EXAM QUANTUM INFORMATION, 2 NOVEMBER 2026, 9–12 HOURS.

1. The states $|0\rangle$, $|1\rangle$, $|2\rangle$ denote orthonormal states of a quantum system, with density matrix ρ.

- *a)* List three conditions that *any* valid density matrix should satisfy.
- *b)* Explain, for each of the matrices ρ_1, ρ_2, ρ_3, if it is a valid density matrix or not:

$$\rho_1 = \tfrac{1}{3}|0\rangle\langle 0| + \tfrac{i}{3}|0\rangle\langle 2| - \tfrac{i}{3}|2\rangle\langle 0| + \tfrac{1}{3}|1\rangle\langle 1| + \tfrac{1}{3}|2\rangle\langle 2|,$$

$$\rho_2 = \tfrac{3}{4}|0\rangle\langle 0| - \tfrac{1}{4}|1\rangle\langle 1| + \tfrac{1}{2}|2\rangle\langle 2|, \quad \rho_3 = \tfrac{1}{2}|0\rangle\langle 0| + \tfrac{1}{2}|1\rangle\langle 1| + \tfrac{1}{2}|2\rangle\langle 2|.$$

- *c)* The condition for a density matrix ρ to represent a pure state can be written in two ways:
  A: A density matrix ρ represents a pure state if and only if $\rho^2 = \rho$.
  B: A density matrix ρ represents a pure state if and only if $\mathrm{Tr}\,\rho^2 = 1$.
  Can you prove that *A* and *B* are equivalent conditions?

2. A quantum system transforms the density matrix ρ of a qubit by the following linear operation:

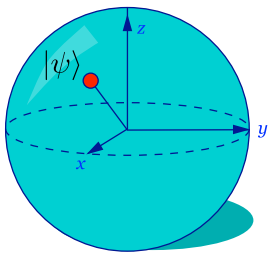
$$\mathcal{L}(\rho) = K_0\,\rho\,K_0^\dagger + K_1\,\rho\,K_1^\dagger, \qquad K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad K_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix},$$

   with $0 \le \gamma \le 1$ a given parameter.

- *a)* Is $\mathcal{L}(\rho)$ Hermitian? Calculate the trace of $\mathcal{L}(\rho)$.
- *b)* Assume that the qubit is initially in the pure state $\psi = \binom{0}{1}$. What is its density matrix after the transformation? Is it still a pure state?
- *c)* Now assume instead that the qubit is initially in the maximally mixed state with density matrix $\rho = \frac{1}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Is it possible to choose the parameter γ such that $\mathcal{L}(\rho)$ is a pure state?

3. A bit-flip operation is a single-qubit gate that switches 0 and 1. The unitary matrix for the bit-flip operation is the σ_x Pauli matrix.

- *a)* The state of a qubit can be represented by a point on the unit sphere, the Bloch sphere. The figure shows one such point in red, corresponding to a state $|\psi\rangle$. Copy the figure and insert the point that corresponds to the state $\sigma_x|\psi\rangle$. (Please mark the angles, such that the position of the new point can be uniquely identified.) Is the state $\sigma_x|\psi\rangle$ orthogonal to $|\psi\rangle$ or not? Explain your answer.



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- *b)* In a classical computer it makes no sense to talk of the "square-root" of a bit flip, but in a quantum computer such a gate exists. What are the elements of a unitary matrix that squares to a bit flip?
- *c)* The SWAP gate is a two-qubit gate that exchanges the states of the first and second qubit: $|01\rangle$ is transformed into $|10\rangle$ and vice versa, while $|00\rangle$ and $|11\rangle$ are unchanged. Can you construct a SWAP gate by combining three CNOT gates? Please show the circuit diagram.

4. Consider the 3-qubit error-correcting code where the state $|0\rangle$ is encoded as $|000\rangle$ and the state $|1\rangle$ is encoded as $|111\rangle$.

- *a)* Given an arbitrary state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, write down its encoded state using the 3-qubit code. How does this state change if a bit-flip error occurs on the second qubit?
- *b)* Outline the procedure for detecting and correcting the bit-flip error.
- *c)* Suppose that instead of a bit flip, a phase-flip error occurs on the first qubit, so $|0\rangle \mapsto |0\rangle$ and $|1\rangle \mapsto -|1\rangle$ for that qubit. What is the resulting state? Will the procedure of part *b)* detect this error?