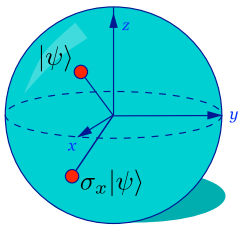
ANSWERS TO THE EXAM QUANTUM INFORMATION, 2 NOVEMBER 2026

each item gives 2 points for a fully correct answer, grade = total $\times 9/24 + 1$

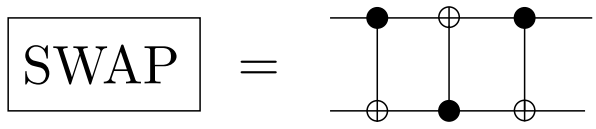
1. *(a)* A valid density matrix must be (i) Hermitian, $\rho = \rho^\dagger$; (ii) of unit trace, $\mathrm{Tr}\,\rho = 1$; (iii) positive semidefinite, so that no eigenvalue is negative.
   (b) ρ_1 is a valid density matrix. It is Hermitian, because the coefficients $\pm\frac{i}{3}$ of $|0\rangle\langle 2|$ and $|2\rangle\langle 0|$ are each other's complex conjugates, and $\mathrm{Tr}\,\rho_1 = \frac{1}{3}+\frac{1}{3}+\frac{1}{3} = 1$. In the basis $\{|0\rangle, |2\rangle\}$ the block $\frac{1}{3}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ has eigenvalues 0 and $\frac{2}{3}$; together with the eigenvalue $\frac{1}{3}$ of $|1\rangle$, no eigenvalue is negative.
   ρ_2 is not a valid density matrix. It is Hermitian with $\mathrm{Tr}\,\rho_2 = 1$, but one eigenvalue, $-1/4$, is negative.
   ρ_3 is not a valid density matrix: it is Hermitian and positive, but $\mathrm{Tr}\,\rho_3 = \frac{3}{2} \neq 1$.
   c) Equivalence in one direction: if $\rho^2 = \rho$ then $\mathrm{Tr}\,\rho^2 = \mathrm{Tr}\,\rho = 1$; in the other direction: if $\mathrm{Tr}\,\rho^2 = 1$, it means that $\sum_n \lambda_n^2 = 1$, with $\lambda_n \in [0,1]$ an eigenvalue of ρ. I also know that $\sum_n \lambda_n = 1$, since $\mathrm{Tr}\,\rho = 1$. Subtract the two sums, $\sum_n \lambda_n(1-\lambda_n) = 0$. This is a sum of positive numbers, it can only vanish if each term in the sum vanishes, so $\lambda_n \in \{0,1\}$, hence $\lambda_n^2 = \lambda_n$ for all n and $\rho^2 = \rho$.

2. *a)* $\mathcal{L}(\rho)^\dagger = \mathcal{L}(\rho^\dagger) = \mathcal{L}(\rho)$, so it's Hermitian; the trace is $\mathrm{Tr}[(K_0^\dagger K_0 + K_1^\dagger K_1)\rho] = \mathrm{Tr}\,\rho = 1$.
   b) $\rho = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so $\mathcal{L}\rho = \left(\begin{smallmatrix} \gamma & 0 \\ 0 & 1-\gamma \end{smallmatrix}\right)$, which is a mixed state.
   c) $\mathcal{L}[\frac{1}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)] = \frac{1}{2}\left(\begin{smallmatrix} 1+\gamma & 0 \\ 0 & 1-\gamma \end{smallmatrix}\right)$, which is a pure state for $\gamma = 1$.

3. *a)* The σ_x operation rotates a point on the Bloch sphere by π around the x-axis. The rotated state is not orthogonal to the original state (an orthogonal state would be located at the inverted point $\vec{r} \mapsto -\vec{r}$).



   b) $\sigma_x = e^{-i\pi/2}e^{i(\pi/2)\sigma_x} \Rightarrow \sqrt{\sigma_x} = e^{-i\pi/4}e^{i(\pi/4)\sigma_x} = \frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix}$

   c) 

4. *a)* The encoded state is $\alpha|000\rangle + \beta|111\rangle$, after the bit flip error it is $\alpha|010\rangle + \beta|101\rangle$
   b) carry out a parity-check measurement (by means of two CNOT gates on a

target ancilla qubit) with the first two qubits as control and another parity-check measurement (using another ancilla as target) with the last two qubits as control; measurement of the two ancilla's reveals which qubit has been flipped; this can then be corrected with a σ_x operation, without knowledge of the value of the qubit

c) The encoded state $\alpha|000\rangle + \beta|111\rangle$ becomes $\alpha|000\rangle - \beta|111\rangle$. The parity checks detect no error, so the code offers no protection against phase errors.