CLASSICAL RELATIVISTIC SPIN-THEORY AND ITS QUANTIZATION

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I. *Introduction.* In the course of the development of modern atomic theory the Z e e m a n-effect has repeatedly played a prominent part. Two outstanding instances hereof may be briefly recalled in the following. Immediately after its discovery in 1896 it proved — on the basis of L o r e n t z' analysis — to lend a most convincing support to the idea, that small, negatively charged particles, identical with those discovered in the cathode rays, were present inside the atom and constituted, through their vibrations, the source of the electro-magnetic disturbances giving rise to spectral lines. The fact, however, that many spectral lines show a Z e e m a n-effect of the so-called anomalous type, remained for a long time a serious difficulty. On the one hand, L o r e n t z' and V o i g t's formal treatments were far from satisfactory from a physical point of view. On the other hand, the development of B o h r's views on the origin spectral lines during the years 1913—1925 was hardly fit to encourage the optimistic view that the anomalous Z e e m a n-effect might be a simple consequence of the quantum laws governing the behaviour of electrons inside the atom.

In 1925 a way out of this difficulty was offered by U h l e n b e c k's and G o u d s m i t's hypothesis of the electronic spin, according to which an electron should possess — besides its mass and charge — an intrinsic rotational moment accompanied by a magnetic moment. Among the experimental facts leading to this hypothesis, the laws of the anomalous Z e e m a n effect, as formulated by L a n d é, ranked first.

II. *Classical spin problem. Aim of this paper.* A point-electron (mass m, charge $-e$), moving in a central field of force gives rise to a magnetic moment equal to its rotational moment multiplied by

$-e/2mc$. In order to explain the anomalous Zeeman-effect Uhlenbeck and Goudsmit had to assume, that the corresponding ratio between magnetic and rotational moment of the electronic spin is twice as large, *viz.* $-e/mc$. The analysis of possible classical models of a rotating electron showed that — although a difference between the two said ratios was to be anticipated — arguments along these lines would be insufficient to predict in an unambiguous way the factor 2 required by experiment.

Dirac's ingenious treatment of the relativistic wave equation of the electron (1928), in which the idea of electronic spin was not primarily introduced, seems to have thoroughly changed the aspect of the theoretical problems involved. In fact, the physical content of Dirac's linear equations — when interpreted in the limit of small velocities — reflects exactly all the properties of the electron including those pertaining to the spin, both the factor 2 and the value $h/2$ for the spinmoment appearing automatically. Thus the optimistic view, to which we alluded in part I, appears to be justified after all, and one is tempted to adopt Dirac's elegant formalism as a primary basis for our description of the electron's behaviour.

Should therefore any investigation which approaches the spin properties from a purely classical point of view, such as for instance Uhlenbeck's and Goudsmit's original treatment, be rejected as inappropriate? There are several reasons which urge us to be cautious with our answer. The famous difficulty of the negative mass — even though it be mitigated to a considerable extent by the hole theory — shows us that even Dirac's theory cannot be considered as a satisfactory foundation. Furthermore we may recall the anomalous value of the magnetic moment of the proton, which was recently discovered by Stern & Frisch. There is no a-priori theoretical reason why the Dirac equations should apply to the electron and not to the proton.

In view of this situation, it is perhaps not without interest that even a consideration, in which the idea of electronic spin is introduced in a purely classical way, affords a simple interpretation of the value of the ratio between the electrons magnetic and rotational moment[1]). The argument rests uniquely on the principle that a consistent set of relativistically invariant equations of motion should be established,

1) H. A. Kramers. On the classical theory of the spinning electron. Physica **1**, 825, 1934.

which — in a system of coordinates moving with the electron — reduces to the well known laws, expressing how only the electric field (in first approximation) governs the acceleration and how only the magnetic field governs the precession of the spin-vector. Considerations pertaining to a detailed classical model of the electron do not enter at all.

In this paper we will show that a classical spin theory developed along these lines is intimately connected with D i r a c's theory of the electron. In fact, if a process of quantization is applied in which the quantum-number of the spin is put equal to $\frac{1}{2}$ and if the classical hamiltonian is chosen in an appropriate way, the result will be identical with D i r a c's formalism.

III. *Equations of motion.* In our paper cited above the equations governing the precession of the spin-vector were written in the relativistically invariant form:

$$\frac{d\mathbf{S}}{d\tau} = \alpha\,[\mathbf{SF}], \tag{1}$$

where $d\tau$ denotes the element of eigenzeit, whereas **S** and **F** are two complex vectors:

$$\mathbf{S} = \mathbf{A} + i\mathbf{B}, \qquad \mathbf{F} = \mathbf{H} + i\mathbf{E}. \tag{2}$$

A and **B**, which characterize the spin, transform under a L o r e n t z-transformation as **H** and **E** (magnetic and electric field strength). A L o r e n t z-transformation corresponds to a (generally complex) orthogonal transformation of the components of **S** and **F**. The condition that **B** always vanishes in an inertial system moving with the electron leads to the relativistically invariant relation:

$$\mathbf{B} = \frac{1}{c}\,[\mathbf{Av}], \tag{3}$$

where **v** is the velocity of the electron.

In a system in which $\mathbf{v} = 0$, the real part of (1) reduces to the unrelativistic classical description of the behaviour of a spinning electron with spin-vector **A** (*i.e.* vector of rotational moment) in a magnetic field **H**, α being the ratio between magnetic and rotational moment:

$$\dot{\mathbf{A}} = \alpha\,[\mathbf{AH}]. \tag{4}$$

The imaginary part of (1) reduces in the same system to:

$$\dot{\mathbf{B}} = \alpha\,[\mathbf{AE}] \tag{5}$$

If the reaction of the spin on the orbital motion may be considered as very small, this motion will — always for $\mathbf{v} = 0$ — obey the law:

$$\dot{\mathbf{v}}m = -e\mathbf{E} \tag{6}$$

so that (5) takes the form:

$$\dot{\mathbf{B}} = -\frac{\alpha m}{e}\,[\mathbf{A}\dot{\mathbf{v}}] \tag{7}$$

If, now, we derive (3) with respect to the time and put $\mathbf{v} = 0$, we obtain a formula which, when comparing with (7), leads immediately to:

$$\alpha = -\frac{e}{mc} \tag{8}$$

If one wishes to take the reaction of the spin on the orbit into account without abandoning the rigorous validity of (3), equation (1) has to be considered as a first approximation only. The procedure to be followed in order to develop a more complete theory along these lines is not unambiguously prescribed. At present we will leave this question apart; we do not know if its treatment will lead to results of physical interest [1]).

IV. *Canonical form.* Before quantizing the equations of motion (6) and (1), it will be necessary first to establish a hamiltonian equation, from which they both can be simultaneously derived. For this purpose let us first consider the equations (4), in which only real vectors occur. They can be written in canonical form if — in agreement with the ordinary treatment of a dipole in a field — the energy is taken to be:

$$H_A = -\alpha\,[\mathbf{AH}) \tag{9}$$

1) The development of the theory to higher approximations seems to require that to the electron, besides an electrical charge (monopole) and a magnetic moment (dipole), should be attributed also an electrical quadrupole, a magnetic octopole, a.s.o. These poles of higher order disappear automatically if a quantization is applied which gives the electron a spin-moment of only $h/2$.

There is only one degree of freedom, and for the canonical coordinates one may choose:

$$p = \mathrm{A}_1, \quad q = \text{arc tg}\,\frac{A_2}{A_3}, \quad (A_2 + iA_3 = \sqrt{A^2 - p^2}\,.\,e^{iq}) \,.$$

This corresponds to the following values for the Poisson-brackets:

$$\{A_1 A_2\} = \frac{\partial A_1}{\partial p}\frac{\partial A_2}{\partial q} - \frac{\partial A_1}{\partial q}\frac{\partial A_2}{\partial p} = -A_3, \; \{A_2 A_3\} = -A_1, \; \{\mathrm{A}_3\mathrm{A}_1\} = -A_2.$$

The equations governing the change of **A**:

$$A_k = -\sum_l \{A_k\, A_l\}\frac{\partial H_A}{\partial A_l}$$

are seen to be identical with (4) if the expression (9) for H_A is adopted.

The equations (1) can be treated in an exactly analogous way:

$$\left.\begin{aligned} H_S &= -\alpha\,(\mathbf{SF}) \\ \{S_1\, S_2\} &= -S_3, \text{ cycl.} \\ \frac{dS_k}{d\tau} &= -\sum_l \{S_k\, S_l\}\frac{\partial H_S}{\partial S_l} \end{aligned}\right\} \tag{10}$$

Formally we have still to do with a system of one degree of freedom, the canonical coordinates being, for instance:

$$p = S_1, \quad q = \text{arc tg}\,\frac{S_2}{S_3},$$

but these coordinates are complex and so is the hamiltonian H_S. This circumstance need not alarm us. Separating everywhere real and imaginary parts:

$$p = p' + ip'', \quad q = q' - iq'', \quad H = H'(p'\,p''\,q'\,q'') + iH''(p'\,p''\,q'\,q''),$$

it is easily verified that the complex equations of motion:

$$\frac{dp}{d\tau} = -\frac{\partial H}{\partial q}, \quad \frac{dq}{d\tau} = \frac{\partial H}{\partial p} \quad (d\tau \text{ real})$$

correspond to real canonical equations of a system of two degrees of freedom in which either H' or H'' is taken as hamiltonian:

$$\begin{aligned} \frac{dp'}{d\tau} &= -\frac{\partial H'}{\partial q'} \quad \frac{dq'}{d\tau} = \frac{\partial H'}{\partial p'} \quad \frac{dp'}{d\tau} = \frac{\partial H''}{\partial q''} \quad \frac{dq''}{d\tau} = -\frac{\partial H''}{\partial p'} \\ &\text{or} \\ \frac{dp''}{d\tau} &= -\frac{\partial H'}{\partial q''} \quad \frac{dq''}{d\tau} = \frac{\partial H'}{\partial p''} \quad \frac{dp''}{d\tau} = -\frac{\partial H''}{\partial q'} \quad \frac{dq'}{d\tau} = \frac{\partial H''}{\partial p''} \end{aligned} \tag{11}$$

This consideration shows that — besides (10) — two alternative ways of deriving the equations from a real hamiltonian offer themselves. For this purpose we have to introduce besides **S** its complex conjugate vector **S***:

$$H'_S = -\alpha \frac{(\mathbf{SF}) + (\mathbf{S^*\,F^*})}{2} \tag{10a}$$

$$\{S_1\, S_2\} = -2S_3, \text{ cycl.}; \quad \{S_1^*\, S_2^*\} = -2S_3^*, \text{ cycl.}; \quad \{S_k\, S_l^*\} = 0$$

$$H''_S = -\alpha \frac{(\mathbf{SF}) - (\mathbf{S^*\,F^*})}{2i} \tag{10b}$$

$$\{S_1\, S_2\} = -2i\, S_3, \text{ cycl.}; \quad \{S_1^*\, S_2^*\} = -2iS_3^* \quad \{S_k\, S_l^*\} = 0$$

$$\frac{dS_k^{(*)}}{d\tau} = -\sum_l \{S_k^{(*)}\, S_l\} \frac{\partial H_S}{\partial S_l} - \sum_l \{S_k^{(*)}\, S_l^*\} \frac{\partial H_S}{\partial S_l^*}$$

where $S_k^{(*)}$ means either S_k or S_k^*.

The system is now explicitely treated as one of two degrees of freedom; the expressions for the P o i s s o n-brackets are found by taking the four real canonical variables (comp. (11)) explicitely into account.

For completeness we might finally mention the alternative of (10) which arises when, in stead of **S**, its complex conjugate **S*** is introduced:

$$\left.\begin{aligned} H_S^* &= -\alpha\, (\mathbf{S^*\,F^*}) \\ \{S_1^*\, S_2^*\} &= -S_3^*, \text{ cycl.} \\ \frac{dS_k^*}{d\tau} &= -\sum_l \{S_k^*\, S_l^*\} \frac{\partial H_S^*}{\partial S_l^*} \end{aligned}\right\} \tag{10c}$$

From (2) we find:

$$(\mathbf{SF}) = \{(\mathbf{AH}) - (\mathbf{BE})\} + i\,\{(\mathbf{AE}) + (\mathbf{BH})\}$$

It would therefore appear most natural to adopt the form H'_S given by (10*a*):

$$H'_S = -\alpha\{(\mathbf{AH}) - (\mathbf{BE})\}$$

since, in an inertial system moving with the electron, it reduces to the familiar real energy expression (9). If we adopt the simpler form H_S or H^*_S given by (10) or (10*c*) it looks at first sight as if we formally introduced an imaginary electrical moment of the electron equal to $-i\alpha\mathbf{A}$ or $+i\alpha\mathbf{A}$ respectively.

The equations of motion (6), in which the spin is neglected, can be derived from the familiar hamiltonian equation:

$$H_0 \equiv \frac{1}{2m}\{-\frac{(\varepsilon + e\Phi)^2}{c^2} + (\mathbf{p} + \frac{e}{c}\mathbf{\Psi})^2\} = -\tfrac{1}{2}mc^2 \qquad (12)$$

$$\frac{d\varepsilon}{d\tau} = \frac{\partial H_0}{\partial t}, \quad \frac{dt}{d\tau} = -\frac{\partial H_0}{\partial \varepsilon}$$

$$\frac{dp_i}{d\tau} = -\frac{\partial H_0}{\partial x_i}, \quad \frac{dx_i}{d\tau} = \frac{\partial H_0}{\partial p_i},$$

where ε is the energy and $\mathbf{p}$ the momentum of the system, Φ the scalar and $\mathbf{\Psi}$ the vector potential of the external field, while $d\tau$ denotes again the element of eigenzeit.

Within the limits of the validity of our classical analysis (*i.e.* $\mathbf{S}$ so small that the reaction of the spin on the orbital motion is negligible) a hamiltonian $\bar{H}$ which simultaneously governs the orbital motion and the spin-precession will be simply obtained by replacing H_0 in (12) by the sum of H_0 and the hamiltonian which governs the spin:

$$\bar{H} \equiv \frac{1}{2m}\{-\frac{(\varepsilon + e\Phi)^2}{c^2} + (\mathbf{p} + \frac{e}{c}\mathbf{\Psi})^2\} + \frac{e}{mc}(\mathbf{SF}) = \tfrac{1}{2}mc^2,$$

or, multiplying by $-2m$:

$$-2m\bar{H} \equiv H \equiv \frac{(\varepsilon + e\Phi)^2}{c^2} - (\mathbf{p} + \frac{e}{c}\mathbf{\Psi})^2 - \frac{2e}{c}(\mathbf{SF}) = m^2c^2 \qquad (13)$$

Here we have based ourselves on (10). Using in stead (10*a*), (10*b*) or (10*c*) we might replace $(\mathbf{SF})$ in (13) by $\frac{1}{2}\{(\mathbf{SF}) + (\mathbf{S^*F^*})\}$, $-\frac{1}{2}i\{(\mathbf{SF}) - (\mathbf{S^*F^*})\}$ or $(\mathbf{S^*F^*})$ respectively. In view of the approximation involved ($\mathbf{S}$ very small), either of these four expressions may be chosen, although the choice $\frac{1}{2}\{(\mathbf{SF}) + (\mathbf{S^*F^*})\}$ might seem the most natural one.

It appears difficult, if not impossible, to establish a hamiltonian equation, in virtue of which the condition (3), which has practically the effect of reducing the two degrees of freedom of the spin to only one, is automatically fulfilled [1]).

V. *Quantization.* In order to quantize the motion governed by the hamiltonian equation (13) or one of its alternatives, H must be considered as an operator H_{op} acting on a wave function ψ:

$$H_{op}\,\psi = m^2c^2\,\psi \tag{14}$$

If the spin is to be given a quantum number $\frac{1}{2}$ and if we restrict ourselves to (13), the following familiar expressions will have to be adopted:

$$\varepsilon = ih\frac{\partial}{\partial t}, \quad p_i = -ih\frac{\partial}{\partial x_i}, \tag{15}$$

$$S_1 = \frac{h}{2}\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}, \quad S_2 = \frac{h}{2}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad S_3 = \frac{h}{2}\begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} \tag{16}$$

The commutation properties of these expressions satisfy the necessary conditions corresponding to the properties of the analogous Poisson-brackets. The introduction of the Pauli spin-matrices for **S** means that ψ, besides on $x_1x_2x_3t$, depends on a spin variable which only can take two values, for instance the eigenvalues $\pm h/2$ of S_1, so that ψ can be represented as a set of two wave components ψ_+, ψ_-.

The relativistic invariance of this choice for **S** is made clear by the investigations of Weyl and van der Waerden; it follows from the fact that to each Lorentz-transformation a unimodular transformation of the wave-components may be assigned in such a way that ψ_+^2, $\psi_+\psi_-$ and ψ_-^2 transform like $-F_2 + iF_3$, F_1 and $F_2 + iF_3$ respectively. With this convention the Pauli matrix components will transform exactly like the components of **F** $(= \mathbf{H} + i\mathbf{E})$. It is interesting to note that the components of **S** remain hermitical even when they undergo a complex orthogonal transformation.

1) This question cannot be settled, anyhow, before a classical system of equations of motion has been established, which allows the condition (3) to be rigorously fulfilled, and not only approximately as in the case of the hamiltonians considered above.

Introducing (15) and (16) into (13) in order to construct H_{op}, the equation (14) assumes exactly the form which D i r a c obtained by „squaring" his linear equations. The latter may be got back from (14) by observing that the H_{op} thus constructed factorizes in the following way:

$$H_{op} = \left\{\frac{\varepsilon + e\Phi}{c} - \frac{2}{h}(\mathbf{p} + \frac{e}{c}\mathbf{\Psi}, \mathbf{S})\right\}\left\{\frac{\varepsilon + e\Phi}{c} + \frac{2}{h}(\mathbf{p} + \frac{e}{c}\mathbf{\Psi}, \mathbf{S})\right\}.$$

Consequently, putting

$$\left\{\frac{\varepsilon + e\Phi}{c} + \frac{2}{h}(\mathbf{p} + \frac{e}{c}\mathbf{\Psi}, \mathbf{S})\right\}\psi = mc\,\chi, \tag{17}$$

χ will satisfy the equation

$$\left\{\frac{\varepsilon + e\Phi}{c} - \frac{2}{h}(\mathbf{p} + \frac{e}{c}\mathbf{\Psi}, \mathbf{S})\right\}\chi = mc\,\psi. \tag{18}$$

Since ψ and χ are both two-component wave functions, (17) and (18) represent a system of four simultaneous equations. They are equivalent with D i r a c's equations.

If, in (13), we had substituted (**S*F***) for (**SF**), we should have had to introduce for the components of **S*** exactly the same matrices (16) as given for those of **S**, but the two-component wave function χ_+, χ_- to be introduced now would have to be such that χ_+^2, $\chi_+\chi_-$ and χ_-^2 transform like $-F_2^* + iF_3^*$, F_1^* and $F_2^* + iF_3^*$ respectively. It is easily verified that the corresponding hamiltonian H_{op}^* factorizes as follows:

$$H_{op}^* = \left\{\frac{\varepsilon + e\Phi}{c} + \frac{2}{h}(\mathbf{p} + \frac{e}{c}\mathbf{\Psi}, \mathbf{S^*})\right\}\left\{\frac{\varepsilon + e\Phi}{c} - \frac{2}{h}(\mathbf{p} + \frac{e}{c}\mathbf{\Psi}, \mathbf{S^*})\right\}$$

and that we consequently are led precisely to the equations (17) and (18) again. Since χ_+^2, $\chi_+\chi_-$ and χ_-^2 transform like $-\psi_-^{*2}$, $\psi_-^*\psi_+^*$, $-\psi_+^{*2}$ we see that χ_+ and χ_- transform like ψ_-^* and $-\psi_+^*$ respectively; this result is well known from v a n d e r W a e r d e n's analysis.

VI. *Concluding remarks.* In the foregoing we have established the intimate connection between D i r a c's linear equations and a purely classical analysis. It would lead us too far to trace this connection in further detail. It may be pointed out, however, that it has been brought about by chosing for the spin-part of the Hamil-

tonian the complex expression (10) or (10*c*). It involves an imaginary electrical moment, but, from a formal point of view, it is much simpler than the real expressions H'_S in (10*a*) and H''_S in (10*b*). From a physical point of view it would perhaps be more natural to chose the hamiltonian (10*a*). This would correspond to the hamiltonian equation:

$$H \equiv \frac{(\varepsilon + e\Phi)^2}{c^2} - (\mathbf{p} + \frac{e}{c}\mathbf{\Psi})^2 - \frac{e}{c}\{(\mathbf{SF}) + (\mathbf{S^*F^*})\} = m_2 c_2.$$

The straightforward quantization of this equation, however, leads not to the simple Dirac-theory, but to a Schrödinger equation of a more complicated type. It would, indeed, lead us to consider Ψ in (4) as depending on two spin-variables, both of which take only two values. Thus Ψ would now be a four-component wave function and its four components would satisfy four simultaneous differential equations of the second degree.

Received 30 April 1935.