Non-relativistic Quantum-Electrodynamics and correspondence Principle

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I.

INTRODUCTION

In Lorentz'theory of electrons we start by introducing some hypothesis regarding the structure of the electron; at some later stage it is then found that many features of the interaction between electrons and radiation field involve only two constants characterizing the electron, viz. its charge e and its mass m. This result is the fundament for the numerous classical applications of electron theory to actual problems and experiments. In all of these the electronic structure plays no explicit part; only e and m appear in the formulae and m which in the theory appears as the sum of the inertial and the electromagnetic mass, may therefore also be denoted as the experimental mass. It is of importance A : to find the simplest way in which the structure-independent part of the assertions of the electron theory can be mathematically expressed, and B : to bring these expressions in the form of canonical equations of motion. In fact, we will thereby have obtained a trustworthy basis for a quantum-theory of interaction between charged particles and field, i. e. for a quantum theory which shows the necessary correspondence with that part of the classical electron theory which is of physical importance.

In literature, the structure-independent assertions of the electron theory are usually presented in a way which involves the application of retarded potentials (compare the well known expression $2e^2/3c^3.\ddot{x}$ for the radiation reaction), whereas for the solution of problem A mentioned above it is preferable that no notions should be introduced which conceal the symmetry of time. This requirement can be fulfilled, however, in a natural way if we introduce the « proper » field of an electron in the way in which it was done in my

monograph on quantum theory ([1]) and in my paper delivered at the Galvani congress in Bologna 1937 ([2]). A definite solution of problem A has been proposed in these papers (in non-relativistic approximation), but the equations of motion were only discussed in some detail as far as the field equations are concerned

The solution in question has been taken as a basis for two papers of 1941. In the first of these, Mr. Serpe ([3]) shows how it can be used for correcting certain points in the quantum theory of a harmonic oscillator emitting a light quantum. In the second Mr. Opechowski ([4]) succeeds in establishing to a certain approximation a Hamiltonian for the equations of motion, i. e. he gives an approximate solution for problem B mentioned above.

In this report we will present an approximate treatment of A and B, i. e. of the problem of structure elimination and discuss, on this basis, certain problems of quantum electrodynamics. The approximations involved refer first of all to the use of Lorentz' old model of the rigid electron. Our treatment will therefore be non-relativistic in this sense that our formulae will only be valid in the region where the velocities of the electron are small compared to the velocity of light. A relativistic treatment would indeed hardly seem possible or promising : on the one hand there exists no relativistic classical electron theory which provides us with a precise and simple model (*) of the contractile electron and its interaction with the radiation field, on the other hand the spin property of the electron and Dirac's theory of 1928 warn us that the requirements of relativity are met in nature in a way which is hardly properly reflected in any classical theory. Another approximation in our treatment will be that we restrict ourselves mostly to the interaction of the electron with the electric dipole radiation only; this means for instance the neglect of radiation pressure on the electron, which may be said to be due to the magnetic dipole radiation and which in quantum theory is reflected in the recoil of the electron from the incident and emitted light quanta.

In several respects our treatment will show similarity with a well known paper by Bloch and Nordsieck ([5]). There, the « bound » light quanta carried along by the moving electron were eliminated by a proper transformation; they are practically identical with what we call the proper field of the electron. It will be seen, however,

(*) The only precise model I know of would necessitate the conception of the electron as a relativistic elastic body along the lines of Herglotz' theory of 1911, which would endow the electron with an infinite number of degrees of freedom.

that our discussion and results move along lines rather different from those followed by the mentioned authors.

It would seem as if our treatment first of all allows an analysis of emission, absorption and scattering of light which supplements that hitherto presented in literature. In the second place it also throws some light on the problem of the divergencies in quantum electrodynamics and on the questions connected with the Lamb-shift, although of course these problems, for their exact treatment require an analysis which goes beyond the approximations which we have imposed upon ourselves.

II.

ELIMINATION OF THE STRUCTURE FROM THE EQUATIONS OF MOTION

Consider one electron in the E, H field, the latter satisfying everywhere the Lorentz equations (light velocity = 1) :

$$\text{rot } \bar{H} = 4\pi \bar{j} + \dot{\bar{E}}, \qquad \text{div } \bar{H} = 0$$

$$\text{rot } \bar{E} = -\dot{\bar{H}}, \qquad \text{div } \bar{E} = 4\pi\rho$$

Take for the electron a rigid body of mass m_0 and charge e. The mass distribution has central symmetry and extends over a region the linear dimensions of which are of the order of magnitude a (« electron radius »). If the wave lengths caracterizing the fields which affect the electron are large compared to a, a structure-independent behaviour of the electron can be expected if its velocity stays small compared to that of light, and we may also assume that the electron does not rotate.

The electron equations of motion are

$$m_0 \ddot{\bar{R}} = e\ddot{\bar{E}} + e\dot{\bar{R}} \wedge \ddot{\bar{H}} - \frac{\partial U}{\partial \bar{R}} \qquad (1)$$

$\bar{R}$ denotes the position of the electron centre and U the potential energy of the electron in a fixed field of force such as might arise from

a fixed distribution of electric charges. If the latter is the case, our E does not include the field strenght due to these charges. The symbol $\tilde{\tilde{Q}}$ is a mean value explained by

$$e\,\tilde{\tilde{Q}} = \int Q\,\rho\,dV$$

With our model we have

$$\dot{\bar{j}} = \rho\,\dot{\bar{R}}$$

We introduce field potentials by

$$\bar{H} = \text{rot}\,\bar{A}, \qquad \bar{E} = -\nabla\,\varphi - \dot{\bar{A}}, \qquad \text{div}\,\bar{A} = 0$$

Thus

$$\Delta\,\varphi = -4\,\pi\,\rho$$

$$\Delta\,\bar{A} - \ddot{\bar{A}} = \Box\,\bar{A} = -4\,\pi Tr\,\bar{j} = -4\,\pi Tr\,\rho\,\dot{\bar{R}} = -4\,\pi\,\rho\,\dot{\bar{R}} + \nabla\,\dot{\varphi} \quad (2)$$

where $Tr\,\bar{B}$ stands as a symbol for the transversal or solenoidal part of a vector field $\bar{B}$.

The Coulomb potential φ is given by

$$\varphi_P = \int (\rho_Q/r_{PQ})\,dV_Q$$

At a distance r from the electronic centre large compared to a we have

$$\varphi \cong e/r$$

We now divide A into two parts

$$\bar{A} = \bar{A}_1 + \bar{A}_0 \quad (3)$$

where $\bar{A}_0$ is the divergence-free solution of

$$\text{rot rot}\,\bar{A}_0 = 4\,\pi\,Tr\,\rho\,\dot{\bar{R}} \quad (4)$$

Thus with neglect of terms of the relative order $\dot{R}^2$, $\bar{A}_0$ is the divergence-free vector potential of a uniformly moving electron in radiation-free space, which at time t has the same $\bar{R}$ and $\dot{\bar{R}}$ as the actual electron. By definition we call $\bar{A}_0$ the vector potential of the « proper » field of the electron; similarly we call $\bar{A}_1$ the vector potential of the « external » field.

The solution of (4) is

$$\bar{A}_{oP} = \int Tr\,(\bar{j}_Q/r_{PQ})dV_Q = \int (\bar{j}_Q/2r_{PQ} + r_{PQ}(\bar{r}_{PQ}\,\bar{j}_Q)/2r_{PQ}^3)dV_Q \quad (5)$$

At distances $r \gg a$ this field reduces to

$$\bar{A}_0 \simeq e\,(\dot{\bar{R}}/2r + \bar{r}(\bar{r}\,\dot{\bar{R}})\,/2r^3) = Tr\,(e\dot{\bar{R}}/r)$$

For $\bar{\bar{A}}_0$ one finds

$$\bar{\bar{A}}_0 = \Theta \frac{e}{a} \dot{\bar{R}} \tag{6}$$

where the numerical factor Θ depends on the ρ distribution (for a surface-charged sphere of radius a we have $\Theta = 2/3$). The same factor appears in the expression for the electromagnetic mass μ :

$$\mu = \Theta \frac{e^2}{a} \tag{7}$$

The electric field strength can be written

$$\bar{E} = \dot{\bar{A}}_1 - \dot{\bar{A}}_0 - \nabla\,\varphi$$

Now $\bar{A}_0$ depends on t both through $\bar{R}$ and $\dot{\bar{R}}$ .Hence

$$-\dot{\bar{A}}_0 = -(\dot{\bar{R}}\,\partial/\partial\,\bar{R})\bar{A}_0 + \bar{F}$$

$$\bar{F} = -(\ddot{\bar{R}}\,\partial/\partial\,\dot{\bar{R}})\bar{A}_0, \quad \bar{F}_P = -\int (\ddot{\bar{R}}/2r_{PQ} + \bar{r}_{PQ}(\bar{r}_{PQ}.\ddot{\bar{R}})/2r_{PQ}^3)\rho_Q\, dV_Q$$

At distances $r \gg a$, $\bar{F}$ reduces to

$$\bar{F} \simeq -Tr(e\ddot{\bar{R}}/r)$$

whereas for $\bar{\bar{F}}$ we find

$$\bar{\bar{F}} = -\Theta \frac{e}{a} \ddot{\bar{R}} = -\frac{\mu}{e} \ddot{\bar{R}} \tag{8}$$

We now define the « proper » electric field as the sum of the Coulomb field $-\nabla\varphi$ and a transversal part $\bar{E}_0$ defined by

$$\bar{E}_0 = -(\dot{\bar{R}}\,\partial/\partial\,\bar{R})\,\bar{A}_0,$$

which at distances $r \gg a$ reduces to

$$\bar{E}_0 \simeq e\,\bar{r}\,(\dot{\bar{R}}^2/2r^3 - 3(\dot{\bar{R}}.\bar{r})^2/2r^5)$$

The proper magnetic field is defined by

$$\bar{H}_0 = \text{rot}\,\bar{A}_0$$

and reduces for distances $r \gg a$ to

$$\bar{H}_0 \simeq e\,\dot{\bar{R}} \wedge \bar{r}/r^3$$

Apart from terms of the relative order $\dot{R}^2$, $\overline{E}_0$ and $\overline{H}_0$ are the (transversal) electric and magnetic field of the uniformly moving electron mentionned above. The averages $\overline{\overline{E}}_0$ and $\overline{\overline{H}}_0$ both vanish; the same holds for $-\overline{\nabla\varphi}$.

The « external » electric and magnetic field are defined by

$$\overline{E}_1 = \overline{E} + \nabla\varphi - \overline{E}_0 = -\dot{\overline{A}}_1 + \overline{F} \tag{10}$$

$$\overline{H}_1 = \overline{H} - \overline{H}_0 = \text{rot}\,\overline{A}_1 \tag{11}$$

Through $\overline{F}$, $\overline{E}_1$ will behave like $\ddot{\overline{R}}/r$ in the neighbourhood of the electron.

Introducing (10) and (11) into (1) we get

$$m_0\,\ddot{\overline{R}} = -e\,\dot{\overline{\overline{A}}}_1 + e\,\overline{\overline{F}} + e\,\dot{\overline{R}} \wedge \overline{\text{rot}\,\overline{A}_1} - \partial U/\partial\,\overline{R}$$

Using (8) and introducing the experimental mass m of the electron by

$$m = m_0 + \mu = m_0 + \Theta e^2/a \tag{12}$$

the equations of motion take the simple form

$$m\,\ddot{\overline{R}} = -e\,\dot{\overline{\overline{A}}}_1 + e\,\dot{\overline{R}} \wedge \overline{\overline{\text{rot}\,\overline{A}_1}} - \partial U/\partial\,\overline{R} \tag{13}$$

For the field equation satisfied by $\overline{A}_1$ we get from (2) and (3)

$$\text{rot rot}\,\overline{A}_1 + \text{rot rot}\,\overline{A}_0 = 4\pi\rho\,\overline{R} - \ddot{\overline{A}}_1 - \ddot{\overline{A}}_0 - \partial\nabla\varphi/\partial\,t$$

The second term on the left hand cancels against the first and the last term on the right, so we are left with

$$\Box\,\overline{A}_1 = \ddot{\overline{A}}_0 = \left\{ (\dddot{\overline{R}}\,\partial/\partial\,\dot{\overline{R}}) + 2(\ddot{\overline{R}}\,\partial/\partial\,\overline{R})\,(\dot{\overline{R}}\,\partial/\partial\,R) + (\dot{\overline{R}}\,\partial/\partial\,\overline{R}) + (\dot{\overline{R}}\,\partial/\partial\,\overline{R})^2 \right\} \overline{A}_0 \tag{14}$$

where the symbolic notation in the last member will be easily understood. It shows that there are three types of terms in $\ddot{\overline{A}}_0$, viz. terms linear in $\dddot{\overline{R}}$, terms linear in $\ddot{\overline{R}}$ and linear in $\dot{\overline{R}}$, and terms cubic in $\dot{\overline{R}}$, the coefficients being still functions of R (and x, y, z).

Until now everything was purely formal. The physical question which arises is whether or — if not — to what approximation (13) and (14) together give us a structure-independent description of the system. Since in (13) the experimental mass itself occurs, the question refers clearly to the behaviour of $\overline{A}_1$. If $\overline{A}_1$ and $\dot{\overline{A}}_1$ are prescribed at $t = 0$ in such a way as to present the necessary singularities (requir-

ed by the form of $\ddot{\bar{A}}_0$) but no infinities at the electrons position will they — at later times — behave in a way which does not depend on the particularities of the charge distribution, such that the motion of the electron and the evolution of the field can be said to depend only on e and m? First of all it is clear that $\bar{A}_1$ and $\dot{\bar{A}}_1$ should be prescribed to be sufficiently smooth functions (characteristic wavelengths all large compared to a), so that the mean values $\dot{\bar{A}}$ and $\widetilde{\text{rot}\ \bar{A}}$ are equal to the values of $\dot{\bar{A}}$ and $\text{rot}\ \bar{A}$ at the centre of the electron. But even so, we can be sure that — rigorously — $\bar{A}$ will in the course of time develop a structure-dependent singularity near the electron. In fact we know that — rigourously — the electromagnetic momentum of the electron apart from the main term $\mu\dot{\bar{R}}$ will contain structure-dependent terms proportional to the higher odd powers of $\dot{\bar{R}}$. This means that $\bar{A}_1$ and $\dot{\bar{A}}_1$ will in short time take such values that (13) accounts at least for this (from relativity point of view erroneous) dependency of momentum on velocity

Still, within a certain approximation, (14) will certainly possess solutions with the structure-independent behaviour which we require. In fact, denoting by *Hom.* a smooth singularity-free solution of the homogeneous equation $\Box\, \bar{A}_1 = 0$ and by *Inh.* the well known time-symmetrical solution of the inhomogeneous equation (5) (half sum of retarded and accelerated potential)

$$Inh._P = \tfrac{1}{2} \int ([\rho_Q \dot{\bar{R}}/r_{PQ}]_{t' = t - r} + [\rho_Q \dot{\bar{R}}/r_{PQ}]_{t' = t + r}) \, dV_Q \qquad (15)$$

the transversal part of which, apart from relativistic corrections, behaves for near the electron like

$$Tr\ Inh. \cong Tr\ [e\, \dot{\bar{R}}/r]_{t' = t} = \bar{A}_0$$

the required solutions of (14) will be

$$\bar{A}_1 = -\bar{A}_0 + Tr\ Inh. + Tr\ Hom. \qquad (16)$$

For later use it will be convenient here to introduce the (transversal) Hertz vector $\bar{Z}_1$ of the external field, defined by

$$\dot{\bar{Z}}_1 = \bar{A}_1 \qquad (17)$$

Comparison with (14) gives

$$\Box\, \bar{Z} = \dot{\bar{A}}_0$$

From this we find

$$\text{rot rot } \bar{Z}_1 = -\Delta \bar{Z}_1 = -\ddot{\bar{Z}}_1 - \dot{\bar{A}}_0 = -\frac{\partial}{\partial t}(\bar{A}_1 + \bar{A}_0)$$
$$= Tr\,\bar{E} = \bar{E}_1 + \bar{E}_0 \qquad (18)$$

This shows that $\bar{Z}_1$, and thereby $\bar{A}_1$ will necessarily show singularities at the electron's position since $\bar{E}_1$ and $\bar{E}_0$ become infinite like $1/r$ and $1/r^2$ respectively; at the same time we see that $\bar{Z}_1$ and $\bar{A}_1$ stay finite and may be sufficiently smooth that $\bar{Z}_{1\,r=0} = \dddot{\bar{Z}}_1$ and $\bar{A}_{1\,r=0} = \dddot{\bar{A}}_1$ are fulfilled. In the electric dipole radiation approximation discussed in the next paragraph $\bar{E}_0$ may be neglected.

III.

ELECTRIC DIPOLE RADIATION ONLY

Consider in particular the case where the influence of retardation and acceleration on ρ in (15) may be neglected, i. e. where as far as r is concerned we may reckon with a fixed position of the electron in space. In that case an analysis of *Hom.* in multipole radiations is indicated and we need only retain the electric dipole radiation. Then (16) assumes the form

$$\bar{A}_1 = e\,Tr\,(\tfrac{1}{2}\dot{\bar{R}}_{t'=t+r} + \tfrac{1}{2}\dot{\bar{R}}_{t'=t-r} - \dot{\bar{R}}_{t'=t} - \tfrac{1}{2}\dot{\bar{S}}_{t'=t+r} + \dot{\bar{S}}_{t'=t-r})/r \qquad (19)$$

where $\dot{\bar{S}}$ is the derivative of a vector the components of which are arbitrary functions of an argument t'. We see that (19) is a solution of (14) in which $\bar{A}_0$ is considered independent of $\bar{R}$. Developing in terms of powers of r we get

$$\bar{A}_1 = e\,Tr(\tfrac{1}{2}\ddot{\bar{R}}\,r + \ldots - \ddot{\bar{S}} - \tfrac{1}{6}\dddot{\bar{S}}\,r^2 - \ldots) \qquad (20)$$

In our particular case we may in the equations of motion (13) neglect the Lorentz force since the magnetic dipole radiation was neglected and we have to use (20) only in order to find the quantity $\dot{\bar{A}}$ which — on our theory — should reduce to the value at the origin $r = 0$. Thus only the term $\ddot{\bar{S}}$ counts in (20), and we find for what from a

simplistic point of view may be called the external electric field acting on the electron (*)

$$-\dot{\bar{A}}_1 = -\dot{\bar{A}}_{1r=o} = \frac{2}{3} e \ddot{\bar{S}}$$

Thus the equation of motion for the electron becomes

$$m \ddot{\bar{R}} = \frac{2}{3} e^2 \dddot{\bar{S}} - \partial U/\partial \bar{R} \qquad (21)$$

where $\dddot{\bar{S}}$ is an arbitrary function of its argument t.

As a first application consider the case where the E.H field does not contain a component of ingoing radiation [represented by the terms $\dot{\bar{R}}_{t+r}$ and $\dot{\bar{S}}_{t+r}$ in (19)]. Then we have clearly $\bar{S}=\bar{R}$ and the terms $\frac{2}{3} e^2 \dddot{\bar{S}}$ in (21) becomes the well known radiation reaction term $\frac{2}{3} e^2 \dddot{\bar{R}}$. More precisely, the condition of no ingoing radiation only requires that $\bar{R}$ equals $\bar{S}$ for all values of the argument above a certain minimum value; the $\dddot{\bar{R}}$ term will then be correct for all t-values larger than this minimum. We can also put the following problem : at $t = 0$ the quantities $\bar{R}$ and $\dot{\bar{R}}$ are prescribed and moreover no radiation field is present outside a sphere with radius $r = r_0$. Then we must clearly put

$\bar{S}(t') = \bar{R}(t')$ for $t' > r_0$ and $\bar{R}(t') = -\bar{S}(t')$ for $t' < -r_0$. In the interval $-r_0 < t' < 0$ we can still choose $\bar{R}(t')$ and in the interval $-r_0 < t' < r_0$ still choose $\bar{S}(t')$ as we like. From $t > r_0$ on the $\dddot{\bar{S}}$ will anyhow be reduced to the $\dddot{\bar{R}}$ value.

Next ask for solutions of our equations of motion which are purely harmonic in time. This will clearly require that $-\partial U/\partial \bar{R}$ in (21) is linear in $\bar{R}$, i. e. that we have a harmonic oscillator or even a free electron. Considering for simplicity an isotropic oscillator with natural frequency k_1, we put

$$\bar{R} = \bar{R}_0 \cos kt, \qquad \bar{S} = \bar{S}_0 \sin kt, \qquad -\partial U/\partial \bar{R} = -mk_1^2 \bar{R}$$

(*) We have made use of $Tr \ddot{\bar{S}} = \frac{2}{3} \ddot{\bar{S}}$. The following formulae allows us to find the transversal part of any vector field $\bar{f}(r)$ the components of which depend only on r. Introducing another vector field $\bar{g}(r)$ the second derivative of which with respect to r is equal to $\bar{rf}$ the formulae runs

$$Tr \bar{f} = Tr \bar{g}''(r)/r = \bar{g}/r^3 - \bar{g}'/r^2 + \bar{g}''/r + \bar{r}(\bar{r}, -3\bar{g}/r^5 + 3\bar{g}'/r^4 - \bar{g}''/r^3)$$
$$= (\bar{g}''/r - \bar{r}(\bar{r}\,\bar{g}'')/r^3) - (r^2 \frac{d}{dr}(\bar{g}/r) - 3\bar{r}(\bar{r}, \frac{d}{dr}(\bar{g}/r)/r^3$$

Then (19) reduces to

$$\bar{A}_1 = k\, e \sin kt\; Tr\{\bar{R}_0 (1-\cos kr) + \bar{S}_0 \sin kr\}/r \qquad (22)$$

and (21) gives us the ratio between $\bar{R}_0$ and $\bar{S}_0$, $-mk^2\bar{R}_0 = -\frac{2}{3} e^2k^3\bar{S}_0 - mk_1^2\bar{R}$.

Introducing a new vector $\bar{T}_0$ and a phase angle η by putting

$$\left.\begin{aligned} &\bar{S}_0 = \bar{T}_0 \cos\eta \qquad \bar{R}_0 = \bar{T}_0 \sin\eta \\ &\operatorname{tg}\eta = \frac{k^3/K}{k^2-k_1^2} \quad \text{harm. osc.} \qquad K^{-1} = \frac{2e^2}{3m} \\ &\operatorname{tg}\eta = k/K \qquad \text{free el.} \end{aligned}\right\} \qquad (23)$$

where K^{-1} is a length of the order of the conventional electronic radius, (22) takes the form

$$\bar{A}^1 = e\, k \sin kt\; Tr\, \bar{T}_0 \frac{\sin(kr-\eta) + \sin\eta}{r} \qquad (24)$$

The electromagnetic field thus behaves like a system of ingoing waves which are reflected at the electron and give rise to outgoing waves. In the wave zone the resulting standing waves are, due to the presence of the electron shifted by the angle η from the standing dipole waves in free space. Fig. 1 gives a sketch of the behaviour of η (k) for a free electron and for a harmonic oscillator.

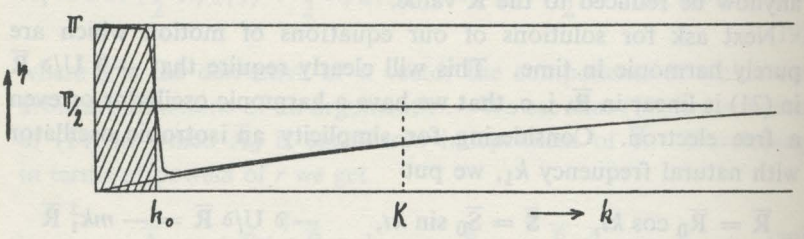

Fig. 1

According to the assumptions in our theory, (24) can only be trusted as long as k is small compared to K (and as long as the amplitude T_0 is sufficiently small). For any particular charge distribution standing oscillations described by (24) will be possible, but the η function will in general differ from (23) as soon as k is no longer

small compared to K. Only in the particular case where ρ is given by

$$\rho = e\, K^2\, e^{-\kappa r}/4\, \eta_1\, r$$

the η_1-function happens to remain unchanged.

We haves till to investigate whether $\bar{R}$ and $\bar{S}$ can change exponentially with time :

$$\bar{R} = \bar{R}_0\, e^{\pm \kappa t}, \qquad \bar{S} = \bar{S}_0\, e^{\pm \kappa t}, \qquad -\partial\, U/\partial\, \bar{R} = -mk_0^2\, \bar{R}$$

Then (19) reduces to

$$\bar{A}_1 = \kappa\, e\, e^{\pm \kappa t}\, Tr \left\{ \pm\, \bar{R}_0\, (Ch\, \kappa\, r - 1) - \bar{S}_0\, Sh\, \kappa\, r \right\}/r$$

Since for physical reasons the field should not increase exponentially with r we have clearly

$$\bar{R}_0 = \pm\, \bar{S}_0$$

On the other hand (21) gives us

$$m\, \kappa^2\, \bar{R}_0 = \pm\, \frac{2}{3}\, e^2\, \kappa^3\, \bar{S}_0 - mk_1^2\, \bar{R}_0$$

Therefore

$$K\, (\kappa^2 + k_1^2) = \kappa^3$$

This equation has one real root which for the free electron is $\kappa = K$ and for the harmonic oscillator still larger. This means that the field (25)

$$\bar{A}^1 = -\, \kappa\, e\, e^{\pm \kappa t}\, Tr\, \bar{S}_0\, (1 - e^{-\kappa r})/r \qquad (26)$$

corresponds to a radiation field confined within a region of linear extension 1/K and cannot, from our point of view, be reckoned to belong to the features of the structure-independent theory looked for. It is of course connected with those well known solutions of the equations of motion with radiation reaction

$$m\, \ddot{\bar{R}} = \frac{2}{3}\, e^2\, \dddot{\bar{R}} - \partial\, U/\partial\, \bar{R}$$

which give rise to an exponential behaviour of $\bar{R}$ with time and which are commonly rejected as non-physical. If in the model we let the electron radius go to zero, μ to $+\infty$ and m_0 to $-\infty$, $\mu + m_0$ remaining constant, we create an infinite source of electric energy and solution (26) becomes understandable.

In this connection the following point has some mathematical interest. For the free electron, tg $\eta_1 = k/K$, the functions

sin $(kr - \eta)$ form an orthogonal set but it is not complete, there being one function, viz. e^{-Kr} orthogonal to them all.

Having thus learnt that, in electric dipole approximation, the classical system : elastically bound electron + field, can be considered as a sum of structure-independent harmonic oscillators, at any rate if we restrict ourselves to states in which the frequencies characterizing the system are small compared to K, it is tempting to quantize the system by assigning to each of these three-dimensional oscillators an energy $(\eta_k + 3/2)\, h$. In the ground state we find for the energy of the system

$$\varepsilon = \sum_K \frac{3}{2} \hbar k \tag{27}$$

We will surround the electron by a reflecting spherical shell of large radius L. The condition that at the wall the tangential component of $\bar{E}$ vanishes leads to

$$kL - \eta(k) = \pi N \qquad (N = 1, 2, 3 \ldots) \tag{28}$$

The allowed k-values are characterized by the integer N and, since

$$\Delta N = \frac{1}{\pi}\left(L - \frac{d\eta}{dk}\right) \Delta k$$

the sum (27) reduces to the integral.

$$\varepsilon = \frac{3\hbar}{2\pi} \int_0 \left(L - \frac{d\eta}{dk}\right) dk$$

For free space, $\eta = 0$, and we get as contribution of the electric dipole radiation to the zero point radiation energy in space

$$\varepsilon = \frac{3\hbar L}{2\pi} k_{max}$$

which of course diverges for $k_{max} \to \infty$.

The influence of the electron on ε adds the contribution

$$\varepsilon_{el.} = \frac{3\hbar}{2\pi}\left(-\int_o^{k_m} k \frac{d\eta}{dk}\, dk + \eta_m k_m\right) = \frac{3\hbar}{2\pi}\int_o^{k_m} \eta\, dk \tag{29}$$

if in the two cases — electron absent and present — we sum over the same number of oscillators. This procedure is of course arbitrary, the difference between two divergent sums not being well defined, but it corresponds to the quantum mechanical formalism

by which contributions of this kind are customarily calculated (adiabatic increase of e from zero to its actual value).

For the « fluctuation » energy of a free electron we therefore get

$$\varepsilon_{\text{el. free}} = \frac{3\hbar}{2\pi}\int_0^{k_m} \operatorname{arctg} k/\mathrm{K}\, dk = \frac{3\hbar}{2\pi}\left\{k.\ \operatorname{arctg} k/\mathrm{K} - \frac{1}{2} lg(1 + k^2/\mathrm{K}^2)\right\}_{k=k_m} \tag{30}$$

For large k_m this expression diverges as $\frac{3\hbar}{4} k_m$ whereas for $k_m \ll \mathrm{K}$ it behaves like

$$\varepsilon_{\text{el. free}} = \frac{9}{8} m \frac{\hbar}{e^2} (k_m/\mathrm{K})^2 \tag{31}$$

This is the well known result (6) if, with customary perturbation methods, one asks for the contribution proportional to e^2. There is no reason to assign structure-independent significance to the exact expression (30) for k-values of the order K or larger. Due to the smallness of $e^2/\hbar$ we see from (31) that on quantum theory we would even have to take k/K small compared to $e/\sqrt{\hbar}$. Indeed, if for k we choose the inverse Compton wave lenght : $\lambda_C^{-1} = \hbar/m$, $\varepsilon_{\text{el.}}$ becomes $\frac{1}{2}\frac{e^2}{\hbar} m$ and from relativity considerations one clearly hesitates to go far beyond this k value. The same caution is indicated by a consideration of the amplitude of the motion of the electron in a classical picture where the oscillators are each given the energy $\frac{3}{2}\hbar k$. One finds that the mean square of the amplitude in a given direction is given by the formal integral

$$\overline{\text{ampl}^2} = \frac{2}{\pi}\frac{e^2}{\hbar}\lambda_C^2 \int \frac{dk}{k}$$

which, on closer investigation, shows that the electric dipole approximation will start to fail when the wave lengths of the radiation field become smaller than the Compton wave length λ_C

A consideration of the harmonic oscillator yields some points of interest. First of all, the application of (29) leads formally to the energy increase

$$\varepsilon_{\text{harm. osc.}} = \frac{3\hbar}{2\pi}\int_0^{k_m} \operatorname{arctg} k^3/\mathrm{K}(k^2 - k_1^2) dk$$

which for $k_m \gg K$ diverges in the same way as (30). The difference between (31) and (29) however converges. Indeed one finds

$$\Delta\varepsilon = \frac{3\hbar}{2\pi}\int_0^{k_m}(\eta_{\text{harm. osc.}} - \eta_{\text{free}})\,dk = \frac{3}{2}\hbar k_1\left\{1 + \frac{1}{\pi}\frac{k_1}{K}(lg\frac{K}{k_1} + C) + \ldots\right\}$$

where the terms neglected contain only the second and higher powers of k_1, and where the correction term depends on k_m and vanishes for $k_m \to \infty$:

$$C = -\frac{1}{2}\,lg(1 + K^2/k_m^2)$$

The main term $\frac{3}{2}\hbar k$, is just what should be expected, namely the energy in the ground state of a harmonic oscillator of natural frequence k_1. It corresponds to the area of the shaded rectangle in Fig. 1, multiplied by $3\hbar/2\pi$. The second term has almost the form which Bethe's original formula (7) gives for the Lamb shift of the levels of a harmonic oscillator (it is the same for all levels), the difference between that in Bethe's formula the K in the numerator of the argument of the logarithm is replaced by the inverse Compton wave length, which we will call k_C. The terms left out are even small compared to the natural line breadth k_1^2/K of the harmonic oscillator. The correction term C vanishes for $k_m \to \infty$ but, if we dare not go beyond $k_m \cong k_C$ in the integration, it becomes pratically $lg\,(k_C/K)$ and we get precisely Bethe's result. In this connection we note that Bethe's formula for the Lamb shift in an arbitrary atom can be written as the sum

$$\Delta\,\varepsilon_{\text{Lamb}} = \sum_j f_j \frac{3}{2\pi}\frac{\hbar k_j^2}{K}\,lg(k_C/k_j)$$

extended over all absorption and emission frequencies k_j and where f_j if the corresponding oscillator strength.

Another point which, this time, lies quite within the domain where our formulae are significant refers to the scattering of light by an elastically bound electron. Consider again our assembly of harmonic oscillators of frequencies k which we may imagine to take the discrete but very finely distributed values determined by (28). Let all of them be in the ground state (energy $\frac{3}{2}\hbar k$) with the exception of one (frequency k') which has the energy $\frac{5}{2}\hbar k'$, the vibration

parallel to — say — the x-axis being excited by one quantum $\hbar k'$. In this situation light of frequency k' coming from all directions is continuously scattered by the electron. By the well known devices of collision theory we are then also able to describe the scattering of a plane wave (with its $\bar{E}$-vector in the x-direction) by the electron. The phase shift η appears from the outset in our formulæ; it is the same as that in classical theory but it is now naturally incorporated in a pure quantum theoretical treatment which — in contrast to the customary Dirac treatment however — corresponds exactly to the classical method where scattering is described as a steady state and no longer as a generation of light in other directions than the direction of incidence. In the theory of particle collisions these steady states never gave difficulties; in the quantum theory of light scattering it seemed difficult to handle them in a rigorous way.

We will explain this steady-state quantum description in some detail. Describe an arbitrary situation of the classical system as a superposition of harmonic oscillations of frequencies by means of the Hertz vector and the vector potential of the external field [compare (24) and (17)]

$$\left.\begin{aligned} \bar{Z}_1 &= -e \sum_k Tr\, \bar{T}_{ok} \frac{\sin(kr-\eta)+\sin\eta}{r} \cos kt \\ \bar{A}_1 &= e\, k \sum_k Tr\, \bar{T}_{ok} \frac{\sin(kr-\eta)+\sin\eta}{r} \sin kt \end{aligned}\right\} \tag{33}$$

From this it will be possible to introduce pairs of canonical variables $\bar{q}_k$ and $\bar{p}_k$ belonging to the x, y and z direction of every oscillator. Normalizing them in such a way that the Hamiltonian becomes

$$\left.\begin{aligned} H &= \frac{1}{2} \Sigma\, k(\bar{p}_k^2 + \bar{q}_k^2) \\ \dot{\bar{q}}_k &= k\, \bar{p}_k\,, \qquad \dot{\bar{p}} = -k\, \bar{q}_k \end{aligned}\right\} \tag{34}$$

we infer from (33) that we may put

$$\bar{T}_{ok} \cos kt = -f(k)\, \bar{q}_k\,, \qquad \bar{T}_{ok} \sin kt = f(k)\, \bar{p}_k$$

The factor f can be found by considering the total energy of the system

$$\begin{aligned} \varepsilon &= \frac{1}{2} m_0 \dot{\bar{R}}^2 + \frac{1}{2} m k_1^2 \bar{R}^2 + \frac{1}{8\pi} \int (H^2 + E^2) dV \\ &= \frac{1}{2} m_0 \dot{\bar{R}}^2 + \frac{1}{2} m k_1^2 \bar{R}^2 + e \dot{\bar{R}} . \tilde{\tilde{\bar{A}}}_1 . + \frac{1}{8\pi} \int (H_1^2 + E_1^2) dV \end{aligned}$$

In deriving this structure-independent expression we have made use of (11), (9), (4) and — as is permitted in electric dipole approximation—neglected $\overline{E}_0 = \overline{E} - \overline{E}'$. From [compare (23)]

$$\left.\begin{aligned} \dot{\overline{R}} &= -\sum_k k\,\overline{T}_{ok} \sin\eta \sin kt = -\sum_k k f(k) \sin\eta\, \bar{p}_k \\ \overline{R} &= -\sum_k f(k) \sin\eta\, \bar{q}_k \end{aligned}\right\} \tag{35}$$

and expressing also $\overline{H}_1$ and $\overline{E}_1$ in the $\bar{p}$'s and $\bar{q}$'s respectively by means of (33), (11), (32), a calculation shows that (34) is justified and that moreover

$$f = \frac{1}{e}\sqrt{\frac{3}{k^3 L}}$$

The standard formulæ for the field become therefore

$$\left.\begin{aligned} \overline{Z}_1 &= Tr \sum_k \sqrt{\frac{3}{k^3 L}}\, \bar{q}_k \frac{\sin(kr-\eta) + \sin\eta}{r}, \quad \overline{E}_1 = \text{rot rot } \overline{Z}_1 \\ \overline{A}_1 &= Tr \sum_k \sqrt{\frac{3}{k^3 L}}\, \bar{p}_k \frac{\sin(kr-\eta) + \sin\eta}{r}, \quad \overline{H}_1 = \text{rot } \overline{A}_1 \end{aligned}\right\} \tag{36}$$

If now we promote p_k and $\bar{q}_k$ to q-numbers satisfying the well known commutation properties, the formalism has become purely quantum mechanical. The $\bar{q}$'s and $\bar{p}$'s describe the light quanta in the external field, i. e. the free light quanta, which are affected by the presence of the electron through the appearance of the phase shift η. We might also call them *phase shifted light quanta*; the free electric dipole light quanta in empty space correspond to $\eta = 0$. In the next paragraph we will give a more general quantum mechanical treatment which does not primarily rest on an analysis of the system in harmonic components.

Returning now to the state considered above where one $\bar{p}, \bar{q}$ oscillator (frequency k') is in the first excited state, all the others in the ground state, we will consider what happens if k' is equal to the natural frequency k_1 or differs from it by an amount small compared to the natural line with k_1^2/K. The phase is then nearly $\pi/2$ and from (23) or (35) we see that the amplitude of the electronic motion (classically or quantum mechanically considered) is much larger than when k' lies outside the line breadth. We have now, indeed, what in customary language would be called an electronic oscillator in its first excited state. It does not jump to the lowest because it is sustained so to say by an incoming free space radiation. By an appropriate quantum mechanical superposition of states of this

kind, all with η very near to $\pi/2$ we could, however, construct a situation in which at $t = 0$ the incoming radiation is zero beyond some distance from the electron, analogous to the choice of $\bar{S}$ discussed on p. 9 and where in the course of time the amplitude of the electronic motion falls exponentially off to zero. This description would be nearly related to the Weisskopf-Wigner treatment of a spontaneous Bohr transition.

If we imagine k' to take successively increasing values and to pass through the region where η changes from π to 0, the customary language which is so well adapted to most physical processes fails in the regions where η is neither near to π or 0, nor to $\pi/2$; there we can neither speak of light which is scattered from the ground state nor of an oscillator in its first excited state. The situation is of course exactly analogous to that in particle collisions but the present rigorous formalism in the case of light quanta which does not rest on development in powers of e^2 fills a gap in the customary methods.

IV.

THE STRUCTURE-INDEPENDENT HAMILTONIAN

It will clearly be of interest to bring the structure-independent classical equations of motion derived in § 2 in Hamiltonian form. We begin with the variational principle which leads to the equations of motion (1) and (2) from which we started in § 2 :

$$\delta \int L \, dt = 0 \tag{37}$$

$$L(\bar{R},\dot{\bar{R}},\bar{A},\dot{\bar{A}}) = \frac{1}{2} m_0 \dot{\bar{R}}^2 + e(\dot{\bar{R}}\,\bar{A}) - U(\bar{R}) - \frac{1}{8\pi} \int (\bar{H}^2 - (Tr\,\bar{E})^2) dV \tag{38}$$

where $\bar{R}$ and $\bar{A}$ are arbitrarily varied subject to δ div $A = 0$. The introduction (3) of a new field $\bar{A}_1$ is not immediately permitted since $\bar{A}_0$ depends both on $\bar{R}$ and $\dot{\bar{R}}$. This difficulty can be overcome if we replace L by a new Lagrangian

$$M = L - \bar{P}.\bar{V} - \dot{\bar{P}}.\bar{R} \tag{39}$$

$$\bar{P} = \partial L/\partial \dot{\bar{R}} \tag{40}$$

where $\bar{V}$ is numerically equal to $\dot{\bar{R}}$ but is expressed, just as M itself, in terms of $\bar{R}$, $\bar{P}$, $\dot{\bar{P}}$, $\bar{A}$, $\dot{\bar{A}}$ by means of (40). It is easily proved that

$$\delta \int M \, dt = 0$$

leads to the same equations of motion as (37). Of course there now appears a redundant variable, the generalized momentum $\partial L/\partial \dot{\bar{R}}$ being identically zero. If $\bar{A}_0$ is now considered as a function of $\bar{R}$, $\bar{P}$, $\bar{A}$ instead of $\bar{R}$, $\dot{\bar{R}}$ the substitution $\bar{A} = \bar{A}_1 + \bar{A}_0$ in (39) is permitted.

From (40) and (38) we have

$$\bar{P} = m_0 \dot{\bar{R}} + e \ddot{\bar{A}}$$

and therefore

$$m_0 \bar{V} = \bar{P} - e \ddot{\bar{A}} \tag{41}$$

The new Lagrangian takes the form

$$M = -\frac{1}{2} m_0 \bar{V}^2 - U - \dot{\bar{P}}.\bar{R} - \frac{1}{8\pi} \int (\bar{H}^2 - (Tr\, \bar{E})^2) dV \tag{42}$$

$\bar{V}$ being explained by (41).

Introducing (6), (7), (12) in (42) we find

$$m \bar{V} = \bar{P} - e \ddot{\bar{A}}_1 \tag{43}$$

Moreover we have from (11), (10)

$$\bar{H} = \bar{H}_1 + \bar{H}_0 = \text{rot}\, \bar{A}_1 + \text{rot}\, \bar{A}_0, \qquad Tr\, \bar{E} = \bar{E}_1 + \bar{E}_0 = -\dot{\bar{A}}_1 + \bar{F} + \bar{E}_0$$

For magnetic and electric field energy we thus get

$$\frac{1}{8\pi} \int H^2 dV = \frac{1}{8\pi} \int \bar{H}_1^2 dV + \frac{1}{4\pi} \int \bar{H}_1 . \bar{H}_0 dV + \frac{1}{8\pi} \int \bar{H}_0^2 dV$$

$$= \frac{1}{8\pi} \int (\text{rot}\, \bar{A}_1)^2 dV + e \ddot{\bar{A}}_1 . \bar{V} + \frac{1}{2} \mu V^2$$

$$\frac{1}{8\pi} \int (Tr\, \bar{E})^2 dV = \frac{1}{8\pi} \int \bar{E}_1^2 \, dV + \frac{1}{4\pi} \int \bar{E}_1 . \bar{E}_0 \, dV + \frac{1}{8\pi} \int \bar{E}_0^2 \, dV$$

As regards $\bar{F}$ and $\bar{E}_0$ we must be cautious. They are obtained by differentiating $\bar{A}_0$ with respect to time but $\bar{A}_0$ will now (outside the electron) be expressed by

$$\bar{A}_0 = Tr\, e\, \bar{V}/r$$

$\bar{V}$ being explained by (43) as a function of $\bar{P}$, A_1 and R. Thus we have

$$\bar{F} = Tr\,\dot{\bar{V}}/r, \qquad \dot{\bar{V}} = \frac{1}{m}\left(\dot{\bar{P}} - e\,\dot{\bar{\bar{A}}}_1 - e\left(\dot{\bar{R}}\,\partial/\partial\,\bar{R}\right)\bar{\bar{A}}_1\right) \tag{44}$$

and $\bar{F}$ depends on $\dot{\bar{P}}, \bar{A}_1, \dot{\bar{A}}_1, \bar{R}, \dot{\bar{R}}$, being linear in all but $\bar{R}$. Similarly we have outside the electron

$$\bar{E}_0 = e\,Tr\,\bar{V}\left(\dot{\bar{R}}\,\frac{\partial}{\partial\,\bar{\bar{R}}}\right)\frac{1}{r}$$

showing that $\bar{E}_0$ depends on $\bar{P}$, $\bar{A}_1$, $\bar{R}$, $\dot{\bar{R}}$, linearly in all but $\bar{R}$.

The integral $\int \bar{E}_1.\bar{E}_0\,dV$ will be structure-independent in spite of the fact that $\bar{E}_1$ through $\bar{F}$ behaves like $1/r$ and $\bar{E}_0$ like $1/r^2$. On the other hand $\int E_0^2\,dV$ diverges if (44) should hold down to $r = 0$ and becomes finite but structure-dependent and proportional to the fourth power of the velocity if the charge-distribution is taken into account. This term must clearly be neglected; it refers to effects which anyhow only a true relativistic theory might describe rightly.

We are thus left with the structure-independent Lagrangian

$$\begin{aligned} M &= -\frac{1}{2}mV^2 - e\,\bar{\bar{A}}_1.\bar{V} - U - \dot{\bar{P}}.\bar{R} - \frac{1}{8\pi}\int (H_1^2 - E_1^2 - 2\bar{E}_1.\bar{E}_0)\,dV \\ &= -P^2/2m + e^2\,\bar{\bar{A}}_1^2/2m - U - \dot{\bar{P}}.\bar{R} - \frac{1}{8\pi}\int (H_1^2 - E_1^2 - 2\bar{E}_1.\bar{E}_0)\,dV \end{aligned} \tag{45}$$

M depends on $\bar{P}$, $\dot{\bar{P}}$, $\bar{A}_1$, $\dot{\bar{A}}_1$, $\bar{R}$, $\dot{\bar{R}}$, but because of the redundant variables we cannot use it immediately for bringing the equations of motion in canonical form since there exists an identical relation between the moments.

From now on we will restrict ourselves to the electric dipole approximation. This means that we can take for $\bar{A}_1$ and $\bar{F}$ (in $\bar{E}_1$) their values for $r = 0$, i. e. neglect their dependency on $\bar{R}_1$, and leave out $\bar{E}_0$ entirely. This means that M no longer depends on $\dot{\bar{R}}$:

$$\frac{\partial\,M}{\partial\,\dot{\bar{R}}} = 0$$

and the canonical form is easily established. Indeed with Lagrangian $M(x_o; x_1, \dot{x}_1; x_2, \dot{x}_2 \ldots x_n, \dot{x}_n)$ the equations of motion are :

$$\frac{\partial\,M}{\partial\,x_o} = 0, \qquad \dot{p}_k = \frac{\partial\,M}{\partial\,x_k}\,(k = 1, 2 \ldots n),\ p_k = \frac{\partial\,L}{\partial\,\dot{x}_k}$$

Therefore

$$\delta\left(\sum_1^n p_k \dot{x}_k - \mathrm{M}\right) = \sum_1^n \dot{x}_k \delta p_k - \sum_1^n \frac{\partial \mathrm{M}}{\partial x_k} \delta x_k = \sum_1^n (\dot{x}_k \delta p_k - \dot{p}_k \delta x_k)$$

If x_o, $\dot{x}_1 \ldots \dot{x}_n$ can be expressed in terms of $x_1 \ldots x_n$, $p_1 \ldots p_n$ by means of $\partial \mathrm{L}/\partial x_o = 0$, $\partial \mathrm{L}/\partial \dot{x}_k = p_k$, the p_k's and x_k's ($k = 1,2 \ldots n$) should satisfy the canonical equations with the Hamiltonian

$$H = \sum_1^n p_k \dot{x}_k - \mathrm{L} = H(p_1 \ldots p_n, x_1 \ldots x_n)$$

Applying this to our case we have first of all to calculate the vector conjugate to $\bar{\mathrm{P}}$ and the vector field conjugate to the vector field $\bar{\mathrm{A}}_1$. From (44) and (45) we find :

$$\gamma_{\bar{\mathrm{P}}} = \frac{\partial \mathrm{M}}{\partial \dot{\bar{\mathrm{P}}}} = -\bar{\mathrm{R}} + \frac{1}{4\pi}\int \bar{\mathrm{E}}_1 \frac{\partial \bar{\mathrm{F}}}{\partial \dot{\bar{\mathrm{P}}}} dV = -\bar{\mathrm{R}} - \frac{e}{4\pi m}\int Tr \frac{\bar{\mathrm{E}}_1}{r} dV$$

Since rot rot $\bar{\mathrm{Z}}_1 = \bar{\mathrm{E}}_1$ and div $\bar{\mathrm{Z}}_1 = 0$ ($\bar{\mathrm{Z}}_1$ Hertz potential) the value of $\bar{\mathrm{Z}}_1$ in the origin is just given by the integral appearing in the fourth member, divided by 4π. Denoting the momentum conjugate to $\bar{\mathrm{P}}$ by $-\bar{\mathrm{R}}_1$ we have therefore

$$-\gamma_{\bar{\mathrm{P}}} \equiv \bar{\mathrm{R}}_1 = \bar{\mathrm{R}} + \frac{e}{m}\bar{\mathrm{Z}}_1 \qquad (46)$$

In order to find the momentum conjugate to $\bar{\mathrm{A}}_1$ in a certain point of space, we have to vary M with respect to $\dot{\bar{\mathrm{A}}}_1$. There is a contribution both from $\dot{\bar{\mathrm{A}}}_1$ and from $\bar{\mathrm{F}}$ in $\bar{\mathrm{E}}_1$:

$$\delta \mathrm{M} = \frac{1}{4\pi}\int \bar{\mathrm{E}}_1 \delta \bar{\mathrm{E}}_1 dV = \frac{1}{4\pi}\int \bar{\mathrm{E}}_{1\mathrm{P}} \Big\} - \delta \dot{\bar{\mathrm{A}}}_1$$

$$+ \frac{e}{m} Tr \int (\rho_Q \delta \dot{\bar{\mathrm{A}}}_{1Q}/r_{PQ}) dV_Q \Big\} dV_P = \int \gamma_{\dot{\bar{\mathrm{A}}}_1} dV$$

Representing the field $\gamma_{\dot{\bar{\mathrm{A}}}}$ by $-\bar{\mathrm{E}}_2/4\pi$ we have thus

$$-\frac{\bar{\mathrm{E}}_2}{4\pi}(\equiv \gamma_{\dot{\bar{\mathrm{A}}}_1}) = -\frac{\bar{\mathrm{E}}_1}{4\pi} + \frac{e\rho}{4\pi m}\int Tr(\bar{\mathrm{E}}_1/r) dV = -\frac{\bar{\mathrm{E}}_2}{4\pi} + \Delta \frac{e^2}{m}\bar{\bar{\mathrm{Z}}}_1 \qquad (47)$$

where we have denoted the scalar field ρ by Δ/e. In structure-independent applications we may take for Δ the Dirac function $\delta(x\text{—X})\,\delta(y\text{—Y})\,\delta(z\text{—Z})$ where X, Y, Z is the position of the electron and x, y, z the position where $\bar{\mathrm{E}}_2$ is required. The field $\bar{\mathrm{E}}_2$ is

determined by the field $\bar{E}_1$ and reversely; $\bar{E}_2$ exibits both a $1/r$ and a Δ singularity at the origin.

The Hamiltonian is easily found from (45) by remarking that terms in M quadratic in $\dot{\bar{P}}$ and $\dot{\bar{A}}_1$ will stay the same, that the term linear in $\dot{\bar{P}}$ will vanish and that the terms independent of $\dot{\bar{P}}$, $\dot{\bar{A}}_1$ will change sign :

$$H = \frac{\mathrm{P}' - e^2 \tilde{\tilde{\bar{A}}}_1^2}{2m} + \mathrm{U}(\bar{\mathrm{R}}) + \frac{1}{8\pi}\int (\mathrm{H}_1^2 + \mathrm{E}_1^2) dV \qquad (48)$$

$$\text{canonical } p-q \text{ pairs} \rightarrow \bar{\mathrm{P}} - \bar{\mathrm{R}}_1 \left(= \bar{\mathrm{R}} + \frac{e}{m}\tilde{\tilde{\bar{Z}}}_1\right)$$

$$\text{and } \bar{\mathrm{A}}_1 - \mathrm{E}_2/4\pi \left(= \bar{\mathrm{E}}_1/4\pi - \Delta \frac{e^2}{m}\tilde{\tilde{\bar{Z}}}_1\right)$$

The only canonical coordinates figuring explicitely in this expression are $\bar{P}$ and $\bar{A}_1$ ($= \mathrm{rot}^{-1}\bar{H}_1$). E_1 is connected with the canonical field $\bar{E}_2$ through (47) and $\bar{R}$ with the canonical variable $\bar{R}_1$ through (46). We still remark that $\bar{P}, -\bar{R}_1$ is here treated as a $q-p$ pair. Analysing the field in multipole components, only the electric dipole component appears in $\tilde{\tilde{\bar{A}}}_1$ and $\tilde{\tilde{\bar{Z}}}_1$. For the other components we have so to say the ordinary A — E conjugation known from radiation-free space. A direct check shows that the canonical equations resulting from (48) are identical with (14) and (13) (with Lorentz force neglected).

We will now explicitely introduce the canonical variables $\bar{R}_1$ and $\bar{E}_2$ into (48). We will thereby formally develop in powers of e and reject all powers higher than the second :

$$H = \frac{\mathrm{P}^2 - e^2 \tilde{\tilde{\bar{A}}}_1^2}{2m} + \mathrm{U}(\bar{\mathrm{R}}_1 - \frac{e}{m}\tilde{\tilde{\bar{Z}}}_2) + \frac{1}{8\pi}\int (\mathrm{H}_1^2 + \mathrm{E}_2^2$$

$$+ \frac{8\pi e^2}{m}\Delta\, \bar{\mathrm{E}}_2 \cdot \tilde{\tilde{\bar{Z}}}_2) dV$$

Omitting, for simplicity, all indices this gives

$$H = \frac{\mathrm{P}^2}{2m} - \frac{e}{m}(\tilde{\tilde{\bar{Z}}}\frac{\partial}{\partial \bar{\mathrm{R}}})\mathrm{U} + \frac{1}{2}\frac{e^2}{m}(\tilde{\tilde{\bar{Z}}}\frac{\partial}{\partial \bar{\mathrm{R}}})^2\mathrm{U} + \frac{e^2}{2m}(-\tilde{\tilde{\bar{A}}}^2 + 2\tilde{\tilde{\bar{E}}}.\tilde{\tilde{\bar{Z}}})$$

$$+ \frac{1}{8\pi}\int (\mathrm{rot}\, \bar{\mathrm{A}}^2 + \bar{\mathrm{E}}^2) dV \qquad (49)$$

$$\text{Canonical } p - q \text{ pairs} \rightarrow \bar{\mathrm{P}} - \bar{\mathrm{R}} \text{ and } \frac{\bar{\mathrm{E}}}{4\pi} - \bar{\mathrm{A}}$$

Let us compare this with the customary non-relativistic Hamiltonian in literature :

$$H = \frac{\mathrm{P}^2}{2m} - \frac{e}{m}\overset{\approx}{\mathrm{A}}.\bar{\mathrm{P}} \qquad + \frac{e^2}{2m}\overset{\approx}{\mathrm{A}}^2 \qquad + \frac{1}{8\pi}\int(\mathrm{rot}\,\bar{\mathrm{A}}^2 + \bar{\mathrm{E}}^2)dV \qquad (50)$$

The term in (49) proportional to e was found by Opechowski in his 1941 paper mentioned on page 1. If we consider it as a perturbation term, proceed in the usual Dirac way and ask for perturbation effects proportional to the first power of e, such as are sufficient for the calculation of emission and absorption of spectral lines (Einstein's A's and B's), it is easily seen that it gives exactly the same result as the corresponding term in (50). In fact, their matrix elements for energy-change zero are the same. This follows for instance from the fact that $\frac{1}{m}\frac{\partial U}{\partial R}$ in the unperturbed motion is equal to $-\ddot{\bar{\mathrm{R}}}$, $\frac{\bar{\mathrm{P}}}{m}$ to $\dot{\bar{\mathrm{R}}}$ and $\dot{\bar{\mathrm{A}}}$ to $\dot{\bar{\mathrm{Z}}}$. Therefore the e-proportional term in (49) equals $e\,\overset{\approx}{\dot{\bar{\mathrm{Z}}}}.\ddot{\bar{\mathrm{R}}}$ and that in (50) $-e\,\overset{\approx}{\dot{\bar{\mathrm{Z}}}}.\bar{\mathrm{R}}$. The difference is a pure time derivative $\frac{d}{dt}(\overset{\approx}{\bar{\mathrm{Z}}}.\bar{\mathrm{R}}.)$ which has no matrix elements for energy change zero. As a third possibility for an e-proportional term with the same matrix elements we mention $e\,\overset{\approx}{\ddot{\bar{\mathrm{Z}}}}.\bar{\mathrm{R}} = -e\,\bar{\mathrm{E}}.\bar{\mathrm{R}}$ ($\bar{\mathrm{E}} = -\dot{\bar{\mathrm{A}}}$ is the electric field in the unperturbed system) which was used by Kramers and Heisenberg in their 1925 work.

For the scattering of light the Dirac perturbation calculus has to be extended to the second power of e^2. The scattering of the free electron is on (50) described by $\frac{e^2}{2m}\overset{\approx}{\mathrm{A}}^2$ only, on (49) by $\frac{e^2}{2m}(-\overset{\approx}{\mathrm{A}}^2 + 2\overset{\approx}{\bar{\mathrm{E}}}.\overset{\approx}{\bar{\mathrm{Z}}})$ and again the no-energy-change matrix elements are required. They are clearly equal in both cases since the difference $\overset{\approx}{\mathrm{A}}^2 - \overset{\approx}{\bar{\mathrm{E}}}.\overset{\approx}{\bar{\mathrm{Z}}} = \overset{\approx}{\dot{\bar{\mathrm{Z}}}}^2 + \overset{\approx}{\ddot{\bar{\mathrm{Z}}}}.\overset{\approx}{\bar{\mathrm{Z}}}$ is a pure time differential in the unperturbed state.

For the scattering from a bound electron a second order calculation must be carried through as regards the term proportional to e. The results are of course not the same on (49) and on (50) but the difference is, as can easily be shown, just compensated by the first order effect of the e^2 term in (49) which has no analogon in (50). Thus the Kramers-Heisenberg formulae result in both cases.

The result of the second order calculation which would formally

lead to the (Lamb) shift of the energy levels is, however, different with the two Hamiltonians. Here we find that the use of ours gives a correction to the use of the customary one which is precisely the correction proposed by Bethe in his (first) derivation of the Lamb shift formula in order to eliminate the divergence introduced by the point model of the electron. This can easily be seen as follows. With (50) the shift of the level m wou'd be (we suppress certain trivial details) :

$$\Delta\,\varepsilon = -\frac{e^2}{m^2}\sum_b\sum_m \frac{(\bar{A}_{ab}\cdot\bar{P}_{mn})\,(\bar{A}_{ba}\cdot\bar{P}_{nm})}{(E_b+E_n)-(E_a+E_m)} \approx$$

$$\approx -\text{const.}\sum_b\sum_m \frac{P_{mn}\,P_{nm}/k}{k+k_{nm}} = -\text{const.}\sum_n P_{mn}\,P_{nm}\int\frac{k\,dk}{k+k_{nm}}$$

Here a denotes the state of no light quantum in empty space, b the state of one light quantum (momentum $\bar{k}$). The summation is over all states b and all other states n of the atom, hk_{nm} being the energy difference between states n and m. Bethe corrects by subtracting the same expression but with k_{nm} everywhere zero and gets

$$\Delta\,\varepsilon = \ldots\int\left(-\frac{k}{k+k_{nm}}+1\right)dk = \ldots\int\frac{k_{nm}}{k+k_{nm}}\,dk$$

It was difficult to make his argument quite rigorous but it had certainly physical plausability.

With our Hamiltonian we get from the e-term

$$\Delta_1\,\varepsilon = -\frac{e^2}{m^2}\sum_b\sum_m \frac{(\bar{Z}_{ab}\cdot\dot{\bar{P}}_{mn})\,(\bar{Z}_{ba}\cdot\bar{P}_{nm})}{(\quad)-(\quad)} \approx$$

$$\approx -\text{const.}\sum_b\sum_n \frac{P_{mn}P_{nm}k_{nm}^2/k^3}{k+k_{nm}} = -\text{const.}\sum_n P_{mn}\,P_{nm}$$

$$\int\frac{k_{nm}^2}{k(k+k_{nm})}\,dk$$

From the e^2-term we must aid

$$\Delta_2\,\varepsilon = \left(\frac{1}{2}\frac{e^2}{m}\sum_{xy} Z_x Z_y U_{xy}\right)_{am,am} = \left(\frac{1}{2}e^2\sum_{xy} Z_x Z_y \frac{P_x\dot{P}_y - \dot{P}_y P_x}{i\hbar}\right)_{am,am}$$

$$\approx \text{const.}\sum_b\sum_n P_{mn}P_{nm}k_{nm}/k^3 = \text{const}\sum_n P_{mn}P_{nm}\int\frac{k_{nm}}{k}\,dk$$

Together

$$\Delta\varepsilon = \text{const} \sum_n P_{mn} P_{nm} \int \left(-\frac{k_{nm}^2}{k(k + k_{nm})} + \frac{k_{nm}}{k} \right) dk = \ldots \int \frac{k_{nm}}{k + k_{nm}} dk$$

On the question of the upper limit of the logarithmically divergent integral this treatment throws of course as little light as Bethe's (first) treatment refered to, the theory being restricted to electric dipole radiation.

V.

FINAL REMARKS

The theory given in the preceding paragraphs has not yet been developed much further. The extension to a system containing more than one electron (always in electric dipole approximation) is straightforward and consists chiefly in the addition of the Coulomb terms and Darwin terms describing the electric and magnetic interactions respectively of the electrons due to the mutual influence of their proper fields. One can say, therefore, that a promising basis is given for a really rigorous scattering theory of atoms in the electric dipole radiation domain. This theory should tell us the exact behaviour of the atoms towards incident light, also within the natural breadths of the absorption and emission lines. We have seen how the notion of the phase shifted light quanta afforded the natural means for the solution of this problem in case of the harmonic oscillator. The problem of the states of steady scattering in arbitrary atoms is a good deal more complicated, of course, not at least due to the Raman effect, but an appropriate choice of phase-shifted light quanta will presumably be helpful also here. There should of course no longer be question of development in powers of e and thus an approximation of the kind involved in (49) will be no help.

A second question is that of the extension of the theory to include also magnetic dipole radiation. Thereby the Compton recoil would be taken care of, be it only in non-relativistic approximation.

In many problems the development of radiation in multipole components will certainly not be appropriate. An example is afforded by the question of a rigorous quantum mechanical dispersion theory of extended matter Here the mathematical handling of the free light quanta will certainly be complicated, even if we restrict ourselves to the model of an assembly of harmonic oscillators.

The present theory shows how the spurious divergencies due to the point model of the electron may be removed to a certain extent. With respect to those due to the zero point fluctuation in the radiation field the situation lies different; with Weisskopf one might say that this problem lies outside the domain of correspondence. The treatment of this question, as well as the establishment of a relativistic theory of charged particles has hitherto been, and will perhaps remain so for some time, a story of artful and ingenious guessing.

The results obtained in this way, from Dirac's theory of 1928 to the beautiful results of the very last year are certainly impressive. Still the fundaments on which they rest are naturally for a large deal derived from « correspondence », so first of all the device of introducing the interaction of radiation with a particle by adding to the momentum p the term $-eA$. Now, we have seen that, in non-relativistic approximation, the elimination of the structure by introducing a « proper » field of bound light quanta does not simply consist in changing this A in A_1, where A_1 represents the free radiation, but involves a much profounder change in the form of the Hamiltonian. Thus one might say that the famous $e \alpha A$ interaction in Dirac's theory does not even ensure that certain simple, unrelativistic effects will be well rendered by the theory, although it is known to work for the very simplest effects like low energy scattering.

It may of course be that point-model and fluctuation-divergencies can and must ultimately be traced to the same source from the point of view of a future complete theory and that one therefore should not think to hard of the device : first quantizing a wrong Hamiltonian and trying to make amends later on. Still something might be learned from a theory of the sort to which the present paper aims and in which it is tried to analyse the structure-independent features of classical theory first and to see whether and to what extent quantization can learn us something new.

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