Then the lagrangian (3.64) assumes the form of an extended free theory.

$$L = L_0 + \kappa L_1 + \kappa^2 L_2 + \kappa^3 L_3 + \ldots \tag{3.70}$$

In this case

$$L_0 = L_0(\text{spin-}2) + L_0(\text{spin-}3/2)$$

where

$$\begin{aligned} L_0(\text{spin-}3/2) &= -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma^5\gamma_\nu\partial_\rho\psi_\sigma \\ &= -\frac{1}{2}(\bar{\not\psi}\not\partial\not\psi - \bar{\psi}_\mu\partial^\mu\not\psi - \bar{\not\psi}\partial^\mu\psi_\mu + \bar{\psi}^\mu\not\partial\psi_\mu) \end{aligned} \tag{3.71}$$

The infinitesimal gauge transformations for h and v follow easily. For ψ they are given by (3.68), (3.69). To lowest order in κ the gauge transformations are

$$\begin{aligned} \delta h_{\mu\nu} &= \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + O(\kappa) \\ \delta\psi_\mu &= \partial_\mu\varepsilon + O(\kappa) \\ \delta v_{a\mu} &= \Omega_{a\mu} + O(\kappa) \end{aligned} \tag{3.72}$$

In the limit $\kappa \to 0$ $h_{\mu\nu}$ and ψ_{μ} transform as free massless spin-2 and spin-3/2 fields, while $v_{a\mu}$ has no physical degrees of freedom left. Thus the extension of the free theory introduces a selfinteraction of the spin-2 field and an interaction between spin-3/2 and spin-2 fields. The interaction contains also the field $v_{a\mu}$ which obeys such a strong gauge transformation that it can be "gauged away" completely i.e. it represents no physical degrees of freedom. It does not enter in L_0.

IV ALGEBRAIC STRUCTURE OF THE GAUGE TRANSFORMATIONS

1. Introduction

The algebraic structure of the proposed gauge theories can be more complicated than one might have hoped for. Therefore possible algebraic structures will be studied in detail.

We will limit ourselves to infinitesimal gauge transformations on a field $\phi(x)$ due to a parameter $\xi(x)$:

(4.1) $$\phi \rightarrow \phi' = \phi + \delta_\xi \phi = \phi + T(\phi,\xi)$$

Here T is linear in the infinitesimal parameter ξ, but does not have to be linear in ϕ. Space-time derivatives of ϕ and ξ are allowed to occur in $T(\phi,\xi)$ as well.

Transformations of the type (4.1) are substitutions and therefore associative. This associativity is expressed by the fact that the infinitesimal transformations satisfy the Jacobi identity

(4.2) $$\sum_{\text{cyclic}} [\delta_{\xi_1}, [\delta_{\xi_2},\delta_{\xi_3}]]\, \phi = 0$$

As we saw in the preceding chapter, the commutator of two transformations which leave an action invariant will again leave that action invariant. Therefore only sets of gauge transformations will be studied with the property that the commutator of two of such transformations belongs to the same set.

(4.3) $$[\delta_{\xi_2},\delta_{\xi_1}]\, \phi = \delta_\alpha \phi$$

The parameter α depends on ξ_1 and ξ_2 and in the most general case on the field ϕ as well.

When one studies the invariances of a particular action, there is an infinite dimensional class of obvious invariances of the type [15]:

(4.4) $$\phi^i \rightarrow \phi^i + C^{ij}(\phi,\xi,x,\ldots)\, L_{,\phi^j}$$

with $C^{ij} = -C^{ji}$, such that

(4.5) $$\delta^C S = \int d^4x\, L_{,\phi^i}\, C^{ij} L_{,\phi^j} = 0$$

Usually one is not interested in such invariances, as they do not reduce the number of physical degrees of freedom of the field, and one excludes these invariances from the set considered. Therefore, when studying the commutator (4.3), the r.h.s has to be determined modulo such invariances and (4.3) is "relaxed" to

(4.6) $$[\delta_{\xi_2},\delta_{\xi_1}]\,\phi^i = \delta_\alpha \phi^i + C^{ij}(\xi_2,\xi_1)\, L_{,\phi^j}$$

where δ_ξ and δ_α are not of the type (4.4). In locally supersymmetric theories, transformations can have commutators like (4.6). This phenomenon is called "on shell closure of the algebra". Yet we want to stress that it is not sufficient that (4.6) reduces to (4.3) for fields which satisfy the equations of motion, but that the antisymmetry of C^{ij} is essential as well.

In the following sections the consequences of discarding such obvious invariances will not be mentioned explicitly. They are easily worked out, and would make the appearance of many formulas more complicated.

2. Field independent commutators and bracket structures

In this section a special type of transformations will be studied, those where α does not depend on the field ϕ. So we may write

(4.7) $$[\delta_{\xi_2},\delta_{\xi_1}]\,\phi = \delta_{g[\xi_2,\xi_1]}\,\phi$$

Here $[\xi_1,\xi_2]$ defines the commutator of the parameters ξ_1, ξ_2:

(4.8) $$g[\xi_2,\xi_1] = \alpha$$

The last expression needs only to hold modulo parameters which induce no change in the field. For simplicity, we assume such parameters do not exist. The Jacobi identity on the fields, reading

$$\sum_{cyclic} \delta_{[\xi_3,[\xi_2,\xi_1]]}\,\phi = 0$$

implies

$$(4.9) \qquad \sum_{cyclic} [\xi_3,[\xi_2,\xi_1]] = 0$$

So we arrive at the following important conclusion. If α in (4.3) does not depend on the field, the parameters form a Lie algebra which is independent of the field. The commutator of parameters (4.8) does not involve the field ϕ.

Next we turn to bracket structures, a special class of transformations with field independent commutators. In a bracket structure the gauge transformation is required to be of the following type

$$(4.10) \qquad \phi \rightarrow \phi' = \phi + \partial\xi - g\{\phi,\xi\}$$

That is, all but the two first terms in the power series expansion (3.16) of $T(\phi,\xi)$ are zero. The bracket $\{\phi,\xi\}$ of a field ϕ and a parameter ξ must be linear in both its arguments. We define

$$(4.11) \qquad \{\phi,\xi\} \equiv -\{\xi,\phi\}$$

The bracket of two parameters is the commutator by definition. Implied by (4.3) are the following properties of brackets

$$(4.12) \qquad \begin{aligned} \partial[\xi_1,\xi_2] &= \{\partial\xi_1,\xi_2\} + \{\xi_1,\partial\xi_2\} \\ \{\phi,[\xi_1,\xi_2]\} + \{\xi_2,\{\phi,\xi_1\}\} + \{\xi_1,\{\xi_2,\phi\}\} &= 0 \\ [\xi_1,[\xi_2,\xi_3]] + [\xi_2,[\xi_3,\xi_1]] + [\xi_3,[\xi_1,\xi_2]] &= 0 \end{aligned}$$

The last property is the Jacobi identity for parameters already found in (4.9). The second one states that $\{\ \}$ forms a representation of $[\ ]$. The first property ensures that the inhomogeneous term $\partial\xi$ can be incorporated in the transformation rule (4.10).

If the gauge transformations of an action have a bracket structure, the following property is very useful. Since the term linear in ϕ in the transformation is directly related to L_1, knowledge of only L_0 and L_1 suffices to determine the complete bracket structure.

3. Field dependent commutators.

Let us consider the most general case, where the parameter α in (4.3) depends on the field ϕ as well.

$$(4.13) \qquad [\delta_{\xi_1},\delta_{\xi_2}]\,\phi = \delta_{C(\phi,\xi_1,\xi_2)}\phi$$

One encounters this type of expression for instance in supergravity theories.

There is a lack of symmetry in (4.13). Two field independent parameters are mapped onto a field dependent parameter α. More symmetrical and elegant expressions are obtained when the transformations which are commuted are allowed to be field dependent as well. Thus we are led to consider

$$(4.14) \qquad \phi \rightarrow \phi + \delta_\pi \phi = \phi + T(\phi,\pi(\phi(x),x))$$

Like in (3.23), the commutator of two such transformations can be written as

$$(4.15) \qquad [\delta_{\pi_1},\delta_{\pi_2}]\,\phi = D_\phi(T(\phi,\pi_1(\phi)),T(\phi,\pi_2(\phi))) - (1 \leftrightarrow 2)$$

where

$$(4.16) \qquad D_\phi(f(\phi),\psi(\phi)) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(f(\phi+\varepsilon\psi(\phi))-f(\phi)\right)$$

A little reflection shows this commutator can be expressed as follows

$$[\delta_{\pi_1},\delta_{\pi_2}]\,\phi = \delta_{[\pi_1,\pi_2]}\,\phi \quad \text{with}$$

$$(4.17) \qquad [\pi_1,\pi_2] = D_\phi(\pi_1,T(\phi,\pi_2)) - D_\phi(\pi_2,T(\phi,\pi_1)) + C(\phi,\pi_1,\pi_2)$$

$[\pi_1,\pi_2]$ involves extra terms besides the function C of (4.13). These extra terms are the derivatives of the first parameter in the direction of the change in ϕ induced by the second one and vice versa.

The Jacobi identity (4.2) implies a Jacobi identity for the field dependent parameters:

$$(4.18) \qquad \sum_{\text{cyclic}} [\pi_3(\phi(x),x),\ [\pi_2(\phi(x),x),\ \pi_1(\phi(x),x)]] = 0$$

For the special case that the gauge parameters do not depend on ϕ, (4.18) reduces to

$$(4.19) \qquad \sum_{\text{cyclic}} \left\{ - D_\phi(C(\phi,\xi_2,\xi_1),\ T(\phi,\xi_3)) + C(\phi,\xi_3,C(\phi,\xi_2,\xi_1)) \right\} = 0$$

As mentioned in section III.3, gauge parameter redefinitions affect the commutator. This gives rise to a concept analoguous that of "fake interactions", discussed in III.3. A commutator is said to have a "fake field dependence" if it is possible to redefine the gauge parameter in such a way that the new commutator is independent of the field.

Like the transformation (4.1) can be written as a power series

$$\phi \to \phi + T(\phi,\xi) = \phi + \partial\xi + g\, T_1(\phi;\xi) + g^2\, T_2(\phi,\phi;\xi) + \ldots$$

every formula in this section can be expanded. Because the first few terms of such series will be needed in later chapters, some attention is given to these expansions in the remainder of this section. The expansion of C in (4.13) is straightforward.

$$(4.20)\qquad C(\phi,\xi_1,\xi_2) = g\, C_0(\xi_2,\xi_1) + g^2\, C_1(\phi;\xi_2,\xi_1) + g^3\, C_2(\phi,\phi;\xi_2,\xi_1) + \ldots$$

The Jacobi identity (4.19) reads to lowest order

$$(4.21)\qquad \sum_{\text{cyclic}} \{C_1(\partial\xi_3,\xi_2,\xi_1) + C_0(C_0(\xi_2,\xi_1),\xi_3)\} = 0$$

For the commutator on the fields, as defined in (3.19), one finds

$$(4.22)\qquad \begin{aligned}[\delta_{\xi_2},\delta_{\xi_1}]\,\phi &= g\, T_1(\partial\xi_1;\xi_2) - g\, T_1(\partial\xi_2;\xi_1)\\ &+ g^2\, T_1(T_1(\phi;\xi_1);\xi_2) - g^2\, T_1(T_1(\phi;\xi_2);\xi_1) + g^2\, T_2(\partial\xi_1,\phi;\xi_2)\\ &+ g^2\, T_2(\phi,\partial\xi_1;\xi_2) - g^2\, T_2(\partial\xi_2,\phi;\xi_1) - g^2\, T_2(\phi,\partial\xi_2;\xi_1)\end{aligned}$$

On the other hand, from (4.13), one finds

$$(4.23)\qquad [\delta_{\xi_2},\delta_{\xi_1}]\,\phi = g\, \delta_{C_0(\xi_2,\xi_1)}\phi + g^2\, \delta_{C_1(\phi;\xi_2,\xi_1)}\phi + \ldots$$

Combining the last two expressions, relations follow between T_1,T_2 and C_0,C_1:

$$(4.24)\qquad \begin{aligned}\partial C_0(\xi_2,\xi_1) &= T_1(\partial\xi_1;\xi_2) - T_1(\partial\xi_2;\xi_1)\\ \partial C_1(\phi;\xi_2,\xi_1) + T_1(\phi;C_0(\xi_2,\xi_1)) &=\end{aligned}$$

$$T_1(T_1(\phi;\xi_1);\xi_2) - T_1(T_1(\phi;\xi_2);\xi_1) + T_2(\partial\xi_1,\phi;\xi_2)$$

$$+ T_2(\phi,\partial\xi_1;\xi_2) - T_2(\partial\xi_2,\phi;\xi_1) - T_2(\phi,\partial\xi_2;\xi_1)$$

Similar higher order formulas can be derived from (4.13) as well.

4. Examples of algebraic structures

We study the algebraic structure of the examples of extended theories which were given in the last section of chapter III.

The gauge transformation for the pure Yang-Mills action (3.34) was given by

$$\delta A_\mu^{\ i} = \partial_\mu \Lambda^i + g\, f^{ijk} A_\mu^{\ j} \Lambda^k \tag{4.25}$$

where the f^{ijk} are fully antisymmetric and satisfy the Jacobi identity $f^{ijk}f^{klm} + f^{ilk}f^{kmj} + f^{imk}f^{kjl} = 0$.

The commutator is independent of the field:

$$[\delta_{\Lambda_2},\delta_{\Lambda_1}]\, A_\mu^{\ i} = \delta_{g[\Lambda_1,\Lambda_2]} A_\mu^{\ i} \tag{4.26}$$

$$[\Lambda_2,\Lambda_1]^i = f^{ijk}\Lambda_2^{\ j}\Lambda_1^{\ k}$$

We define the bracket of A_μ with Λ as

$$\{A_\mu,\Lambda\}^i = g\, f^{ijk}\Lambda^j A_\mu^{\ k} \qquad \{\Lambda,A_\mu\}^i = -\{A_\mu,\Lambda\}^i \tag{4.27}$$

Then $\{\cdot,\cdot\}$ and $[\cdot,\cdot]$ form a bracket structure like (4.12). Taking the special case a=1,2,3 we can write:

$$A_\mu = A_\mu^{\ j}\sigma^j \qquad \Lambda = \Lambda^j\sigma^j \tag{4.28}$$

where the σ^j are the Pauli matrices. Normalizing $-gf^{ijk} = \varepsilon^{ijk}$ this leads to the transformation rule

$$\delta A_\mu = \partial_\mu \Lambda - \{A_\mu,\Lambda\} \tag{4.29}$$

where the bracket denotes the usual commutator of matrices. The identity

(4.30) $$\left[\delta_{\Lambda_2},\delta_{\Lambda_1}\right] A_\mu = \delta_{g[\Lambda_2,\Lambda_1]} A_\mu$$

is a trivial consequence of matrix properties now.

The gauge transformation (3.48) of the Einstein-Hilbert action (3.40) can be cast by means of a field redefinition in the form

(4.31) $$\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + \kappa\left[(\partial_\rho h_{\mu\nu})\xi^\rho - h_{\rho\mu}\partial^\rho\xi_\nu - h_{\rho\nu}\partial^\rho\xi_\mu\right]$$

For the commutator we find

(4.32) $$\left[\delta_{\xi_2},\delta_{\xi_1}\right] h_{\mu\nu} = \delta_{\kappa[\xi_2,\xi_1]} h_{\mu\nu}$$

with $$[\xi_2,\xi_1]^\mu = (\partial_\rho\xi_1{}^\mu)\xi_2{}^\rho - (\partial_\rho\xi_2{}^\mu)\xi_1{}^\rho$$

This is the commutator of general coordinate transformations

(4.33) $$x^\mu \rightarrow x^\mu - \xi^\mu(x)$$

Again the transformations (4.31) form a bracket structure, when we define

(4.34) $$\{\xi,h\}_{\mu\nu} = \xi_\rho\partial^\rho h_{\mu\nu} - h_{\rho\mu}\partial^\rho\xi_\nu - h_{\rho\nu}\partial^\rho\xi_\mu$$

$$\{h,\xi\}_{\mu\nu} \equiv -\{\xi,h\}_{\mu\nu}$$

In the Yang-Mills case these two brackets were aspects of the same thing, namely matrix commutators (or more properly speaking, commutators of Lie algebra valued quantities). In the gravitational case both brackets are Lie derivatives. $\{h,\xi\}$ is the Lie derivative of the tensor h by the vector ξ, and $[\xi_2,\xi_1]$ is the Lie derivative of the vector ξ_2 by ξ_1.

It is of interest that in both the Yang-Mills and the gravitational case these bracket structures can be extended to much wider structures incorporating brackets between tensors of arbitrary rank. In the Yang-Mills case this is accomplished trivially by defining for all $T^i{}_{\mu_1\ldots\mu_t}$ and $S^i{}_{\mu_1\ldots\mu_s}$

$$(4.35) \qquad \{S,T\}^i{}_{\mu_1\ldots\mu_{s+t}} = f^{ijk}\, T^j{}_{\mu_1\ldots\mu_t}\, S^k{}_{\mu_{t+1}\ldots\mu_s}$$

In the gravitational case we adopt the following definition, due to Schouten [16]. For all fully symmetric tensors $T_{\mu_1\ldots\mu_t}$ and $S_{\mu_1\ldots\mu_s}$

$$(4.36) \qquad \{S,T\}_{\mu_1\ldots\mu_{s+t-1}} = \frac{s!\ t!}{(s+t-1)!} \sum_\mu^t S_{\rho\mu_{t+1}\ldots\mu_{t+s-1}}\, \partial^\rho\, T_{\mu_1\ldots\mu_t} - \frac{s!\ t!}{(s+t-1)!} \sum_\mu^s T_{\rho\mu_{s+1}\ldots\mu_{s+t-1}}\, \partial^\rho\, S_{\mu_1\ldots\mu_s}$$

where $\sum_\mu^s$ indicates a symmetrized sum of $\binom{s+t-1}{s}$ terms. These generalised brackets satisfy

$$(4.37) \qquad \{S,\{T,U\}\} + \{T,\{U,S\}\} + \{U,\{S,T\}\} = 0$$

$$\partial\{S,T\} = \{\partial S,T\} + \{S,\partial T\}$$

The bracket (4.36) is related to the Poisson bracket

$$(4.38) \qquad [f(x,k),\ g(x,k)] = \frac{\partial f}{\partial k_\lambda}\frac{\partial g}{\partial x_\lambda} - \frac{\partial f}{\partial x_\lambda}\frac{\partial g}{\partial k_\lambda}$$

When $f(x,k) = \sum_{n=0}^\infty k_{\lambda_1}\ldots k_{\lambda_s} F^{(n)}_{\lambda_1\ldots\lambda_s}(x)$ and a similar expression holds for $g(x,k)$ then we have

$$(4.39) \qquad [f(x,k),\ g(x,k)] = \sum_{s=0}^\infty k_{\lambda_1}\ldots k_{\lambda_s} \sum_{n+m=s+1} \{F^{(n)},\ G^{(m)}\}_{\lambda_1\ldots\lambda_s}$$

Next we turn to the gauge transformations of the action of a gravitationally interacting spin-½ field. As we saw in chapter III, the transformation rules are

$$(4.40) \qquad \delta\psi = \kappa\, \xi^\lambda \partial_\lambda \psi + \tfrac{1}{2}\kappa\, \Omega^{ab} \sigma_{ab}\psi$$

$$(4.41) \qquad \delta e_{a\mu} = \kappa(\, \xi^\lambda \partial_\lambda e_{a\mu} + e_{a\lambda}\partial_\mu \xi^\lambda + \Omega_{ab} e^b{}_\mu)$$

Let us write

$$(4.42) \qquad e_{a\mu} = \eta_{a\mu} + \tfrac{1}{2}\kappa\, b_{a\mu}$$

The antisymmetric part of b is equal to the field v of chapter III, and the symmetric part is related by a field redefinition to h. From now on both world and Lorentz indices will be denoted by Greek characters. The transformation rule for $b_{\mu\nu}$ is easily seen to be

(4.43) $$\delta b_{\mu\nu} = 2\,\partial_\nu \xi_\mu + 2\,\Omega_{\mu\nu} + \kappa(\xi^\lambda \partial_\lambda b_{\mu\nu} + b_{\mu\lambda}\partial_\nu \xi^\lambda + \Omega_{\mu\lambda} b^\lambda{}_\nu)$$

We could write (4.40) and (4.43) with brackets

(4.44) $$\delta\psi = -\kappa\{\psi,\xi\} - \kappa\{\psi,\Omega\}$$

$$\delta b_{\mu\nu} = 2\partial_\nu \xi_\mu + 2\Omega_{\mu\nu} - \kappa\{b,\xi\}_{\mu\nu} - \kappa\{b,\Omega\}_{\mu\nu}$$

The commutators are given by

(4.45) $$[\xi_1,\xi_2]^\nu = \xi_1{}^\rho \xi_2{}^\nu{}_{,\rho} - \xi_2{}^\rho \xi_1{}^\nu{}_{,\rho}$$

$$[\Omega_1,\Omega_2]^{\mu\nu} = \Omega_1{}^\mu{}_\alpha \Omega_2{}^{\alpha\nu} - \Omega_1{}^\nu{}_\alpha \Omega_2{}^{\alpha\mu}$$

$$[\xi,\Omega]^{\mu\nu} = \xi_\lambda \Omega^{\mu\nu,\lambda} \qquad [\Omega,\xi]^{\mu\nu} = -[\xi,\Omega]^{\mu\nu}$$

These are the commutators of the local Poincaré group

(4.46) $$x^\mu \to x'^\mu = x^\mu - \xi^\mu(x)$$

$$dx^\mu \to dx'^\mu = dx^\mu - \Omega^{\mu\nu}(x)\, dx_\nu$$

The brackets satisfy properties which are a generalisation of (4.12):

(4.47) $$\{\{\phi,x_1\},x_2\} + \{\{x_1,\phi\},x_2\} + \{[x_1,x_2],\phi\} = 0$$

for $\phi = \psi$ or b, $\qquad x = \Omega$ or ξ

and

(4.48) $$\partial[\xi_1,\xi_2] = \{\partial\xi_1,\xi_2\} + \{\xi_1,\partial\xi_2\}$$
$$[\Omega_1,\Omega_2] = \{\Omega_1,\Omega_2\} + \{\Omega_1,\Omega_2\}^T$$
$$[\xi,\Omega] = \{\partial\xi,\Omega\} + \{\xi,\Omega\}$$

where the "transposed bracket" is denoted as $\{\Omega,b\}^T_{\mu\nu} \equiv -\{b,\Omega\}_{\mu\nu}$.
(remember that b and ω are not of the same tensorial type, the latter being antisymmetric, the former not)
These properties ensure

$$[\delta_{x_1},\delta_{x_2}]\,\phi = \delta_{\kappa[x_1,x_2]}\,\phi \tag{4.49}$$

Note that the commutator (4.45) of a local Lorentz transformation and a general coordinate transformation is a Lorentz transformation again, with parameter $\xi_\lambda\Omega^{\mu\nu,\lambda}$. This is to be expected since $h_{\mu\nu}$ is invariant for local Lorentz transformations (see section III.4).

Finally we consider the transformations of the supergravity lagrangian (3.64). The transformation rules are given by

$$\delta e_{a\mu} = \kappa\{\xi^\lambda(\partial_\lambda e_{a\mu}) + (\partial_\mu\xi^\lambda)\, e_{a\lambda}\} + \kappa\{\Omega^{ab}e_{b\mu}\} + \frac{1}{8}\kappa^2\{\bar{\varepsilon}\gamma_a\psi_\mu\} \tag{4.50}$$

and

$$\begin{aligned}\delta\psi_\mu &= \kappa\{\xi^\lambda(\partial_\lambda\psi_\mu) + (\partial_\mu\xi^\lambda)\psi_\lambda\} + \kappa\{\tfrac{1}{2}\Omega^{ab}\sigma_{ab}\psi_\mu\} \\ &\quad + \{(\partial_\mu - \tfrac{1}{2}\omega_\mu{}^{ab}\sigma_{ab})\,\varepsilon\}\end{aligned} \tag{4.51}$$

For the notation we refer to chapter III. When one calculates the commutators, one finds the following. The commutators of ξ and Ω are the same as for the spin-½ coupled to gravity case. The commutators of a supersymmetry transformation with a general coordinate transformation or a local Lorentz transformation are given by

$$[\xi,\varepsilon] = \xi^\lambda\partial_\lambda\varepsilon \tag{4.52}$$

$$[\Omega,\varepsilon] = \tfrac{1}{2}\sigma^{ab}\,\Omega_{ab}\,\varepsilon$$

Note the resemblance to the transformation rules for a spin-½ field ϕ. The commutator of two supersymmetry transformation gives to lowest order a general coordinate transformation. Higher order terms involve a local Lorentz part and a supersymmetry part as well:

$$(4.53) \qquad [\varepsilon_1,\varepsilon_2] = [\varepsilon_1,\varepsilon_2]^G_\mu + [\varepsilon_1,\varepsilon_2]^L_{ab} + [\varepsilon_1,\varepsilon_2]^S$$

$$= -\frac{i}{8}\bar{\varepsilon}_1\gamma^\mu\varepsilon_2 - \frac{i}{8}\bar{\varepsilon}_1\gamma^\lambda\varepsilon_2\,\omega_\lambda{}^{ab} - \frac{i}{8}\bar{\varepsilon}_1\gamma^\lambda\varepsilon_2\psi_\lambda$$

We note this commutator is field dependent. This field dependence can not be removed by means of field parameter redefinitions. In this respect the algebraic structure of supergravity theories is different from the bracket structures of Yang-Mills or gravity theories.

If one calculates the commutator of two supersymmetry transformations on the spin-3/2 field, one encounters extra terms of the type discussed in section IV.1 which "vanish on shell".

V THE CONSTRUCTION OF FIRST ORDER LAGRANGIANS

1. Introduction

Basically new gauge theories can be constructed in two ways. One way is to consider a global symmetry of the action and to try to extend this symmetry to a local symmetry. The other way is to try to extend a local invariant free action to a full action describing interacting fields. In this thesis the second approach is used. This has been done because there was no global symmetry avaiblable as an obvious candidate for extension, while it was natural to search for theories of interacting massless spin-5/2 or spin-3 fields, such theories being known for fields of spin 1, 3/2 and 2.

The idea of extending free actions goes back a long time. Thus far it has been applied to the case of gravitation only, although it was realized that one could search for other theories in a similar way. The method was developed by Gupta, Feynmann, Wyss, Deser and others [17-22]. Our version [23] is very similar to Fronsdal's [22].

If one is willing to give up manifest Lorentz invariance, there is another way to search for theories of interacting massless higher spins. Using the "light front formulation of dynamics" [24] first order interactions between massless higher spins have been constructed. Indeed, some of the essential properties of these theories are the same as in our formulation, which is manifestly Lorentz covariant like all standard field theories.

Schematically our search for lagrangians of interacting massless fields like

$$L = L_0 + gL_1 + g^2L_2 + \dots \tag{5.1}$$

proceeds as follows. Firstly, lagrangians are selected to describe the free massless fields. Here we choose the set of free lagrangians discussed in chapter II. Thus a candidate L_0 is established. L_0 is gauge invariant for commuting free gauge transformations. Next one searches for a candidate first order interaction lagrangian L_1. Such a candidate L_1 must have the property that it is possible to find a first order extension of the free gauge transformation

$$\phi \to \phi' = \phi + \partial\xi + gT_1(\phi;\xi) \tag{5.2}$$

such that the first order extended gauge transformation leaves the sum of free and first order lagrangian invariant modulo terms of order g^2:

$$(5.3) \qquad L_0 + gL_1 \rightarrow L_0 + gL_1 + O(g^2) \qquad \text{for} \quad \phi \rightarrow \phi'$$

In the next section a simple algorithm will be given to search for L_1 's which are suitable candidates.

In principle one could iterate this procedure. However this will not be done, for several reasons. Presumably one would have to iterate ad infinitum. Next, we expect ambiguities in second and higher order, as there may be several not completely equivalent solutions (here we are thinking about the difference between the geometric and the Cartan version of general relativity). Furthermore, to calculate a second order lagrangian, one has to know all first order lagrangians involving fields occurring in the searched for second order interaction. Finally, the calculation of second order lagrangians becomes an order of magnitude more cumbersome than the calculation of the first order ones.

Here we try to arrive at a complete theory in another way. We attempt to determine from the first order transformation rules which algebraic structure underlies the full theory. This is impossible to do if merely the free transformations are known, because free transformations commute. But the first order transformation rules determine quite a lot about the possible algebraic structure of the full theory.

Finally, having succeeded in all this, one could try to search for a complete lagrangian.

In this thesis we will be concerned mainly with the construction of first order interactions. As for structures of complete theories, it will be established that some particular elegant structures cannot be realized for massless particles with spin higher than 2.

2. Searching for first order lagrangians

As was established in chapter III, a source constraint (3.5) is associated with every lagrangian of massless fields like (5.1). To first order in g this source constraint was shown to imply

$$(5.4) \qquad \partial L_{1,\phi} + B_1(\phi; L_{0,\phi}) = 0 \qquad \text{for all } \phi(x)$$

Remember that $L_{,\phi}$ stands for the l.h.s. of the equation of motion as derived from the lagrangian L. Because B_1 in (5.4) is linear in its second argument, we may conclude

(5.5) $$\partial L_{1,\phi} = 0 \text{ for } \phi \text{ which satisfy } L_{0,\phi} = 0$$

We look for first order lagrangians with the above property, by means of an exhaustive search. The construction algorithm consists of several steps.

(i) We state the type of first order lagrangian searched for. The type is given by specifying which fields participate in L_1 and how many derivatives are involved. In this thesis we are only looking for interaction lagrangians which are cubic in the fields.

Example: We look for an interaction lagrangian linear in a spin-2 field $h_{\mu\nu}$ and quadratic in a spin-1 field A_μ, and with two derivatives, i.e. L_1 is of the type $h_{..,.}A_{.,.}A_{.}$ The two derivatives may be distributed in any way.

(ii) List all interaction lagrangians of a given type modulo total divergences.

Example: For type $h_{..,.}A_{.,.}A_{.}$ we find 13 independent possibilities, like*) $h_{\rho\rho,\sigma}A^{\sigma}{}_{,\lambda}A_{\lambda}$ and $h_{\rho\lambda}A_{\rho,\sigma}A_{\lambda,\sigma}$ etc.

(iii) Determine, if any, which combinations satisfy (5.5).

Example: Of the 13 lagrangians mentioned above, 5 linear combinations satisfy (5.5).

(iv) Take field redefinitions into account in order to eliminate "fake first order interactions".

Example: Precisely one of the 5 possibilities remains. The other four can be obtained from L_0 through the field redefinitions

$A_\mu \rightarrow A_\mu + k_1 A_\mu h_{\rho\rho} + k_2 A_\rho h_{\mu\rho}$ and

$h_{\mu\nu} \rightarrow h_{\mu\nu} + k_3 A_\mu A_\nu + k_4 \eta_{\mu\nu} A_\lambda A_\lambda$, where the k_i are arbitrary constants.

*) For typographical reasons upper and lower indices will not always be properly matched.

3. The determination of the first order gauge transformation and commutator

To every candidate first order lagrangian which satisfies (5.5) corresponds a first order gauge transformation and a first order commutator. They can be obtained in a straightforward way.

First we calculate $\partial L_{1\,,\phi}$ for arbitrary $\phi(x)$.

According to (5.4) this can be viewed as

$$(5.6) \qquad B_1(\phi;\, L_{0\,,\phi}) = -\,\partial L_{1\,,\phi} \quad \text{for arbitrary } \phi(x)$$

where B(.;.) is some operator linear in its second argument. B is of the same tensorial type as the gauge parameter ξ of the free gauge transformation. B is not determined completely by (5.6) because one may add pieces which yield zero when its second argument is $L_{0\,,\phi}$. This possibility exists because $L_{0\,,\phi}$ satisfies the free source constraint $\partial L_{0\,,\phi} = 0$ and so one can determine B only up to contributions of the form

$$(5.7) \qquad B_1(\phi;\, L_{0\,,\phi}) = b_1(\phi;\, \partial L_{0\,,\phi})$$

The operator B_1 corresponds directly to the first order contribution T_1 of the first order gauge transformation

$$(5.8) \qquad \phi \rightarrow \phi' = \phi + \partial\xi + gT_1(\phi;\xi)$$

The correspondence involves some partial integrations of the action, as was explained in chapter III. Let us give this rule for going from B_1 to T_1 in a less abstract notation. A term in B_1

$$B^{\mu_1\mu_2..\mu_{s-1}} = C^{\mu_1...\mu_{s-1}\alpha_1...\alpha_k}_{\nu_1...\nu_s\;\beta_1...\beta_\ell\gamma_1...\gamma_s}(\partial^{\beta_1}..\partial^{\beta_\ell}\phi_{\gamma_1..\gamma_s})\partial^{\alpha_1}..\partial^{\alpha_k}(L_{0\,,\phi})_{\nu_1..\nu_s}$$

corresponds to the following term in T_1:

$$T^{\nu_1..\nu_s} = (-1)^k\, C^{\mu_1..\mu_{s-1}\alpha_1..\alpha_k}_{\nu_1..\nu_s\;\beta_1..\beta_\ell\gamma_1..\gamma_s}\partial^{\alpha_1}..\partial^{\alpha_k}((\partial^{\beta_1}..\partial^{\beta_\ell}\phi_{\gamma_1..\gamma_s})\xi^{\mu_1..\mu_{s-1}})$$

Here C is some constant tensor, i.e. a combination of Minkowski metric tensors. The ambiguous terms in B_1 of type (5.7) correspond to terms in T_1 of the type

$\partial^{\mu_s}\,(\Xi(\phi))^{\mu_1...\mu_{s-1}}$. Such terms can be regarded as coming from gauge parameter redefinitions $\xi \to \xi + g\Xi$.

Finally the lowest order commutator can be calculated. If the action is invariant modulo terms of order g^2 for transformations of the type

$$\phi \to \phi + \delta_\xi \phi$$

it also must be invariant modulo terms of order g^2 for the commutator transformation

$$\phi \to \phi + \delta_C \phi = \phi + \left[\delta_{\xi_2}, \delta_{\xi_1}\right] \phi \tag{5.9}$$

This is a consequence of the analoguous statement for exact invariances proven in chapter III. In the present case a simplification occurs. Since $\delta_\xi \phi = \partial\xi + gT_1(\phi,\xi)$ it follows

$$\delta_C \phi = \left[\delta_{\xi_2}, \delta_{\xi_1}\right] \phi = g\left(T_1(\partial\xi_1;\xi_2) - T_1(\partial\xi_2;\xi_1) \right) + O(g^2) \tag{5.10}$$

The commutator on the field is already of order g. Therefore only the change in L_0 need to be considered:

$$\delta_C(L_0 + gL_1) = O(g^2) \quad \Leftrightarrow \quad \delta_C(L_0) = O(g^2) \tag{5.11}$$

Comparing (5.10) and (5.11) one concludes that the part proportional to g of $\delta_C \phi$ must be an exact invariance of L_0. That is

$$T_1(\partial\xi_1;\xi_2) - T_1(\partial\xi_2;\xi_1) = \partial C_0(\xi_2,\xi_1) \tag{5.12}$$

Here C_0 is an antisymmetric function of ξ_1 and ξ_2 of the same tensorial type as ξ. C_0 is the first order commutator.

As mentioned in chapter III, field redefinitions affect the form of the gauge transformation, but not of the commutator. Gauge parameter redefinitions do affect both the transformation and the commutator.

VI RESULTS

Introduction

We have performed a systematic search for first order lagrangians cubic in the fields [25]. Of course one can always construct interactions of the form

$$(6.1) \qquad L_1 = F(\phi_1)\, F(\phi_2)\, F(\phi_3)$$

where $F(\phi_i)$ denotes the field strength corresponding to the field ϕ_i. Such interactions we call trivial. Note that these interactions are invariant for the same free field transformations as the free field theory. Furthermore, L_1 must contain at least $s_1+s_2+s_3$ derivatives if all these fields are bosonic, and at least $s_1+s_2+s_3-1$ if two of the three are fermionic.

We are not interested here in these trivial interactions, but we want rather to find interactions where at least one of the fields does not enter through its field strength. Such interactions which give rise to non-abelian gauge transformations we call non-trivial.

For every combination of spins with all spins less or equal three, we have determined the first order lagrangians with the minimum number of derivatives. The following pattern emerges. Non-trivial interaction lagrangians exist in case two or all three of the fields participating carry the same spin s (except for 0-0-s). Modulo field redefinitions these interactions are unique. But when all three spins are different no non-trivial interaction lagrangians exist.

The first order interactions we found are listed in Appendix C, together with their transformation rules.

Some of the basic properties of the first order lagrangians depend in a very simple way on the values of the spins involved. For instance, the dimension of a first order lagrangian is determined by the spins, as is the symmetry in the fields which have the same spin. These properties are the same as found in reference [24], which uses a approach completely different from ours. There first order interactions are constructed in a non-manifest Lorentz covariant way, using only the physical components of the fields. Classifying first order lagrangians according to whether some of the fields enter only through their field strength, again a simple dependence on the value of the spin is found.

However when we try for $s>2$ to determine the full algebraic structure of the theory, we must conclude it is not of the form discussed in chapter IV. A way out may be the inclusion of extra interactions of even higher spin fields.

Detailed examples of results and the calculations that lead to them, will be presented in the following chapters. In chapter VII the cases 1-1-1, 2-2-2, 2-½-½ and 2-3/2-3/2 are discussed, in chapter VIII the spin-3 selfinteraction 3-3-3, and in chapter IX matter-gauge interactions of the type s-0-0.

2. Regularities in dimension and in symmetry properties

First we have to specify what we mean with the expression "dimension of an interaction lagrangian". Consider the action S

(6.2) $$S = \int d^4x \, L$$

S is a scalar. Hence the lagrangian (density) L has the dimenion of an inverse length to the fourth power. If we put

(6.3) $$\hbar = c = 1$$

the lagrangian L has the same dimension as a mass to the fourth power. This we will express simply as "the lagrangian has dimension four" or

(6.4) $$[L] = 4$$

Using the same notation we have

(6.5) $$[\text{boson field}] = 1; \quad [\text{fermion field}] = 3/2; \quad [\text{derivative}] = 1;$$
$$[\text{field strength of spin-s field}] = s+1$$

So for the first order interaction lagrangian and the coupling constant g one has

(6.6) $$[gL_1] = 4$$

Let us define D

(6.7) $$D = -[g]$$

Let us call a cubic interaction lagrangian between fields with spins s_1, s_2 and s_3 a "s_1-s_2-s_3-interaction". Two cases can be distinguished. Either L_1 is cubic in bosonic fields or L_1 is linear in a bosonic field and quadratic in fermion fields. From (6.6) we see, if L_1 involves d derivatives, that

(6.8) $$D = d-1 \quad \text{for } s_1, s_2, s_3 \text{ all integer}$$
$$D = d \quad \text{for } s_1 \text{ integer}, \; s_2, s_3 \text{ half integer}$$

In Table I we have plotted for every s_1, s_2 the smallest D for which it is possible to find a lagrangian of type s_1-s_2-s_2. In all cases considered, this lagrangian turned out to be unique modulo field redefinitions. While all lagrangians with $D<1$ have been known since the discovery of supergravity theories, only the present investigation has lead to Lorentz covariant cubic interaction lagrangians with $D=2$. Where in Table 1 no entries are given, we have not tried to construct explicit Lagrangians. From the following discussion it will be clear what we expect for those entries.

Table I

D	s_2 → 0	1/2	1	3/2	2	5/2	3
s_1 ↓ 0	-1	0	1	2	3		
1	0	0	0	1	2		
2	1	1	1	1	1	2	
3	2	2	2	2	2	2	2

Table I strongly suggests the following formula for D as a function of s_1 and s_2:

(6.9) $$D = s_1 - 1 \quad \text{if } s_2 < s_1$$
$$D = s_1 - 1 + 2(s_2 - s_1) \quad \text{if } s_2 > s_1$$

Let us allow for the possibility that multiple species of fields carry the same spin, or, equivalently, that the fields can have internal degrees of freedom. Such degrees of freedom will be labeled by latin indices. Then the symmetry properties in internal indices of first order lagrangians can be studied. In case all spins are equal, s-s-s, we write

(6.10) $$L_1 = f_{ijk}\, L_1(\phi^i,\phi^j,\phi^k)$$

and if two of the three fields carry the same spin we write

(6.11) $$L_1 = c_{ij}\, L_1(\phi(s_1),\ \phi^i(s_2),\ \phi^j(s_2))$$

For the cases investigated, the symmetry of the coefficient f_{ijk} and c_{ij} is given by

(6.12) $$\begin{array}{ll} f_{ijk} & \text{fully (anti)symmetric if } s \text{ even (odd)} \\ c_{ij} & \text{(anti)symmetric if } s_1 \text{ even (odd)} \end{array}$$

Here we emphasize we work always with real boson and Majorana fermion fields. For example if we have s=1, (6.12) states that a massless vector field cannot couple to <u>one</u> type of scalar field, but coupling to a <u>complex</u> scalar is allowed:

(6.13) $$\begin{aligned} L_1 &= \tfrac{1}{2}\{(\partial_\mu\phi^*)(-ieA_\mu\phi)+(\partial_\mu\phi)(-ieA_\mu\phi)^*\} \\ &= e(\partial_\mu\phi_1)A_\mu\phi_2 - e(\partial_\mu\phi_2)A_\mu\phi_1 \qquad \text{if } \phi = \phi_1+i\phi_2 \end{aligned}$$

In general we may conclude that even spin is associated with symmetry in internal indices, while odd spin is associated with antisymmetry.

3. Classification

Some of the main properties of the first order interaction lagrangians do not depend on their detailed structure. In this respect we can distinguish three types and some subtypes of cubic first order interactions.

The classification of the interactions we found is shown in <u>Table II</u>

Table II Classification of first order lagrangians (see text below for explanation)

type	s_2 → 0	1/2	1	3/2	2	5/2	3
s_1 ↓ 0			I^A	I^A	I^A		
1	I^B	I^B	II	III^B	III^B		
2	I^B	I^B	III^A	III^C	II	III^B	
3	I^B	I^B	III^A	III^A	III^C	III^C	II

First we discuss the interactions between gauge and matter fields. Here matter field refers to spin-0 and spin-½ fields, which do not transform under a free gauge transformation. When we present in the following the structure of the lagrangians we do not show explicitly the derivatives working on the fields.

IA In case the lagrangian is linear in the matter field (so it has to be a scalar field) and quadratic in the gauge field, we only find the trivial possibility

(6.14) $$L_1^{IA} = \phi\, F(A(s))\, F(A(s))$$

where $F(A(s))$ denotes the field strength of the spin-s field, given in chapter II (2.17). These interactions involve 2s or 2s-1 derivatives if s is integer or half integer respectively. There are no first order corrections to the free field transformation.

IB In case the lagrangian is quadratic in the matter field and linear in the gauge field, things are more interesting. Now the lagrangian is given by

(6.15) $$L_1^{IB} = c_{ij}\, A(s)\, \phi^i\, \phi^j$$

where ϕ denotes the spin-0 or spin-½ field and A(s) the spin-s gauge field. Although s derivatives are involved if ϕ is a scalar, A(s) does not enter through its field strength (for "fake interactions" it does). To first order, the transformations which leave L_0+gL_1 invariant are for scalar ϕ

(6.16) $$\delta A_{\mu_1\dots\mu_s} = \partial_{\mu_1}\xi_{\mu_2\dots\mu_s}, \quad \delta\phi^i = c^{ij}\xi_{\mu_1\dots\mu_{s-1}}\partial^{\mu_1}\dots\partial^{\mu_{s-1}}\phi^j$$

For further details we refer to chapter IX.

II The next category is formed by interactions between fields which all carry the same spin.

(6.17) $$L_1^{II} = f^{ijk}\, A^i(s)\, A^j(s)\, A^k(s)$$

Clearly s must be an integer. As mentioned before L_1 involves s derivatives and f_{ijk} is fully (anti)symmetric if s is even(odd). The first order gauge transformations and first order commutator have the following structure

(6.18) $$\delta A_{\mu 1 \cdots \mu s} = \partial_{\mu 1}\, \xi_{\mu 2 \cdots \mu s} + g\, T_1(A,\xi)$$

$$C_0^{(s-1)}(\xi_2(s-1),\xi_1(s-1)) \neq 0$$

The (s-1) denotes the number of indices of C_0 and ξ. Spin 1-1-1 and 2-2-2 lagrangians will be dealt with in chapter VII, the spin 3-3-3 interaction in chapter VIII.

The last main group consists of interactions between gauge fields with different spins. Here we have three subtypes.

IIIA These lagrangians have the structure

(6.19) $$L_1^{III} = A(s_1)\, F(A(s_2))\, F(A(s_2))$$

The examples we found are 2-1-1, 3-1-1 and 3-3/2-3/2. Probably for all combinations s_1-s_2-s_2 with $s_2 < \frac{1}{2} s_1$ the first order lagrangians have this structure. First order terms do affect the gauge transformations

(6.20) $$\delta A_{\mu 1 \cdots \mu s_1} = \partial_{\mu 1}\, \xi_{\mu 2 \cdots \mu s_1}$$

$$\delta A_{\mu 1 \cdots \mu s_2} = \partial_{\mu 1}\, \xi_{\mu 2 \cdots \mu s_2} + g\, T_1(F(A(s_2)),\xi(s_1-1))$$

But these first order transformations do not lead to a non vanishing lowest order commutator.

IIIB For 1-3/2-3/2, 1-2-2 and 2-5/2-5/2 we have found first order lagrangians with a structure like

(6.21) $$L_1^{IIIB} = F(A(s_1))\, A(s_2)\, A(s_2)$$

We expect such structures for all $s_2>s_1$. Not only the first order terms in the gauge transformations are non zero

(6.22) $$\delta A_{\mu_1\cdots\mu_{s_1}} = \partial_{\mu_1}\xi_{\mu_2\cdots\mu_{s_1}} + g\, T_1(A(s_2),\xi(s_2-1))$$

$$\delta A_{\mu_1\cdots\mu_{s_2}} = \partial_{\mu_1}\xi_{\mu_2\cdots\mu_{s_2}} + g\, T_1(F(A(s_1)),\xi(s_2-1))$$

But also the first order commutator of two $\xi(s_2-1)$ transformations is non-trivial, and gives a lowest order spin-s_1 transformation. This can be seen from consideration of the commutator on $A(s_1)$

(6.23) $$C_0^{(s_1-1)}(\xi_1(s_2-1),\xi_2(s_2-1)) \neq 0$$

<u>IIIC</u> Finally we come to the type of interaction lagrangians with the most intricate structure. These lagrangians have a structure like

(6.24) $$L_1^{IIIC} = A(s_1)\, A(s_2)\, A(s_2)$$

and is realized if $\frac{1}{2}s_1<s_2<s_1$. We have obtained such interactions for 2-3/2-3/2, 3-2-2 and 3-5/2-5/2. Here the first order gauge transformations look like

(6.25) $$\delta A_{\mu_1\cdots\mu_{s_1}} = \partial_{\mu_1}\xi_{\mu_2\cdots\mu_{s_1}} + g\, T_1(A(s_2),\xi(s_2-1))$$

$$\delta A_{\mu_1\cdots\mu_{s_2}} = \partial_{\mu_1}\xi_{\mu_2\cdots\mu_{s_2}} + g\, T_1(A(s_1),\xi(s_2-1)) + g\, T_1(A(s_2),\xi(s_1-1))$$

In general (6.25) leads to two non trivial first order commutators:

(6.26) $$C_0^{(s_1-1)}(\xi_1(s_2-1),\xi_2(s_2-1)) \neq 0$$

$$C_0^{(s_2-1)}(\xi_1(s_1-1),\xi_2(s_2-1)) \neq 0$$

Concluding, each type of interaction lagrangian belongs to a specific region in Table 2. Whether the first order commutator vanishes or not follows from the simple rules

(6.27) $$C_0^{(s_1-1)}(\xi_1(s_2-1),\xi_2(s_2-1)) \neq 0 \quad \text{if} \quad s_2 > \tfrac{1}{2}s_1$$

$$C_0^{(s_1-1)}(\xi_1(s_1-1),\xi_2(s_2-1)) \neq 0 \quad \text{if} \quad \tfrac{1}{2}s_1 < s_2 < s_1$$

4. Algebraic structure

We have indicated that for any combination of spin s_1-s_2-s_2 an non-trivial unique first order interaction can be found. Also the first order gauge transformations and commutators were obtained. The next step in our search for interacting higher spin theories is to guess from this first order information the full algebraic structure of the theory.

Here we ran into trouble for all interactions involving higher spins. The basic problem is in each case the same. If we study the commutator on the field of two first order transformations

(6.28) $$[\delta_{\xi_2},\delta_{\xi_1}]\,\phi = g\,T_1(\partial\xi_1;\xi_2) - g\,T_1(\partial\xi_2,\xi_1)$$
$$+\ g^2\,T_1(T_1(\phi;\xi_1);\xi_2) - g^2\,T_1(T_1(\phi;\xi_2);\xi_1)$$

it appears for all $s>2$ that the order g^2 contribution contains terms with more space-time derivatives acting on ϕ than in the original transformation. These terms are not of the type that can be cancelled by higher order terms in the gauge transformation or commutator. A way out may be the inclusion of an extra higher spin interaction to account for these terms with too many derivatives, leading to an infinite tower of higher spin interactions. We will come back to these problems later when discussing 3-3-3 interactions in chapter VIII and s-0-0 interactions in chapter IX.

VII REOBTAINING STANDARD THEORIES INVOLVING MASSLESS SPIN-1 AND SPIN-2 FIELDS

1. Introduction

The purpose of this chapter is to illustrate the method for constructing theories of interacting massless fields as presented in chapter V. We will start with the simplest example [25], that of interacting massless spin-1 fields, leading to the Yang-Mills theory. Next gravitational selfinteractions will be discussed [22, 23, 25]. This is followed by determining the coupling of spin-½ fields to gravity, to make clear both how we incorporate matter-gauge interactions in our scheme and how some subtleties turn up when fermions are involved. Finally we study the possibility of an interaction between massless spin-2 and spin-3/2 fields. We show the type of algebraic structure to be different from that of the Yang-Mills theory and the theories of gravitation, because the supergravity 2-3/2-3/2 interaction leads to field dependent commutators.

2. Interacting massless spin-1 fields

Right from the beginning the possibility of more than one species of spin-1 field will be included. Our starting point is the free lagrangian *)

$$(7.1) \qquad L_0 = -\tfrac{1}{2} A^i_{\ \mu,\nu} A^i_{\ \mu,\nu} + \tfrac{1}{2} A^i_{\ \mu,\nu} A^i_{\ \nu,\mu}$$

where a summation over the internal index i is implied from i=1 to some integer N.

Next we search for a suitable first order interaction, using the algorithm sketched in section V.2. An odd number of derivatives is needed, since the interaction lagragian has to be a Lorentz scalar. The simplest possibility is one derivative. Modulo total divergencies there is only one candidate

$$(7.2) \qquad L_1 = f^{abc} A^a_{\ \mu,\nu} A^b_{\ \mu} A^c_{\ \nu}$$

Next we have to check whether this L_1 can satisfy the constraint (see (5.5))

*) Here and in the following we denote space-time derivatives by means of comma's. Moreover, for typographical reasons upper and lower indices are not always properly matched.

(7.3) $$\partial^\mu L_{1,A_\mu} = 0 \quad \text{for } A_\mu \text{ which satify } A_{\mu,\nu\nu} - A_{\nu,\nu\mu} = 0$$

Direct calculation shows this is possible provided

(7.4) $$f^{abc} = -f^{acb} = -f^{bac}$$

At least three species of spin-1 fields are needed.

Having succeeded in obtaining a suitable first order interaction, the first order transformation which leaves L_0+gL_1 invariant can be determined, as explained in section V.3. The first order transformation reads

(7.5) $$A^a_\mu \rightarrow A^a_\mu + \partial_\mu \Lambda^a + g f^{abc} A^b_\mu \Lambda^c$$

leading to a first order commutator

(7.6) $$[\delta_{\Lambda_2}, \delta_{\Lambda_1}] A_\mu = \delta_{gC_0(\Lambda_2,\Lambda_1)} A_\mu + O(g^2)$$

$$C_0(\Lambda_2,\Lambda_1)^a = f^{abc} \Lambda_1{}^b \Lambda_2{}^c$$

It is not difficult to find a full algebraic structure which could underlie the gauge transformations. For if the f^{ijk} satisfy the Jacobi identity

(7.7) $$f^{abc} f^{cde} + f^{aec} f^{cbd} + f^{adc} f^{ceb} = 0$$

then C_0 satisfies the Jacobi identity as well, and the first order transformations form a realisation of this commutator C_0. Thus we have a bracket structure. Comparing the above results to the Yang-Mills theory as described in sections III.4 and IV.4 we conclude that, applying our formalism to find an extended theory of massless spin-1 fields, we are led directly to the Yang-Mills theory. The full theory can now be obtained by searching for a lagrangian which is invariant for (7.5).

3. Selfinteracting massless spin-2 fields

Let us study the possibility of selfinteractions for massless spin-2 fields. The free lagrangian is given by

(7.8) $$L_0 = \tfrac{1}{2} h_{\mu\nu,\rho} h_{\mu\nu,\rho} - h_{\mu\nu,\rho} h_{\mu\rho,\nu} + h_{\mu\nu,\mu} h_{\rho\rho,\nu} - \tfrac{1}{2} h_{\rho\rho,\mu} h_{\sigma\sigma,\mu}$$

The free field equation satisfies the source constraint

$$\partial^{\mu} L_{0,h_{\mu\nu}} = 0 \quad \text{for all } h_{\mu\nu}(x) \tag{7.9}$$

Cubic interaction lagrangians must have an even number of derivatives. The simplest possibility would be to have no derivatives at all, but no such lagrangian can satisfy the requirement

$$\partial^{\mu}(L_{1,h_{\mu\nu}}) = 0 \quad \text{for all } h_{\mu\nu} \text{ which satisfy} \tag{7.10}$$

$$L_{0,h_{\mu\nu}} = 0 \quad \Leftrightarrow \quad h_{\mu\nu,\rho\rho} = h_{\mu\rho,\rho\nu} + h_{\nu\rho,\rho\mu} - h_{\rho\rho,\mu\nu} \tag{7.11}$$

The next simplest possibility is to look for interactions with two derivatives. Modulo total divergencies 15 Lorentz scalars can be constructed, e.g. $L_1{}^1 = h_{,\mu} h_{,\mu} h$ $\quad L_1{}^2 = h_{\mu\nu,\rho} h_{\mu\sigma,\rho} h_{\sigma\nu}$ etc, where $h = h_{\lambda\lambda}$.

On calculating the 15 quantities

$$\partial^{\mu} \left(L_1{}^{i}{}_{,h_{\mu\nu}}\right) \qquad i = 1, \ldots 15$$

one concludes that 5 linear combinations can be made to satisfy the requirement (7.10). Four of these are "fake interactions", related to the four field redefinitions

$$h_{\mu\nu} \to h_{\mu\nu} + c_1 h_{\mu\rho} h_{\rho\nu} + c_2 h_{\rho\rho} h_{\mu\nu} + c_3 \eta_{\mu\nu} h_{\rho\sigma} h_{\rho\sigma} + c_4 \eta_{\mu\nu} h_{\rho\rho} h_{\sigma\sigma} \tag{7.12}$$

Modulo these field redefinitions we are left with precisely one possibility. One of the equivalent forms is given in appendix C. It coincides with the first order term in the series expansion (3.46) of the Einstein-Hilbert lagrangian (3.40) for pure gravitation.

The next step is to obtain the first order gauge transformation and the first order commutator. One finds for the gauge transformation

$$\begin{aligned} \delta h_{\mu\nu} = {} & \xi_{\mu,\nu} + \xi_{\nu,\mu} + \kappa(h_{\mu\nu,\rho}\xi_{\rho} + \xi_{\rho,\mu}h_{\rho\nu} + \xi_{\rho,\nu}h_{\rho\mu}) \\ & + \kappa(d_1 h_{\mu\rho}\xi_{\rho} + d_2 h_{\rho\rho}\xi_{\mu})_{,\nu} + \kappa(d_1 h_{\nu\rho}\xi_{\rho} + d_2 h_{\rho\rho}\xi_{\nu})_{,\mu} \\ & + \kappa c_1(h_{\mu\rho}\xi_{\rho,\nu} + h_{\mu\rho}\xi_{\nu,\rho} + h_{\nu\rho}\xi_{\rho,\mu} + h_{\nu\rho}\xi_{\mu,\rho}) + \kappa c_2(h_{\rho\rho}\xi_{\mu,\nu} + h_{\rho\rho}\xi_{\nu,\mu} + 2h_{\mu\nu}\xi_{\rho,\rho}) \\ & + \kappa c_3 \eta_{\mu\nu} h_{\rho\sigma}\xi_{\rho,\sigma} + \kappa c_4 \eta_{\mu\nu} h_{\rho\rho}\xi_{\sigma,\sigma} \end{aligned} \tag{7.13}$$

Here the parameters c_1, c_2, c_3, c_4 arise from field redefinitions and d_1 and d_2 arise from the possibility to redefine the gauge parameter ξ as

(7.14) $$\xi_\mu \rightarrow \xi_\mu + \kappa d_1 h_{\mu\rho} \xi_\rho + \kappa d_2 h_{\rho\rho} \xi_\mu$$

To each form of the first order lagrangian belonging to the equivalence class of solutions corresponds a particular choice of the parameters c_i. For example, for the form given in appendix C they are all zero. The first order commutator does not depend on the parameters c_i:

(7.15) $$[\delta_{\xi_2}, \delta_{\xi_1}]\, h_{\mu\nu} = \delta_{\kappa C_0(\xi_2,\xi_1)}\, h_{\mu\nu} + O(\kappa^2)$$

$$C_0(\xi_2,\xi_1)^\mu = [\xi_2{}^{\mu,\rho}\xi_1{}^\rho + d_1(\xi_1{}^{\mu,\rho}\xi_2^\rho + \xi_1{}^{\rho,\mu}\xi_2{}^\rho) + 2d_2\xi_1{}^{\rho,\rho}\xi_2{}^\mu] - [1 - 2]$$

Next we search for an algebraic structure. Let us first try whether a bracket structure can be found which agrees with the above expressions for the gauge transformation and commutator. The requirement that C_0 has to satisfy the Jacobi identity forces $d_1 = d_2 = 0$. The requirement that the first order term of the gauge transformation forms a representation of this commutator C_0 determines the coefficients c_i. So one finds

(7.16) $$d_1 = d_2 = 0; \quad c_2 = c_3 = c_4 = 0; \quad c_1 = -1 \text{ or } c_1 = 0$$

Because the coefficients d_i are completely fixed, we are unambiguously led to the commutator of general coordinate transformations

(7.17) $$[\xi_1, \xi_2]^\mu = \xi_2{}^{\mu,\rho}\xi_1{}^\rho - \xi_1{}^{\mu,\rho}\xi_2{}^\rho$$

The freedom in the coefficient c_1 stems from the fact that we can still choose at this point whether $h_{\mu\nu}$ is covariant or contravariant under general coordinate transformations.

Having established the inevitable link between a theory of interacting massless spin-2 fields and general coordinate invariance, the full Einstein-Hilbert lagrangian may be found from the latter invariance.

The reader may wonder: why doesn't the cosmological constant turn up? The answer is that this phenomenon lies outside the scope of our formulation, since the bilinear part of the lagrangian with a cosmological constant is not of the form of the free massless field lagrangians considered in chapter II. To include a cosmological constant, the lowest order lagrangian should be modified with a linear part as follows

$$(7.18) \qquad L_0 \rightarrow L_0' = L_0 + \frac{\Lambda}{\kappa} h_{\mu\mu}$$

Note we have an extra independent coupling constant. The extra piece satisfies the lowest order source constraint (7.9) as well, since a constant has zero divergence.

4. Spin-½ fields coupled to gravity

In this section the possibility for interactions between massless spin-2 and spin-½ fields will be investigated. An unique first order interacion will be obtained, and we will show different ways to arrive at the full algebraic structure of the theory.

Of course it is well known that spin-½ fields can have gravitational interactions, so we know a first order interaction has to exist. On the other hand, the usual form of this interaction involves vierbeins rather than the metric, as explained in chapters III and IV. As we will show here, it is possible to incorporate the extra (unphysical) degrees of freedom of the vierbein in the theory. An alternative, equally valid from the point of view of extended massless theories, is not to switch to a vierbein but allow for spin-½ fields as metric dependent realisations of the general coordinate transformations. A similar conclusion was recently obtained by Woodard [23].

As usual we start with a free lagrangian

$$(7.19) \qquad L_0 = L_0(\text{spin-}2) + L_0(\text{spin-}\tfrac{1}{2})$$

which is the sum of the free spin-2 lagrangian (7.8) and the free lagrangian of a massless Majorana spinor

$$(7.20) \qquad L_0(\text{spin-}\tfrac{1}{2}) = \frac{1}{2} \bar{\psi} \not{\partial} \psi$$

For convenience we have taken the spin-½ fields massless as well, but this is not essential and all results of this section carry over to the massive case. Again the free lagrangian satisfies the free source constraint (7.9). No analogue exists for the l.h.s of the spin-½ equation of motion.

Next we have to specify the form of the first order lagrangian we will search for. The simplest possibility involves one derivative. Only two independent lagrangians have this form

$$(7.21)\qquad L_1{}^1 = i\, h_{\mu\nu}\, \bar\psi\, \gamma_\mu\, \psi_{,\nu} \qquad\qquad L_1{}^2 = i\, h_{\rho\rho}\, \bar\psi\, \gamma_\sigma\, \psi_{,\sigma}$$

In order that $L_0+\kappa L_1$ has the correct dimension, κ must have dimension -1, like in the case of gravitational selfinteraction. A suitable first order lagrangian must satisfy the criterion

$$(7.22)\qquad \partial^\mu (L_{1,h_{\mu\nu}}) = 0 \qquad\qquad \text{for fields which satisfy}$$

$$(7.23)\qquad \not\partial\psi = 0$$

Both lagrangians $L_1{}^1$ and $L_1{}^2$ satisfy the above criterion. However the second one is related to a field redefinition

$$(7.24)\qquad L_0 \to L_0 + \kappa\, L_1{}^2 + O(\kappa^2) \qquad \text{if } \psi \to \psi + \kappa h_{\rho\rho}\psi$$

So we are left with one possibility and the first order interaction between massless spin-2 and spin-½ fields is unique.

The most general gauge transformation corresponding to this first order interaction is

$$(7.25)\qquad \delta\psi = \kappa\xi_\nu \psi_{,\nu} + \tfrac{1}{2}\kappa\xi_{\nu,\rho}\sigma_{\rho\nu}\psi + \kappa x_1 \xi_{\nu,\nu}\psi + \kappa x_2 \not\xi \not\partial\psi$$

where x_1 and x_2 are arbitrary constants. The possibility to redefine the field gives rise to the x_1 term, while the last one vanishes on shell in the sense discussed in section IV.1.

Formally the first order commutator of two successive transformations can be anything, since only terms of order κ^2 are present:

$$(7.26)\qquad [\delta_{\xi_2}, \delta_{\xi_1}]\,\psi = O(\kappa^2) = \delta_{\kappa C_0(\xi_2,\xi_1)}\psi + O(\kappa^2)$$

So no clue to the algebraic structure of the gauge transformations is obtained.

However, it is reasonable to insist on a common algebraic structure for both spin 2-2-2 and spin 2-½-½ interactions. So we require

(7.27) $$[\delta_{\xi_2},\delta_{\xi_1}]\,\phi = \delta_{\kappa[\xi_2,\xi_1]}\,\phi \quad \text{with} \quad [\xi_2,\xi_1]^\mu = \xi_1{}^{\mu,\rho}\xi_2{}^\rho - \xi_2{}^{\mu,\rho}\xi_1{}^\rho$$

Let us therefore calculate the commutator on ϕ using the transformation rule (7.25). We find

(7.28) $$[\delta_{\xi_2},\delta_{\xi_1}]\,\phi = \delta_{\kappa[\xi_2,\xi_1]}\phi$$
$$+ \kappa^2 \;(\xi_2{}^{\mu,\rho}\xi_1{}^{\lambda,\mu}+\xi_2{}^{\mu,\rho}\xi_1{}^{\mu,\lambda}+\xi_2{}^{\rho,\mu}\xi_1{}^{\lambda,\mu}+\xi_2{}^{\rho,\mu}\xi_1{}^{\mu,\lambda})\;\tfrac{1}{4}\,\sigma^{\lambda\rho}\,\phi$$
$$+ \text{ terms proportional to } \; x_1 \text{ and } x_2$$

This expression does not agree with (7.27). A piece, proportional to $\sigma^{\rho\lambda}\phi$ appears in the commutator. This situation persists for all possible choices of x_1 and x_2.

Basically there are two options to save the situation. The first, and most natural in the scheme used so far, is to see whether higher order terms in the transformation can account for a correct behaviour under commutation of two transformations. The second option, suggested by our knowledge that natural descriptions of gravitationally interacting spin-½ fields use a vierbein rather than a metric formulation, is to introduce a second gauge parameter in the transformation of ϕ.

In the first option one adds a second order term to (7.25)

(7.29) $$\tfrac{1}{4}\,\kappa^2\,(\xi_{\mu,\rho} + \xi_{\rho,\mu})\,h_{\mu\lambda}\,\sigma_{\rho\lambda}\,\phi$$

This second order term in the gauge tansformation gives rise to a contribution to the lowest order commutator because of its dependence on $h_{\mu\nu}$. This contribution just cancels the undesired term in (7.28). In this way one can go on searching for a $h_{\mu\nu}$-dependent realisation of the general coordinate transformations on ϕ. Ultimately one finds

(7.30) $$\phi \rightarrow \phi + \kappa\,\xi_\nu\,\phi_{,\nu} + \tfrac{1}{2}\kappa\,\Omega(h)_{\mu\nu}\sigma_{\mu\nu}\phi$$

where

(7.31) $$\Omega_{\mu\nu}(h) = \tfrac{1}{2}\,\xi_{\nu,\mu} - \tfrac{1}{2}\,\xi_{\mu,\nu} +$$

$$\tfrac{1}{2}\sum_{n=1}^{\infty}\left(\frac{-\kappa}{4}\right)^n \sum_{k=0}^{n}\left(-1+\frac{2k}{n}\right)\binom{n}{k} w_{\mu\mu_1}\cdots w_{\mu_{k-1}\mu_k}(\xi_{\mu_k,\mu_{k+1}}+\xi_{\mu_{k+1},\mu_k})w_{\mu_{k+1}\mu_{k+2}}\cdots w_{\mu_{n-1}\mu_n}$$

Here $w_{\mu\nu}(h)$ is uniquely determined as follows:

(7.32)
(a) $(\eta_{\mu\lambda}+\tfrac{1}{2}\kappa w_{\mu\lambda})(\eta_{\lambda\nu}+\tfrac{1}{2}\kappa w_{\lambda\nu})=(\eta_{\mu\nu}+\kappa h_{\mu\nu})$
(b) $w_{\mu\nu} = w_{\nu\mu}$

Note that $\eta_{\mu\nu}+\tfrac{1}{2}\kappa b_{\mu\nu}$ has signature (+−−−), like $\eta_{\mu\nu}+\kappa h_{\mu\nu}$.

Actually, the solution (7.31) was suggested by the second option, as we will indicate at the end of this section.

The second approach anticipates the occurrence of a vierbein field. We view (7.28) as the expression of the fact that there is another type of invariance, which we have overlooked so far. This extra invariance becomes manifest because the commutator of two general coordinate transformations not only gives another GCT transformation but a transformation of another type as well. So we suppose the first order gauge transformation of ψ must be extended to

(7.33) $$\delta\psi = \xi_\nu\,\psi_{,\nu} + \tfrac{1}{2}\,\xi_{\nu,\lambda}\,\sigma_{\lambda\nu}\,\psi + c\varepsilon_{\alpha\beta}\,\sigma_{\alpha\beta}\,\psi$$

where $\varepsilon^{\alpha\beta}$ is an antisymmetric tensor field and c an arbitrary constant. We recognize the new transformations to be the local Lorentz transformations.

On changing the above transformation rule, we are forced to make many additional changes as well, in order to keep the description consistent. We show now how to achieve this.

The $\varepsilon_{\alpha\beta}$ dependent piece in the transformation rule of ψ gives a contribution to the $\varepsilon_{\alpha\beta}$ dependent piece of the variation of L_0. A new interaction lagrangian is needed to cancel this contribution.

Instead of a symmetric field $h_{\mu\nu}$, consider a non-symmetric field $b_{\mu\nu}$ with $(\eta_{\mu\lambda}+\tfrac{1}{2}\kappa b_{\mu\lambda})(\eta_{\lambda\nu}+\tfrac{1}{2}\kappa b_{\lambda\nu})=(\eta_{\mu\nu}+\kappa h_{\mu\nu})$ and not necessarily $b_{\mu\nu}=b_{\nu\mu}$. Let the lowest order gauge transformation be

(7.34) $$\delta b_{\mu\nu} = \xi_{\mu,\nu} + \xi_{\nu,\mu} + \varepsilon_{\mu\nu}$$

The antisymmetric part of $b_{\mu\nu}$ can be gauged away completely. No free lagrangian corresponds to the antisymmetric part. For a first order interaction between ψ and $h_{\mu\nu}$ both

$$\text{(7.35)} \qquad \partial^{\mu}(L_{1,b_{\mu\nu}} + L_{1,b_{\nu\mu}}) = 0$$

$$L_{1,b_{\mu\nu}} - L_{1,b_{\nu\mu}} = 0 \qquad \text{for } \psi \text{ with } \partial\!\!\!/\psi = 0$$

are required. Now we have four such first order interactions and two field redefinitions. Modulo these field redefinitions there is again precisely one first order interaction lagrangian which satisfies (7.35). One of the equivalent forms is given by

$$\text{(7.36)} \qquad L_1 = i\left(\tfrac{1}{4}h_{\rho\rho}\,\bar{\psi}\,\partial\!\!\!/\,\psi - \tfrac{1}{2}h_{\mu\nu}\,\bar{\psi}\,\gamma_{\mu}\,\psi_{,\nu} + \tfrac{1}{4}b_{\mu\nu}\,\bar{\psi}\,\sigma_{\mu\nu}\,\partial\!\!\!/\,\psi\right)$$

The corresponding first order transformation reads indeed

$$\delta\psi = \xi_{\nu}\psi_{,\nu} + \tfrac{1}{4}\,\xi_{\nu,\lambda}\,\sigma_{\lambda\nu}\psi + \tfrac{1}{4}\,\varepsilon_{\nu\lambda}\,\sigma_{\nu\lambda}\psi$$

To simplify somewhat, we redefine the antisymmetric gauge parameter, setting

$$\text{(7.37)} \qquad \Omega^{\mu\nu} = \tfrac{1}{2}\left(\varepsilon^{\mu\nu} - \xi^{\mu,\nu} + \xi^{\nu,\mu}\right)$$

Then we find for the transformation rule of ψ

$$\text{(7.38)} \qquad \delta\psi = \xi_{\nu}\,\psi_{,\nu} + \tfrac{1}{2}\,\Omega_{\mu\nu}\,\sigma_{\mu\nu}\psi$$

This is just the transformation rule we were after. All commutators on ψ are related to commutators on parameters:

$$\text{(7.39)} \qquad [\delta^{P_2},\delta^{P_1}]\,\psi = \delta^{[P_2,P_1]} \quad \text{with}$$

$$[\xi_1,\xi_2]^{\nu} = \xi_1{}^{\rho}\xi_2{}^{\nu,\rho} - \xi_2{}^{\rho}\xi_1{}^{\nu,\rho}$$

$$[\Omega_1,\Omega_2] = \Omega_1{}^{\mu\alpha}\Omega_2{}^{\alpha\nu} - \Omega_1{}^{\nu\alpha}\Omega_2{}^{\alpha\mu}$$

$$[\xi,\Omega]^{\mu\nu} = \xi_{\lambda}\Omega^{\mu\nu,\lambda} \qquad\qquad [\Omega,\xi]^{\mu\nu} = -\,[\xi,\Omega]^{\mu\nu}$$

and these commutators satisfy the appropriate Jacobi identities. What remains to do is to find a transformation rule for $b_{\mu\nu}$. In fact we have already a representation for $h_{\mu\nu}$, the transformation rule of the previous section where the Ω-dependence can be regarded to be represented trivially. By means of field redefinitions other representations for $h_{\mu\nu}$ can be obtained. As for the antisymmetric part of $b_{\mu\nu}$, every first order term is admissible in the first order transformation since they all can be regarded as gauge parameter redefinitions. After some puzzling a suitable transformation law for $b_{\mu\nu}$ can be obtained.

(7.40) $$\delta b_{\mu\nu} = 2\xi_{\mu,\nu} + 2\Omega_{\mu\nu} + \kappa(b_{\mu\nu,\lambda}\xi^{\lambda} + b_{\mu\lambda}\xi^{\lambda}{}_{,\nu} + \Omega_{\mu\lambda}b_{\lambda\nu})$$

With the above transformation law $b_{\mu\nu}$ forms an exact representation of the algebra (7.39). The connection with the vierbein formalism is made by means of the identification

(7.41) $$e_{a\mu} = \eta_{a\mu} + \tfrac{1}{2}\kappa\, b_{a\mu}$$

From the vierbein option one can go over to the no vierbein case as follows. Instead of letting Ω be an independent gauge parameter we choose it to be dependent on ξ and h in such a way that a symmetric b remains symmetric if a transformation (7.40) is applied. In other words we fix the Ω-gauge by requiring b to be symmetric. Then b is a function of h and the resulting transformation law for ψ is modulo field redefinitions equivalent to (7.30).

5. Supergravity

Supergravity theories are all centered around the interaction between massless spin-3/2 fields and gravitation. The possibility of this interaction was found only when the the theories for free massless spin-3/2 and spin-2 fields were known for quite a time [12,13]. In this section we will show that in the extended massless field theory formalism a straightforward calculation gives a first order interaction. Disentangling the algebraic structure from the first order theory is much harder, however. As we will show, one of the commutators has to become field dependent. Hence the commutators are not longer directly determined by the first order interaction, as they are for the Yang-Mills theory and gravitation.

The free lagrangian which is our starting point is in this case:

$$(7.42) \qquad L_0 = L_0(\text{spin-}2) + L_0(\text{spin-}3/2)$$

The lagrangian for free massless spin-2 fields is given by (7.8), while the Rarita-Schwinger lagrangian of free massless spin-3/2 fields is given by

$$(7.43) \qquad L_0(\text{spin-}3/2) = \tfrac{1}{2}(-\bar{\psi}_\nu \partial\!\!\!/ \psi_\nu + \bar{\psi}_\nu \partial_\nu \psi\!\!\!/ + \bar{\psi\!\!\!/} \partial_\nu \psi_\nu - \bar{\psi\!\!\!/} \partial\!\!\!/ \psi\!\!\!/)$$

ψ_μ is a Majorana vector-spinor. The equations of motion of the free lagrangian satisfy

$$(7.44) \qquad \partial^\mu (L_{0,h_{\mu\nu}}) = 0 \qquad \text{for all } h_{\mu\nu}(x)$$

$$\partial^\mu (L_{0,\bar{\psi}_\mu}) = 0 \qquad \text{for all } \psi_\mu(x)$$

The simplest cubic interactions we can search for have the generic form

$$(7.45) \qquad L_1 = h_{..}\, \bar{\psi}_{.}\, \gamma_{.}\, \psi_{.,.}$$

Modulo total divergencies, 13 lagrangians of the above type exist. To ensure gauge invariance, constraints are imposed as follows

$$(7.46) \qquad \partial^\mu (L_{1,h_{\mu\nu}}) = 0 \qquad \text{for all } \psi_\mu \text{ which satisfy } L_{0,\bar{\psi}_\mu} = 0$$

$$\partial^\mu (L_{1,\bar{\psi}_\mu}) = 0 \qquad \text{for all } \psi_\mu,\ h_{\mu\nu} \text{ which satisfy}$$

$$L_{0,\bar{\psi}_\mu} = 0 \text{ and } L_{0,h_{\mu\nu}} = 0$$

Six linear combinations of first order lagrangians satisfy the above constraints. However, five of these are related to field redefinitions and thus are mere "fake interactions". So again there is an unique first order interaction. One of its equivalent forms can be found in appendix C.

Having obtained a first order interaction lagrangian, the corresponding first order gauge transformation can be determined. One finds

$$(7.47) \qquad \delta h_{\rho\sigma} = \xi_{\rho,\sigma} + \xi_{\sigma,\rho} - i \frac{\kappa}{8} (\bar{\varepsilon}\gamma_\rho \psi_\sigma + \bar{\varepsilon}\gamma_\sigma \psi_\rho)$$

$$\delta\psi_\mu = \varepsilon_{,\mu} + \tfrac{1}{2}\kappa\, \sigma_{\rho\lambda} h_{\mu\rho,\lambda}\varepsilon + \kappa(\psi_\nu \xi_{\nu,\mu} + \psi_{\mu,\nu}\xi_\nu - \tfrac{1}{2}\sigma_{\nu\lambda}\psi_\mu \xi_{\nu,\lambda})$$

Redefinitions of fields and gauge parameters permit us to cast these transformation rules in different forms.

It appears that two of the lowest order commutators are non-trivial. First, the commutator of a GCT (spin-2) and a super (spin-3/2) transformation gives a first order contribution to the variation of ψ_μ. Therefore it must have the form of a lowest order super transformation. Indeed we find

$$(7.48) \qquad [\delta_\xi, \delta_\varepsilon]\, \psi_\mu = \partial_\mu [\xi,\varepsilon] + O(\kappa^2) \quad \text{with}$$

$$[\xi,\varepsilon] = \kappa\, \xi_\nu \varepsilon_{,\nu} + \tfrac{1}{2}\kappa\, \xi_{\rho,\lambda}\, \sigma_{\lambda\rho}\varepsilon$$

Note this commutator is the same as a GCT transformation on a spin-$\frac{1}{2}$ field, as found in the previous section.

The other commutator which gives rise to a first order variation of one of the fields is of two supertransformations on $h_{\mu\nu}$:

$$(7.49) \qquad [\delta_{\varepsilon_2}, \delta_{\varepsilon_1}]\, h_{\mu\nu} = \partial_\nu [\varepsilon_2,\varepsilon_1]_\mu + \partial_\mu [\varepsilon_2,\varepsilon_1]_\nu + O(\kappa^2)$$

$$\text{with} \quad [\varepsilon_2,\varepsilon_1]_\mu = -\frac{i}{8}\, \bar{\varepsilon}_2 \gamma_\mu \varepsilon_1$$

This is the famous commutator of supergravity: the commutator of two supertransformations is a general coordinate transformation (GCT). Note we have not an anticommutator here but a commutator, because ε_1 and ε_2 are the parameters of the gauge transformation, not the generators. All other commutators, not mentioned above, give only second order contributions to the variations of the fields. Because they could be influenced by second order gauge transformations as well, which are not yet determined, we are not able to calculate these commutators without making additional assumptions as to the nature of these second order contributions.

It is much harder to modify the gauge transformations in such a way that they form a Lie algebra than in the case of gravitational interactions alone.

First we meet the same problem as in the previous section: the commutator on the ψ_μ field of two GCT transformations contains terms proportional to $\sigma^{\rho\lambda}\psi_\mu$ and for the same reason the Jacobi identity between ξ_1, ξ_2 and ε is not satisfied. One remedy is to make the commutator $[\varepsilon,\xi]$ field dependent, and to include higher order terms in the transformation law for ψ_μ. The other possibility is to introduce a "pure gauge" antisymmetric field $v_{\mu\nu}$ together with an antisymmetric gauge parameter $\varepsilon^{\alpha\beta}$, similar to what has been done in the previous section.

Next we turn to the commutators involving supertransformations. Here it will not be attempted to find full transformation laws, we merely want to find transformations and commutators satisfying

$$[\delta_{P_1},\delta_{P_2}]\, F = \delta_{\kappa[P_1,P_2]}\, F + O(\kappa^2) \qquad (7.50)$$

where P_1 and P_2 stand for gauge parameters and F for a field. Moreover we want the commutators to satisfy the lowest order Jacobi identity. For the commutator of a GCT or local Lorentz transformation with a supertransformation, (7.50) holds provided we extend the ψ_μ transformation with a suitable second order term.

For the commutator of two super transformations this is only possible if in addition we make the commutator field dependent i.e. C_1 (see chapter IV) is non zero.

Finally we find for the gauge transformations, modulo field and gauge parameter redefinitions, and neglecting terms of order κ^3

$$\begin{aligned} \delta\psi_\mu = {} & \varepsilon_{,\mu} + \kappa(\xi_\nu\psi_{\mu,\nu}+\xi_{\nu,\mu}\psi_\nu+\tfrac{1}{2}\sigma_{\nu\lambda}\Omega_{\nu\lambda}\psi_\mu+\tfrac{1}{2}\sigma_{\rho\lambda}(h_{\mu\rho,\lambda}-\tfrac{1}{2}v_{\rho\lambda,\mu})\varepsilon \\ & + \kappa^2\,\frac{1}{8}\,\{(b_{\sigma\mu}b_{\sigma\rho,\lambda}-b_{\sigma\lambda}b_{\mu\rho,\sigma}-b_{\sigma\lambda}b_{\rho\mu,\sigma}+b_{\sigma\lambda}b_{\mu\sigma,\rho}+b_{\sigma\lambda}b_{\rho\sigma,\mu})+ \\ & +(\tfrac{1}{2}\bar{\psi}_\rho\gamma_\mu\psi_\lambda-\bar{\psi}_\mu\gamma_\rho\psi_\lambda)\}\sigma^{\rho\lambda}\varepsilon \end{aligned} \qquad (7.51)$$

$$\delta b_{\mu\nu} = 2\xi_{\mu,\nu}+2\Omega_{\mu\nu} + \kappa(b_{\mu\nu,\lambda}\xi^\lambda+b_{\mu\lambda}\xi^\lambda{}_{,\nu}+\Omega_{\mu\lambda}b_{\lambda\nu}-\tfrac{1}{2}\,\bar{\varepsilon}\gamma_\mu\psi_\nu)$$

The commutators are given by, neglecting terms of order κ^2,

$$[\xi,\varepsilon] = \xi_\nu\varepsilon_{,\nu} \qquad (7.52)$$

$$[\Omega,\varepsilon] = \tfrac{1}{2}\,\sigma_{\alpha\beta}\Omega_{\alpha\beta}\,\varepsilon$$

and

$$[\varepsilon_1,\varepsilon_2] = -\frac{1}{8} X_\mu + \frac{\kappa}{16} X_\lambda h_{\lambda\mu} + \frac{\kappa}{16} X_\lambda (v_{\mu\nu,\lambda} + h_{\lambda\nu,\mu} - h_{\lambda\mu,\nu}) + \frac{\kappa}{8} X_\lambda \psi_\lambda$$

with $X_\mu = \bar{\varepsilon}_1 \gamma_\mu \varepsilon_2$

With the above transformation rules not only (7.50) is satisfied which states that the fields form realisations of the algebra (7.39),(7.52), apart from higher order terms. Moreover in lowest order all Jacobi identies (4.21) are satisfied as well.

VIII TOWARDS A SPIN-3 SELF INTERACTION

1. Introduction

A first step towards theories of massless spin-3 fields is made in this chapter. It is shown that a non-trivial cubic first order interaction exists for massless spin-3 fields [23]. This interaction is cubic in space-time derivatives implying that the coupling constant has the dimension of an inverse mass squared. At least three different spin species are needed to form this interaction lagrangian. The first order gauge transformation is shown to be noncommuting. The algebraic structure of the theory appears to be different from the Yang-Mills theory or the theory of general relativity in the following sense. No extended massless theory of only spin-3 fields exists [25]. The first order commutator on the fields is suggestive of an algebraic structure involving massless fields of every higher spin.

2. The first order lagrangian

The lagrangian for a free spin-3 field $\phi_{\alpha\beta\gamma}$, symmetric in α,β and γ has been given in chapter II. It can be written in the form

$$(8.1) \qquad L_0 = -\tfrac{1}{2}\,\phi_{\alpha\beta\gamma,\rho}\phi_{\alpha\beta\gamma,\rho} + \frac{3}{2}\,\phi_{\alpha\beta\lambda,\lambda}\phi_{\alpha\beta\rho,\rho} + \frac{3}{2}\,\phi_{\lambda\lambda\alpha,\beta}\phi_{\rho\rho\alpha,\beta} + \frac{3}{4}\,\phi_{\lambda\lambda\rho,\rho}\phi_{\alpha\alpha\beta,\beta} - 3\,\phi_{\lambda\lambda\beta,\gamma}\phi_{\beta\gamma\rho,\rho}$$

This lagrangian is invariant under the free gauge transformation

$$(8.2) \qquad \phi_{\alpha\beta\gamma} \rightarrow \phi_{\alpha\beta\gamma} + \partial_\alpha\xi_{\beta\gamma} + \partial_\beta\xi_{\gamma\alpha} + \partial_\gamma\xi_{\alpha\beta}$$

where the gauge parameter must be traceless

$$(8.3) \qquad \xi_{\rho\rho} = 0$$

The free source constraint related to this gauge transformation states that the traceless divergence of the l.h.s of the equation of motion vanishes

$$(8.4) \qquad \partial^\gamma\left(L_{0,\phi_{\alpha\beta\gamma}} - \tfrac{1}{4}\,\eta_{\alpha\beta}\, L_{0,\phi_{\rho\rho\gamma}}\right) = 0 \qquad \text{for all } \phi_{\alpha\beta\gamma}(x)$$

Here $L_{0,\phi_{\rho\rho\gamma}}$ is an abbreviation for $\eta_{\mu\nu}L_{0,\phi_{\mu\nu\gamma}}$. As explained in chapter V, this implies a suitable first order interaction has to satisfy

(8.5) $$\partial^{\gamma}\left(L_{1,\phi_{\alpha\beta\gamma}} - \tfrac{1}{2}\,\eta_{\alpha\beta}\, L_{1,\phi_{\rho\rho\gamma}}\right) = 0$$

for all $\phi_{\alpha\beta\gamma}(x)$ which satisfy $L_{0,\phi_{\alpha\beta\gamma}} = 0$

Here the l.h.s of the field equation is given by

(8.6) $$L_{0,\phi_{\alpha\beta\gamma}} = \phi_{\alpha\beta\gamma,\lambda\lambda} - \underset{\alpha\beta\gamma}{\Sigma^1}\,\phi_{\lambda\beta\gamma,\alpha\lambda} + \underset{\alpha\beta\gamma}{\Sigma^2}\,\phi_{\lambda\lambda\gamma,\alpha\beta}$$

Since we want to find a trilinear interaction for the fields $\phi_{\alpha\beta\gamma}$, the interaction should contain an odd number of derivatives. The simplest possibility is to take only one derivative, implying the coupling parameter is dimensionless. However no interaction of this type which fulfills (8.5) can be found, even if one allows for different spin-3 fields to be present.

The next simplest case contains three derivatives. Then the coupling parameter has the dimension of an inverse squared mass. Right from the beginning we will include the possibility of several spin-3 fields $\phi^a_{\alpha\beta\gamma}$. This means that the lowest order lagrangian is the sum of the lowest order lagrangians of every spin-3 field.

(8.7) $$L_0 = \sum_{a=1}^{n} L_0(\phi^a_{\alpha\beta\gamma})$$

The gauge transformations and source constraints are generalised accordingly. The generalisation of (8.5) is

(8.8) $$\partial^{\gamma}\left(L_{1,\phi^a_{\alpha\beta\gamma}} - \tfrac{1}{2}\,\eta_{\alpha\beta}\, L_{1,\phi^a_{\rho\rho\gamma}}\right) = 0$$

for all $\phi^c_{\rho\sigma\tau}$ which satisfy $L_{0,\phi^c_{\rho\sigma\tau}} = 0$

A typical first order lagrangian with three derivatives has the form

(8.9) $$L_1 = f^{abc}\,\phi^a_{\dots,\dots}\,\phi^b_{\dots,\cdot}\,\phi^c_{\dots}$$

where the dots represent indices, and a summation over a,b,c, from 1 to n is implied. The derivatives on such a lagrangian may be distributed in any other way as well.

Table III The 48 lagrangians of type $\phi^a_{\ldots,\ldots}\phi^b_{\ldots,\cdot}\phi^c_{\ldots}$

$$L_1^{1abc} = \phi^a_{\lambda\lambda\rho}\phi^b_{\lambda\lambda\alpha,\beta}\phi^c_{\sigma\sigma\alpha,\beta\rho}$$
$$L_1^{2abc} = \phi^a_{\lambda\lambda\rho,\rho\sigma}\phi^b_{\lambda\lambda\beta}\phi^c_{\alpha\alpha\beta,\sigma}$$
$$L_1^{3abc} = \phi^a_{\lambda\lambda\rho,\sigma}\phi^b_{\alpha\alpha\beta}\phi^c_{\beta\gamma\gamma,\rho\sigma}$$
$$L_1^{4abc} = \phi^a_{\alpha\beta\gamma}\phi^b_{\lambda\lambda\alpha,\rho}\phi^c_{\sigma\sigma\beta,\gamma\rho}$$
$$L_1^{5abc} = \phi^a_{\alpha\beta\gamma,\gamma\sigma}\phi^b_{\lambda\lambda\alpha}\phi^c_{\beta\rho\rho,\sigma}$$
$$L_1^{6abc} = \phi^a_{\alpha\beta\gamma,\sigma}\phi^b_{\lambda\lambda\alpha}\phi^c_{\rho\rho\beta,\gamma\sigma}$$
$$L_1^{7abc} = \phi^a_{\alpha\alpha\beta}\phi^b_{\rho\sigma\tau,\lambda}\phi^c_{\rho\sigma\tau,\lambda\beta}$$
$$L_1^{8abc} = \phi^a_{\alpha\alpha\beta,\beta\lambda}\phi^b_{\rho\sigma\tau}\phi^c_{\rho\sigma\tau,\lambda}$$
$$L_1^{9abc} = \phi^a_{\alpha\alpha\beta,\lambda}\phi^b_{\rho\sigma\tau}\phi^c_{\rho\sigma\tau,\beta\lambda}$$
$$L_1^{10abc} = \phi^a_{\alpha\beta\gamma}\phi^b_{\alpha\rho\sigma,\lambda}\phi^c_{\beta\rho\sigma,\gamma\lambda}$$
$$L_1^{11abc} = \phi^a_{\alpha\beta\lambda,\lambda\gamma}\phi^b_{\alpha\rho\sigma}\phi^c_{\beta\rho\sigma,\gamma}$$
$$L_1^{12abc} = \phi^a_{\alpha\beta\gamma,\lambda}\phi^b_{\alpha\rho\sigma}\phi^c_{\beta\rho\sigma,\gamma\lambda}$$
$$L_1^{13abc} = \phi^a_{\alpha\beta\gamma}\phi^b_{\alpha\beta\delta,\lambda}\phi^c_{\sigma\sigma\delta,\lambda\gamma}$$
$$L_1^{14abc} = \phi^a_{\alpha\beta\gamma}\phi^b_{\alpha\beta\delta,\gamma\lambda}\phi^c_{\rho\rho\delta,\lambda}$$
$$L_1^{15abc} = \phi^a_{\alpha\beta\gamma,\gamma\lambda}\phi^b_{\alpha\beta\rho,\lambda}\phi^c_{\rho\sigma\sigma}$$
$$L_1^{16abc} = \phi^a_{\alpha\beta\gamma,\lambda}\phi^b_{\alpha\beta\rho,\gamma\lambda}\phi^c_{\rho\sigma\sigma}$$
$$L_1^{17abc} = \phi^a_{\alpha\beta\gamma,\lambda}\phi^b_{\alpha\beta\rho}\phi^c_{\rho\sigma\sigma,\gamma\lambda}$$
$$L_1^{18abc} = \phi^a_{\alpha\beta\gamma,\gamma\lambda}\phi^b_{\alpha\beta\rho}\phi^c_{\rho\sigma\sigma,\lambda}$$
$$L_1^{19abc} = \phi^a_{\rho\rho\alpha,\beta\gamma}\phi^b_{\sigma\sigma\beta}\phi^c_{\tau\tau\gamma,\alpha}$$
$$L_1^{20abc} = \phi^a_{\rho\rho\alpha,\beta}\phi^b_{\sigma\sigma\beta,\gamma}\phi^c_{\tau\tau\gamma,\alpha}$$
$$L_1^{21abc} = \phi^a_{\alpha\beta\gamma,\alpha\beta}\phi^b_{\rho\rho\sigma}\phi^c_{\lambda\lambda\sigma,\gamma}$$
$$L_1^{22abc} = \phi^a_{\alpha\beta\gamma}\phi^b_{\rho\rho\sigma,\alpha}\phi^c_{\lambda\lambda\sigma,\beta\gamma}$$
$$L_1^{23abc} = \phi^a_{\alpha\alpha\beta,\beta}\phi^b_{\lambda\lambda\rho,\sigma}\phi^c_{\rho\sigma\gamma,\gamma}$$
$$L_1^{24abc} = \phi^a_{\alpha\alpha\beta}\phi^b_{\lambda\lambda\rho,\sigma}\phi^c_{\rho\sigma\gamma,\gamma\beta}$$
$$L_1^{25abc} = \phi^a_{\lambda\lambda\alpha,\alpha\beta}\phi^b_{\rho\rho\gamma,\delta}\phi^c_{\beta\gamma\delta}$$
$$L_1^{26abc} = \phi^a_{\lambda\lambda\rho,\sigma}\phi^b_{\alpha\alpha\beta,\gamma}\phi^c_{\beta\gamma\sigma,\rho}$$
$$L_1^{27abc} = \phi^a_{\lambda\lambda\alpha,\alpha\beta}\phi^b_{\rho\rho\sigma}\phi^c_{\beta\sigma\gamma,\gamma}$$
$$L_1^{28abc} = \phi^a_{\lambda\lambda\alpha,\beta}\phi^b_{\rho\rho\gamma}\phi^c_{\beta\gamma\sigma,\sigma\alpha}$$
$$L_1^{29abc} = \phi^a_{\alpha\alpha\beta}\phi^b_{\delta\epsilon\lambda,\lambda}\phi^c_{\delta\epsilon\rho,\rho\beta}$$
$$L_1^{30abc} = \phi^a_{\alpha\alpha\beta,\gamma}\phi^b_{\rho\sigma\lambda,\lambda\beta}\phi^c_{\gamma\rho\sigma}$$
$$L_1^{31abc} = \phi^a_{\alpha\alpha\beta}\phi^b_{\rho\sigma\delta,\epsilon}\phi^c_{\sigma\rho\epsilon,\delta\beta}$$
$$L_1^{32abc} = \phi^a_{\alpha\alpha\lambda,\alpha\beta}\phi^b_{\rho\sigma\alpha}\phi^c_{\rho\sigma\beta,\lambda}$$
$$L_1^{33abc} = \phi^a_{\alpha\beta\lambda,\lambda}\phi^b_{\delta\delta\alpha,\rho}\phi^c_{\beta\rho\sigma,\sigma}$$
$$L_1^{34abc} = \phi^a_{\alpha\beta\gamma}\phi^b_{\alpha\lambda\lambda,\rho}\phi^c_{\beta\rho\sigma,\sigma\gamma}$$
$$L_1^{35abc} = \phi^a_{\alpha\beta\lambda,\lambda\gamma}\phi^b_{\alpha\rho\rho,\sigma}\phi^c_{\beta\gamma\sigma}$$
$$L_1^{36abc} = \phi^a_{\alpha\beta\gamma,\lambda}\phi^b_{\alpha\delta\delta,\rho}\phi^c_{\beta\lambda\rho,\gamma}$$
$$L_1^{37abc} = \phi^a_{\alpha\beta\gamma,\lambda\gamma}\phi^b_{\alpha\rho\rho}\phi^c_{\beta\sigma\lambda,\sigma}$$
$$L_1^{38abc} = \phi^a_{\alpha\beta\gamma,\lambda}\phi^b_{\sigma\sigma\alpha}\phi^c_{\beta\lambda\rho,\rho\gamma}$$
$$L_1^{39abc} = \phi^a_{\alpha\beta\rho,\lambda\sigma}\phi^b_{\beta\gamma\lambda}\phi^c_{\alpha\gamma\sigma,\rho}$$
$$L_1^{40abc} = \phi^a_{\alpha\beta\lambda,\rho}\phi^b_{\beta\gamma\rho,\sigma}\phi^c_{\alpha\gamma\sigma,\lambda}$$
$$L_1^{41abc} = \phi^a_{\alpha\beta\gamma,\alpha\beta}\phi^b_{\rho\sigma\tau}\phi^c_{\rho\sigma\tau,\gamma}$$
$$L_1^{42abc} = \phi^a_{\alpha\beta\gamma}\phi^b_{\rho\sigma\tau,\alpha}\phi^c_{\rho\sigma\tau,\beta\gamma}$$
$$L_1^{43abc} = \phi^a_{\alpha\beta\lambda,\lambda}\phi^b_{\alpha\beta\rho,\sigma}\phi^c_{\rho\sigma\gamma,\gamma}$$
$$L_1^{44abc} = \phi^a_{\alpha\beta\gamma}\phi^b_{\alpha\beta\rho,\lambda}\phi^c_{\rho\lambda\sigma,\sigma\gamma}$$
$$L_1^{45abc} = \phi^a_{\alpha\beta\gamma,\gamma\sigma}\phi^b_{\alpha\beta\rho,\lambda}\phi^c_{\rho\lambda\sigma}$$
$$L_1^{46abc} = \phi^a_{\alpha\beta\gamma,\lambda}\phi^b_{\alpha\beta\rho,\sigma}\phi^c_{\rho\sigma\lambda,\gamma}$$
$$L_1^{47abc} = \phi^a_{\alpha\beta\gamma,\gamma\lambda}\phi^b_{\alpha\beta\rho}\phi^c_{\rho\lambda\sigma,\sigma}$$
$$L_1^{48abc} = \phi^a_{\alpha\beta\gamma,\lambda}\phi^b_{\alpha\beta\rho}\phi^c_{\lambda\rho\sigma,\sigma\gamma}$$

There are altoghether 48 lagrangians of the above type, modulo total divergencies. A possible choice is listed in Table III.

The Euler-Lagrange equations for such an interaction lagrangian read

$$(8.10)\qquad L_{,\phi_{\alpha\beta\gamma}} = \frac{\partial L}{\partial\phi_{\alpha\beta\gamma}} - \partial_\delta \frac{\partial L}{\partial\phi_{\alpha\beta\gamma,\delta}} + \partial_\delta\partial_\varepsilon \frac{\partial L}{\partial\phi_{\alpha\beta\gamma,\delta\varepsilon}} - \partial_\delta\partial_\varepsilon\partial_\zeta \frac{\partial L}{\partial\phi_{\alpha\beta\gamma,\delta\varepsilon\zeta}}$$

As explained in chapter III, field redefinitions give rise to fake interactions. There are 18 field redefinitions which give rise on substitution in L_0 to a first order interaction of type (8.9). They are listed in Table IV.

Table IV Field redefinitions $\phi_{\alpha\beta\gamma} \rightarrow \phi_{\alpha\beta\gamma}+g\bar{\phi}_{\alpha\beta\gamma}$.
Symmetrisation in α,β and γ is implied.

$\bar{\phi}^1_{\alpha\beta\gamma} = f_1^{abc}\phi^b_{\varepsilon\delta\delta,\varepsilon}\phi^c_{\alpha\beta\gamma}$	$\bar{\phi}^{10}_{\alpha\beta\gamma} = f_{10}^{abc}\phi^b_{\alpha\delta\delta}\phi^c_{\beta\varepsilon\varepsilon,\gamma}$
$\bar{\phi}^2_{\alpha\beta\gamma} = f_2^{abc}\phi^b_{\varepsilon\delta\delta}\phi^c_{\alpha\beta\gamma,\varepsilon}$	$\bar{\phi}^{11}_{\alpha\beta\gamma} = f_{11}^{abc}\eta_{\alpha\beta}\phi^b_{\delta\delta\varepsilon,\varepsilon}\phi^c_{\gamma\zeta\zeta}$
$\bar{\phi}^3_{\alpha\beta\gamma} = f_3^{abc}\phi^b_{\alpha\delta\delta,\varepsilon}\phi^c_{\beta\gamma\varepsilon}$	$\bar{\phi}^{12}_{\alpha\beta\gamma} = f_{12}^{abc}\eta_{\alpha\beta}\phi^b_{\varepsilon\delta\delta}\phi^c_{\gamma\zeta\zeta,\varepsilon}$
$\bar{\phi}^4_{\alpha\beta\gamma} = f_4^{abc}\phi^b_{\alpha\delta\delta}\phi^c_{\beta\gamma\varepsilon,\varepsilon}$	$\bar{\phi}^{13}_{\alpha\beta\gamma} = f_{13}^{abc}\eta_{\alpha\beta}\phi^b_{\delta\delta\zeta,\varepsilon}\phi^c_{\zeta\varepsilon\gamma}$
$\bar{\phi}^5_{\alpha\beta\gamma} = f_5^{abc}\phi^b_{\delta\delta\varepsilon,\alpha}\phi^c_{\beta\gamma\varepsilon}$	$\bar{\phi}^{14}_{\alpha\beta\gamma} = f_{14}^{abc}\eta_{\alpha\beta}\phi^b_{\delta\delta\zeta}\phi^c_{\zeta\varepsilon\gamma,\varepsilon}$
$\bar{\phi}^6_{\alpha\beta\gamma} = f_6^{abc}\phi^b_{\varepsilon\delta\delta}\phi^c_{\beta\gamma\varepsilon,\alpha}$	$\bar{\phi}^{15}_{\alpha\beta\gamma} = f_{15}^{abc}\eta_{\alpha\beta}\phi^b_{\delta\delta\varepsilon,\gamma}\phi^c_{\varepsilon\zeta\zeta}$
$\bar{\phi}^7_{\alpha\beta\gamma} = f_7^{abc}\phi^b_{\alpha\delta\varepsilon,\delta}\phi^c_{\beta\gamma\varepsilon}$	$\bar{\phi}^{16}_{\alpha\beta\gamma} = f_{16}^{abc}\eta_{\alpha\beta}\phi^b_{\zeta\delta\varepsilon,\zeta}\phi^c_{\delta\varepsilon\gamma}$
$\bar{\phi}^8_{\alpha\beta\gamma} = f_8^{abc}\phi^b_{\delta\varepsilon\alpha}\phi^c_{\beta\gamma\varepsilon,\delta}$	$\bar{\phi}^{17}_{\alpha\beta\gamma} = f_{17}^{abc}\eta_{\alpha\beta}\phi^b_{\delta\varepsilon\zeta}\phi^c_{\delta\varepsilon\gamma,\zeta}$
$\bar{\phi}^9_{\alpha\beta\gamma} = f_9^{abc}\phi^b_{\alpha\delta\varepsilon}\phi^c_{\beta\delta\varepsilon,\gamma}$	$\bar{\phi}^{18}_{\alpha\beta\gamma} = f_{18}^{abc}\eta_{\alpha\beta}\phi^b_{\delta\varepsilon\zeta}\phi^c_{\delta\varepsilon\zeta,\gamma}$

However, not all field redefinitions will give rise to independent fake interactions, because of the gauge invariance of the free lagrangian. In particular,

$$(8.11)\qquad \bar{\phi}^{10} - \tfrac{1}{2}\bar{\phi}^{15} \quad \text{and} \quad \bar{\phi}^9 - \tfrac{1}{2}\bar{\phi}^{18}$$

correspond to a gauge transformations if the coefficients satisfy

$$(8.12) \qquad f_{10}{}^{abc} = f_{15}{}^{abc} = f_{10}{}^{acb} = f_{15}{}^{acb}$$

and similarly for f_9 and f_{18}. Furthermore

$$(8.13) \qquad \bar{\phi}^5 + \bar{\phi}^6 - \tfrac{1}{2}\, \bar{\phi}^{15}$$

is a gauge transformation if

$$(8.14) \qquad f_5{}^{abc} = f_6{}^{abc} \quad \text{and} \quad f_{15}{}^{abc} = f_5{}^{abc} + f_5{}^{acb}$$

Having done all the work of listing all possible fake interactions, they can be used to eliminate elements of our base of 48 lagrangians of type (8.9). Of the remaining elements one and only one combination can be found which satisfies the requirement (8.8). The coefficient f_{abc} of this combination is fully antisymmetric.

Thus we have established the following. Modulo field redefinitions, there is one and only one cubic interaction lagrangian for massless spin-3 fields which is trilinear in space-time derivatives. In terms of the lagrangians of Table III, it can be given as

$$(8.15) \qquad L_1 = \sum_{i=1}^{48} f^{abc}\, a_i\, L_1{}^i{}_{abc}$$

where f^{abc} is fully antisymmetric and

$a_1 = -9/2$	$a_{13} = 0$	$a_{25} = 3/2$	$a_{37} = 0$
$a_2 = -3$	$a_{14} = 12$	$a_{26} = 6$	$a_{38} = 0$
$a_3 = -3$	$a_{15} = -3$	$a_{27} = 3$	$a_{39} = -12$
$a_4 = -6$	$a_{16} = -3$	$a_{28} = 3$	$a_{40} = -10$
$a_5 = 0$	$a_{17} = -9$	$a_{29} = 6$	$a_{41} = -4$
$a_6 = 0$	$a_{18} = -9$	$a_{30} = -15$	$a_{42} = -1$
$a_7 = 3/2$	$a_{19} = -9/4$	$a_{31} = -21/2$	$a_{43} = 12$
$a_8 = -1$	$a_{20} = -5/4$	$a_{32} = 9$	$a_{44} = 12$
$a_9 = -7$	$a_{21} = 0$	$a_{33} = 6$	$a_{45} = 3$
$a_{10} = 6$	$a_{22} = 3$	$a_{34} = 12$	$a_{46} = -6$
$a_{11} = 9$	$a_{23} = 9/2$	$a_{35} = -6$	$a_{47} = 9$
$a_{12} = 9$	$a_{24} = 9$	$a_{36} = -6$	$a_{48} = 9$

That the above lagrangian cannot be generated by field redefinitions can be checked as follows. Lagrangian number 42 from Table III appears in the expression (8.15) but cannot be obtained by means of a field redefinition from the lowest order lagrangian.

3. The first order transformation and commutator

To obtain the first order gauge transformation, we first write the traceless divergence of the l.h.s. of the field equation as

$$(8.16) \qquad \partial^{\gamma}\left(L_{1,\phi^{a}_{\alpha\beta\gamma}} - \tfrac{1}{4}\,\eta_{\alpha\beta}\,L_{1,\phi^{a}_{\rho\rho\gamma}}\right) = B_{1}{}^{a}{}_{b}{}^{\alpha\beta}{}_{\rho\sigma\tau}(\phi)\; L_{0,\phi^{b}_{\rho\sigma\tau}}$$

For the case considered here, this is the explicit form of (5.6) of chapter V. Having done this, it is easy to obtain by means of use of "partial integrations", as explained in chapter V, the first order gauge transformation

$$(8.17) \qquad \phi^{a}_{\alpha\beta\gamma} \to \phi^{a}_{\alpha\beta\gamma} + \sum_{\alpha\beta\gamma} \xi^{a}_{\alpha\beta,\gamma} + gT_{1}{}^{a}{}_{b}{}^{\alpha\beta\gamma}{}_{\rho\sigma}(\phi)\,\xi^{b}_{\rho\sigma}$$

There are many equivalent forms of the first order term. Field redefinitions, gauge parameter redefinitions and contributions "vanishing on shell" in the sense explained in section IV.1 all can modify the gauge transformation. The freedom due to field redefinitions is fixed once a specific form of the first order lagrangian is chosen. The particular choice (8.15) was made to obtain a simple transformation law. The three possible "on shell vanishing terms" are listed in Table V, and the twelve possible gauge parameter redefinitions in Table VI. Only the gauge parameter redefinitions do affect the commutator.

Using this freedom the first order gauge transformation can be cast in the relatively simple form

$$(8.18) \qquad \phi^{a}_{\alpha\beta\gamma} \to \phi^{a}_{\alpha\beta\gamma} + \sum_{\text{cyclic}}\Big\{\xi^{a}_{\alpha\beta,\gamma} + \tfrac{1}{3}\big(\phi^{b}_{\alpha\beta\gamma,\rho\sigma}\xi^{c}_{\rho\sigma} + 3\,\phi^{b}_{\alpha\rho\sigma}\xi^{c}_{\beta\gamma,\rho\sigma}$$

$$-2\,\phi^{b}_{\alpha\beta\rho,\sigma}\xi^{c}_{\gamma\rho,\sigma} - 4\,\phi^{b}_{\alpha\beta\rho,\sigma}\xi^{c}_{\gamma\sigma,\rho} + \eta_{\alpha\beta}\,\phi^{b}_{\rho\sigma\tau,\gamma}\xi^{c}_{\rho\sigma,\tau}\big)\Big\}$$

Table V On shell vanishing terms.

$$\phi^a_{\alpha\beta\gamma} \rightarrow \phi^a_{\alpha\beta\gamma} + f^{abc}\,\xi^b_{\alpha\lambda}\,L_{0,\phi^c_{\lambda\beta\gamma}}$$

$$\phi^a_{\alpha\beta\gamma} \rightarrow \phi^a_{\alpha\beta\gamma} + f^{abc}\,\eta_{\alpha\beta}\,\xi^b_{\gamma\lambda}\,L_{0,\phi^c_{\lambda\rho\rho}}$$

$$\phi^a_{\alpha\beta\gamma} \rightarrow \phi^a_{\alpha\beta\gamma} + f^{abc}\,(\eta_{\alpha\beta}\,\xi^b_{\rho\sigma}\,L_{0,\phi^c_{\rho\sigma\gamma}} + \xi^b_{\alpha\beta}\,L_{0,\phi^c_{\gamma\lambda\lambda}})$$

Table VI Gauge parameter redefinitions. Implied is that only the parts symmetric and traceless in μ and ν are being considered.

gauge parameter redefinition	change induced in $C_0(\xi_2,\xi_1)$
$c_1 f^{abc}\xi^b_{\mu\nu}\phi^c_{\delta\delta\varepsilon,\varepsilon}$	$\{c_1 2f^{abc}\xi_2{}^b_{\mu\nu}\xi_1{}^c_{\delta\varepsilon,\delta\varepsilon}\} + \{\xi_2{}^b - \xi_1{}^c\}$
$c_2 f^{abc}\xi^b_{\mu\nu,\varepsilon}\phi^c_{\delta\delta\varepsilon}$	$\{c_2 f^{abc}\xi_2{}^b_{\mu\nu,\varepsilon}\xi_1{}^c_{\delta\varepsilon,\delta}\} + \{\xi_2{}^b - \xi_1{}^c\}$
$c_3 f^{abc}\xi^b_{\delta\varepsilon}\phi^c_{\mu\nu\delta,\varepsilon}$	$\{c_3 f^{abc}\xi_2{}^b_{\delta\varepsilon}(\xi_1{}^c_{\mu\nu,\delta\varepsilon}+2\xi_1{}^c_{\mu\varepsilon,\delta\nu})\} + \{\xi_2{}^b - \xi_1{}^c\}$
$c_4 f^{abc}\xi^b_{\delta\varepsilon,\varepsilon}\phi^c_{\mu\nu\delta}$	$\{c_4 f^{abc}\xi_2{}^b_{\delta\varepsilon,\varepsilon}(\xi_1{}^c_{\mu\nu,\delta}+2\xi_1{}^c_{\mu\delta,\nu})\} + \{\xi_2{}^b - \xi_1{}^c\}$
$c_5 f^{abc}\xi^b_{\mu\delta}\phi^c_{\nu\delta\varepsilon,\varepsilon}$	$\{c_5 f^{abc}\xi_2{}^b_{\mu\delta}(\xi_1{}^c_{\nu\delta,\varepsilon\varepsilon}+\xi_1{}^c_{\delta\varepsilon,\varepsilon\nu}+\xi_1{}^c_{\nu\varepsilon,\varepsilon\delta})\}$ $+ \{\xi_2{}^b - \xi_1{}^c\}$
$c_6 f^{abc}\xi^b_{\mu\delta,\varepsilon}\phi^c_{\nu\delta\varepsilon}$	$\{c_6 f^{abc}\xi_2{}^b_{\mu\delta,\varepsilon}(\xi_1{}^c_{\nu\delta,\varepsilon}+\xi_1{}^c_{\delta\varepsilon,\nu}+\xi_1{}^c_{\varepsilon\nu,\delta})\}$ $+ \{\xi_2{}^b - \xi_1{}^c\}$
$c_7 f^{abc}\xi^b_{\mu\varepsilon}\phi^c_{\delta\delta\nu,\varepsilon}$	$\{2c_7 f^{abc}\xi_2{}^b_{\mu\varepsilon}\xi_1{}^c_{\delta\nu,\delta\varepsilon}\} + \{\xi^b{}_2 - \xi^c{}_1\}$
$c_8 f^{abc}\xi^b_{\mu\varepsilon,\varepsilon}\phi^c_{\delta\delta\nu}$	$\{2c_8 f^{abc}\xi_2{}^b_{\mu\varepsilon,\varepsilon}\xi_1{}^c_{\nu\delta,\delta}\} + \{\xi_2{}^b - \xi_1{}^c\}$
$c_9 f^{abc}\xi^b_{\mu\delta}\phi^c_{\delta\varepsilon\varepsilon,\nu}$	$\{2c_9 f^{abc}\xi_2{}^b_{\mu\delta}\xi_1{}^c_{\delta\varepsilon,\varepsilon\nu}\} + \{\xi_2{}^b - \xi_1{}^c\}$
$c_{10} f^{abc}\xi^b_{\mu\delta,\nu}\phi^c_{\delta\varepsilon\varepsilon}$	$\{2c_{10} f^{abc}\xi_2{}^b_{\mu\delta,\nu}\xi_1{}^c_{\delta\varepsilon,\varepsilon}\} + \{\xi_2{}^b - \xi_1{}^c\}$
$c_{11} f^{abc}\xi^b_{\varepsilon\delta}\phi^c_{\varepsilon\delta\mu,\nu}$	$\{c_{11} f^{abc}\xi_2{}^b_{\varepsilon\delta}(\xi_1{}^c_{\varepsilon\delta,\mu\nu}+2\xi_2{}^c_{\mu\varepsilon,\delta\nu})\} + \{\xi_2{}^b - \xi_1{}^c\}$
$c_{12} f^{abc}\xi^b_{\varepsilon\delta,\nu}\phi^c_{\varepsilon\delta\mu}$	$\{c_{12} f^{abc}\xi_2{}^b_{\varepsilon\delta,\nu}(\xi_1{}^c_{\varepsilon\delta,\mu}+2\xi_1{}^c_{\mu\varepsilon,\delta})\} + \{\xi_2{}^b - \xi_1{}^c\}$

Detailed analysis of all possible forms of the first order gauge transformation shows that the following term is always present:

$$S_{\alpha\beta\gamma} = f^{abc}\, \phi^{b}{}_{\alpha\beta\gamma,\rho\sigma}\xi^{c}{}_{\rho\sigma} \tag{8.19}$$

To (8.17) corresponds the following form of the commutator

$$C_0(\xi_2,\xi_1)^a{}_{\mu\nu} = f^{abc}\left(\xi_1{}^b{}_{\mu\nu,\rho\sigma}\xi_2{}^c{}_{\rho\sigma} - \tfrac{2}{3}\,\xi_1{}^b{}_{\mu\rho,\sigma}\xi_2{}^c{}_{\nu\rho,\sigma} - \tfrac{4}{3}\,\xi_1{}^b{}_{\mu\rho,\sigma}\xi_2{}^c{}_{\nu\sigma,\rho}\right) + f^{abc}\left(\xi_1{}^b - \xi_2{}^c\right) - \tfrac{1}{2}\eta_{\mu\nu}\,\mathrm{Tr}\, f^{abc}\left(\quad \right) \tag{8.20}$$

Note $C_0{}^a{}_{\mu\nu}$ is traceless, like $\xi_1{}^b$ and $\xi_2{}^c$. The reader may check that the above transformation law (8.18) and first order commutator (8.20) satisfy

$$[\delta_{\xi_2},\delta_{\xi_1}]\,\phi^a{}_{\alpha\beta\gamma} = g \sum_{\text{cyclic}} \partial_\alpha C_0{}^a{}_{\beta\gamma}(\xi_2,\xi_1) + O(g^2) \tag{8.21}$$

4. No spin-3 analogue of General Relativity

In section IV.4 it was shown that Yang-Mills theory and the theory of General Relativity were very similar. They are selfcontained in the sense that one can limit such theories to be theories of the particles interacting with themselves or external sources. They can be viewed as extended massless theories and their actions are invariant for field transformations which form a realisation of a non-trivial algebra

$$[\delta_{\xi_2},\delta_{\xi_1}]\,\phi = \delta_{g[\xi_2,\xi_1]}\,\phi \tag{8.22}$$

where the commutator $[\xi_2,\xi_1]$ does not depend on ϕ. For spin three this is not possible, because of the following. The term $S_{\alpha\beta\gamma}$ of (8.19) in the first order gauge transformation gives rise in the commutator on $\phi_{\alpha\beta\gamma}$ to a term

$$[\delta_{\xi_2},\delta_{\xi_1}]\,\phi = g^2\left(f^{abc}f^{bde}+f^{abe}f^{bcd}\right)\phi^d{}_{\alpha\beta\gamma,\rho\sigma\mu\nu}\xi_1{}^e{}_{\rho\sigma}\xi_2{}^c{}_{\mu\nu} \tag{8.23}$$

However, no such term can be present in $\delta_{g[\xi_2,\xi_1]}\phi$, since $T_1(\phi;C_0)$ contains no terms with four derivatives acting on ϕ. Thus it is impossible for an extended free spin-3 theory to have a bracket structure like General Relativity or Yang-Mills.

Even an stronger result can be proved. Also when we allow for higher order terms $T_2(\phi,\phi;\xi)$ in the gauge transformation and $C_1(\xi_1,\xi_2;\phi)$ in the commutator we cannot cancel (8.23). Contributions from T_2 terms will have at least one derivative acting on a gauge parameter while $\partial_\alpha C_{1\beta\gamma}$ has a different index structure. We are thus led to the following conclusion. No self contained "extended" theory, based on the free spin-3 lagrangian (8.1) of interacting massless spin-3 fields exists. For what we mean by "extended", we refer to chapter III.

5. Does the algebraic structure involve every higher spin?

One way out of the impasse of the preceding sections, is to start anew, and search for other lagrangians describing free massless spin-3 fields instead of (8.1). On the other hand we have not ruled out every theory involving higher massless interacting spin-3 fields, based on the free lagrangian (8.1).

Maybe a solution can be found along the following lines. Note that if the f^{abc}'s satisfy the Jacobi identity, (8.23) reduces to

(8.24) $$[\delta_{\xi_2},\delta_{\xi_1}]\,\phi_{\alpha\beta\gamma} = f^{adb}\,\phi^{d}{}_{\alpha\beta\gamma,\rho\sigma\mu\nu}(f^{bec}\xi_1{}^{e}{}_{\rho\sigma}\xi_2{}^{c}{}_{\mu\nu})$$

This is very suggestive. Making a bold extrapolation of the transformation laws obtained for various spin-3 first order interactions (see Appendix C), one would conclude that the effect of a spin-5 transformation on $\phi^{a}{}_{\alpha\beta\gamma}$ contains a term like

(8.25) $$\delta\phi^{a}{}_{\alpha\beta\gamma} = g\,f^{abc}\,\phi^{d}{}_{\alpha\beta\gamma,\rho\sigma\mu\nu}\xi^{c}{}_{\rho\sigma\mu\nu} + \text{ other terms}$$

Now compare this to (8.24). The commutator of two spin-3 transformations not only gives again a spin-3 transformation, parametrized by C_0 of (8.20), but a spin-5 transformation as well, parametrized by

(8.26) $$[\xi_2,\xi_1]^{(5)a}{}_{\alpha\beta\gamma\delta} = f^{abc}\,\xi_1{}^{b}{}_{\alpha\beta}\xi_2{}^{c}{}_{\gamma\delta}$$

where the above expression is to be symmetrized in α,β,γ and δ. Even if the assumptions made above could all be justified, we are still left with some problems. If there are spin-5 fields around, there must be a free spin-5 lagrangian and most likely a cubic spin-5 selfinteraction as well. Not only this, but when two such spin-5 transformtions are commuted, terms with 8 derivatives will appear. Reasoning as before, they would be interpreted as spin-9 transformations. So one could go one and one. Infinite different higher spins are expected, with more and more derivatives. Presumably all intermediate spins could be needed also.

So we can envisage the following structure. Let us group all different symmetric tensor fields toghether into one hyperfield Φ

$$\Phi \;=\; \phi + \phi_{\mu} + \phi_{\mu\nu} + \phi_{\mu\nu\rho} + \phi_{\mu\nu\rho\sigma} + \cdots$$

and all gauge parameters into one hyperparameter Ξ. The hyperfield and hyperparameter could form a bracket structure like

(8.27) $$\delta\Phi \;=\; \partial\Phi \;-\; g\;\{\Phi,\Xi\}$$

satifying

(8.28) $$[\delta_{\Xi_2},\delta_{\Xi_1}]\;\Phi \;=\; \delta_{g[\Xi_2,\Xi_1]}\;\Phi$$

for some properly defined commutator $[\Xi_2,\Xi_1]$.

We stress all considerations in this sections are mere speculations for the moment, to be verified by later research.

IX INTERACTION BETWEEN A GAUGE FIELD AND SCALAR FIELDS

Introduction

The main function of the gauge fields we know in nature is to mediate interactions between matter fields. Such interactions can also give, for instance in case of a spin-2 gauge field, essential information about the selfinteraction of the gauge field.

In this section the matter fields are represented by massless scalar fields. First order interactions quadratic in a scalar field ϕ and linear in a spin-s gauge field $A_{\mu_1 \ldots \mu_s}$ are studied [25]. We begin by discussing the familiar cases where A carries spin-1 or spin-2. Next we show that for any spin-s gauge boson an analoguous interaction exists, involving s derivatives and unique modulo field redefinitions. An other way to arrive at the first order interaction is to start considering the global symmetries of the lagrangian of a free scalar field. Finally it is pointed out that when trying to disentangle the algebraic structure of the theory, one encouters problems very similar to those of the preceding chapter.

2. Coupling scalar to massless spin-1 or spin-2 fields

When a single type of spin-1 field is taken, the simplest first order lagrangian quadratic in ϕ and linear in the spin-1 field A_μ reads

(9.1) $$L_1 = - c^{ij} \phi^i_{,\mu} \phi^j A_\mu$$

where ϕ^i is a massless real scalar field with N "internal" components.

The requirement that the first order lagrangian satisfies the source constraint

(9.2) $$\partial^\mu (L_{1,A_\mu}) = 0 \quad \text{for all } \phi^i \text{ which satisfy } L_{0,\phi^i} = \phi^i_{,\mu\mu} = 0$$

gives

(9.3) $$c^{ij} = - c^{ji} \qquad i=1,\ldots N$$

The scalar field must have at least two components. The corresponding first order gauge transformations are given by

(9.4) $$\delta\phi^i = g\, c^{ij}\, \phi^j\, \Lambda \qquad i=1,\ldots N$$

$$\delta A_\mu = \Lambda_{,\mu}$$

These transformations commute and the abelian group of the transformations of a single spin-1 field is not changed. In case N=2, the two real components of the scalar field may be combined into one complex scalar with the transformation rule

(9.5) $$\delta\phi = i\, \phi\, \Lambda$$

In case there are several species of spin-1 fields A^a_μ, a=1,...n, one finds

(9.6) $$L_1 = -\, c^{ija}\, \phi^i_{,\mu}\, \phi^j\, A^a_\mu$$

with

(9.7) $$c^{ija} = -\, c^{jia}$$

The first order transformation of ϕ reads

(9.8) $$\delta\phi^i = g\, c^{ija}\, \phi^j\, \Lambda^a \qquad i=1,\ldots N$$

The commutator of two such transformations on the scalar field is

(9.9) $$[\delta_{\Lambda_2}\, \delta_{\Lambda_1}]\, \phi^i = g^2\, (c^{ija}c^{jkb} - c^{ijb}c^{jka})\, \Lambda_2{}^a\, \Lambda_1{}^b\, \phi^k$$

Requiring the r.h.s. of (9.9) to be of the same form as (9.8) gives

(9.10) $$(c^{ija}c^{jkb} - c^{ijb}c^{jka}) = c^{ikc}f^{cab} \qquad \text{for } i,k=1,\ldots N \text{ and } a,b=1,\ldots n$$

for some constants f^{abc}. So we arrive at

(9.11) $$[\Lambda_2,\Lambda_1]^a = g\, f^{abc}\, \Lambda_2{}^b\, \Lambda_1{}^a$$

The f^{abc} are the structure constants. If these are non-zero then a self-interaction between the spin-1 fields as described in chapter VII must be present as well, in order that the transformation rule of the spin-1 field is modified in such a way as to agree with the commutator (9.11).

A first guess for the first order interaction of a scalar field with a massless spin-2 field might be $L_1 = h_{\rho\rho}\phi\phi$, this however does not obey the source constraint

(9.12) $$\partial (L_{1,h_{\mu\nu}}) = 0 \qquad \text{for all } \phi \text{ which satisfy } L_{1,\phi} = \phi_{,\mu\mu} = 0$$

There is an first order interaction which contains two derivatives. This interaction is unique modulo field redefinitions. One form is given by

(9.13) $$L_1 = -\tfrac{1}{2}(h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h_{\rho\rho})\, \phi_{,\mu}\, \phi_{,\nu}$$

If the scalar field ϕ has an internal index as well, the interaction must be symmetric in these internal indices. The corresponding gauge transformation is

(9.14) $$\delta\phi = g\, \phi_{,\nu}\, \xi^{\nu}$$

On calculating the lowest order commutator, the second order terms in the gauge transformation should be taken into account. Assume that up to second order in g one has

(9.15) $$\delta\phi = g\, \phi_{,\nu}\, \xi^{\nu} + g^2(x_1\phi\, h_{\rho\rho,\nu}\xi^{\nu} + x_2\phi\, h_{\rho\nu,\nu}\xi^{\rho})$$

Other terms of order g^2 amount to spin-2 gauge parameter redefinitions or field redefinitions. The lowest order commutator is

(9.16) $$[\delta^{\xi_2},\delta^{\xi_1}]\,\phi = g^2\{\phi_{,\nu}({\xi_1}^{\nu,\lambda}{\xi_2}^{\lambda} - {\xi_2}^{\nu,\lambda}{\xi_1}^{\lambda})$$
$$+x_1\phi(2{\xi_1}^{\rho,\rho\nu}{\xi_2}^{\nu}-2{\xi_2}^{\rho,\rho\nu}{\xi_1}^{\nu}) + x_2\phi(({\xi_1}^{\rho,\nu\nu}+{\xi_1}^{\lambda,\lambda\rho}){\xi_2}^{\rho}-({\xi_2}^{\rho,\nu\nu}+{\xi_2}^{\nu,\nu\rho}){\xi_1}^{\rho})\}$$

If we require the r.h.s of (9.16) to be of the same form as (9.15), one must have $x_1 = x_2 = 0$. So we find

(9.17) $$[\delta_{\xi_2},\delta_{\xi_1}]\,\phi = g^2\,\phi_{,\nu}\,(\xi_1{}^{\nu,\lambda}\xi_2{}^{\lambda}-\xi_2{}^{\nu,\lambda}\xi_1{}^{\lambda}) = g^2\,\phi_{,\nu}\,[\xi_2,\xi_1]^{\nu}$$

and $[\xi_2,\xi_1]$ is the familiar commutator of general coordinate transformations

(9.18) $$[\xi_2,\xi_1]^{\nu} = \xi_1^{\nu,\lambda}\,\xi_2{}^{\lambda} - \xi_2{}^{\nu,\lambda}\xi_1{}^{\lambda}$$

To make the transformation rule of the scalar field consistent with the transformation rule of the spin-2 field, one is forced to include a spin-2 selfinteraction as discussed in section (VII.3). The scalar field and the spin-2 field then obey the same commutator.

Finally one can consider the change in ϕ when one commutes a spin-1 and a spin-2 transformation. Calculating the commutator on the scalar field, denoting the spin-1 coupling strength with g, and the spin-2 one with g', we find

(9.19) $$[\delta^{\xi},\delta^{\Lambda}]\,\phi^{i} = g\,g'\,c^{ija}\,\phi^{j}\,\Lambda^{a}{}_{,\nu}\,\xi^{\nu}$$

Hence the commutator of a "charge" transformation and a general coordinate transformation is again a charge transformation. The charge transformation parameter transforms as a scalar under general coordinate transformations.

(9.20) $$[\delta^{\xi},\delta^{\Lambda}]\,\phi^{i} = \delta^{g'[\xi,\Lambda]}\,\phi^{i}$$

$$[\xi,\Lambda] = \Lambda^{a}{}_{,\nu}\,\xi^{\nu}$$

To lowest order, this commutator can be cancelled if we add a second order term to the gauge transformation of ϕ

(9.21) $$g\,g'\,c^{ija}\,\Lambda^{a}{}_{\mu}\,\xi^{\mu}$$

This second order term amounts to gauge parameter redefintion of the gauge parameter Λ.

3. Coupling to a gauge field of arbitrary spin

Let us try to find a first order interaction linear in a spin-s gauge boson field $A_{\mu_1\cdots\mu_s}$, and quadratic in a massless scalar field ϕ^{i}. A suitable first order interaction lagrangian L_1 should satisfy the constraint

$$(9.22) \qquad \partial^{\rho}\left(L_{1,A_{\rho\mu_2\ldots\mu_s}} - \frac{1}{2(s-1)} \sum_{\mu}^{2} \eta_{\mu_2\mu_3} L_{1,A\lambda_{\lambda\rho\mu_4\ldots\mu_s}}\right) = 0$$

$$\text{for all } \phi(x) \text{ which satisfy } L_{0,\phi} = \phi_{,\nu\nu} = 0$$

How many derivatives do we need? First we note that since the gauge field is double traceless, at least s-2 derivatives are needed. Even s-2 derivatives are not enough as follows from a straightforward calculation. But for s derivatives a (modulo field redefinitions) unique first order interaction exists. We briefly indicate how to obtain this lagrangian. By means of partial intergrations, all derivatives acting on the gauge field may be shifted away. Then all lagrangians involving $\phi_{,\lambda\lambda}$ or $\phi^{i}_{,\lambda}\phi^{j}_{,\lambda}$ are related to field redefinitions. So only lagrangians of the type

$$(9.23) \qquad L_1 = A_{\mu_1\ldots\mu_s} \phi^{i}_{,\mu_1\ldots\mu_k} \phi^{j}_{,\mu_{k+1}\ldots\mu_s}$$

need to be considered.

One and only one combination of lagrangians of type (9.23) satisfies the constraint (9.22). It is given by

$$(9.24) \qquad L_1 = c^{ij} A_{\mu_1\ldots\mu_s} \sum_{k=0}^{s} \binom{s}{k} (-1)^k \phi^{i}_{,\mu_1\ldots\mu_k} \phi^{j}_{,\mu_{k+1}\ldots\mu_s}$$

Implied by (9.24) is the symmetry character of the coefficients c_{ij}:

$$(9.25) \qquad c^{ij} = (-1)^s c^{ji}$$

By means of field redefinitions one can modify this first order interaction to an equivalent one which corresponds to a particular simple transformation rule. This equivalent first order lagrangian is

$$(9.26) \qquad L_1 = N(s) A_{\mu_1\ldots\mu_s} c^{ij}\Big\{ \sum_{k=0}^{s}(-1)^k \binom{s}{k} \phi^{i}_{,\mu_1\ldots\mu_k} \phi^{j}_{,\mu_{k+1}\ldots\mu_s}$$

$$+ \sum_{k=0}^{s-2} \sum_{\mu}^{2} a_k \eta_{\mu_1\mu_2} \phi^{i}_{,\lambda\lambda\mu_3\ldots\mu_{k+2}} \phi^{j}_{\mu_{k+3}\ldots\mu_s}\Big\}$$

with

$$c^{ij} = (-1)^s c^{ji} ,$$

(9.27)

$$a_k = (-1)^k s \left\{ \sum_{m=0}^{k} \binom{s-1}{m} - 2^{s-1} \right\} ,$$

$$N(s) = \left[-s\ 2^s\right]^{-1}$$

The first order transformation rule of the scalar field which corresponds to the above interaction lagrangian is simply given by

$$(9.28) \qquad \delta\phi^i = g\ c^{ij}\ \phi^j{}_{,\mu_1\cdots\mu_{s-1}}\ \xi^{\mu_1\cdots\mu_{s-1}}$$

We have established in this section the following. A first order interaction between a spin-s gauge field and two massless scalar fields involves at least s derivatives. For s derivatives this interaction is unique modulo field redefinitions. Field redefinitions allow us to cast either the lagrangian or the transformation rule for the scalar in a simple form.

4. Starting from a global symmetry

It is of interest that the first order interaction between massless scalars and a spin-s gauge field found in the preceding section, also can be obtained by different means. As we mentioned before, looking for new theories can be done in two ways. Either start from a known global symmetry, or start from a locally invariant free theory. In this thesis we have been pursuing the latter approach, which is the easiest when one looks for selfinteractions. However, to make contact with the other apporoach, we show in this section how the first order interactions between scalars and gauge field are related to global invariances of the action for massless scalar fields

$$(9.29) \qquad S_0 = \tfrac{1}{2} \int d^4x\ \phi^i{}_{,\mu}\ \phi^i{}_{,\mu}$$

It is well known that the action (9.29) is invariant for infinitesimal "phase transformations" and infinitesimal "translations":

$$(9.30) \qquad \begin{aligned} \phi^i(x) &\to \phi^i(x) + \xi^{ij}\ \phi^j(x) \qquad \text{where } \xi^{ij} = -\xi^{ji} \\ \phi^i(x) &\to \phi^i(x) + \xi^{\lambda}\ \phi^i{}_{,\lambda}(x) \end{aligned}$$

Here the parameters ξ are not space-time dependent. The above invariance can be easily generalized. The action (9.29) is invariant for all global transformations

$$(9.31)\quad \phi^i(x) \rightarrow \phi^i(x) + \xi^{ij}_{\mu_1\cdots\mu_{s-1}}\, \phi^j_{,\mu_1\cdots\mu_{s-1}}(x) \qquad \text{where } \xi^{ij} = (-1)^s \xi^{ji}$$

It is a natural question to ask whether these global invariances can generalized to local ones. If we calculate the variation of the action (9.29) for space-time dependent transformations, one finds

$$(9.32)\qquad \delta S_0 = \int d^4x \,\big(\phi^i_{,\lambda}\, \xi(x)^{ij}_{\mu_1\cdots\mu_{s-1},\lambda}\, \phi^j_{,\mu_1\cdots\mu_{s-1}}$$

$$+ \text{ terms involving } \partial\cdot\xi \text{ or } \Box\xi\big)$$

Assume there is a gauge field $A^{ij}_{\mu_1\cdots\mu_s}$ which is fully antisymmetric and obeys to lowest order the transformation rule

$$(9.33)\qquad \delta A^{ij}_{\mu_1\cdots\mu_s} = \Sigma^1_{\mu}\, \partial_{\mu_1}\, \xi^{ij}_{\mu_2\cdots\mu_s}$$

Then the first order lagrangian

$$(9.34)\qquad L_1 = \frac{1}{s}\, A^{ij}_{\mu_1\cdots\mu_s}\, \phi^i\, \phi^j_{\mu_1\cdots\mu_s}$$

will cancel δS_0 modulo terms involving $\partial\cdot\xi$ and $\Box\xi$. The latter can all be obtained by field redefinitions. Working out the details, and defining

$$(9.35)\qquad A^{ij}_{\mu_1\cdots\mu_s}(x) = c^{ij}\, A_{\mu_1\cdots\mu_s}(x)$$

it appears we have found the same class of interactions as in the previous section. Actually it was by means of the "global invariace" method sketched here, that we obtained the scalar-gauge interactions first. So far, it has been neither required that the gauge field is double traceless, nor that the gauge parameter is traceless. These constraints only enter when one introduces the free lagrangians of chapter II in order to give the gauge field the correct dynamical degrees of freedom.

5. Towards an algebraic structure?

Let us study the commutator of two gauge transformations on a massless scalar. The transformation rule is given by

$$(9.36)\qquad \delta\phi^i = g\, c^{ija}\, \phi^j_{,\mu_1\cdots\mu_{s-1}}\, \xi^a_{\mu_1\cdots\mu_{s-1}}$$

which is the same as (9.27), apart from the allowance for several gauge parameters ξ^a.

Calculating the commutator on the field we find

$$(9.37) \quad [\delta^{\xi_2}, \delta^{\xi_1}]\ \phi^i = \{c^{ijk}\ c^{jkb}(\phi^k_{,\mu_1\cdot\cdot\mu_{s-1}}\ \xi_1^{b\mu_1\cdot\cdot\mu_{s-1}})_{,\nu_1\cdot\cdot\nu_{s-1}}\ \xi_2^{a\nu_1\cdot\cdot\nu_{s-1}}\}$$

$$- \{\ \xi_2 \ - \ \xi_1\ \}$$

Here the same phenomenon occurs as we already met in our discussion of the algebraic structure of the spin-3 selfinteraction. The commutator on the field of two spin-s transformations contains terms with more derivatives than the original spin-s transformation.

In principle quartic terms in the lagrangian can also give rise to terms proportional to g^2 in the transformation rule. When bilinear in A and ϕ, they can contribute to the lowest order commutator. However, they cannot cancel the terms with the highest or next highest number of derivatives acting on ϕ in (9.37).

Again this is very suggestive of an algebraic structure involving all higher spins together.

Perhaps one should not study the transformations of a scalar matter field only, but of an infinite set of matter fields of every spin. Then it may be advantageous to make a study of two-parameter functions f(x,k) of two four vectors x and k. By means of making a Taylor series expansion in k one recovers formulas applying to tensor fields. In this formalism the transformation law of the matter field would read

$$(9.38) \qquad \phi(x,k) \rightarrow \phi(x,k) + \{\phi(x,k), \xi(x,k)\}$$

where we still have to find an explicit expression for the bracket.

An example of a bracket which satisfies the Jacobi identity is the following generalisation of the Poisson Bracket

$$(9.39) \qquad \{f,\ g\} = \sum_k \frac{k^n}{n!} \left(\frac{\partial^n}{\partial x^n} f\right)\left(\frac{\partial^n}{\partial k^n} g\right) - (g \ - \ f)$$

For scalar fields this yields:

$$(9.40) \qquad \phi(x) \rightarrow \phi(x) + \sum_{s} \frac{k^n}{n!} \phi_{,\lambda_1 \cdots \lambda_s} \xi^{\lambda_1 \cdots \lambda_{s-1}}$$

This look more or less all right, but we must emphasize here it is not correct, as we did not include the tracelessness of the gauge parameter. Nor does it reproduces the lowest order term of the commutator we have found for the spin-3 selfinteraction.

A more far fetched possibility, which remains to be studied, is that the full transformation rule does involve non-local operators. This would mean the scheme sketched in chapters III and IV is no longer valid. Then the transformation rule (9.36) would be a first order approximation which happens to contain only local operators, but it would not be possible to make higher order approximations using only local operators.

APPENDIX A: NOTATIONS AND CONVENTIONS

Most of our notations and conventions are the same as those of Bjorken & Drell and of Itzykson & Zuber [8].

Unless otherwise stated indices are raised and lowered with the Minkowski metric

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Summation over repeated indices of any kind is implied, unless otherwise stated.

For convenience sake, upper and lower indices are not always properly matched if no confusion can arise.

Derivatives will often denoted by a comma: $f_{,\mu} \equiv \partial_\mu f \equiv \frac{\partial f}{\partial x^\mu}$.

The gamma-matrices satisfy $\{\gamma^\mu,\gamma^\nu\} = \gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$.
The definitions of γ^5 and $\sigma^{\mu\nu}$ are $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu,\gamma^\nu]$.
The commutator of the latter is $[\sigma_{\mu\nu},\sigma_{\alpha\beta}] = \eta_{\mu\beta}\sigma_{\nu\alpha} + \eta_{\nu\alpha}\sigma_{\mu\beta} - \eta_{\nu\beta}\sigma_{\mu\alpha} - \eta_{\mu\alpha}\sigma_{\nu\beta}$.

We work most of the time with <u>real</u> bose fields and anticommuting <u>Majorana</u> fermion fields. Useful identies for Majorana spinors ψ_1 and ψ_2 are:

$$\bar\psi_1\psi_2 = \bar\psi_2\psi_1$$
$$\bar\psi_1\gamma^\mu\psi_2 = -\bar\psi_2\gamma^\mu\psi$$
$$\bar\psi_1\gamma^\mu\gamma^\nu\psi_2 = \bar\psi_2\gamma^\nu\gamma^\mu\psi_1 \qquad \text{etc.}$$

The abbreviations "l.h.s." and "r.h.s" stand for "left hand side" and "right hand side".

Throughout this thesis we use units in which $\hbar = c = 1$.

With the symbol $L_{,\phi}$ we denote the l.h.s of the equation of motion as derived from a lagrangian (density) $L=L(\phi,\partial\phi,\partial\partial\phi,\partial\partial\partial\phi,\ldots)$:

$$L_{,\phi} = \frac{\partial L}{\partial\phi} - \partial_\mu \frac{\partial L}{\partial\phi_{,\mu}} + \partial_\mu\partial_\nu \frac{\partial L}{\partial\phi_{,\mu\nu}} - \partial_\mu\partial_\nu\partial_\rho \frac{\partial L}{\partial\phi_{,\mu\nu\rho}} + \ldots$$

APPENDIX B: LAGRANGIANS FOR FREE MASSLESS FIELDS OF SPIN 0, 1/2, 1, 3/2, 2, 5/2 AND 3

$$L_0(0) = \tfrac{1}{2}\,\phi_{,\mu}\,\phi_{,\mu}$$

$$L_0(1/2) = i\tfrac{1}{2}\,\bar{\psi}\not\partial\psi$$

$$L_0(1) = \tfrac{1}{2}(-A_{\mu,\nu}A_{\mu,\nu} + A_{\mu,\nu}A_{\nu,\mu})$$

$$L_0(3/2) = i\tfrac{1}{2}(-\bar{\psi}_\mu\not\partial\psi_\mu + \bar{\psi}_\mu\not\psi_{,\mu} + \bar{\not\psi}\psi_{\mu,\mu} - \bar{\not\psi}\not\partial\not\psi)$$

$$L_0(2) = \tfrac{1}{2}(h_{\mu\nu,\rho}h_{\mu\nu,\rho} - 2\,h_{\mu\nu,\rho}h_{\mu\rho,\nu} + 2\,h_{\mu\nu,\mu}h_{\rho\rho,\nu} - h_{\rho\rho,\mu}h_{\nu\nu,\mu})$$

$$L_0(5/2) = i\tfrac{1}{2}(\bar{\psi}_{\mu\nu}\not\partial\psi_{\mu\nu} - \tfrac{1}{2}\,\bar{\psi}_{\rho\rho}\not\partial\psi_{\sigma\sigma} + 2\,\bar{\not\psi}_\mu\not\partial\not\psi_\mu + 2\,\bar{\psi}_{\rho\rho}\not\psi_{\lambda,\lambda} - 4\,\bar{\not\psi}_\mu\psi_{\mu\lambda,\lambda})$$

$$L_0(3) = \tfrac{1}{2}(-\phi_{\alpha\beta\gamma,\lambda}\phi_{\alpha\beta\gamma,\lambda} + 3\,\phi_{\alpha\beta\gamma,\gamma}\phi_{\alpha\beta\lambda,\lambda} + 3\,\phi_{\alpha\alpha\gamma,\rho}\phi_{\beta\beta\gamma,\rho}$$

$$+ 3/2\,\phi_{\alpha\alpha\lambda,\lambda}\phi_{\gamma\gamma\rho,\rho} - 6\,\phi_{\alpha\beta\gamma,\gamma}\phi_{\alpha\lambda\lambda,\beta})$$

APPENDIX C: CUBIC FIRST ORDER INTERACTIONS, TRANSFORMATIONS AND COMMUTATORS

spin 0-0-0 $\quad D = -1$ (D is the minus the dimension of the coupling constant, see section VI.2)

First order lagrangian:

$$L_1 = \phi\,\phi\,\phi$$

No first order gauge transformation.

No non-zero first order commutator.

spin 0-1/2-1/2 $\quad D = 0$

First order lagrangian:

$$L_1 = i\,c^{ij}\,\phi\,\bar{\psi}^i\psi^j$$

with $c^{ij} = c^{ji}$.

No first order gauge transformation.

No non-zero first order commutator.

spin 0-1-1 $\quad D = 1$

First order lagrangian:

$$L_1 = c^{ij}\,\phi\,F^i_{\mu\nu}\,F^j_{\mu\nu}$$

with $c^{ij} = c^{ji}$ and $F^i_{\mu\nu} = A^i_{\mu,\nu} - A^i_{\nu,\mu}$.

Gauge transformation:

$$A^i_\mu \rightarrow A^i_\mu + \partial_\mu\Lambda^i$$

No non-zero first order commutator.

spin 0-3/2-3/2 D = 2

First order lagrangian:

$$L_1 = i\, c^{ij}\, \phi\, \bar{\Gamma}^i_{\mu\nu}\, \bar{\Gamma}^j_{\mu\nu}$$

with $c^{ij} = c^{ji}$ and $\Gamma^i_{\mu\nu} = \psi^i_{\mu,\nu} - \psi^i_{\nu,\mu}$.

Gauge transformation:

$$\psi^i_\mu \to \psi^i_\mu + \partial_\mu \varepsilon$$

No non-zero first order commutator.

spin 0-2-2 D = 3

First order lagrangian:

$$L_1 = c^{ij}\, \phi\, R^i_{\lambda\mu\nu\kappa}\, R^j_{\lambda\mu\nu\kappa}$$

with $c^{ij} = c^{ji}$ and $R^i_{\lambda\mu\nu\kappa} = h^i_{\lambda\nu,\kappa\mu} - h^i_{\mu\nu,\lambda\kappa} - h^i_{\lambda\kappa,\mu\nu} + h^i_{\mu\kappa,\nu\lambda}$

Gauge transformation:

$$h^i_{\mu\nu} \to h^i_{\mu\nu} + \xi^i_{\mu,\nu} + \xi^i_{\nu,\mu}$$

No non-zero first order commutator.

spin 1-0-0 D = 0

First order lagrangian:

$$L_1 = -c^{ij}\, A_\mu\, \phi^i_{,\mu}\, \phi^j$$

with $c^{ij} = -c^{ji}$.

Gauge transformation:

$$A_\mu \to A_\mu + \partial_\mu \Lambda$$

$$\phi^i \rightarrow \phi^i + g\, c^{ij}\, \phi^j\, \Lambda$$

No non-zero first order commutator.

spin 1-1/2-1/2 D = 0

First order lagrangian:

$$L_1 = -\frac{i}{2}\, c^{ij}\, A_\mu\, \bar{\psi}^i\, \gamma_\mu\, \psi^j$$

with $c^{ij} = -c^{ji}$.

Gauge transformation:

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$$

$$\psi^i \rightarrow \psi^i + g\, c^{ij}\, \psi^j\, \Lambda$$

No non-zero first order commutator.

spin 1-1-1 D = 0

First order lagrangian:

$$L_1 = f^{ijk}\, A^i_{\mu,\nu}\, A^j_\mu\, A^k_\nu$$

with $f^{ijk} = -f^{jik} = -f^{ikj}$.

Gauge tranformation:

$$A^i_\mu \rightarrow A^i_\mu + \partial_\mu \Lambda^i + g\, f^{ijk}\, A^j_\mu\, \Lambda^k$$

First order commutator:

$$[\Lambda_2, \Lambda_1]^i = f^{ijk}\, \Lambda_1{}^j\, \Lambda_2{}^k$$

spin 1-3/2-3/2 D = 1

First order lagrangian:

$$L_1 = i\, c^{ij} F_{\mu\nu}\left(\bar{\psi}^i_\mu \psi^j_\nu - \bar{\psi}^i_\rho \gamma_\rho \gamma_\mu \psi^j_\nu - \tfrac{1}{2}\bar{\psi}^i_\lambda \sigma_{\mu\nu} \psi^j_\lambda + \tfrac{1}{4}\bar{\psi}^i_\rho \gamma_\rho \sigma_{\mu\nu} \gamma_\lambda \psi^j_\lambda\right)$$

with $c^{ij} = -c^{ji}$ and $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$.

Gauge transformation:

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda - i\, g\, c^{ij}\, \bar{\varepsilon}^i \psi^j_\nu$$

$$\psi^i_\mu \rightarrow \psi^i_\mu + \partial_\mu \varepsilon^i - \tfrac{1}{2}\, g\, c^{ij} F_{\lambda\rho}\, \sigma_{\lambda\rho}\, \gamma_\mu\, \varepsilon^j$$

Non-zero first order commutator:

$$[\delta_{\varepsilon_2}, \delta_{\varepsilon_1}]\, A_\mu = \delta_{g[\varepsilon_2,\varepsilon_1]}\, A_\mu + O(g^2)$$

$$\text{with} \quad [\varepsilon_2, \varepsilon_1] = i\, c^{ij}\, \bar{\varepsilon}_1^i\, \varepsilon_2^j$$

<u>spin 1-2-2</u> $D = 2$

First order lagrangian:

$$L_1 = c^{ij} F_{\mu\nu}\Big(\tfrac{1}{2} h^i_{\mu\rho,\sigma} h^j_{\nu\rho,\sigma} - h^i_{\mu\rho,\sigma} h^j_{\nu\sigma,\rho} + h^i_{\rho\sigma,\mu} h^j_{\nu\rho,\sigma}$$
$$- \tfrac{1}{2} h^i_{\rho\sigma,\mu} h^j_{\rho\sigma,\nu} + \tfrac{1}{2} h^i_{\mu\rho,\rho} h^j_{\nu\sigma,\sigma} - h^i_{\rho\rho,\mu} h^j_{\nu\sigma,\sigma}$$
$$+ \tfrac{1}{2} h^i_{\rho\rho,\mu} h^j_{\sigma\sigma,\nu} + h^i_{\rho\sigma,\sigma} h^j_{\rho\mu,\nu} - h^i_{\sigma\sigma,\rho} h^j_{\rho\mu,\nu}\Big)$$

with $c^{ij} = -c^{ji}$ and $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$.

Gauge transformation:

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda + g\, c^{ij} h^i_{\mu\nu,\sigma}(\xi^j_{\nu,\sigma} - \xi^j_{\sigma,\nu})$$

$$h^i_{\mu\nu} \rightarrow h^i_{\mu\nu} + \xi^i_{\mu,\nu} + \xi^i_{\nu,\mu} + g\, c^{ij}\, \Sigma' \left[F_{\mu\lambda}(\xi^j_{\nu,\lambda} - \xi^j_{\lambda,\nu}) - \tfrac{1}{2}\eta_{\mu\nu} F_{\lambda\sigma}\, \xi^j_{\lambda,\sigma}\right]$$

Here $\Sigma' H_{\mu\nu}$ stands for $\tfrac{1}{2}(H_{\mu\nu} + H_{\nu\mu})$.

Non-zero first order commutator:

$$[\delta_{\xi_2}, \delta_{\xi_1}]\, A_\mu = \delta_{g[\xi_2,\xi_1]}\, A_\mu + O(g^2)$$

$$\text{with} \quad [\xi_2, \xi_1] = c^{ij}\, (\xi_1{}^i_{\nu,\sigma} \xi_2{}^j_{\nu,\sigma} - \xi_1{}^i_{\nu,\sigma} \xi_2{}^j_{\sigma,\nu})$$

<u>spin 2-0-0</u> D = 1

First order lagrangian:

$$L_1 = -\tfrac{1}{2} c^{ij} (h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h_{\rho\rho}) \phi^i_{,\mu} \phi^j_{,\nu}$$

with $c^{ij} = c^{ji}$.

Gauge transformation:

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}$$

$$\phi^i \rightarrow \phi^i + g\, c^{ij} \phi^j_{,\lambda} \xi_\lambda$$

No non-zero first order commutator.

<u>spin 2-1/2-1/2</u> D = 1

First order lagrangian:

$$L_1 = -\frac{i}{4} c^{ij} (h_{\mu\nu} \bar{\psi}^i \gamma_\mu \psi^j_{,\nu} - h_{\rho\rho} \bar{\psi}^i \gamma_\lambda \psi^j_{,\lambda})$$

with $c^{ij} = c^{ji}$.

Gauge transformation:

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}$$

$$\psi^i \rightarrow \psi^i + g\, c^{ij} (\psi^j_{,\nu} \xi_\nu - \tfrac{1}{2}\xi_{\nu,\lambda} \sigma_{\nu\lambda} \psi^j)$$

No non-zero first order commutator.

<u>spin 2-1-1</u> D = 1

First order lagrangian:

$$L_1 = \tfrac{1}{2} c^{ij} (h_{\mu\nu} F^i_{\mu\rho} F^j_{\nu\rho} - \tfrac{1}{4} h_{\lambda\lambda} F^i_{\mu\nu} F^j_{\mu\nu})$$

with $c^{ij} = c^{ji}$ and $F^i_{\mu\nu} = A^i_{\mu,\nu} - A^i_{\nu,\mu}$.

Gauge transformation:

$$h_{\mu\nu} \to h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}$$

$$A^i_{\ \mu} \to A^i_{\ \mu} + \partial_\mu \Lambda^i + g\, c^{ij}\, F^j_{\ \mu\nu}\xi_\nu$$

No non-zero first order commutator (modulo gauge parameter redefinitions).

<u>spin 2–3/2–3/2</u> $\quad D = 1$

First order lagrangian:

$$L_1 = c^{ij}\big(-\tfrac{1}{4}h_{\rho\rho}\,\bar\psi^i_{\ \nu}\gamma_\lambda\psi^j_{\ \nu,\lambda} + \tfrac{1}{4}h_{\rho\rho}\,\bar\psi^i_{\ \nu}\gamma_\lambda\psi^j_{\ \lambda,\nu} - \tfrac{1}{4}h_{\rho\rho}\,\bar\psi^i_{\ \lambda}\gamma_\lambda\gamma_\rho\gamma_\sigma\psi^j_{\ \sigma,\rho}$$
$$+ \tfrac{1}{4}h_{\mu\nu}\,\bar\psi^i_{\ \lambda}\gamma_\mu\psi^j_{\ \lambda,\nu} - \tfrac{1}{2}h_{\mu\nu}\,\bar\psi^i_{\ \lambda}\gamma_\mu\psi^j_{\ \nu,\lambda} + \tfrac{1}{2}h_{\mu\nu}\bar\psi^i_{\ \mu}\gamma_\lambda\psi^j_{\ \nu,\lambda}$$
$$- \tfrac{1}{4}h_{\mu\nu}\,\bar\psi^i_{\ \lambda}\gamma_\lambda\psi^j_{\ \mu,\nu} + \tfrac{1}{4}h_{\mu\nu}\,\bar\psi^i_{\ \mu}\gamma_\nu\gamma_\rho\gamma_\sigma\psi^j_{\ \sigma,\rho} + \tfrac{1}{4}h_{\mu\nu}\,\psi^i_{\ \lambda}\gamma_\lambda\gamma_\mu\gamma_\rho\psi^j_{\ \rho,\nu}$$
$$- 3/4\; h_{\mu\nu}\,\bar\psi^i_{\ \mu}\gamma_\lambda\psi^j_{\ \lambda,\nu} + \tfrac{1}{4}h_{\mu\nu}\,\bar\psi^i_{\ \lambda}\gamma_\lambda\gamma_\rho\gamma_\mu\psi^j_{\ \nu,\rho}\big)$$

with $c^{ij} = c^{ji}$.

Gauge transformation:

$$h_{\rho\sigma} \to h_{\rho\sigma} + \xi_{\rho,\sigma} + \xi_{\sigma,\rho} - g\, c^{ij}\,\tfrac{1}{8}\,(\bar\varepsilon^i\gamma_\rho\psi^j_{\ \sigma} + \bar\varepsilon^i\gamma_\sigma\psi^j_{\ \rho})$$

$$\psi^i_{\ \mu} \to \psi^i_{\ \mu} + \varepsilon_{,\mu} + \tfrac{1}{2}g\, c^{ij}\, h_{\mu\rho,\lambda}\,\sigma_{\rho\lambda}\varepsilon^j + g\, c^{ij}(\xi_{\nu,\mu}\psi^j_{\ \nu} + \xi_\nu\psi^j_{\ \mu,\nu} - \tfrac{1}{2}\xi_{\nu,\lambda}\sigma_{\nu\lambda}\psi^j_{\ \mu})$$

Non-zero first order commutators:

$$[\delta_{\varepsilon_2},\delta_{\varepsilon_1}]\, h_{\mu\nu} = \delta_{g[\varepsilon_2,\varepsilon_1]}\, h_{\mu\nu} + O(g^2)$$

$$\text{with} \quad [\varepsilon_2,\varepsilon_1]_\mu = -\, c^{ij}\,\tfrac{1}{8}\,\bar\varepsilon_2\,\gamma_\mu\,\varepsilon_1$$

$$[\delta_\xi,\delta_\varepsilon]\,\psi^i_{\ \mu} = \delta_{g[\xi,\varepsilon]}\,\psi^i_{\ \mu} + O(g^2)$$

$$\text{with} \quad [\xi,\varepsilon]^i = c^{ij}\,\xi_\nu\,\varepsilon^j_{\ ,\nu} - \tfrac{1}{4}\, c^{ij}\,\sigma_{\rho\lambda}\,\xi_{\rho,\lambda}\,\varepsilon^j$$

<u>spin 2–2–2</u> $\quad D = 1$

First order lagrangian:

$$L_1 = h_{\mu\lambda,\lambda} h_{\mu\rho,\rho} h_{\sigma\sigma} - 3/2\, h_{\mu\nu,\rho} h_{\mu\rho,\nu} h_{\sigma\sigma} + \tfrac{1}{4} h_{\mu\nu,\rho} h_{\mu\nu,\rho} h_{\sigma\sigma} +$$

$$\tfrac{1}{2} h_{\mu\lambda,\lambda} h_{\rho\rho,\mu} h_{\sigma\sigma} - \tfrac{1}{4} h_{\lambda\lambda,\mu} h_{\rho\rho,\mu} h_{\sigma\sigma} - 2\, h_{\lambda\lambda,\mu} h_{\mu\rho,\sigma} h_{\rho\sigma} + h_{\mu\nu} h_{\mu\sigma,\rho} h_{\nu\rho,\sigma}$$

$$+ 2\, h_{\mu\nu} h_{\nu\rho,\sigma} h_{\rho\sigma,\mu} - h_{\mu\nu} h_{\mu\rho,\sigma} h_{\nu\rho,\sigma} - \tfrac{1}{2} h_{\mu\nu} h_{\rho\sigma,\mu} h_{\rho\sigma,\nu}$$

$$+ \tfrac{1}{2} h_{\mu\nu} h_{\lambda\lambda,\mu} h_{\rho\rho,\nu} - h_{\lambda\mu,\lambda} h_{\rho\sigma,\mu} h_{\rho\sigma} + h_{\lambda\lambda,\mu} h_{\rho\sigma,\mu} h_{\rho\sigma})$$

Gauge transformation:

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} + g\,(h_{\mu\nu,\rho}\xi_\rho + h_{\rho\mu}\xi_{\rho,\nu} + h_{\rho\nu}\xi_{\rho,\mu})$$

Non-zero first order commutator:

$$[\xi_2,\xi_1]_\mu = \xi_2^{\rho}\, \xi_1^{\mu,\rho} - \xi_2^{\mu,\rho} \xi_1^{\rho}$$

<u>spin 2-5/2-5/2</u> D = 2

First order lagrangian:

$$L_1 = c^{ij}\, \frac{i}{8}\, \{R_{\mu\nu\lambda\kappa}\, (4\bar\psi^i_{\mu\nu}\psi^j_{\lambda\kappa} - \bar\psi^i_{\rho\nu}\gamma_\mu\gamma_\lambda\psi^j_{\kappa\rho} - 2\bar\psi^i_{\rho\lambda}\gamma_\rho\gamma_\kappa\psi^j_{\mu\nu}$$

$$-2\bar\psi^i_{\mu\nu}\gamma_\kappa\gamma_\rho\psi^j_{\rho\lambda} + 2\bar\psi^i_{\rho\nu}\gamma_\rho\gamma_\mu\gamma_\lambda\gamma_\sigma\psi^j_{\sigma\kappa})$$

$$+ R_{\mu\nu}\, (6\bar\psi^i_{\mu\lambda}\psi^j_{\nu\lambda} - 6\bar\psi^i_{\mu\nu}\psi^j_{\lambda\lambda} - 2\psi^i_{\mu\rho}\gamma_\rho\gamma_\lambda\psi^j_{\lambda\nu} + 4\bar\psi^i_{\lambda\rho}\gamma_\rho\gamma_\nu\psi^j_{\mu\lambda})$$

$$+ R\, (-2\bar\psi^i_{\mu\lambda}\psi^j_{\mu\lambda} + 3/2\bar\psi^i_{\mu\mu}\psi^j_{\lambda\lambda} - \bar\psi^i_{\lambda\rho}\gamma_\rho\gamma_\sigma\psi^j_{\sigma\lambda})\}$$

with $c^{ij} = c^{ji}$ and $R_{\mu\nu\lambda\kappa} = h_{\mu\nu,\lambda\kappa} + h_{\lambda\kappa,\mu\nu} - \tfrac{1}{2}(h_{\mu\kappa,\nu\lambda} + h_{\nu\lambda,\mu\kappa} + h_{\nu\kappa,\mu\lambda} + h_{\mu\lambda,\nu\kappa})$

Gauge transformation $(\gamma_\mu\varepsilon_\mu = 0)$:

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} + i\, g\, c^{ij}\, \Sigma'\, (2\bar\psi^i_{\mu\lambda}\varepsilon^j_{\lambda,\nu} + \bar\psi^i_{\mu\nu,\lambda}\varepsilon^j_\lambda)$$

$$\psi^i_{\mu\nu} \rightarrow \psi^i_{\mu\nu} + \varepsilon^i_{\mu,\nu} + \varepsilon^i_{\nu,\mu} - g\, c^{ij}\, \Sigma'\, (R_{\mu\nu\lambda\kappa}\gamma_\lambda\varepsilon^j_\kappa - \tfrac{1}{4}R_{\mu\sigma\lambda\kappa}\gamma_\nu\gamma_\sigma\gamma_\kappa\varepsilon^j_\lambda)$$

Here $\Sigma'\, H_{\mu\nu}$ stands for $\tfrac{1}{2}(H_{\mu\nu} + H_{\nu\mu})$.

N.B. To show that these are the correct gauge transformations one needs to use the so-called Schouten identity. This identity simply states that in four dimensions fully antisymmetric tensors vanish if their rank is higher than or equal to five.

Non-zero first order commutator:

$$[\delta_{\varepsilon_2},\delta_{\varepsilon_1}]\, h_{\mu\nu} = \delta_{g[\varepsilon_2,\varepsilon_1]}\, h_{\mu\nu} + O(g^2)$$

$$\text{with} \quad [\varepsilon_2,\varepsilon_1]_\mu = i\, c^{ij} (\bar{\varepsilon}_1{}^i{}_{\mu,\lambda}\varepsilon_2{}^j{}_\lambda - \bar{\varepsilon}_2{}^j{}_{\mu,\lambda}\varepsilon_1{}^i{}_\lambda)$$

spin 3-0-0 D = 2

First order lagrangian:

$$L_1 = \frac{1}{24} c^{ij} \{\Phi_{\alpha\beta\gamma} (\phi^i{}_{,\alpha\beta\gamma}\phi^j - 3\phi^i{}_{,\alpha\beta}\phi^j{}_{,\gamma} + 3\phi^i{}_{,\alpha}\phi^j{}_{,\beta\gamma} - \phi^i\phi^j{}_{,\alpha\beta\gamma}) - 3\,\Phi_{\alpha\rho\rho} (\phi^i{}_{,\alpha\lambda\lambda}\phi^j - 3\phi^i{}_{,\lambda\lambda}\phi^j{}_{,\alpha})\}$$

with $c^{ij} = -c^{ji}$.

Gauge transformation ($\xi_{\rho\rho}=0$) :

$$\Phi_{\alpha\beta\gamma} \to \Phi_{\alpha\beta\gamma} + \xi_{\alpha\beta,\gamma} + \xi_{\beta\gamma,\alpha} + \xi_{\gamma\alpha,\beta}$$

$$\phi^i \to \phi^i + g\, c^{ij}\, \phi^j{}_{,\alpha\beta}\xi_{\alpha\beta}$$

No non-zero first order commutator.

spin 3-1/2-1/2 D = 2

First order lagrangian:

$$L_1 = \frac{i}{6} c^{ij} \{ \phi_{\alpha\beta\gamma}\, \bar{\psi}^i{}_{,\alpha}\gamma_\beta\psi^j{}_{,\gamma} - \Phi_{\rho\rho\alpha}(\tfrac{1}{4}\bar{\psi}^i{}_{,\lambda}\gamma_\alpha\psi^j{}_{,\lambda} + 2\bar{\psi}^i{}_{,\lambda}\gamma_\lambda\psi^j{}_{,\alpha} + \bar{\psi}^i{}_{,\lambda}\gamma_\lambda\gamma_\alpha\gamma_\rho\psi^j{}_{,\rho})\}$$

with $c^{ij} = -c^{ji}$.

Gauge transformation ($\xi_{\rho\rho}=0$) :

$$\phi_{\alpha\beta\gamma} \to \phi_{\alpha\beta\gamma} + \xi_{\alpha\beta,\gamma} + \xi_{\gamma\alpha,\beta} + \xi_{\beta\gamma,\alpha}$$

$$\psi^i \to \psi^i + g\, c^{ij} (\xi_{\alpha\beta}\, \psi^j{}_{,\alpha\beta} - 2/3\, \sigma_{\beta\delta}\xi_{\beta\alpha,\delta}\psi^j{}_{,\alpha})$$

No non-zero first order commutator.

spin 3-1-1 D = 2

First order lagrangian:

$$L_1 = c^{ij} \phi_{\alpha\beta\gamma} (F^i{}_{\mu\alpha} F^j{}_{\mu\beta,\gamma} - \tfrac{1}{4}\eta_{\alpha\beta} F^i{}_{\mu\nu} F^j{}_{\mu\nu,\gamma} + \eta_{\alpha\beta} F^i{}_{\mu\nu,\nu} F^j{}_{\mu\gamma})$$

with $c^{ij} = -c^{ji}$ and $F^i{}_{\mu\nu} = A^i{}_{\mu,\nu} - A^i{}_{\nu,\mu}$.

Gauge transformation ($\xi_{\rho\rho}=0$) :

$$\phi_{\alpha\beta\gamma} \rightarrow \phi_{\alpha\beta\gamma} + \xi_{\alpha\beta,\gamma} + \xi_{\beta\gamma,\alpha}\ \xi_{\gamma\alpha,\beta}$$

$$A^i{}_\mu \rightarrow A^i{}_\mu + \partial_\mu \Lambda^i + g\ c^{ij}\ (3\xi_{\alpha\beta} F^j{}_{\mu\alpha,\beta} - \xi_{\mu\alpha,\beta} F^j{}_{\beta\alpha})$$

No non-zero first order commutator.

spin 3-3/2-3/2 D = 2

First order lagrangian:

$$L_1 = i\ c^{ij}\ \phi_{\alpha\beta\gamma} (\bar{\Gamma}^i{}_{\alpha\mu} \gamma_\beta \Gamma^j{}_{\gamma\mu} - \tfrac{1}{4}\eta_{\beta\gamma} \bar{\Gamma}^i{}_{\mu\nu} \gamma_\alpha \Gamma^j{}_{\mu\nu})$$

with $c^{ij} = -c^{ji}$ and $\Gamma^i{}_{\mu\nu} = \psi^i{}_{\mu,\nu} - \psi^i{}_{\nu,\mu}$.

Gauge transformation ($\xi_{\rho\rho}=0$) :

$$\phi_{\alpha\beta\gamma} \rightarrow \phi_{\alpha\beta\gamma} + \xi_{\alpha\beta,\gamma} + \xi_{\beta\gamma,\alpha} + \xi_{\gamma\alpha,\beta}$$

$$\psi^i{}_\mu \rightarrow \psi^i{}_\mu + \partial_\mu \varepsilon^i + g\ c^{ij}\ (Y^j{}_\mu - \tfrac{1}{2}\ \gamma_\mu \gamma_\sigma Y^j{}_\sigma) \quad \text{with}$$

$$Y^j{}_\mu = (\xi_{\rho\sigma}(4\eta_{\lambda\rho} + 4\sigma_{\lambda\rho})\Gamma^j{}_{\sigma\mu})_{,\lambda} - (\xi_{\rho\sigma}(3\eta_{\mu\rho} + 2\sigma_{\mu\rho})\Gamma_{\sigma\lambda})_{,\lambda}$$

No non-zero first order commutator.

spin 3-2-2 D = 2

First order lagrangian:

$$L_1 = c^{ij}\,\phi_{\alpha\beta\gamma}\Big(-\tfrac{1}{8} h^i_{\mu\nu,\alpha\beta\gamma} h^j_{\mu\nu} + \tfrac{3}{8} h^i_{\mu\nu,\alpha\beta} h^j_{\mu\nu,\gamma} + \tfrac{1}{8} h^i_{\mu\mu,\alpha\beta\gamma} h^j_{\nu\nu}$$

$$- \tfrac{3}{8} h^i_{\mu\mu,\alpha\beta} h^j_{\nu\nu,\gamma} - h^i_{\mu\lambda,\lambda} h^j_{\mu\alpha,\beta\gamma} + 2\, h^i_{\mu\lambda,\beta} h^j_{\mu\alpha,\lambda\gamma} + 2\, h^i_{\alpha\lambda,\beta} h^j_{\beta\rho,\rho\lambda}$$

$$+ h^i_{\alpha\lambda,\beta\rho} h^j_{\gamma\rho,\lambda} - \tfrac{1}{2} h^i_{\alpha\lambda,\lambda\rho} h^j_{\beta\gamma,\rho} - h^i_{\alpha\lambda,\rho} h^j_{\beta\gamma,\rho\lambda} + h^i_{\lambda\rho} h^j_{\alpha\beta,\gamma\lambda\rho}$$

$$+ 3/2\, h^i_{\rho\lambda,\lambda} h^j_{\alpha\beta,\gamma\rho} + \tfrac{1}{2} h^i_{\lambda\rho,\lambda\rho} h^j_{\alpha\beta,\gamma} + \tfrac{1}{2} h^i_{\alpha\lambda} h^j_{\rho\rho,\lambda\beta\gamma} - h^i_{\alpha\lambda,\lambda} h^j_{\rho\rho,\beta\gamma}$$

$$+ \eta_{\alpha\beta}\big(- h^i_{\mu\lambda,\sigma} h^j_{\gamma\mu,\lambda\sigma} + h^i_{\gamma\lambda,\lambda} h^j_{\rho\rho,\sigma\sigma} - h^i_{\gamma\lambda,\sigma} h^j_{\mu\mu,\lambda\sigma} - \tfrac{1}{2} h^i_{\gamma\lambda} h^j_{\mu\mu,\lambda\sigma\sigma}$$

$$- 3/2\, h^i_{\mu\lambda,\lambda\gamma} h^j_{\mu\rho,\rho} + 3/8\, h^i_{\mu\lambda,\lambda\gamma\rho} h^j_{\mu\rho} - 7/8\, h^i_{\mu\lambda,\gamma} h^j_{\mu\rho,\rho\lambda} - 3/16\, h^i_{\lambda\rho,\lambda\rho\gamma} h^j_{\sigma\sigma}$$

$$+ 3/16\, h^i_{\lambda\rho,\gamma} h^j_{\sigma\sigma,\lambda\rho} + 9/16\, h^i_{\lambda\rho,\lambda\rho} h^j_{\sigma\sigma,\gamma} + 7/16\, h^i_{\lambda\rho} h^j_{\sigma\sigma,\gamma\lambda\rho} - h^i_{\lambda\rho,\sigma\lambda} h^j_{\gamma\sigma,\rho}$$

$$- \tfrac{1}{2} h^i_{\lambda\rho,\sigma} h^j_{\gamma\sigma,\lambda\rho} - 2\, h^i_{\lambda\rho,\rho} h^j_{\gamma\sigma,\lambda\sigma} - 2\, h^i_{\lambda\rho} h^j_{\gamma\sigma,\sigma\lambda\rho} \big) \Big)$$

with $c^{ij} = -c^{ji}$.

Gauge transformation $(\xi_{\rho\rho}=0)$:

$$\phi_{\alpha\beta\gamma} \rightarrow \phi_{\alpha\beta\gamma} + \xi_{\alpha\beta,\gamma} + \xi_{\beta\gamma,\alpha} + \xi_{\gamma\alpha,\beta}$$

$$-\tfrac{g}{16} c^{ij}\, \Sigma'' \big(\eta_{\alpha\beta} h^i_{\lambda\rho} \zeta^j_{\lambda,\rho\gamma} - 4\, \eta_{\alpha\beta} h^i_{\lambda\rho,\rho} \zeta^j_{\lambda,\gamma} - \eta_{\alpha\beta} h^i_{\rho\rho} \zeta^j_{\lambda,\lambda\gamma} - 8 \eta_{\alpha\beta} h^i_{\gamma\rho,\lambda\lambda} \zeta^j_{\rho}$$

$$- 8\, \eta_{\alpha\beta} h^i_{\rho\rho,\lambda\lambda} \zeta^j_{\gamma} + 8\, h^i_{\alpha\beta,\lambda\lambda} \zeta^j_{\gamma} + 8 \eta_{\alpha\beta} h^i_{\lambda\rho} \zeta^j_{\gamma,\lambda\rho} - 5\, \eta_{\alpha\beta} h^i_{\lambda\rho,\gamma} \zeta^j_{\lambda,\rho}$$

$$+ 32\, h^i_{\alpha\lambda,\gamma} \zeta^j_{\beta,\lambda} + 4\, \eta_{\alpha\beta} h^i_{\gamma\lambda} \zeta^j_{\rho,\rho\lambda} - 12\, \eta_{\alpha\beta}\, h^i_{\gamma\lambda,\rho} \zeta^j_{\rho,\lambda} - 4\, \eta_{\alpha\beta}\, h^i_{\gamma\rho,\lambda} \zeta^j_{\rho,\lambda}$$

$$+ 4\, \eta_{\alpha\beta} h^i_{\lambda\rho,\rho\gamma} \zeta^j_{\lambda} + 16\, h^i_{\alpha\lambda,\beta\gamma} \zeta^j_{\lambda} - 2\, \eta_{\alpha\beta}\, h^i_{\rho\rho,\lambda} \zeta^j_{\gamma,\lambda} + 8\, h^i_{\alpha\beta,\lambda} \zeta^j_{\gamma,\lambda}$$

$$+ 24\, h^i_{\rho\rho,\alpha\beta} \zeta^j_{\gamma} + 8\, \eta_{\alpha\beta}\, h^i_{\lambda\rho,\lambda\rho} \zeta^j_{\gamma} - 16\, h^i_{\alpha\lambda,\lambda\beta} \zeta^j_{\gamma} + 9\, \eta_{\alpha\beta} h^i_{\rho\rho,\gamma} \zeta^j_{\lambda,\lambda}$$

$$- 6\, \eta_{\alpha\beta} h^i_{\rho\rho,\gamma\lambda} \zeta^j_{\lambda} + 8\, \eta_{\alpha\beta} h^i_{\gamma\lambda,\lambda\rho} \zeta^j_{\rho} - 16\, h^i_{\alpha\beta,\gamma} \zeta^j_{\lambda,\lambda} - 24\, h^i_{\alpha\beta,\lambda\gamma} \zeta^j_{\lambda}\big)$$

$$h^i_{\mu\nu} \rightarrow h^i_{\mu\nu} + \zeta^i_{\mu,\nu} + \zeta^i_{\nu,\mu}$$

$$- \tfrac{g}{8} c^{ij}\, \Sigma' \big(8\, \phi_{\alpha\beta\mu} \zeta^j_{\nu,\alpha\beta} + 12\, \phi_{\rho\rho\gamma,\mu} \zeta^j_{\nu,\gamma} + 16\, \phi_{\mu\alpha\beta,\nu} \zeta^j_{\alpha,\beta} - 4\, \phi_{\mu\nu\gamma} \zeta^j_{\lambda,\lambda\gamma}$$

$$+ 4\, \eta_{\mu\nu} \phi_{\rho\rho\gamma} \zeta^j_{\lambda,\lambda\gamma} - 12\, \phi_{\mu\nu\gamma,\lambda} \zeta^j_{\lambda,\gamma} + 7\, \phi_{\rho\rho\lambda,\lambda\nu} \zeta^j_{\mu} - 8\, \phi_{\rho\rho\mu} \zeta^j_{\nu,\lambda\lambda}$$

$$+ 8\, \phi_{\rho\rho\mu,\lambda\lambda} \zeta^j_{\nu} - 8\, \phi_{\alpha\beta\mu,\alpha\beta} \zeta^j_{\nu} + 4\, \phi_{\rho\rho\lambda,\mu\nu} \zeta^j_{\lambda} - 8\, \phi_{\mu\lambda\rho,\lambda\nu} \zeta^j_{\rho}$$

$$+ 8\, \phi_{\rho\rho\mu} \zeta^j_{\lambda,\lambda\nu} + 8\, \phi_{\rho\rho\mu,\lambda\nu} \zeta^j_{\lambda} + 4\, \eta_{\mu\nu} \phi_{\rho\rho\gamma} \zeta^j_{\gamma,\lambda\lambda} - 4\, \phi_{\mu\nu\gamma} \zeta^j_{\gamma,\lambda\lambda}$$

$$- 4\, \phi_{\mu\nu\gamma,\rho} \zeta^j_{\gamma,\rho} - 12\, \eta_{\mu\nu} \phi_{\rho\rho\gamma,\lambda\lambda} \zeta^j_{\gamma} + 4\, \phi_{\mu\nu\gamma,\lambda\lambda} \zeta^j_{\gamma} + 12\, \eta_{\mu\nu} \phi_{\alpha\beta\gamma,\alpha\beta} \zeta^j_{\gamma}$$

$$- 6\, \eta_{\mu\nu}\phi_{\rho\rho\lambda,\lambda\rho}\zeta^j{}_\rho - 4\, \phi_{\mu\nu\lambda,\lambda\rho}\zeta^j{}_\rho)$$

$$+ \frac{g}{8}\, c^{ij}\, \Sigma'\, (12\, h^j{}_{\mu\nu,\rho\sigma}\xi_{\rho\sigma} + 12\, h^j{}_{\mu\nu,\rho}\xi_{\rho\sigma,\sigma} + 3\, h^j{}_{\mu\nu}\xi_{\rho\sigma,\rho\sigma} - 16\, h^j{}_{\nu\lambda,\sigma}\xi_{\mu\sigma,\lambda}$$

$$+ 16\, h^j{}_{\nu\lambda,\lambda}\xi_{\mu\sigma,\sigma} - 16\, h^j{}_{\nu\rho,\sigma\mu}\xi_{\rho\sigma} + 4\, h^j{}_{\delta\varepsilon,\varepsilon}\xi_{\mu\nu,\delta} + 8\, h^j{}_{\delta\varepsilon}\xi_{\mu\nu,\delta\varepsilon}$$

$$- 8\, h^j{}_{\sigma\lambda,\nu\lambda}\xi_{\mu\sigma} + 8\, h^j{}_{\sigma\lambda,\nu}\xi_{\mu\sigma,\lambda} - 8\, \eta_{\mu\nu}h^j{}_{\rho\lambda,\lambda}\xi_{\rho\sigma,\sigma})$$

Here $\Sigma'\, H_{\mu\nu}$ stands for $\frac{1}{2}(H_{\mu\nu}+H_{\nu\mu})$ and

$\Sigma''\, F_{\alpha\beta\gamma}$ for $1/6(F_{\alpha\beta\gamma}+F_{\beta\gamma\alpha}+F_{\gamma\alpha\beta}+F_{\gamma\beta\alpha}+F_{\beta\alpha\gamma}+F_{\alpha\gamma\beta})$.

Non-zero first order commutators: (modulo gauge parameter redefinitions)

$$[\delta_{\zeta_2},\delta_{\zeta_1}]\, \phi_{\alpha\beta\gamma} = \delta_{g[\zeta_2,\zeta_1]}\, \phi_{\alpha\beta\gamma} + O(g^2)$$

$$\text{with} \quad [\zeta_2,\zeta_1]_{\alpha\beta} = \frac{2}{3}\, c^{ij}\, (\zeta_1{}^i{}_{\alpha,\lambda}\zeta_2{}^j{}_{\beta,\lambda} - \tfrac{1}{4}\eta_{\alpha\beta}\, \zeta_1{}^i{}_{\rho,\lambda}\zeta_2{}^j{}_{\rho,\lambda})$$

$$[\delta_\zeta,\delta_\xi]\, h^i{}_{\mu\nu} = \delta_{g[\zeta,\xi]}\, h^i{}_{\mu\nu} + O(g^2)$$

$$\text{with} \quad [\zeta,\xi]^i{}_\mu = g\, c^{ij}\, \xi_{\mu\rho,\lambda}\zeta^j{}_{\rho,\lambda}$$

spin 3-5/2-5/2 D = 2

First order lagrangian:

(The following notation is used:

$$\Phi=\gamma_\alpha\phi_{\alpha\rho\rho},\ \Phi_\alpha=\phi_{\alpha\rho\rho},\ \Phi_{\alpha\beta}=\gamma_\rho\phi_{\alpha\beta\rho},\ \Psi=\Psi_{\rho\rho},\ \Psi_\mu=\gamma_\rho\psi_{\mu\rho};$$

of course the γ-matrix is thought to be between ψ^i and ψ^j.)

$$L_1 = c^{ij}\{\, \phi_{\alpha\beta\gamma}(\tfrac{1}{2}\bar{\Psi}^i{}_{\lambda,\lambda\alpha}\psi^j{}_{\beta\gamma} + \bar{\Psi}^i{}_{\lambda,\alpha}\psi^j{}_{\beta\gamma,\lambda} + \tfrac{1}{2}\, \bar{\Psi}^i{}_{\lambda,\lambda}\psi^j{}_{\beta\gamma,\alpha}$$

$$- \bar{\Psi}^i{}_{\alpha,\beta\lambda}\psi^j{}_{\gamma\lambda} - \Psi^i{}_{\alpha,\lambda}\psi^j{}_{\gamma\lambda,\beta} + \tfrac{1}{2}\, \bar{\Psi}^i{}_{\lambda,\alpha\beta}\psi^j{}_{\lambda\gamma} + \bar{\Psi}^i{}_\lambda\psi^j{}_{\beta\gamma,\alpha\lambda}$$

$$- \tfrac{1}{2}\, \bar{\Psi}^i{}_\lambda\psi^j{}_{\lambda\gamma,\alpha\beta} + \tfrac{1}{4}\, \bar{\Psi}^i{}_{\alpha,\beta\gamma}\Psi^j - \tfrac{1}{4}\, \bar{\Psi}^i{}_\alpha\Psi^j{}_{,\beta\gamma})$$

$$+ \Phi_{\alpha\beta}\, (\bar{\psi}^i{}_{\alpha\lambda,\lambda\rho}\psi^j{}_{\beta\rho} + \bar{\psi}^i{}_{\alpha\lambda,\rho}\psi^j{}_{\rho\rho,\lambda} - \bar{\psi}^i{}_{\alpha\beta,\lambda\rho}\psi^j{}_{\lambda\rho} - \bar{\psi}^i{}_{\alpha\beta,\lambda}\psi^j{}_{\lambda\rho,\rho}$$

$$- \tfrac{1}{4}\, \bar{\psi}^i{}_{\alpha\beta}\psi^j{}_{\lambda\rho,\lambda\rho} + \bar{\psi}^i{}_{\lambda\rho,\lambda}\psi^j{}_{\alpha\rho,\beta} - \bar{\psi}^i{}_{\lambda\rho,\beta}\psi^j{}_{\alpha\rho,\lambda} + \bar{\psi}^i{}_{\lambda\rho}\psi^j{}_{\alpha\rho,\beta\lambda} - \tfrac{1}{2}\, \bar{\psi}^i{}_{\alpha\rho,\beta}\Psi^j{}_{,\rho}$$

$$+ \tfrac{1}{2}\, \bar{\psi}^i{}_{\alpha\rho,\rho}\Psi^j{}_{,\beta} - \tfrac{1}{2}\, \bar{\psi}^i{}_{\alpha\rho}\Psi^j{}_{,\beta\rho} - \tfrac{1}{4}\, \bar{\psi}^i{}_{\lambda\rho,\alpha\beta}\psi^j{}_{\lambda\rho} + \tfrac{1}{4}\, \bar{\psi}^i{}_{\lambda\rho,\alpha}\psi^j{}_{\lambda\rho,\beta} - \tfrac{1}{4}\, \bar{\Psi}^i{}_{,\alpha}\Psi^j{}_{,\beta})$$

$$+ \Phi_\alpha \Big(-\tfrac{1}{2}\bar\Psi^i{}_{\rho,\lambda\lambda}\psi^k{}_{\alpha\rho} - \tfrac{1}{4}\,\bar\Psi^i{}_{\alpha,\lambda\lambda}\Psi^j - \bar\Psi^i{}_{\lambda,\lambda}\psi^j{}_{\alpha\rho,\rho}$$

$$- \bar\Psi^i{}_{\lambda,\rho}\psi^j{}_{\alpha\rho,\lambda} - 3/2\,\bar\Psi^i{}_{\lambda}\psi^j{}_{\alpha\rho,\rho\lambda} - \tfrac{1}{2}\,\bar\Psi^i{}_{\lambda,\rho\alpha}\psi^j{}_{\lambda\rho} + \bar\Psi^i{}_{\lambda,\rho}\psi^j{}_{\lambda\rho,\alpha} - 3/4\,\bar\Psi^i{}_{\lambda,\alpha}\psi^j{}_{\lambda\rho,\rho}$$

$$+ 3/4\,\bar\Psi^i{}_{\lambda}\psi^l{}_{\lambda\rho,\alpha\rho} - 1/8\,\bar\Psi^i{}_{\lambda,\lambda\alpha}\Psi^j + 3/8\,\bar\Psi^i{}_{\lambda,\lambda}\Psi^j{}_{,\alpha} + 1/8\,\bar\Psi^i{}_{\lambda,\alpha}\Psi^j{}_{,\lambda}$$

$$+ 1/8\,\bar\Psi^i{}_{\lambda}\Psi^j{}_{,\lambda\alpha} + \bar\Psi^i{}_{\alpha,\lambda\rho}\psi^j{}_{\lambda\rho} + \bar\Psi^i{}_{\alpha,\lambda}\psi^j{}_{\lambda\rho,\rho} + \tfrac{1}{4}\,\bar\Psi^i{}_{\alpha}\psi^j{}_{\lambda\rho,\lambda\rho}\Big)$$

$$+ \tfrac{1}{2}\,\Phi\,\Big(-\,\bar\psi^i{}_{\rho\lambda,\lambda\sigma}\psi^j{}_{\rho\sigma} - \bar\phi^i{}_{\rho\lambda,\lambda}\psi^j{}_{\rho\sigma,\sigma} + \bar\psi^i{}_{\rho\lambda,\lambda}\Psi^j{}_{,\rho} + \bar\psi^i{}_{\rho\lambda}\Psi^j{}_{,\rho\lambda}\Big)\ \Big\}$$

with $c^{ij} = -c^{ji}$.

Gauge transformation $(\xi_{\rho\rho}=0$ and $\gamma_\mu\varepsilon_\mu=0)$:

$$\phi_{\alpha\beta\gamma} \rightarrow \phi_{\alpha\beta\gamma} + \xi_{\alpha\beta,\gamma} + \xi_{\beta\gamma,\alpha} + \xi_{\gamma\alpha,\beta}$$

$$+ g\,c^{ij}\,\Sigma''\,\Big\{\tfrac{1}{2}\,\bar\varepsilon^i{}_{\lambda,\lambda}\gamma_\alpha\psi^j{}_{\beta\gamma} + \tfrac{1}{2}\,\bar\varepsilon^i{}_{\alpha,\beta}\gamma_\gamma\Psi^j + \bar\varepsilon^i{}_{\lambda}\gamma_\alpha\psi^j{}_{\beta\gamma,\lambda} - \bar\varepsilon^i{}_{\alpha,\lambda}\gamma_\beta\psi^j{}_{\gamma\lambda}$$

$$+ \bar\varepsilon^i{}_{\lambda,\alpha}\gamma_\beta\psi^j{}_{\lambda\gamma} + \eta_{\alpha\beta}\Big(-1/8\,\bar\varepsilon^i{}_{\rho,\lambda}\gamma_\lambda\psi^j{}_{\rho\gamma} + \tfrac{1}{4}\,\bar\varepsilon^i{}_{\rho}\gamma_\lambda\psi^j{}_{\rho\gamma,\lambda} + 1/8\,\bar\varepsilon^i{}_{\rho,\lambda}\gamma_\gamma\psi^j{}_{\rho\lambda}$$

$$- \tfrac{1}{4}\,\bar\varepsilon^i{}_{\gamma}\gamma_\lambda\psi^j{}_{\lambda\rho,\rho} - 1/8\,\bar\varepsilon^i{}_{\gamma,\rho}\gamma_\lambda\psi^j{}_{\lambda\rho} - 1/8\,\bar\varepsilon^i{}_{\lambda,\lambda}\gamma_\rho\psi^j{}_{\gamma\rho} - \tfrac{1}{4}\,\bar\varepsilon^i{}_{\lambda}\gamma_\rho\psi^j{}_{\rho\gamma,\lambda}$$

$$- 1/16\,\bar\varepsilon^i{}_{\gamma,\lambda}\gamma_\lambda\Psi^j + 1/8\,\bar\varepsilon^i{}_{\gamma}\gamma_\lambda\Psi^j{}_{,\lambda} - 1/8\,\bar\varepsilon^i{}_{\lambda,\gamma}\gamma_\rho\psi^j{}_{\lambda\rho} - \tfrac{1}{4}\,\bar\varepsilon^i{}_{\lambda}\gamma_\rho\psi^j{}_{\lambda\rho,\gamma}$$

$$+ 1/8\,\bar\varepsilon^i{}_{\lambda,\lambda}\gamma_\gamma\Psi^j\Big)\ \Big\}$$

$$\psi^i{}_{\mu\nu} \rightarrow \psi^i{}_{\mu\nu} + \varepsilon^i{}_{\mu,\nu} + \varepsilon^i{}_{\nu,\mu}$$

$$+ g\,c^{ij}\,\Sigma'\,\Big\{ -(\Phi_{\mu\lambda,\nu}\gamma_\rho\varepsilon^j{}_\lambda)_{,\rho} + (\Phi_{\mu\nu,\lambda}\gamma_\rho\varepsilon^j{}_\lambda)_{,\rho} + \tfrac{1}{2}\,(\Phi_{\mu\nu}\gamma_\rho\varepsilon^j{}_{\lambda,\lambda})_{,\rho}$$

$$-(\Phi_{\mu\lambda}\gamma_\rho\varepsilon^j{}_{\nu,\lambda})_{,\rho} - \tfrac{1}{2}\,(\Phi_{,\mu}\gamma^\rho\varepsilon^j{}_\nu)_{,\rho} - \tfrac{1}{2}\,(\Phi_\mu\gamma_\rho\gamma_\lambda\varepsilon^j{}_{\nu,\lambda})_{,\rho} + 2\,(\phi_{\mu\rho\lambda,\nu}\varepsilon^j{}_\lambda)_{,\rho}$$

$$- \phi_{\mu\nu\lambda,\lambda\rho}\varepsilon^j{}_\rho - 2\,\phi_{\mu\nu\lambda,\rho}\varepsilon^j{}_{\rho,\lambda} - \tfrac{1}{2}\,\phi_{\mu\nu\lambda,\lambda}\varepsilon^j{}_{\rho,\rho} - \phi_{\mu\nu\lambda}\varepsilon^j{}_{\rho,\rho\lambda} - (\phi_{\mu\rho\lambda,\lambda}\varepsilon^j{}_\nu)_{,\rho}$$

$$+ 2\,(\phi_{\mu\rho\lambda}\varepsilon^j{}_{\nu,\rho})_{,\lambda} + 2\,\Phi_{\lambda,\lambda\mu}\varepsilon^j{}_\nu + 5/2\,\Phi_{\lambda,\mu}\varepsilon^j{}_{\nu,\lambda} + \tfrac{1}{2}\,\Phi_\lambda\varepsilon^j{}_{\nu,\lambda\mu} + \tfrac{1}{4}\,\Phi_{\lambda,\lambda}\varepsilon^j{}_{\nu,\mu}$$

$$+ (\Phi_{\mu,\lambda}\varepsilon^j{}_\nu)_{,\lambda} + \phi_{\mu\rho\lambda,\lambda}\varepsilon^j{}_{\rho,\nu} + \tfrac{1}{4}\,\Phi_{\lambda,\mu\nu}\varepsilon^j{}_\lambda - \tfrac{1}{2}\,\Phi_{\lambda,\mu}\varepsilon^j{}_{\lambda,\nu} + \tfrac{1}{4}\,\Phi_\lambda\varepsilon^j{}_{\lambda,\mu\nu}$$

$$- 3/2\,\Phi_{\mu\gamma,\nu\lambda}\varepsilon^j{}_\lambda - \tfrac{1}{2}\,\Phi_{\mu,\nu}\varepsilon^j{}_{\lambda,\lambda} - \tfrac{1}{2}\,\Phi_{\mu,\lambda}\varepsilon^j{}_{\lambda,\nu} + \tfrac{1}{2}\,\Phi_\mu\varepsilon^j{}_{\lambda,\lambda\nu}$$

$$+ \tfrac{1}{4}\,\gamma_\nu\,\Big(-\,\Phi_{\mu\lambda,\rho}\gamma_\rho\gamma_\sigma\varepsilon^j{}_{\lambda,\sigma} + \Phi_{\mu\lambda,\rho}\varepsilon^j{}_{\rho,\lambda} - \Phi_{\lambda\rho}\varepsilon^j{}_{\mu,\lambda\rho} + \Phi_{\mu\rho,\lambda}\varepsilon^j{}_{\rho,\lambda}$$

$$+ 2\,\phi_{\mu\lambda\rho}\gamma_\sigma\varepsilon^j{}_{\lambda,\rho\sigma} - 3/2\,\Phi_{\rho\lambda,\mu}\varepsilon^j{}_{\rho,\lambda} - \tfrac{1}{2}\,\Phi_{\rho\lambda}\varepsilon^j{}_{\rho,\lambda\mu} - 2\,(\Phi_{\lambda\rho,\rho}\varepsilon^j{}_\lambda)_{,\mu}$$

$$+ \phi_{\mu\lambda\rho,\rho}\gamma_\sigma\varepsilon^j{}_{\lambda,\sigma} - \tfrac{1}{2}\,\Phi_{,\lambda}\varepsilon^j{}_{\mu,\lambda} + 3/2\,\Phi_{,\mu\lambda}\varepsilon^j{}_\lambda + \Phi_{,\lambda}\varepsilon^j{}_{\lambda,\mu} - \Phi_{\lambda,\rho}\gamma_\rho\varepsilon^j{}_{\mu,\lambda}$$

$$+ \tfrac{1}{2} \Phi_{\lambda}\gamma_{\rho}\varepsilon^{j}_{\mu,\lambda\rho} - \tfrac{1}{2} \Phi_{\mu,\lambda}\gamma_{\rho}\varepsilon^{j}_{\lambda,\rho} + 7/4\ \Phi_{\lambda,\mu\rho}\gamma_{\rho}\varepsilon^{j}_{\lambda} - \tfrac{1}{2} \Phi_{\lambda}\gamma_{\rho}\varepsilon^{j}_{\lambda,\mu\rho}$$

$$+ 3/4\ \Phi_{\lambda,\rho}\gamma_{\rho}\varepsilon^{j}_{\lambda,\mu} + 9/2\ \Phi_{\lambda,\mu}\gamma_{\rho}\varepsilon^{j}_{\lambda,\rho} + \Phi\varepsilon^{j}_{\mu,\lambda\lambda} - \Phi_{,\lambda\lambda}\varepsilon^{j}_{\mu} + \Phi_{\lambda\rho,\lambda\rho}\varepsilon^{j}_{\mu}$$

$$- \tfrac{1}{2} \Phi_{\lambda,\lambda\rho}\gamma_{\rho}\varepsilon^{j}_{\mu} + 3/4\ \Phi_{\lambda,\lambda}\gamma_{\rho}\varepsilon^{j}_{\mu,\rho} - \tfrac{1}{2} \Phi\varepsilon^{j}_{\lambda,\lambda\mu} - \Phi_{\mu,\lambda}\gamma_{\lambda}\varepsilon^{j}_{\rho,\rho} - 3/2\ \Phi_{\mu}\gamma_{\rho}\varepsilon^{j}_{\lambda,\rho\lambda})$$

$$+ \tfrac{1}{4} \eta_{\mu\nu} \Big(\tfrac{1}{2} (\Phi_{\rho\lambda}\gamma_{\sigma}\varepsilon^{j}_{\rho,\lambda})_{,\sigma} - \tfrac{1}{2} (\Phi\gamma_{\lambda}\varepsilon^{j}_{\sigma,\sigma})_{,\lambda} + 3/4\ (\Phi_{\lambda}\gamma_{\rho}\gamma_{\sigma}\varepsilon^{j}_{\lambda,\sigma})_{,\rho} - \Phi_{\lambda,\sigma}\varepsilon^{j}_{\lambda,\sigma}$$

$$-(\Phi_{\lambda,\rho}\varepsilon^{j}_{\rho})_{,\lambda} + \Phi_{\lambda}\varepsilon^{j}_{\rho,\rho\lambda} + 2\phi_{\lambda\rho\sigma,\lambda\rho}\varepsilon^{j}_{\sigma} - \phi_{\lambda\rho\sigma}\varepsilon^{j}_{\sigma,\lambda\rho} - 2\ \Phi_{\lambda,\sigma\sigma}\varepsilon^{j}_{\lambda} + 5/4\ \Phi_{\lambda,\lambda}\varepsilon^{j}_{\rho,\rho})\}$$

$$+ g\ c^{ij}\ \Sigma'\ \{ - (\xi_{\mu\lambda,\sigma}\gamma_{\lambda}\gamma_{\rho}\psi^{j}_{\sigma\nu})_{,\rho} + (\xi_{\sigma\lambda}\gamma_{\lambda}\gamma_{\rho}\psi^{j}_{\mu\nu,\sigma})_{,\rho}$$

$$+ \tfrac{1}{2} (\xi_{\sigma\lambda,\lambda}\gamma_{\sigma}\gamma_{\rho}\psi^{j}_{\mu\nu})_{,\rho} - (\xi_{\lambda\sigma}\gamma_{\lambda}\gamma_{\rho}\psi^{j}_{\sigma\mu,\nu})_{,\rho} - \tfrac{1}{2}(\xi_{\mu\lambda}\gamma_{\lambda}\gamma_{\rho}\Psi^{j}_{,\nu})_{,\rho} + (\xi_{\mu\nu,\lambda}\gamma_{\rho}\Psi^{j}_{\lambda})_{,\rho}$$

$$+ \tfrac{1}{2} (\xi_{\mu\nu}\gamma_{\rho}\Psi^{j}_{\lambda,\lambda})_{,\rho} - (\xi_{\mu\lambda}\gamma_{\rho}\Psi^{j}_{\nu})_{,\rho\lambda} + (\xi_{\mu\lambda}\gamma_{\rho}\Psi^{j}_{\lambda,\nu})_{,\rho} + 3\ (\xi_{\mu\lambda,\rho}\psi^{j}_{\rho\nu})_{,\lambda}$$

$$+ (\xi_{\mu\lambda}\psi^{j}_{\rho\nu,\lambda})_{,\rho} - 5/4\ \xi_{\rho\lambda,\rho\lambda}\psi^{j}_{\mu\nu} - 4\ \xi_{\rho\lambda,\lambda}\psi^{j}_{\mu\nu,\rho} - 3\ \xi_{\rho\lambda}\psi^{j}_{\mu\nu,\rho\lambda}$$

$$+ 3\ (\xi_{\rho\lambda}\psi^{j}_{\rho\mu,\nu})_{,\lambda} - \xi_{\rho\lambda,\nu}\psi^{j}_{\rho\mu,\lambda} + 3/2\ \xi_{\mu\lambda,\lambda}\Psi^{j}_{,\nu} - \tfrac{1}{2} \xi_{\mu\lambda,\nu}\Psi^{j}_{,\lambda} + \tfrac{1}{2} \xi_{\mu\lambda}\Psi^{j}_{,\lambda\nu}$$

$$- (\xi_{\mu\nu,\rho}\psi^{j}_{\rho\lambda})_{,\lambda} - \tfrac{1}{4} \xi_{\mu\nu}\psi^{j}_{\rho\lambda,\rho\lambda} + (\xi_{\mu\lambda,\nu}\psi^{j}_{\lambda\rho})_{,\rho} - \xi_{\mu\lambda,\rho}\psi^{j}_{\lambda\rho,\nu} + \tfrac{1}{2} \xi_{\rho\lambda,\mu}\psi^{j}_{\rho\lambda,\nu}$$

$$- \tfrac{1}{4} \xi_{\rho\lambda,\mu\nu}\psi^{j}_{\rho\lambda} - \tfrac{1}{4} \xi_{\rho\lambda}\psi^{j}_{\rho\lambda,\mu\nu}$$

$$+ \tfrac{1}{4} \gamma_{\mu} \Big(- 2\ (\xi_{\nu\lambda,\sigma}\gamma_{\lambda}\gamma_{\rho}\Psi^{j}_{\sigma})_{,\rho} + \xi_{\nu\lambda,\rho}\gamma_{\rho}\gamma_{\sigma}\Psi^{j}_{\lambda,\sigma} - (\xi_{\rho\lambda}\gamma_{\rho}\gamma_{\sigma}\Psi^{j}_{\nu,\lambda})_{,\sigma}$$

$$+ \xi_{\rho\lambda,\sigma}\gamma_{\rho}\gamma_{\sigma}\gamma_{\varepsilon}\psi^{j}_{\nu\lambda,\varepsilon} + \tfrac{1}{2} \xi_{\nu\lambda,\rho}\gamma_{\lambda}\gamma_{\rho}\gamma_{\varepsilon}\Psi^{j}_{,\varepsilon} - (\xi_{\nu\lambda}\gamma_{\lambda}\gamma_{\sigma}\Psi^{j}_{\varepsilon,\varepsilon})_{,\sigma} - \xi_{\nu\lambda,\sigma}\Psi^{j}_{\lambda,\sigma}$$

$$+ \tfrac{1}{2} \xi_{\rho\lambda,\sigma\nu}\gamma_{\sigma}\psi^{j}_{\rho\lambda} + 3/2\ \xi_{\rho\lambda,\sigma}\gamma_{\sigma}\psi^{j}_{\rho\lambda,\nu} - \tfrac{1}{2} \xi_{\rho\lambda,\nu}\gamma_{\sigma}\psi^{j}_{\rho\lambda,\sigma} + 5/2\ \xi_{\rho\lambda}\gamma_{\sigma}\psi^{j}_{\rho\lambda,\nu\sigma}$$

$$+ \xi_{\nu\rho,\lambda\sigma}\gamma_{\sigma}\psi^{j}_{\rho\lambda} + 3\ \xi_{\nu\rho,\lambda}\gamma_{\sigma}\psi^{j}_{\rho\lambda,\sigma} - 3/2\ \xi_{\rho\sigma,\nu\lambda}\gamma_{\sigma}\psi^{j}_{\rho\lambda} - \tfrac{1}{2} \xi_{\rho\sigma,\lambda}\gamma_{\sigma}\psi^{j}_{\rho\lambda,\nu}$$

$$- \xi_{\rho\lambda,\sigma}\gamma_{\sigma}\psi^{j}_{\rho\nu,\lambda} - 5\ (\xi_{\rho\lambda}\gamma_{\sigma}\psi^{j}_{\nu\rho,\sigma})_{,\lambda} - 5/2\ (\xi_{\rho\lambda}\Psi^{j}_{\rho,\nu})_{,\lambda} + \tfrac{1}{2} \xi_{\rho\lambda,\nu}\Psi^{j}_{\rho,\lambda}$$

$$+ (\xi_{\sigma\nu,\lambda}\gamma_{\sigma}\psi^{j}_{\rho\lambda})_{,\rho} - \xi_{\nu\rho,\lambda}\Psi^{j}_{\lambda,\rho} - (\xi_{\sigma\rho,\lambda}\gamma_{\sigma}\psi^{j}_{\nu\lambda})_{,\rho} + 7\ \xi_{\rho\lambda}\Psi^{j}_{\nu,\rho\lambda}$$

$$- \xi_{\rho\sigma,\lambda}\gamma_{\sigma}\psi^{j}_{\rho\nu,\lambda} - 2\ \xi_{\sigma\lambda}\gamma_{\sigma}\psi^{j}_{\lambda\rho,\rho\nu} + (\xi_{\sigma\lambda}\gamma_{\sigma}\psi^{j}_{\nu\rho,\rho})_{,\lambda} + \xi_{\nu\rho}\Psi^{j}_{\lambda,\lambda\rho} + 6\ \xi_{\rho\lambda,\lambda}\Psi^{j}_{\nu,\rho}$$

$$- \xi_{\nu\rho,\sigma}\gamma_{\sigma}\psi^{j}_{\rho\lambda,\lambda} - 2\ \xi_{\sigma\lambda,\nu}\gamma_{\sigma}\psi^{j}_{\lambda\rho,\rho} + \tfrac{1}{2} \xi_{\nu\lambda,\sigma}\gamma_{\sigma}\Psi^{j}_{,\lambda} - \tfrac{1}{2} \xi_{\nu\lambda}\gamma_{\sigma}\Psi^{j}_{,\lambda\sigma}$$

$$+ \tfrac{1}{2} \xi_{\lambda\sigma}\gamma_{\sigma}\Psi^{j}_{,\lambda\nu} - 2\ \xi_{\lambda\rho,\rho\sigma}\gamma_{\sigma}\psi^{j}_{\lambda\nu} - \tfrac{1}{2} \xi_{\nu\sigma,\lambda}\gamma_{\sigma}\Psi^{j}_{,\lambda} - \tfrac{1}{2} \xi_{\lambda\rho,\rho\nu}\Psi^{j}_{\lambda} + \xi_{\lambda\sigma,\nu}\gamma_{\sigma}\Psi^{j}_{,\lambda}$$

$$+ \tfrac{1}{2} \xi_{\nu\sigma}\gamma_{\sigma}\psi^{j}_{\rho\lambda,\rho\lambda} - \xi_{\lambda\sigma,\lambda}\gamma_{\sigma}\Psi^{j}_{,\nu} + 3/2\ \xi_{\rho\lambda,\rho\lambda}\Psi^{j}_{\nu} - 5/8\ \xi_{\nu\lambda,\lambda}\gamma_{\sigma}\Psi^{j}_{,\sigma}$$

$$- \xi_{\nu\lambda,\lambda\sigma}\gamma_{\sigma}\Psi^{j}_{\sigma} + 2\ \xi_{\nu\lambda,\lambda}\Psi^{j}_{\sigma,\sigma} \Big) +$$

$$+ \tfrac{1}{2} \eta_{\mu\nu} \Big(3/2 (\xi_{\rho\lambda,\lambda}\gamma_\sigma\Psi^j{}_\rho)_{,\sigma} + \tfrac{1}{2} (\xi_{\rho\lambda,\sigma}\gamma_\rho\gamma_\varepsilon\psi^j{}_{\lambda\sigma})_{,\varepsilon}$$

$$+ \tfrac{1}{2} (\xi_{\rho\lambda}\gamma_\varepsilon\Psi^j{}_{\rho,\lambda})_{,\varepsilon} - \tfrac{1}{2} (\xi_{\rho\lambda,\lambda}\gamma_\rho\gamma_\varepsilon\Psi^j)_{,\varepsilon} - (\xi_{\rho\lambda}\psi^j{}_{\rho\sigma,\lambda})_{,\sigma} + \tfrac{1}{2} \xi_{\rho\lambda}\Psi^j{}_{,\rho\lambda}$$

$$+ \xi_{\rho\lambda,\lambda}\Psi^j{}_{,\rho} + 5/4\, \xi_{\rho\lambda,\rho\lambda}\Psi^j - \tfrac{1}{2} \xi_{\rho\lambda,\sigma}\psi^j{}_{\rho\lambda,\sigma} + \tfrac{1}{4} \xi_{\rho\lambda,\sigma\sigma}\psi^j{}_{\rho\lambda} + \tfrac{1}{4} \xi_{\rho\lambda}\psi^j{}_{\rho\lambda,\sigma\sigma}$$

$$- 2\, \xi_{\rho\lambda,\lambda\sigma}\psi^j{}_{\rho\sigma}\Big) \Big\}$$

Here $\Sigma' H_{\mu\nu}$ stands for $\tfrac{1}{2}(H_{\mu\nu}+H_{\nu\mu})$ and

$\Sigma'' F_{\alpha\beta\gamma}$ for $1/6(F_{\alpha\beta\gamma}+F_{\beta\gamma a}+F_{\gamma\alpha\beta}+F_{\gamma\beta\alpha}+F_{\beta\alpha\gamma}+F_{\alpha\gamma\beta})$.

Non-zero first order commutators: (modulo gauge parameter redefinitions)

$$[\delta_{\varepsilon_2},\delta_{\varepsilon_1}]\, \phi_{\alpha\beta\gamma} = \delta_{g[\varepsilon_2,\varepsilon_1]}\, \phi_{\alpha\beta\gamma} + O(g^2)$$

$$\text{with} \quad [\varepsilon_2,\varepsilon_1]_{\alpha\beta} = 1/3\, c^{ij} \Big(\bar{\varepsilon}_1{}^i{}_\lambda\gamma_\alpha\varepsilon_2{}^j{}_{\beta,\lambda} + \bar{\varepsilon}_1{}^i{}_\lambda\gamma_\beta\varepsilon_2{}^j{}_{\alpha,\lambda}$$

$$- \bar{\varepsilon}_1{}^i{}_{\alpha,\lambda}\gamma_\beta\varepsilon_2{}^j{}_\lambda - \bar{\varepsilon}_1{}^i{}_{\beta,\lambda}\gamma_\alpha\varepsilon_2{}^j{}_\lambda - \tfrac{1}{2}\eta_{\alpha\beta} (\bar{\varepsilon}_1{}^i{}_\lambda\gamma_\rho\varepsilon_2{}^j{}_{\rho,\lambda} - \bar{\varepsilon}_1{}^i{}_{\rho,\lambda}\gamma_\rho\varepsilon_2{}^j{}_\lambda) \Big)$$

$$[\delta_\varepsilon,\delta_\xi]\, \psi^i{}_{\mu\nu} = \delta_{g[\delta_\varepsilon,\delta_\xi]}\, \psi^i{}_{\mu\nu} + O(g^2)$$

$$\text{with} \quad [\varepsilon,\xi]^i{}_\mu = c^{ij} (-\xi_{\rho\sigma,\mu\lambda}\gamma_\rho\gamma_\lambda\varepsilon^j{}_\sigma + \xi_{\rho\sigma,\lambda}\gamma_\rho\gamma_\lambda\varepsilon^j{}_{\sigma,\mu}$$

$$+ 2\, \xi_{\mu\sigma,\lambda}\varepsilon^j{}_{\sigma,\lambda}) - \tfrac{1}{4}\gamma_\mu\, \gamma\cdot(\quad)$$

<u>spin 3-3-3</u> D = 2

N.B. The expression given below is related by means of a field redefinition transformation to the first order lagrangian given in [23].
First order lagrangian:

$$L_1 = f^{abc} \Big(-9/2\, \phi^a{}_{\lambda\lambda\rho}\phi^b{}_{\sigma\sigma\alpha,\beta}\phi^c{}_{\tau\tau\alpha,\beta\rho} + 3\, \phi^a{}_{\lambda\lambda\rho,\beta}\, \phi^b{}_{\sigma\sigma\alpha,\rho}\, \phi^c{}_{\tau\tau\alpha,\beta}$$

$$-6\, \phi^a{}_{\alpha\beta\lambda}\phi^b{}_{\alpha\rho\rho,\tau}\phi^c{}_{\beta\sigma\sigma,\tau\lambda} + 3/2\, \phi^a{}_{\lambda\lambda\rho}\phi^b{}_{\alpha\beta\gamma,\sigma}\phi^c{}_{\alpha\beta\gamma,\sigma\rho} - \phi^a{}_{\lambda\lambda\rho,\rho\sigma}\phi^b{}_{\alpha\beta\gamma}\phi^c{}_{\alpha\beta\gamma,\sigma}$$

$$- 7\, \phi^a{}_{\lambda\lambda\rho,\sigma}\phi^b{}_{\alpha\beta\gamma}\phi^c{}_{\alpha\beta\gamma,\rho\sigma} - 6\, \phi^a{}_{\alpha\beta\rho}\phi^b{}_{\alpha\gamma\lambda,\sigma}\phi^c{}_{\beta\gamma\lambda,\rho\sigma} - 9\, \phi^a{}_{\alpha\beta\lambda,\tau}\phi^b{}_{\alpha\rho\sigma,\lambda}\phi^c{}_{\beta\rho\sigma,\tau}$$

$$+ 12\, \phi^a_{\alpha\beta\rho} \phi^b_{\alpha\beta\gamma,\rho\sigma} \phi^c_{\gamma\lambda\lambda,\sigma} + 3\, \phi^a_{\alpha\beta\lambda,\tau} \phi^b_{\alpha\beta\gamma,\tau} \phi^c_{\gamma\rho\rho,\lambda} + 9\, \phi^a_{\alpha\beta\lambda,\tau} \phi^b_{\alpha\beta\gamma,\lambda} \phi^c_{\gamma\rho\rho,\tau}$$

$$-9/4\, \phi^a_{\lambda\lambda\alpha,\beta\gamma} \phi^b_{\rho\rho\beta} \phi^c_{\sigma\sigma\gamma,\alpha} - 5/4\, \phi^a_{\lambda\lambda\alpha,\beta} \phi^b_{\rho\rho\beta,\gamma} \phi^c_{\sigma\sigma\gamma,\alpha} + 3\, \phi^a_{\alpha\beta\gamma} \phi^b_{\lambda\lambda\rho,\alpha} \phi^c_{\sigma\sigma\rho,\beta\gamma}$$

$$+9/2\, \phi^a_{\lambda\lambda\rho,\rho} \phi^b_{\alpha\alpha\beta,\gamma} \phi^c_{\beta\gamma\sigma,\sigma} + 9\, \phi^a_{\lambda\lambda\rho} \phi^b_{\alpha\alpha\beta,\gamma} \phi^c_{\beta\gamma\sigma,\sigma\rho} + 3/2\, \phi^a_{\lambda\lambda\rho,\rho\sigma} \phi^b_{\alpha\alpha\beta,\gamma} \phi^c_{\beta\gamma\sigma}$$

$$+ 6\, \phi^a_{\lambda\lambda\rho,\sigma} \phi^b_{\alpha\alpha\beta,\gamma} \phi^c_{\beta\gamma\sigma,\rho} - 3\, \phi^a_{\lambda\lambda\sigma,\beta} \phi^b_{\rho\rho\alpha,\sigma} \phi^c_{\alpha\beta\gamma,\gamma} + 6\, \phi^a_{\lambda\lambda\rho} \phi^b_{\alpha\beta\gamma,\gamma} \phi^c_{\alpha\beta\sigma,\sigma\rho}$$

$$- 15\, \phi^a_{\lambda\lambda\rho,\sigma} \phi^b_{\alpha\beta\gamma,\gamma\rho} \phi^c_{\alpha\beta\sigma} - 21/2\, \phi^a_{\lambda\lambda\rho} \phi^b_{\alpha\beta\gamma,\sigma} \phi^c_{\alpha\beta\sigma,\gamma\rho} + 9\, \phi^a_{\lambda\lambda\rho,\gamma\sigma} \phi^b_{\alpha\beta\gamma} \phi^c_{\alpha\beta\sigma,\rho}$$

$$+ 6\, \phi^a_{\alpha\beta\gamma,\gamma} \phi^b_{\rho\rho\alpha,\sigma} \phi^c_{\beta\sigma\lambda,\lambda} + 12\, \phi^a_{\alpha\beta\gamma} \phi^b_{\rho\rho\alpha,\sigma} \phi^c_{\beta\sigma\lambda,\lambda\gamma} + 6\, \phi^a_{\alpha\beta\gamma,\sigma} \phi^b_{\beta\lambda\lambda,\rho\alpha} \phi^c_{\gamma\sigma\rho}$$

$$- 12\, \phi^a_{\alpha\beta\lambda,\rho\sigma} \phi^b_{\beta\gamma\rho} \phi^c_{\gamma\alpha\sigma,\lambda} - 10\, \phi^a_{\alpha\beta\lambda,\rho} \phi^b_{\beta\gamma\rho,\sigma} \phi^c_{\gamma\alpha\sigma,\lambda} - 4\, \phi^a_{\lambda\rho\sigma,\lambda\rho} \phi^b_{\alpha\beta\gamma} \phi^c_{\alpha\beta\gamma,\sigma}$$

$$- \phi^a_{\lambda\rho\sigma} \phi^b_{\alpha\beta\gamma,\lambda} \phi^c_{\alpha\beta\gamma,\rho\sigma} - 12\, \phi^a_{\alpha\beta\gamma} \phi^b_{\alpha\beta\lambda,\gamma\rho} \phi^c_{\lambda\rho\sigma,\sigma} + 3\, \phi^a_{\alpha\beta\lambda,\lambda\sigma} \phi^b_{\alpha\beta\gamma,\rho} \phi^c_{\gamma\rho\sigma}$$

$$- 6\, \phi^a_{\alpha\beta\lambda,\sigma} \phi^b_{\alpha\beta\gamma,\rho} \phi^c_{\gamma\rho\sigma,\lambda} - 9\, \phi^a_{\alpha\beta\lambda,\sigma} \phi^b_{\alpha\beta\gamma,\lambda} \phi^c_{\gamma\rho\sigma,\rho})$$

with $f^{abc}=f^{bca}=-f^{bac}$.

Gauge transformation ($\xi_{\rho\rho}=0$) :

$$\phi^a_{\alpha\beta\gamma} \rightarrow \phi^a_{\alpha\beta\gamma} + \xi^a_{\alpha\beta,\gamma} + \xi^a_{\beta\gamma,\alpha} + \xi^a_{\gamma\alpha,\beta}$$

$$+ g\, f^{abc} \sum_{\text{cyclic}} \big(\phi^b_{\alpha\beta\gamma,\rho\sigma} \xi^c_{\rho\sigma} + 3\, \phi^b_{\alpha\rho\sigma} \xi^c_{\beta\gamma,\rho\sigma}$$

$$- 2\, \phi^b_{\alpha\beta\rho,\sigma} \xi^c_{\gamma\rho,\sigma} - 4\, \phi^b_{\alpha\beta\rho,\sigma} \xi^c_{\gamma\sigma,\rho} + \eta_{\alpha\beta}\, \phi^b_{\rho\sigma\tau,\gamma} \xi^c_{\rho\sigma,\tau}\big)$$

Here $\sum_{\text{cyclic}} F_{\alpha\beta\gamma}$ stands for $F_{\alpha\beta\gamma}+F_{\beta\gamma\alpha}+F_{\gamma\alpha\beta}$.

Non-zero first order commutator:

$$[\delta_{\xi_2,\xi_1}]\, \phi^a_{\alpha\beta\gamma} = \delta_{g[\xi_2,\xi_1]}\, \phi^a_{\alpha\beta\gamma} + O(g^2)$$

$$\text{with} \quad [\xi_2,\xi_1]^a_{\mu\nu} = f^{abc}\big(\xi_1{}^b_{\mu\nu,\rho\sigma} \xi_2{}^c_{\rho\sigma} - 2/3\, \xi_1{}^b_{\mu\rho,\lambda} \xi_2{}^c_{\nu\rho,\lambda}$$

$$- 4/3\, \xi_1{}^b_{\mu\rho,\lambda} \xi_2{}^c_{\nu\lambda,\rho} + \xi_2{}^c_{\mu\nu,\rho\sigma} \xi_1{}^b_{\rho\sigma} - 2/3\, \xi_2{}^c_{\mu\rho,\lambda} \xi_1{}^b_{\nu\rho,\lambda}$$

$$-4/3\, \xi_2{}^c_{\mu\rho,\lambda} \xi_1{}^b_{\nu\lambda,\rho} - 1/3\, \eta_{\mu\nu}(\xi_1{}^b_{\rho\sigma,\lambda} \xi_2{}^c_{\rho\sigma,\lambda} + 2\, \xi_1{}^b_{\rho\sigma,\lambda} \xi_2{}^c_{\rho\lambda,\sigma})\big)$$

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SAMENVATTING

Elementaire-deeltjesfysica probeert de natuur op een zo fundamenteel mogelijk niveau te beschrijven. In de huidige beschrijving hebben alle soorten krachten te maken met de uitwisseling van massaloze deeltjes tussen bronnen die krachten op elkaar uitoefenen. Er zijn verschillende soorten massaloze deeltjes. Een opzicht waarin ze van elkaar kunnen verschillen is hun spin. De meeste van de bekende krachten, waaronder de elektromagnetische, zijn met massaloze spin-1 deeltjes geassocieerd. Gravitatie is met spin 2 geassocieerd. Een van de manieren waarin dit verschil van spin zich uit is dat gelijknamige elektrische ladingen elkaar afstoten terwijl door gravitatie massa's elkaar juist aantrekken.

Experimenteel zijn krachten ten gevolge van hogere spin deeltjes niet waargenomen. Verrassend is dit niet, want we verwachten dat hogere spin krachten, als die bestaan, veel zwakker zijn dan de gravitatie krachten die met spin 2 te maken hebben, zoals gravitatiekrachten weer veel zwakker zijn dan elektrische krachten die met spin 1 te maken hebben.

Theoretisch zouden zulke hogere spin krachten toch interessant kunnen zijn doordat zij eventueel de afzonderlijke theorieën van de fundamentele krachten in een breder kader zouden kunnen plaatsen of interne inconsistenties van de huidige theorieën kunnen oplossen. Wij denken hierbij aan supergravitatie, een theorie waarin massaloze spin 3/2 deeltjes een grote rol spelen. Het is goed mogelijk dat om gravitatie te quantiseren, supergravitatie onontbeerlijk is. Misschien zouden eventuele hogere-spintheorieën ook voor de oplossing van dergelijke problemen van belang zijn.

Tot voor kort waren er alleen theorieën die vrije massaloze hogere spin deeltjes beschreven, en zelfs deze theorieën zijn er nog niet eens zo lang. In dit proefschrift worden een stel eerste orde interacties geconstrueerd tussen velden met hogere spin. Het wordt duidelijk dat in het algemeen inderdaad zulke eerste orde interacties bestaan. Wel is het zo dat hoe hoger de spin, hoe meer ruimte-tijd-afgeleides nodig zijn in de interactielagrangiaan.

Is het nu zo, dat voor hogere spins net zulke lagrangianen te vinden zijn als voor massaloze spin-1 en spin-2 deeltjes? Ondanks het bestaan van eerste orde interacties blijkt dit niet het geval te zijn. De spin-1 en spin-2 theorieën hebben elk de eigenschap dat als men alle andere soorten deeltjes

wegdenkt, er toch nog een consistente theorie overblijft. Dit is niet meer het geval voor hogere spins. Voor spin 3 wordt dit expliciet aangetoond in dit proefschrift. Het wordt aannemelijk gemaakt dat oneindig veel soorten hogere spins tegelijk een rol zouden moeten spelen in een consistente theorie van massaloze hogere spins met interactie. In het hier gebruikte formalisme zou dit een zeer ingewikkelde theorie worden. Het oneindig worden van het aantal ruimte-tijd-afgeleides lijkt er op te wijzen dat de theorie niet-lokaal wordt. Toch reduceert in de eerste orde benadering in de koppelingsconstante deze ingewikkelde theorie tot die van de relatief eenvoudige interacties die in dit proefschrift zijn beschreven.

CURRICULUM VITAE

Gerrit Burgers, geboren op 17 januari 1957, behaalde in 1975 het einddiploma Gymnasium-β aan het Gymnasium Haganum in Den Haag. Daarna studeerde hij natuurkunde aan de Rijksuniversiteit te Leiden. Dit resulteerde in 1978 in het kandidaatsexamen Natuurkunde en Scheikunde met Wiskunde (N4). Het doctoraal examen Natuurkunde met bijvak Wiskunde volgde in 1980. Tijdens zijn studie verrichtte hij experimenteel werk onder leiding van dr. E. Mazur in de groep Moleculaire Natuurkunde van prof.dr. J.J.M. Beenakker en prof.dr. H.F.P. Knaap. Vanaf eind 1980 werkt hij als wetenschappelijk medewerker in Leiden onder leiding van prof.dr. F.A. Berends op het gebied van de veldentheorie en de hoge-energiefysica. Eerst in dienst van de Rijksuniversiteit Leiden, nu binnen de werkgroep H-th-L van de Stichting voor Fundamenteel Onderzoek der Materie. Een deel van dit onderzoek vond zijn weerslag in dit proefschrift.

Aan het onderwijs droeg hij bij door het verzorgen van diverse werkcolleges en het maken van een dictaat bij het college "Very dense states of matter in particle physics and early cosmology" van de Lorentz-hoogleraar prof.dr. L. van Hove. Ter ondersteuning van het onderzoek werden zomerscholen in Cargèse (1981) en Edinburgh (1983) bezocht.

NAWOORD

Velen hebben bijgedragen aan het tot stand komen van dit proefschrift. In het bijzonder wil ik hier vermelden de stimulerende samenwerking met prof. H. van Dam, en waardevolle discussies met dr.ir. F.A. Bais en dr. M. de Roo.

Tot de promotie wordt met het oog op de beperkte ruimte in de senaatskamer uitsluitend toegang verleend op vertoon van een uitnodiging.

Receptie na afloop van de promotie in het Academiegebouw, Rapenburg 73, Leiden

N.B. Met tijdrovende parkeermoeilijkheden bij het Universiteitsgebouw moet nog altijd rekening worden gehouden

STELLINGEN

1. Niettegenstaande het feit dat er geen representaties van GL(4) zijn die zich gedragen als spinoren onder de ondergroep van Lorentztransformaties, zijn er wel formele representaties van algemene coordinatentransformaties met deze eigenschap.
Dit proefschrift, Hoofdstuk VII.

2. De Bel-Robinsontensor*) kan in verband gebracht worden met een globale invariantie van de Einstein-Hilbertlagrangiaan onder een transformatie van de metriek, die in gelineariseerde vorm wordt gegeven door

$\delta h_{\mu\nu} = \xi^{\rho\sigma\tau}\partial_\rho \partial_\sigma \partial_\tau h_{\mu\nu}$.

*) C.W. Misner et al., Gravitation, Freeman (1973), Ex. 15.2.

3. Beschouw de QED bijdrage tot de 2^e-orde-in-α vertexcorrectie van het elektron (massa m, lading e) ten gevolge van de vacuumpolarisatie van een ander soort fermionen (massa M, lading q.e). Als t de invariante impulsoverdracht op het elektron in de vertex is (met t>0 voor t tijdachtig), dan is de natuurlijke variabele om deze bijdrage in uit te drukken

$$x(t) = \frac{[1+4(M^2-m^2)/t]^{\frac{1}{2}} - [1-4m^2/t]^{\frac{1}{2}}}{[1+4(M^2-m^2)/t]^{\frac{1}{2}} + [1-4m^2/t]^{\frac{1}{2}}} .$$

4. In de limiet $t \gg M^2 \gg m^2$ wordt de QED bijdrage van een ander soort fermionen op de 2^e-orde-in-α vertexcorrectie van het elektron (zie stelling 3) gegeven door

$$\gamma^\mu \left(\frac{\alpha}{\pi}\right)^2 q^2 \left\{\frac{1}{36} u^3 + \frac{19}{72} u^2 + \left(\frac{265}{216} + \frac{1}{6}\frac{\pi^2}{6}\right) u - \frac{1}{3}\zeta(3) + \frac{19}{36}\frac{\pi^2}{6} + \frac{3355}{1296}\right\} .$$

Hier is $u = -\ln(t/M^2) + i\pi$.
Cf. R. Barbieri et al., Nuovo Cimento <u>11A</u> (1972) 824,865.

5. Er is een natuurlijke generalisatie van het fotonenergiespectrum van begintoestand-stralingsprocessen, zoals $e^+e^- \to \mu^+\mu^-\gamma$, wanneer men hogere orde stralingscorrecties beschouwt. Deze generalisatie wordt gegeven door de differentiele werkzame doorsnede $\frac{d\sigma}{ds'}$, waar s' het kwadraat van de invariante massa van het geproduceerde muonpaar is.

6. Bij de toekenning van kernspintoestanden aan de rotatie-energieniveaus van een molecuul in een kristalveld hoeft men met de spiegelsymmetrieën van de rotatiepotentiaal geen rekening te houden.
D. van der Putten en N.J. Trappeniers, Physica 129A (1985) 327.

7. Voor de simulatie van suspensies kan men beter een algoritme in termen van de twee-deeltjefrictie dan in termen van de twee-deeltjesmobiliteit kiezen.

8. Beschouw een stochastische wandeling op een oneindig puntrooster met random geplaatste valpunten. Laat q de dichtheid van de valpunten zijn, η de kans op ontsnapping bij aankomst in een valpunt, en laat er tevens een verlieskans α per stap zijn. Voor de totale verlieskans $L(q,\alpha,\eta)$ geldt:

$$\lim_{\substack{q\to 0\\ \alpha\to 0}} L(q,\alpha,\eta) = \left\{1 + \frac{q}{\alpha}\left[\frac{1}{1-F} + \frac{\eta}{1-\eta}\right]^{-1}\right\}^{-1}$$

waarbij F de kans op terugkeer is van de stochastische wandeling naar de oorsprong in afwezigheid van vallen en verlieskans.

9. Het type algebra's van transformatieregels van velden dat nu als "op de massaschil sluitende algebra's" pleegt te worden aangeduid, had men beter "algebra's voor een bepaalde actie" kunnen noemen.
Dit proefschrift, Hoofdstuk IV.

10. Kenmerkend voor de relatie tussen een drogreden D van de categorie *argumentum ad misericordiam* en de conclusie C die D pretendeert te beargumenteren, is dat D weliswaar niet C ondersteunt, maar wel een door C ondersteunde conclusie C'. Aan het *argumentum ad misericordiam* ligt dus de drogreden van de bevestiging van de consequens ten grondslag.

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