Linearized Theory: Sound Waves

In the linearized limit, $\Lambda_{i\alpha}$ becomes $\delta_{i\alpha}$, and the distinction between the reference and target spaces effectively vanishes.

$$u_{\alpha\beta} \to (\partial_i u_j + \partial_j u_i)/2 \qquad u_i(\mathbf{x}) \sim e^{i\mathbf{q}.\mathbf{x}} = e^{i(q_x x + q_y y)}$$

$$F = \frac{1}{2}\int \frac{d^D q}{(2\pi)^D} u_i(-\mathbf{q}) K_{ij}(\mathbf{q}) u_j(\mathbf{q})$$

$$K_{ij}(q) = \mu q^2 \delta^T_{ij} + (\lambda + 2\mu) q^2 \hat{q}_i \hat{q}_j$$

$$\rho \ddot{u}_i = f_i = \partial_j \sigma_{ij} = (\lambda + \mu)\partial_i \partial_j u_j + \mu \nabla^2 u_i$$

$$\rho \omega^2 u_i = (\lambda + 2\mu) q_i q_j u_j + \mu q^2 \delta^T_{ij} u_j = K_{ij}(\mathbf{q}) u_j$$

$K_{ij}(\mathbf{q})$: Rigidity matrix

q_i: wavenumber

$\delta^T_{ij} = \delta_{ij} - \hat{q}_i \hat{q}_j$

$u_{Li} = \hat{q}_i u_L : u_L = \hat{q}_i u_i$

$u_{Ti} = \delta^T_{ij} u_j$

$$\omega_L^2 = c_L^2 q^2; \quad \omega_T^2 = c_T^2 q^2$$

$$c_L^2 = (\lambda + 2\mu)/\rho = [B + 2(D-1)(\mu/D)]/\rho$$

$$c_T^2 = \mu/\rho$$

Note c_L = c_T in 2D when $B=0$

Dynamical matrix and Waves in 2D

$$D_{ij}(\mathbf{q}) = \tfrac{1}{\rho} K_{ij}(\mathbf{q}) = \frac{1}{\rho}\begin{pmatrix} (\lambda+2\mu)q_x^2 + \mu q_y^2 & (\lambda+\mu)q_x q_y \\ (\lambda+2\mu)q_x q_y & (\lambda+2\mu)q_y^2 + \mu q_x^2 \end{pmatrix}$$

D_{ij}: Continuum dynamical matrix $\quad \omega^2 u_i = D_{ij} u_j$

$$\omega_L^2 = c_L^2 q^2 : \quad u_L = (\hat{q}_x, \hat{q}_y); \hat{\mathbf{q}} = \mathbf{q} / q$$

$$\omega_T^2 = c_T^2 q^2 : \quad u_T = (\hat{q}_y, -\hat{q}_x)$$

2D uniaxial system

$$f = \tfrac{1}{2}\left(K_1 u_{xx}^2 + 2K_2 u_{xx} u_{yy} + K_3 u_{yy}^2 + 2K_5 u_{xy}^2\right)$$

$$D_{ij}(\mathbf{q}) = \frac{1}{\rho}\begin{pmatrix} K_1 q_x^2 + K_5 q_y^2 & (K_2 + K_5)q_x q_y \\ (K_2 + K_5)q_x q_y & K_3 q_y^2 + K_5 q_x^2 \end{pmatrix}$$

Surface (Rayleigh) waves

$$\sigma_{yy} = (\lambda + 2\mu)u_{yy} + \lambda u_{xx}$$

$$\sigma_{xy} = 2\mu u_{xy}$$

$$u_i(x,y) \propto A e^{iqx-\kappa y}$$

$$q_x^2 + q_y^2 \to q^2 - \kappa^2$$

Convert q_y to $i\kappa$

$$u_x(x,y) = e^{iqx}\left(qA_L e^{-\kappa_L y} - i\kappa_T A_T e^{-\kappa_T y}\right)$$

$$\kappa_L^2 = q^2 - (\omega^2 / c_L^2) > 0$$

$$u_y(x,y) = e^{iqx}\left(i\kappa_L A_L e^{-\kappa_L y} + qA_T e^{-\kappa_T y}\right)$$

$$\kappa_T^2 = q^2 - (\omega^2 / c_T^2) > 0$$

$$\sigma_{yy}(y=0) = e^{iqx}[i(c_L^2(q^2-\kappa_L^2) - 2q^2c_T^2)A_L - 2c_T^2 q\kappa_T A_T] = 0$$

$$\sigma_{xy}(y=0) = \mu e^{iqx}[-2q\kappa_L A_L + i(q^2+\kappa_T^2)A_T] = 0$$

$$\begin{bmatrix} -2q\kappa_L & i(q^2+\kappa_L^2) \\ i\left(c_L^2\left(q^2-\kappa_L^2\right) - 2q^2c_T^2\right) & -2c_T^2 q\kappa_T \end{bmatrix} \begin{bmatrix} A_L \\ A_T \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4q^2c_T^2\kappa_T\kappa_L + (q^2+\kappa_T^2)[c_L^2(q^2-\kappa_L^2) - 2q^2c_T^2] = 0$$

Rayleigh Wave Speed

$$[2q^2 - (\omega^2 / c_T^2)][\omega^2 - 2c_T^2 q^2] = 4q^2 \sqrt{[q^2 - (\omega^2 / c_L^2)][q^2 - (\omega^2 / c_T^2)]}$$

$$\omega^2 = c_T^2 s^2 q^2$$

$$s^6 - 8s^4 + 8s^2[3 - 2(c_T^2 / c_L^2)] - 16[1 - (c_T^2 / c_L^2)] = 0$$

Solve subject to the constraint that s^2, $\kappa_L{}^2$, and $\kappa_T{}^2$ be greater than zero: only one solution.

$$c_T = c_L \qquad s^6 - 8s^4 + 8s^2 = 0$$

Solutions: $s = 0;\ s^2 = 4(1 \pm \sqrt{1/2}) > 1$

Only first soln satisfies constraints.

Response and Fluctuations

$$f_i^{\text{int}}(\mathbf{x}) = -\frac{\delta \mathcal{F}}{\delta u_i(\mathbf{x})} = -\int d^D x' K_{ij}(\mathbf{x},\mathbf{x}') u_j(\mathbf{x}') = -f_i^{\text{ext}}(\mathbf{x})$$

$$\chi_{ij}(\mathbf{x},\mathbf{x}') = \frac{\delta u_i(\mathbf{x})}{\delta f_j(\mathbf{x}')} = K_{ij}^{-1}(\mathbf{x},\mathbf{x}')$$

Note: χ diverges when any eigenmode frequency vanishes

$$\chi_{ij}(\mathbf{q}) = K_{ij}^{-1}(\mathbf{q}) = \frac{\hat{q}_i \hat{q}_j}{(\lambda + 2\mu) q^2} + \frac{\delta_{ij} - \hat{q}_i \hat{q}_j}{\mu q^2}$$

$$S_{ij}(\mathbf{x},\mathbf{x}') = \left\langle u_i(\mathbf{x}) u_j(\mathbf{x}') \right\rangle - \left\langle u_i(\mathbf{x}) \right\rangle \left\langle u_j(\mathbf{x}') \right\rangle = T \chi_{ij}(\mathbf{x},\mathbf{x}')$$

$$S_{ij}(\mathbf{q}) = T \chi_{ij}(\mathbf{q}) : \quad \left\langle \mathbf{u}^2(\mathbf{x}) \right\rangle \approx \int \frac{d^D q}{(2\pi)^D} \frac{T}{K q^2} \xrightarrow[D \le 2]{} \infty$$

$$\left\langle \rho(\mathbf{x}) \right\rangle = \sum\nolimits_{\mathbf{G}} \left\langle e^{i\mathbf{G}\cdot(\mathbf{x} - \mathbf{u}(\mathbf{x}))} \right\rangle \rho_{\mathbf{G}}(\mathbf{x}); \qquad \left\langle e^{-i\mathbf{G}\cdot\mathbf{u}(\mathbf{x}))} \right\rangle \approx e^{-G_i \langle u_i u_j \rangle G_j / 2}$$

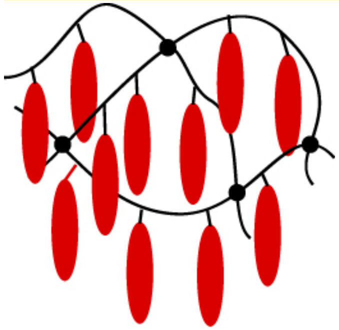
No Long-Range Crystalline Order for $D \le 2$

What are Nematic Gels?

- Homogeneous Elastic media with broken rotational symmetry (uniaxial, biaxial)
- Most interesting - systems with broken symmetry that develop spontaneously from a homogeneous, isotropic elastic state

Examples of LC Gels

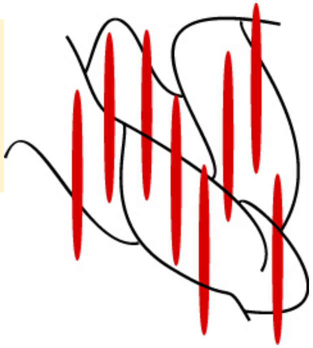
1. Liquid Crystal Elastomers - Weakly crosslinked liquid crystal polymers



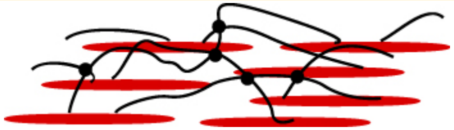
Nematic

Smectic-*C*

2. Tanaka gels with hard-rod dispersion



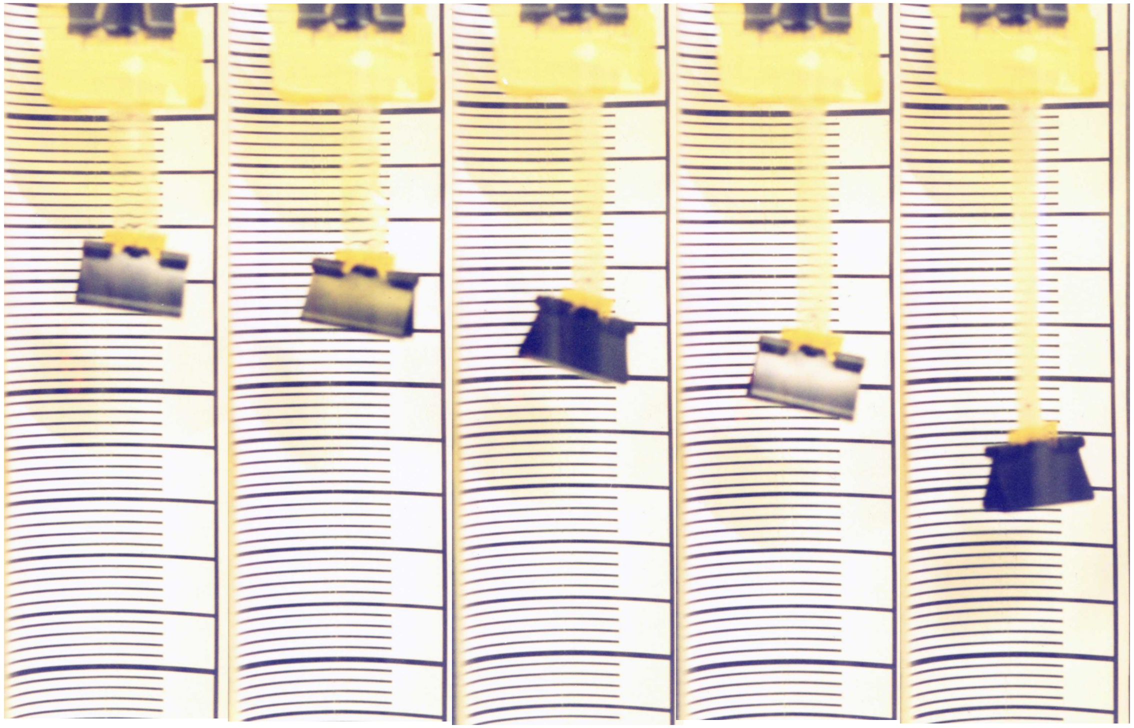
3. Anisotropic membranes



4. Glasses with orientational order

Properties I

- **Large thermoelastic effects** - Large thermally induced strains - artificial muscles

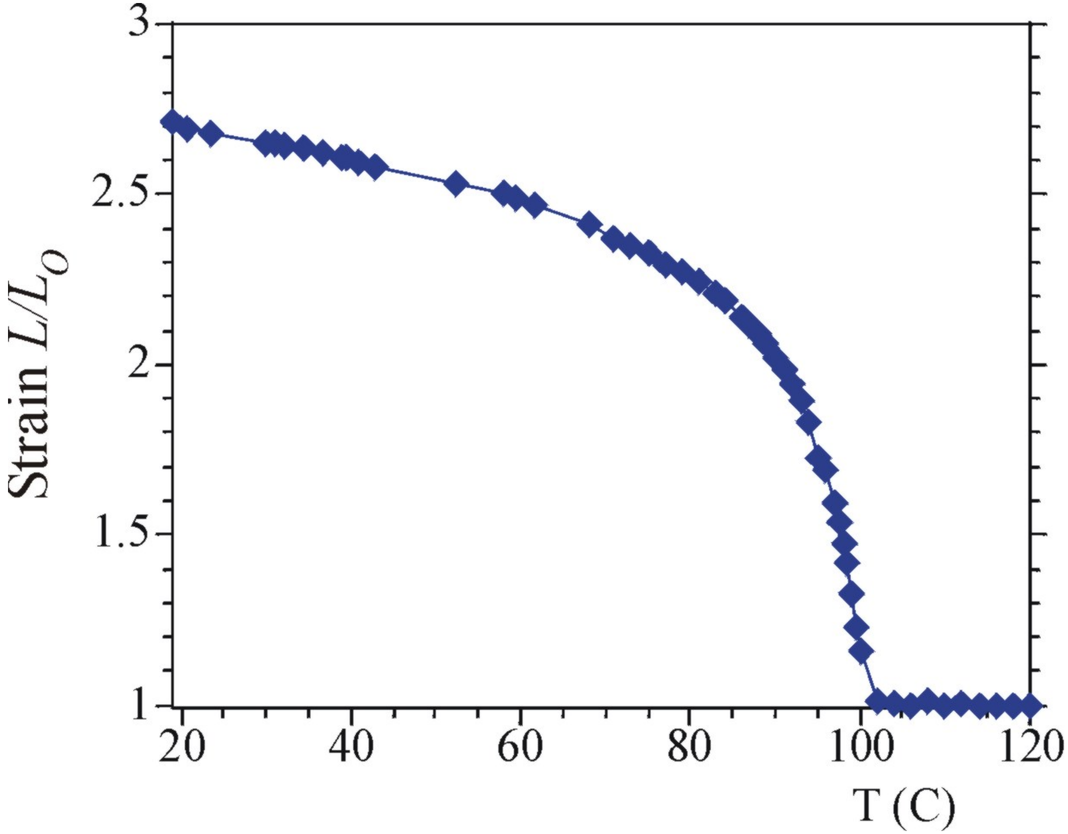


Courtesy of Eugene Terentjev

300% strain

Properties II

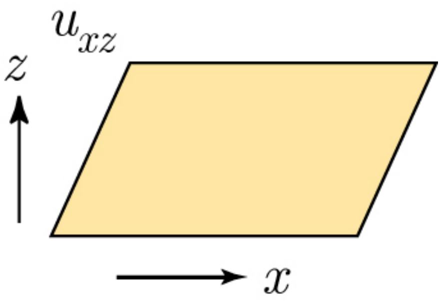
Large strain in small temperature range



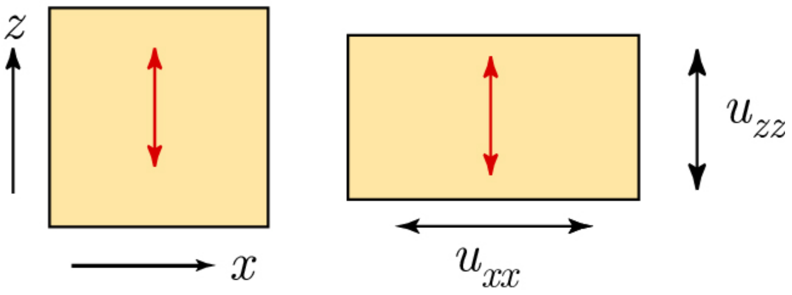
Terentjev

Properties III

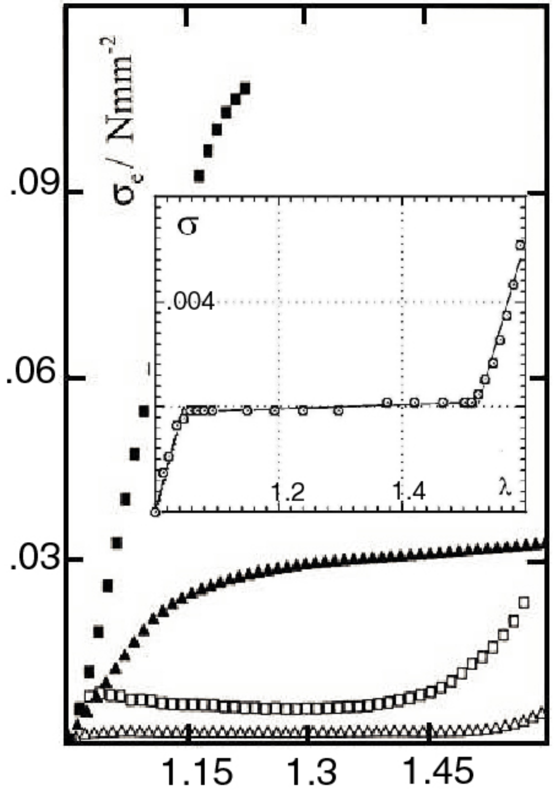
- Soft or “Semi-soft” elasticity



Vanishing xz shear modulus

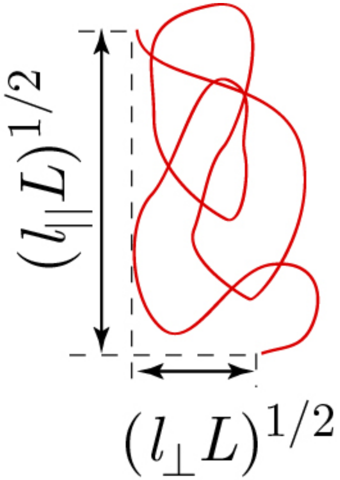


Soft stress-strain for stress perpendicular to order



Warner Finkelmann

Neoclassical Elastomer Theory



Warner and Terentjev

Gel: random walks between crosslinks with probability distribution

$$P(R) = \left[\frac{\det l}{(2\pi L)^3}\right]^{1/2} \exp(-3R_i l^{-1}_{ij} R_j \,/\, 2L)$$

$$\left[R_{0\alpha}R_{0\beta}\right]_{\rm av} = \tfrac{1}{3}\ell_{0,\alpha\beta}L$$

$$\ell_{ij} = \ell_{||}n_i n_j + \ell_{\perp}(\delta_{ij} - n_i n_j) \qquad R_i = \Lambda_{i\alpha}R^0_\alpha$$

$$P_0(R_0) = \left[\frac{\det l_0}{(2\pi L)^3}\right]^{1/2} \exp(-3R_{0\alpha} l^{-1}_{0,\alpha\beta} R_{0\beta} \,/\, 2L)$$

l =anisotropic step-length tensor

l_0 =tensor at time of crosslinking

$$\left[R_{0\alpha}R_{0\beta}\right]_{\rm av} = \tfrac{1}{3}\ell_{0,\alpha\beta}L$$

Stretch from Step Anisotropy

Free energy density

$$\begin{aligned} f &= -nT[\ln P(R)]_{\text{av}} \\ &= \frac{3}{2L} nT \Lambda_{i\alpha} l^{-1}_{ik} \Lambda_{k\beta} [R_{0\alpha} R_{0\beta}]_{\text{av}} \\ &= \frac{1}{2} nT \,\mathrm{Tr} \underline{\Lambda} \underline{l}_0 \underline{\Lambda}^T \underline{l}^{-1} \end{aligned}$$

Uniaxial constant volume stretch

$$\underset{\sim}{\Lambda} = \begin{pmatrix} \Lambda^{-1/2} & 0 & 0 \\ 0 & \Lambda^{-1/2} & 0 \\ 0 & 0 & \Lambda \end{pmatrix}$$

Minimize over Λ at fixed ℓ's

$$f = \tfrac{1}{2} nT[\Lambda^2 l_{0,||} / l_{||} + 2\Lambda^{-1} l_{0,\perp} / l_{\perp}]$$

$$\Lambda = \left(\frac{\ell_{0,\perp} \ell_{||}}{\ell_{0,||} \ell_{\perp}} \right)^{1/3} \rightarrow \left(\frac{\ell_{||}}{\ell_{\perp}} \right)^{1/3}$$

$\ell_{||} > \ell_{\perp}$: $\Lambda > 1$ as expected

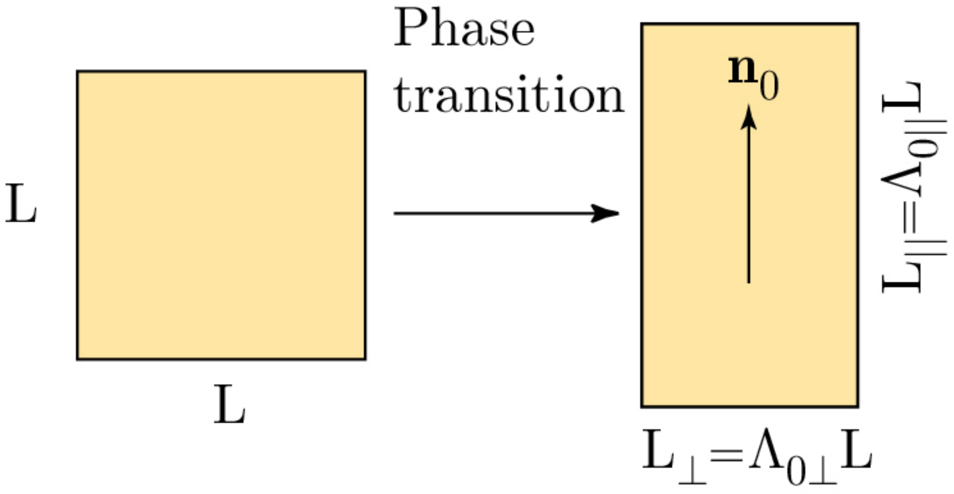
Model for Isotropic-Nematic Trans.

$$\begin{aligned}\bar{u}_{\alpha\beta} &= u_{\alpha\beta} - \tfrac{1}{3}\delta_{\alpha\beta}u_{\gamma\gamma} \\ &= \Psi(n_\alpha n_\beta - \tfrac{1}{3}\delta_{\alpha\beta})\end{aligned}$$

Deviatoric strain: like the order parameter for the isotropic-to-nematic transition

$$f = \tfrac{1}{2}Bu_{\alpha\alpha}^2 + \mu \mathrm{Tr}\underset{\sim}{\bar{u}}^2 - C\mathrm{Tr}\underset{\sim}{\bar{u}}^3 + D\left(\mathrm{Tr}\underset{\sim}{\bar{u}}^2\right)^2 - Eu_{\alpha\alpha}\mathrm{Tr}\underset{\sim}{\bar{u}}^2$$

μ approaches zero signals a transition to a nematic state with a nonvanishing Ψ. $u_{\alpha\alpha} \sim \Psi^2$



$$\begin{aligned}\underset{\sim}{u}_0 &= \tfrac{1}{2}\left(\underset{\sim}{\Lambda}_0^T\underset{\sim}{\Lambda}_0 - \underline{\delta}\right) \\ &= \underset{\sim}{\bar{u}}_0 + \tfrac{1}{3}\underset{\sim}{\delta}u_{\alpha\alpha}\end{aligned}$$

Direction of $\mathbf{n}_0$ is arbitrary: Broken continuous symmetry!

Broken Continuous Symmetry: Goldstone Modes

Goldstone Theorem: an ordered state that breaks a continuous symmetry (e.g. a rotational symmetry) necessarily has one or more zero energy modes (excitations)

$\mathbf{m}$: 2D magnetization; $f(\mathbf{m})$: invariant under

$$\begin{pmatrix} m_x \\ m_y \end{pmatrix} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} m_x \\ m_y \end{pmatrix} \approx \begin{pmatrix} m_x - \theta m_y \\ \theta m_x + m_y \end{pmatrix}$$

$$\frac{df}{d\theta} = -\frac{\partial f}{\partial m_x} m_y + \frac{\partial f}{\partial m_y} m_x = 0$$

$$= -h_x m_y + h_y m_x = 0$$

$$\left.\frac{\partial}{\partial h_y}\frac{df}{d\theta}\right|_{m_y, h_y = 0} = -h_x \frac{\partial m_y}{\partial h_y} + m_x = 0$$

$$\chi_{yy} = \frac{\partial m_y}{\partial h_y} = \frac{m_x}{h_x} \xrightarrow[h_x \to 0]{} \infty$$

There must be a zero mode: It is the trivial rigid rotation of the magnetization.

Strain of New Phase

$$\begin{aligned} R_i(\mathbf{x}) &= \Lambda_{0ij} x_j + \delta u_i(\mathbf{x}) \\ &= x'_i + u'_i(\mathbf{x}') \end{aligned}$$

$\underline{u}'$ is the strain relative to the new state at points $\mathbf{x}'$

$$\Lambda_{ij} = \frac{\partial R_i}{\partial x_j} = \frac{\partial R_i}{\partial x'_k}\frac{\partial x'_k}{\partial x_j} = \Lambda'_{ik}\Lambda_{0kj}$$

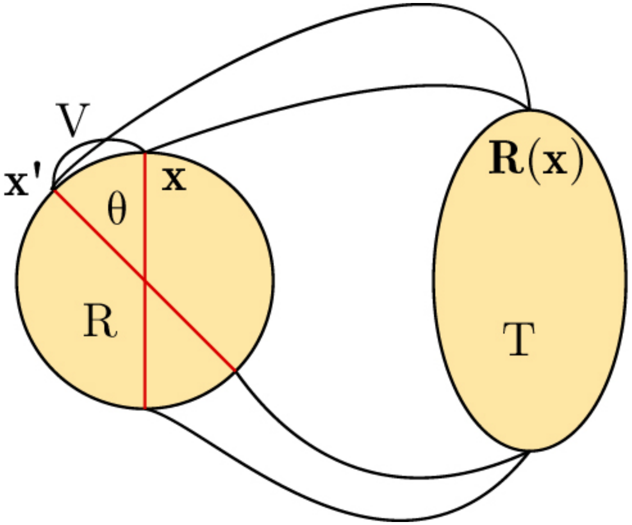
$$\begin{aligned} \delta \underset{\sim}{u} &= \underset{\sim}{u} - \underset{\sim}{u}_0 \\ &= \tfrac{1}{2}\left(\underset{\sim}{\Lambda}^T \underset{\sim}{\Lambda} - \underset{\sim}{\Lambda}_0^T \underset{\sim}{\Lambda}_0\right) \\ &= \underset{\sim}{\Lambda}_0^T \underset{\sim}{u}' \underset{\sim}{\Lambda}_0 \end{aligned}$$

$\delta\underline{u}$ is the deviation of the strain relative to the original reference frame R from $\underline{u}_0$, and half the difference between the original metric and the new one.

$$\underline{u}' = \tfrac{1}{2}\left(\underline{\Lambda}'^T \underline{\Lambda}' - \underline{\delta}\right) \approx \tfrac{1}{2}\left(\underset{\sim}{\eta}' + \underset{\sim}{\eta}'^T\right)$$

$\delta\underline{u}$ is linearly proportional to $\underline{u}'$

Elasticity of New Phase



Rotation of anisotropy direction costs no energy

$$r = \frac{\Lambda_{0||}^{2}}{\Lambda_{0\perp}^{2}} > 1$$

$$\underline{u}' = (\underline{\Lambda}_0^T)^{-1}\left(\underline{V}\,\underline{u}_0\underline{V}^{-1} - \underline{u}_0\right)\underline{\Lambda}_0^{-1}$$

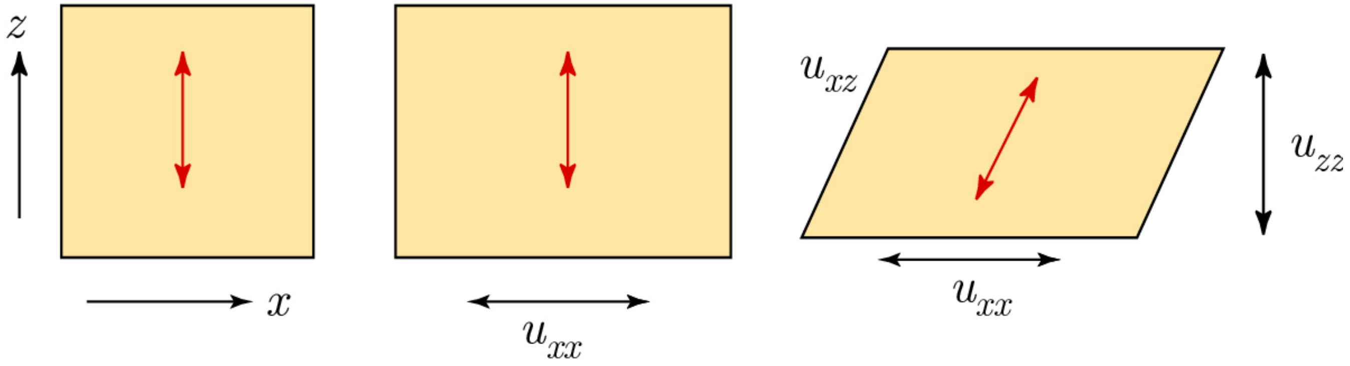
$$u'_{xz} \sim \frac{(r-1)}{2\sqrt{r}}\theta$$

$$= \tfrac{1}{4}(r-1)\begin{pmatrix} 1-\cos 2\theta & \frac{1}{\sqrt{r}}\sin 2\theta \\ \frac{1}{\sqrt{r}}\sin 2\theta & -\frac{1}{r}\left(1-\cos 2\theta\right) \end{pmatrix}$$

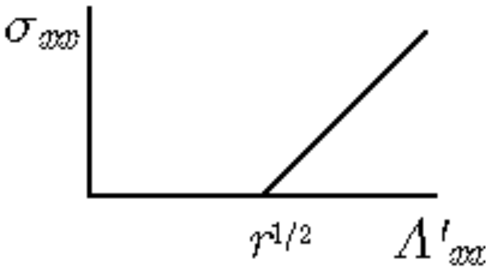
$C_5 = 0$ because of rotational invariance

$$f_{\mathrm{el}} = \tfrac{1}{2}K_1 u'^2_{zz} + K_2 u'_{zz}u'_{\nu\nu} + \tfrac{1}{2}K_3 u'_{\nu\nu}u'_{\nu\nu} + 2K_4 u'_{\nu\tau}u'_{\nu\tau} \xcancel{+ 2K_5} u'_{z\nu}u'_{z\nu}$$

Soft Extensional Elasticity



$$\underline{u} = \frac{1}{4}(r-1)\begin{pmatrix} 1-\cos 2\theta & \frac{1}{\sqrt{r}}\sin 2\theta \\ \frac{1}{\sqrt{r}}\sin 2\theta & -\frac{1}{r}\left(1-\cos 2\theta\right) \end{pmatrix}$$



$$u_{zz} = -\frac{1}{r}u_{xx}$$

$$u_{xz} = \frac{1}{2\sqrt{r}}\sqrt{u_{xx}(r-1-2u_{xx})}$$

Strain u_{xx} can be converted to a zero energy rotation by developing strains u_{zz} and u_{xz} until $u_{xx} = (r-1)/2$

Frozen anisotropy: Semi-soft

System is now uniaxial – why not simply use uniaxial elastic energy? This predicts linear stress-stain curve and misses lowering of energy by reorientation:

$$f = \tfrac{1}{2}K_1 u_{zz}^2 + K_2 u_{zz} u_{\nu\nu} + \tfrac{1}{2}K_3 u_{\nu\nu}^2 + K_4 u_{\nu\tau}^2 + 2K_5 u_{\nu z}^2$$

Model Uniaxial system: Produces harmonic uniaxial energy for small strain but has nonlinear terms – reduces to isotropic when $h=0$

$$\frac{\partial f}{\partial u_{\alpha\beta}} = \sigma_{\alpha\beta}^{II} = \sigma_{\alpha\beta}$$

$f(u)$: isotropic

Rotation $$u \rightarrow u' = u + \theta \begin{pmatrix} -2u_{xz} & u_{xx} - u_{zz} \\ u_{xx} - u_{zz} & 2u_{xz} \end{pmatrix}$$

Semi-soft stress-strain

$$\frac{df}{d\theta}=\frac{\partial f}{\partial u_{xz}}\frac{du_{xz}}{d\theta}+\frac{\partial f}{\partial u_{zx}}\frac{du_{zx}}{d\theta}+\frac{\partial f}{\partial u_{xx}}\frac{du_{xx}}{d\theta}+\frac{\partial f}{\partial u_{zz}}\frac{du_{zz}}{d\theta}=0$$

$$\frac{df}{d\theta}=2\sigma_{xz}(u_{xx}-u_{zz})+2(\sigma_{zz}-\sigma_{xx})u_{xz}=0$$

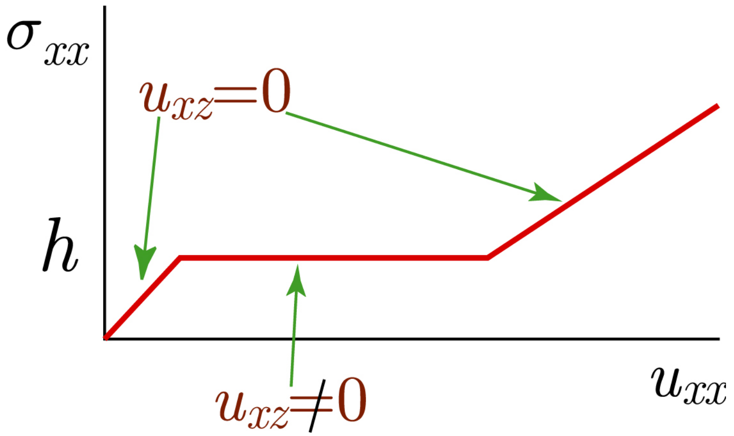
Ward Identity

$$\sigma_{xz}=\frac{(\sigma_{xx}-\sigma_{zz})u_{xz}}{u_{xx}-u_{zz}}=0\Rightarrow u_{xz}=0 \text{ or } \sigma_{xx}=\sigma_{zz}$$

$$\sigma_{\alpha\beta}=\frac{\partial f}{\partial u_{\alpha\beta}}$$

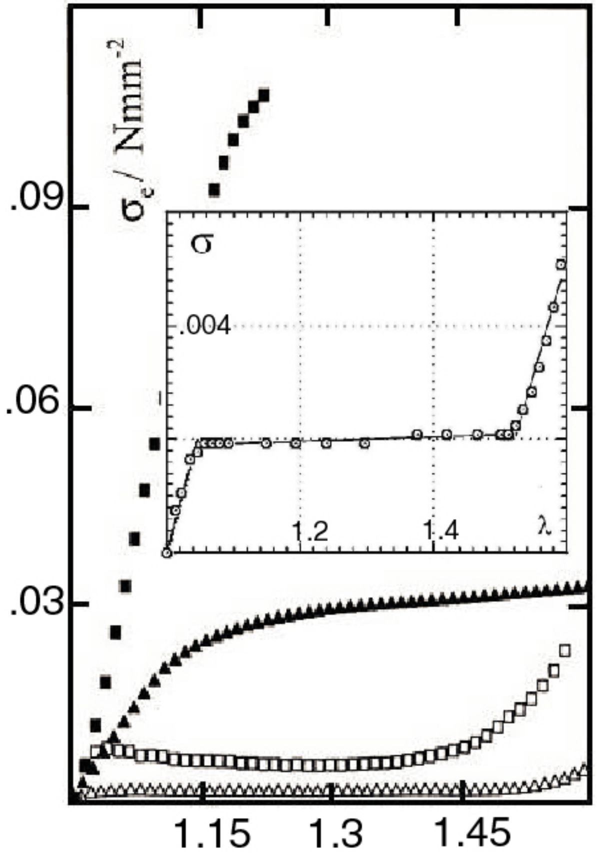
Second PK stress tensor.

σ_{zz}: internal stress;
σ_{xx} : 2nd PK form of external stress;
σ_{xz} =0.

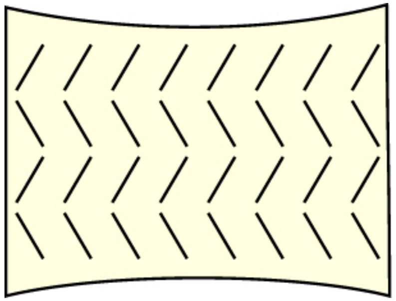


Semi-soft Extensions

Break rotational symmetry



Stripes form in real systems: semi-soft, BC



Not perfectly soft because of residual anisotropy arising from crosslinking in the the nematic phase - **semi-soft.** length of plateau depends on magnitude of spontaneous anisotropy r.

Warner-Terentjev

Note: Semi-softness only visible in nonlinear properties

Finkelmann, et al., J. Phys. II **7**, 1059 (1997);
Warner, J. Mech. Phys. Solids **47**, 1355 (1999)

Elastic Membranes

Elastic membranes are two-dimensional manifolds that live in a three dimensional space. Thus, not only are the reference and target spaces different, they have different dimensionalities as well. It is useful to generalize this concept to D-dimensional manifolds fluctuating in a d-dimensional space

$$u_{\alpha\beta} = \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha + \partial_\alpha \mathbf{u} \cdot \partial_\beta \mathbf{u} + \partial_\alpha \mathbf{h} \cdot \partial_\beta \mathbf{h}\right)$$

$$\mathcal{H} = \tfrac{1}{2} \int d^D x \left[\lambda u_{\alpha\alpha}^2 + 2\mu u_{\alpha\beta} u_{\alpha\beta} + \kappa \left(\partial^2 \mathbf{h}\right)^2 \right]$$

Foppl-von Karman number

$$\gamma = Y_{2D} L^2 / \kappa$$

$$S_{hh}(\mathbf{q}) = \frac{T}{\kappa q^4}; \qquad S_{uu}(\mathbf{q}) \approx \frac{T}{\mu q^2}$$

Harmonic order

$$\left\langle h^2(\mathrm{x}) \right\rangle = \int_{L^{-1}}^{q_c} \frac{d^D q}{(2\pi)^D} \frac{T}{\kappa q^4} \approx L^{2\zeta} = L^{4-D}$$

$\zeta = (4\text{-}D)/2$ is the wandering exponent

Nonlinearities

$$\mathcal{H}_{\mathrm{nl}} \approx \tfrac{1}{2}\int d^D x[\lambda(\partial_\alpha u_\alpha \partial_\beta \mathbf{h}\cdot\partial_\beta \mathbf{h} + \tfrac{1}{4}(\partial_\beta \mathbf{h}\cdot\partial_\beta \mathbf{h})^2) +$$

$$2\mu(\partial_\alpha u_\beta \partial_\alpha \mathbf{h}\cdot\partial_\beta \mathbf{h} + \tfrac{1}{4}(\partial_\alpha \mathbf{h}\cdot\partial_\beta \mathbf{h})^2]$$

Integrate over **h** (or use your favorite diagrammatic technique): Divergent contribution to μ for $D<4$.

Expect new critical exponents

$$\mu' \approx \mu\left(1 - \tfrac{\mu}{T}\int d^D q\, q^4 S_{hh}^2(q)\right)$$

$$\approx \mu\left(1 - \frac{\mu T}{\kappa^2}\int_{q_L}^{q_U}\frac{q^4 d^D q}{q^8}\right) \approx \mu\left(1 - \frac{A\mu T}{(4-D)\kappa^2 q_L^{4-D}}\right)$$

$$S_{hh}(\mathbf{q}) \sim q^{-(4-\eta)}; \qquad \mathbf{h}(\mathbf{q}) = \zeta_h \mathbf{h}(b\mathbf{q})$$

$$\left\langle \mathbf{h}(\mathbf{q}_1)\cdot\mathbf{h}(\mathbf{q}_2)\right\rangle = S_{hh}(\mathbf{q}_1)(2\pi)^D\delta(\mathbf{q}_1+\mathbf{q}_2) = \zeta_h^2\left\langle \mathbf{h}(b\mathbf{q}_1)\cdot\mathbf{h}(b\mathbf{q}_2)\right\rangle$$

$$= \zeta_h^2 S_{hh}(b\mathbf{q}_1)(2\pi)^D\delta(b\mathbf{q}_1+b\mathbf{q}_2) = \zeta_h^2 b^{-D} S_{hh}(b\mathbf{q}_1)(2\pi)^D\delta(\mathbf{q}_1+\mathbf{q}_2)$$

$$\Rightarrow S_{hh}(\mathbf{q}) = \zeta_h^2 b^{-D} S_{hh}(b\mathbf{q}) \sim \zeta_h^2 b^{-D}(bq)^{-(4-\eta)} \Rightarrow \zeta_h = b^{(D+4-\eta)/2}$$

Constraints of Rotational Invariance

$$S_{uu}(\mathbf{q}) \sim q^{-(2-\eta_u)}; \quad \mathbf{u}(\mathbf{q}) = \zeta_u \mathbf{u}(b\mathbf{q}) \Rightarrow \zeta_u = b^{(D+2+\eta_u)/2}$$

$$\mathbf{h}(\mathbf{x}) = \int \frac{d^D q}{(2\pi)^D} e^{i\mathbf{q}\cdot\mathbf{x}} \mathbf{h}(\mathbf{q}) = \int \frac{d^D q'}{(2\pi)^D} b^{-D} e^{i\mathbf{q}'\cdot(\mathbf{x}/b)} \zeta_h \mathbf{h}(\mathbf{q}') = b^{-(D-4+\eta)/2}\mathbf{h}(\mathbf{x}\,/\,b)$$

$$\mathbf{u}(\mathbf{x}) = b^{-(D-2-\eta_u)/2}\mathbf{u}(\mathbf{x}\,/\,b) \Rightarrow S_{hh}(\mathbf{x},0) \sim |\,\mathbf{x}\,|^{-(D-4+\eta)};\ S_{uu}(\mathbf{x},0) \sim |\,\mathbf{x}\,|^{-(D-2-\eta_u)}$$

Rotational invariance requires all terms in H scale the same way

$$\mathcal{H}_{\rm nl} \approx \tfrac{1}{2}\int d^D x[2\mu(\partial_\alpha u_\beta \partial_\alpha u_\beta + \partial_\alpha u_\beta \partial_\alpha \mathbf{h}\cdot\partial_\beta \mathbf{h} + \tfrac{1}{4}(\partial_\alpha \mathbf{h}\cdot\partial_\beta \mathbf{h})^2 + \ldots]$$

$$(\partial u)^2 \longrightarrow b^{-2} b^{-(D-2-\eta_u)} (\partial u)^2$$

$$\partial u(\partial h)^2 \longrightarrow b^{-3} b^{-(D-2-\eta_u)/2} b^{-(D-4+\eta)}\, \partial u(\partial h)^2$$

$$(\partial h)^4 \longrightarrow b^{-4} b^{-2(D-4+\eta)} (\partial h)^4$$

Equate
Exponents

$$\eta_u + 2\eta = 4 - D$$

Results

Expansion in $\varepsilon = 4 - D$:

$$\eta = \frac{12\,\varepsilon}{24 + d_c}; \quad \eta_u = \frac{d_c\,\varepsilon}{24 + d_c}; \quad d_c = d - D$$

Aronowitz and TCL, PRL **60**, 2132 (1988)

Self-Consistent Field theory, $D = 2, d = 3$

$$\eta = 0.821; \quad \eta_u = 0.358$$

Le Doussal, Radzhhovsky, PRL **69**, 1209 (1992)

Comments: κ stiffens and μ softens with wavenumber

$$\kappa(q) \sim (\xi_G q)^{-\eta}$$

$$\mu(q) \sim (\xi_G q)^{\eta_u}$$

$\xi_G =$ Ginzburg length

$$x = L_x \tilde{x}; \quad u = L_u \tilde{u}; \quad h = u L_h \tilde{h}$$

$$\underline{u} \sim \partial u + (\partial h)^2 = L_x^{-1} L_u \tilde{\partial} \tilde{u} + L_x^{-2} L_h^2 (\tilde{\partial} \tilde{h})^2$$

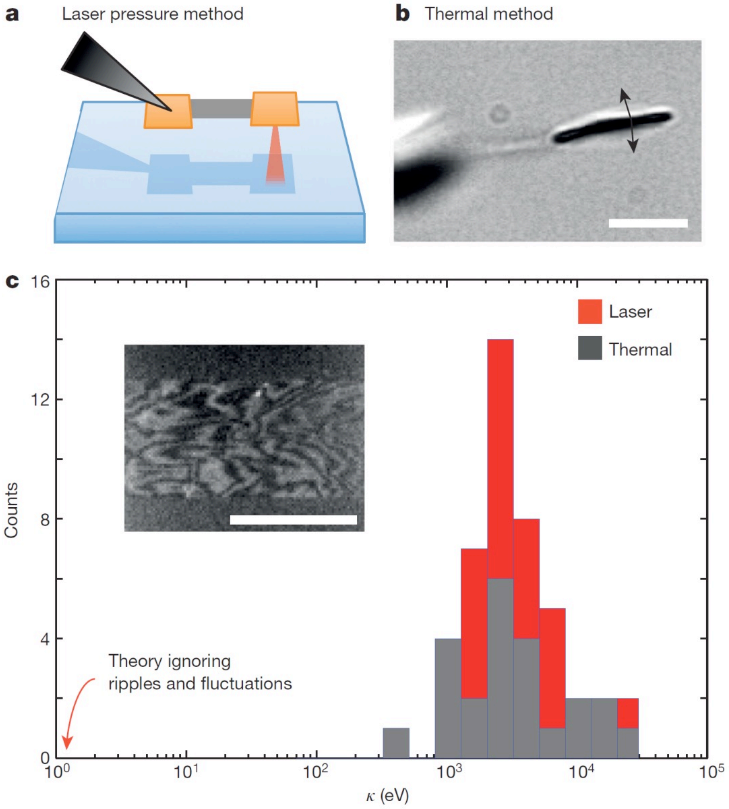
$$\underline{u} \sim L_x^{-1} L_u \underline{\tilde{u}} \text{ when } L_h^2 = L_x L_u$$

$$\int d^D x [\tfrac{\mu}{T} \underline{u}^2 + \tfrac{\kappa}{T} (\partial^2 h)^2] = \int d^D \tilde{x} L_x^D [L_x^{-2} L_u^2 \tfrac{\mu}{T} \underline{u}^2 + L_x^{-4} L_h^2 \tfrac{\kappa}{T} (\partial^2 h)^2]$$

Choose: $L_x^{D-2} L_u^2 \frac{\mu}{T} = L_x^{D-4} L_h^2 \frac{\kappa}{T} = 1 \Rightarrow L_x = \left(\frac{\kappa^2}{\mu T} \right)^{1/(4-D)} = \xi_G$

Experiments

Blees, … McEuen, Nature **524**, Issue 7564,204 (2015);
Kosmrlj and Nelson, PRB **93**, 125431 (2016);
Wan, Bowick,Nelson, PRB **96**, 014106 (2017)



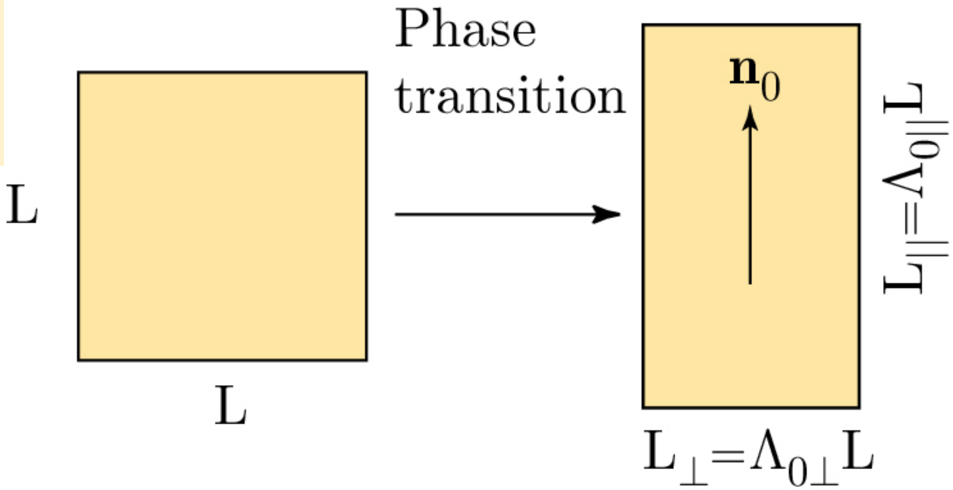
Measured κ is ~4000 times larger than microscopic value of κ_0=1.2 eV. Measurements done by direct measurements of displacement in response to force and of cantilever spring constant.

$$\frac{\kappa_{\text{eff}}}{\kappa_0} = \left(\frac{W}{\xi_G}\right)^{\eta}; \qquad \xi_G^2 = \frac{32\pi^2\kappa_0^2}{3Y_{2D}k_BT}$$

$$Y_{2D} = \frac{4B\mu}{B+\mu}: \quad 2D \text{ Young's modulus}$$

Spontaneous Symmetry Breaking

Phase transition to anisotropic state as μ goes to zero



$$\underline{u}_0 = \tfrac{1}{2}\left(\underline{\Lambda}_0^T \underline{\Lambda}_0 - \underline{\delta}\right)$$

$$\underline{\Lambda}_0 = \sqrt{\underline{\delta} + 2\underline{u}_0}$$

Direction of $\mathbf{n}_0$ is arbitrary

$$\begin{aligned}\tilde{u}_{\alpha\beta} &= \tilde{u}_{0\alpha\beta} \\ &= \Psi(n^0_\alpha n^0_\beta - \tfrac{1}{3}\delta_{\alpha\beta})\end{aligned}$$

Symmetric-Traceless part

$$u_{\alpha\alpha} \sim \Psi^2$$