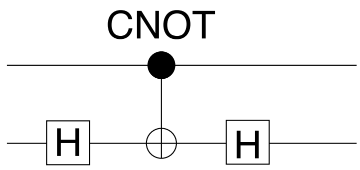
ANSWERS TO THE EXAM QUANTUM INFORMATION, 17 DECEMBER 2021

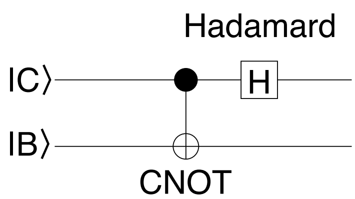
each item gives 2 points for a fully correct answer, grade = total $\times 9/24 + 1$

1. *a)* The coefficient matrix of the state $|\Psi\rangle$ is $c = \begin{pmatrix} 2^{-1/2} & 1/2 \\ 0 & 1/2 \end{pmatrix}$, with determinant $2^{-3/2}$, so the concurrence is $2^{-1/2} \neq 0$, the state is entangled.
   b) $|\psi_0\rangle_B = (3/4)^{-1/2}(2^{-1/2}|0\rangle_B + 2^{-1}|1\rangle_B)$, $|\psi_1\rangle_B = |1\rangle_B$ (including the normalization constant)
   c) the state $|\psi_0\rangle_B$ is found with probability $p_0 = 3/4$, the state $|\psi_1\rangle_B$ with probability $p_1 = 1/4$, the reduced density matrix is $\rho_B = p_0|\psi_0\rangle\langle\psi_0| + p_1|\psi_1\rangle\langle\psi_1|$. This is the same as before the measurement, $\tilde{\rho}_B = \rho_B$.

2. *a)* $\mathrm{H} = 2^{-1/2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\mathrm{H} = \mathrm{H}^\dagger$, $\mathrm{H} = \mathrm{H}^{-1}$, so this operation is both Hermitian and unitary.
   b) $\mathrm{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\mathrm{HXH} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \mathrm{Z}$, so $\alpha|0\rangle + \beta|1\rangle \mapsto \alpha|0\rangle - \beta|1\rangle$.
   c) the CNOT operation is a controlled-X operation, so to obtain the controlled-Z we can use the Hadamard operation on the target qubit to transform the X-operation in a Z-operation:

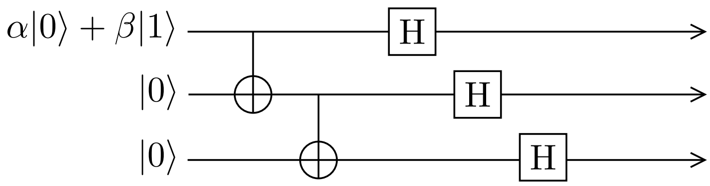


3. *a)* The circuit contains a CNOT operation (with the qubit shared with Charlie as the control and the qubit shared with Bob as the target), followed by a Hadamard operation (on the qubit shared with Charlie). See the diagram, where $|B\rangle$ indicates the qubit shared with Bob and $|C\rangle$ the qubit shared with Charlie.



   b) Alice's qubit has been teleported to Bob, so the final state is $|0\rangle_A|0\rangle_A(\alpha|0\rangle_B|0\rangle_C + \beta|1\rangle_B|1\rangle_C)$.
   c) Yes, the qubits of Bob and Charlie are entangled, if $\alpha\beta \neq 0$.

4. *(a)* use a combination of CNOT and Hadamard operations



   (b) $S_1|\psi_1\rangle = |\psi_1\rangle$, $S_2|\psi_2\rangle = (-1)^2|\psi_2\rangle = |\psi_2\rangle$, because $\sigma_x(|0\rangle + |1\rangle) =$

$|0\rangle + |1\rangle$ and $\sigma_x(|0\rangle - |1\rangle) = -(|0\rangle + |1\rangle)$. Hence $\alpha|\psi_0\rangle + \beta|\psi_1\rangle$ is an eigenstate of both S_1 and S_2 with eigenvalue 1.
(c) For a σ_z error on the first qubit a measurement of S_1 gives outcome -1 and a measurement of S_2 gives outcome $+1$; for an error on the second qubit the outcomes are $-1, -1$ and for an error on the third qubit the outcomes are $+1, -1$. The erroneous states remain eigenstates of S_1 and S_2, so the measurement does not perturb them.