Gravitational Radiation:

1. The Physics of Gravitational Waves and their Generation

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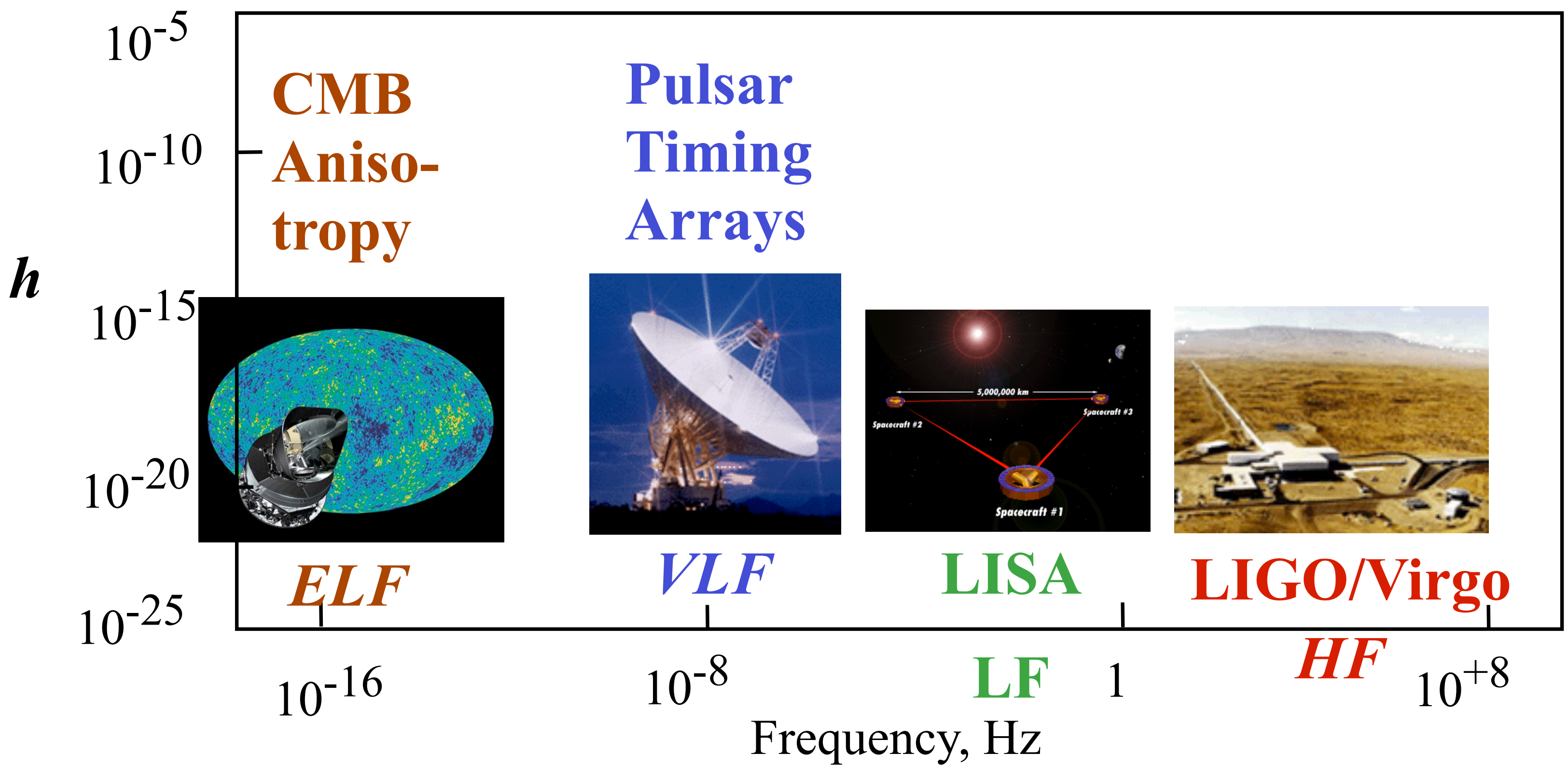
Lorentz Lectures, University of Leiden, September 2009

PDFs of lecture slides are available at
http://www.cco.caltech.edu/~kip/LorentzLectures/
each Thursday night before the Friday lecture

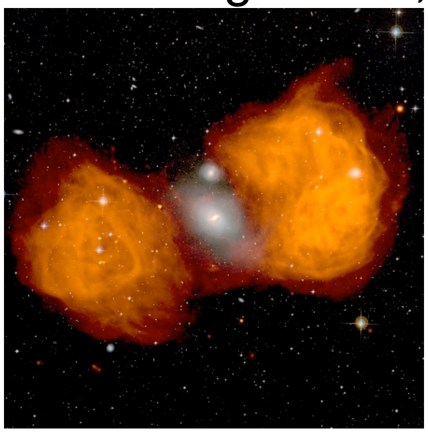
Introduction

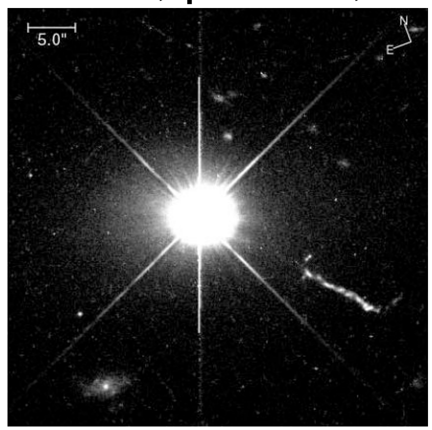
- **The Gravitational-Wave window onto the universe is likely to be opened**
  - in the next decade
  - in four widely different frequency bands, spanning 22 decades:

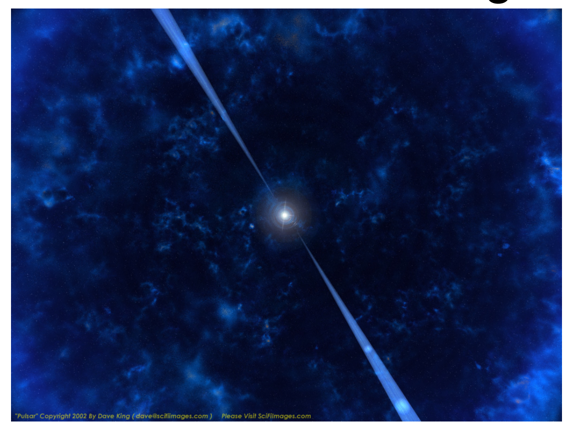


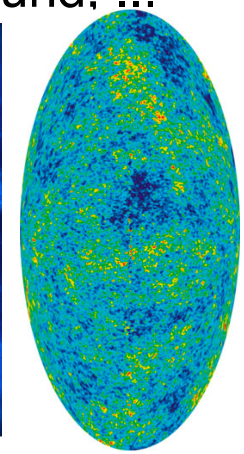
Radical New Windows ➠ Great Surprises

- **Radio Window: 1940s & 50s**
  - 10^4 x lower frequency than optical
  - radio galaxies, quasars, pulsars, cosmic microwave background, ...

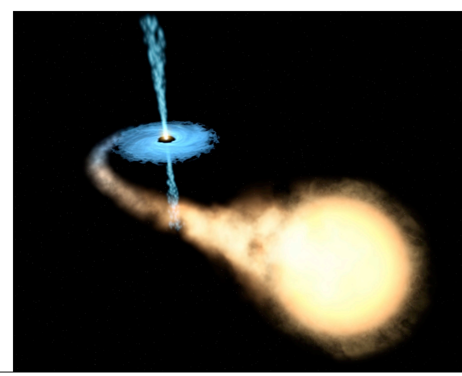


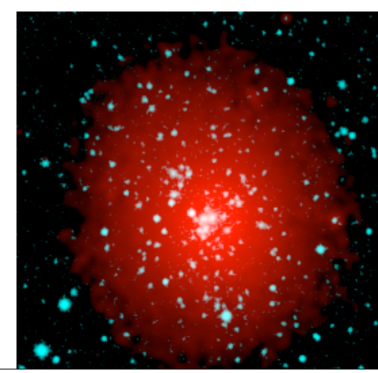






- **X-Ray Window: 1960s & 70s**
  - 10^3 x higher frequency than optical
  - black holes, accreting neutron stars, hot intergalactic gas, ...





Radical New Windows ➠ Great Surprises

- **Gravitational Waves are far more radical than Radio or X-rays**

Completely new form of radiation!

Frequencies to be opened span 22 decades

$$f_{HF} / f_{ELF} \sim 10^{22}$$

- **What will we learn from Gravitational Waves?**
  - ***"Warped side of the universe"***
    - our first glimpses, then in-depth studies
  - ***The nonlinear dynamics of curved spacetime***
  - ***Answers to astrophysical & cosmological puzzles:***
    - How are supernovae powered?
    - How are gamma-ray bursts powered?
    - What was the energy scale of inflation? ...
  - ***Surprises***

Growth of GW Community

- **1994: LIGO Approved for Construction: ~ 30 scientists**
- **Today: ~ 1500 scientists**
  - influx from other fields
  - needed for success
  - drawn by expected science payoffs

These Lectures

1. **The Physics of Gravitational Waves and their Generation**
   Today

2. **Astrophysical and Cosmological Sources of Gravitational Waves, and the Information they Carry**
   Next Friday, Sept 25

3. **Gravitational Wave Detection: Methods, Status, and Plans**
   Following Friday, Oct 2

These Lectures

- ***Prerequisites for these lectures:***
  - Knowledge of physics at advanced undergraduate level
  - Especially special relativity and Newtonian gravity
  - Helpful to have been exposed to General Relativity; not necessary
- ***Goals of these lectures:***
  - Overview of gravitational-wave science
  - Focus on physical insight, viewpoints that are powerful
- ***Pedagogical form of these lectures:***
  - Present key ideas, key results, without derivations
  - Give references where derivations can be found

These Lectures

- ***Pedagogical references that cover this lecture's material:***
  - **LH82**: K.S. Thorne, "Gravitational Radiation: An Introductory Review" in *Gravitational Radiation,* proceedings of the 1982 Les Houches summer school, eds. N. Deruelle and T. Piran (North Holland, 1983) - requires some knowledge of general relativity
  - **NW89**: K.S. Thorne, *Gravitational Waves: A New Window onto the Universe* (unpublished book, 1989), available on Web at http://www.its.caltech.edu/~kip/stuff/Kip-NewWindow89.pdf - does not require prior knowledge of general relativity, except in Chaps. 5 & 6.
  - **BT09**: R.D. Blandford and K.S. Thorne, Applications of Classical Physics (near ready for publication, 2009), available on Web at http://www.pma.caltech.edu/Courses/ph136/yr2008/ - contains an introduction to general relativity.

Resources

- ***The best introductory textbook on general relativity:***
  - James B. Hartle, *Gravity an Introduction to General Relativity* (Addison Wesley, 2003)

- ***The best course-length introduction to gravitational-wave science:***
  - *Gravitational Waves, a Web-Based Course* (including videos of lectures, readings, problem sets, problem solutions): http://elmer.caltech.edu/ph237/

Outline of This Lecture

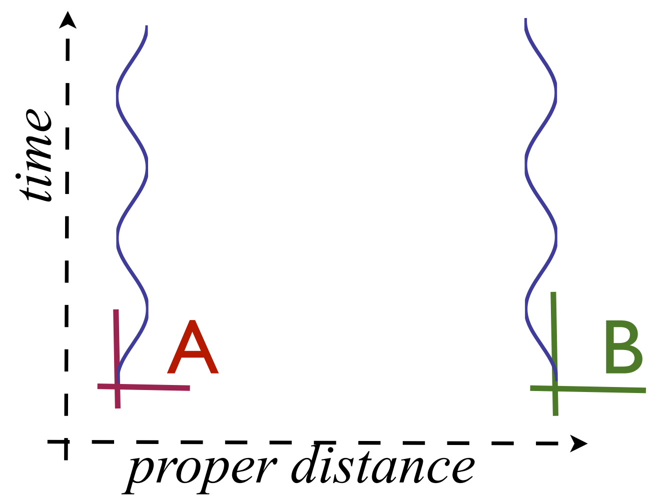
1. **Gravitational waves (GWs) in the language of tidal gravity**
2. **GWs in the linearized approximation to general relativity**
3. **GW generation**
   a. Linearized sources
   b. Slow-motion sources
   c. Nonlinear, highly dynamical sources: Numerical relativity
4. **GWs in curved spacetime; geometric optics; GW energy**
5. **Interaction of GWs with matter and EM fields**

1. Gravitational Waves in the Language of Tidal Gravity

Relative Motion of Inertial Frames

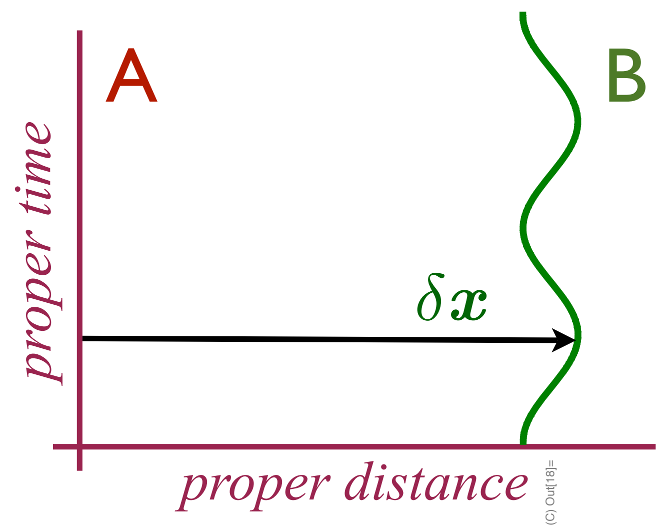
GW-induced motion of Local Inertial Frames

Non-meshing of local inertial frames ⇒ Curvature of Spacetime



Local Inertial Frame of A

Tidal Gravity



$$\ddot{x}_j = \frac{1}{2}\ddot{h}^{\rm GW}_{jk} x_k$$

GW field

Riemann Tensor $-R_{jtkt}$

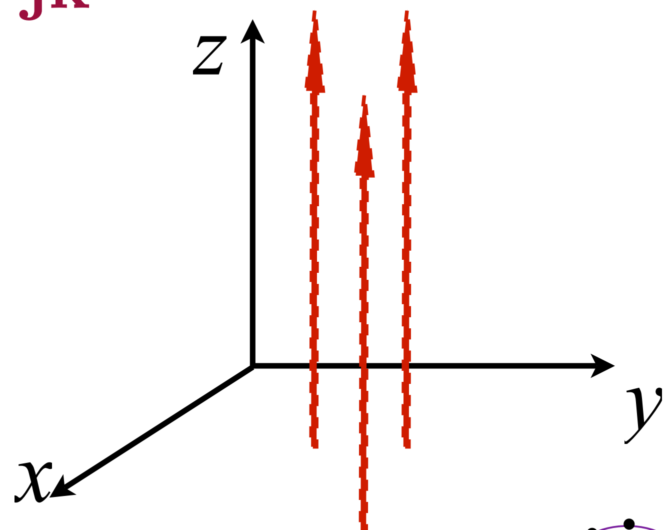
like $\ddot{x}_j = \dfrac{\partial g_j}{\partial x_k} x_k$ $\dfrac{-\partial^2 \Phi}{\partial x_j \partial x_k} x_k$

$$\delta x_j = \frac{1}{2} h^{\rm GW}_{jk} x_k$$

analogous to $E_j = -\dot{A}_j^{\rm T}$

(transverse Lorenz gauge)

The GW field $\mathbf{h}_{\mathbf{jk}}^{\mathbf{GW}}$



- The gravitational-wave field, h_{jk}^{GW}

Symmetric, transverse, traceless (TT); two polarizations: +, x

- \+ Polarization

$$h_{xx}^{\mathrm{GW}} = h_+(t - z/c) = h_+(t - z)$$

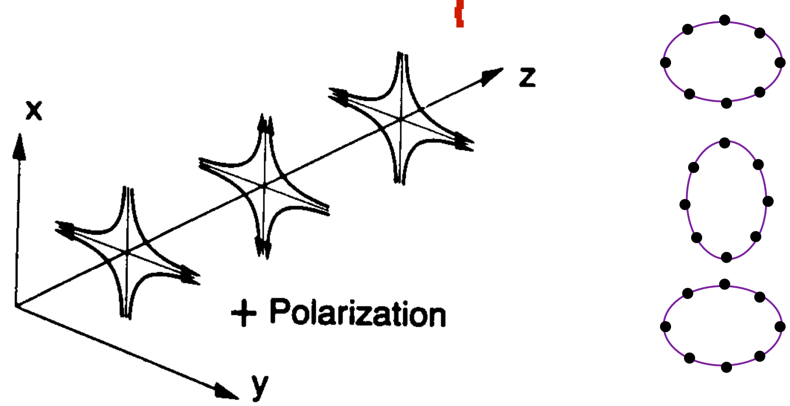
$$h_{yy}^{\mathrm{GW}} = -h_+(t - z)$$

Lines of force

$$\ddot{x}_j = \frac{1}{2}\ddot{h}_{jk}^{\mathrm{GW}} x_k$$

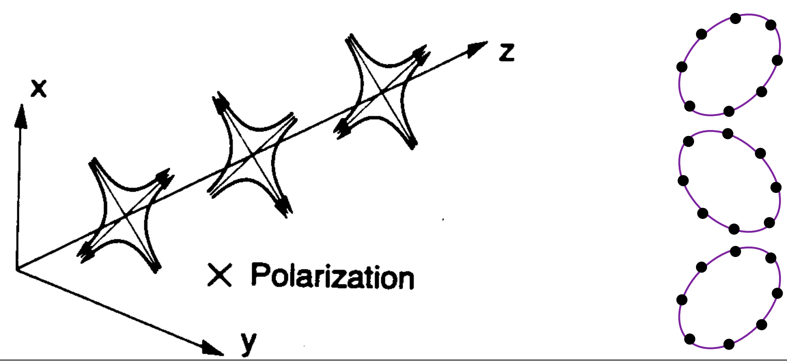
$$\ddot{x} = \ddot{h}_+ x$$

$$\ddot{y} = -\ddot{h}_+ y$$



- x Polarization

$$h_{xy}^{\mathrm{GW}} = h_{yx}^{\mathrm{GW}} = h_\times(t - z)$$

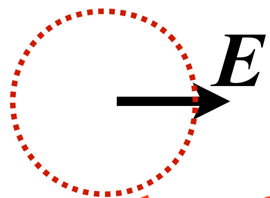


Gravitons

- **Quantum spin and rest mass: imprint on classical waves**

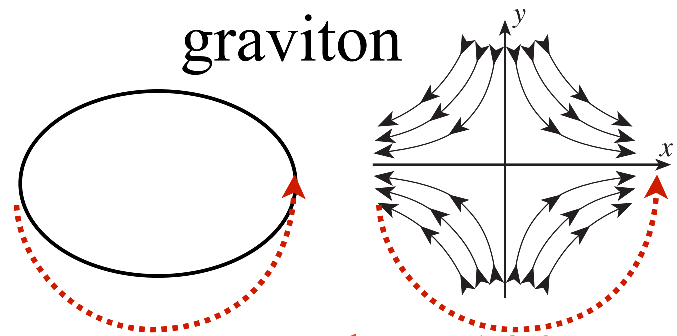
$$\text{spin} = \frac{180^{\text{o}}}{\text{return angle}}$$

photon



return angle = 360º
spin=1

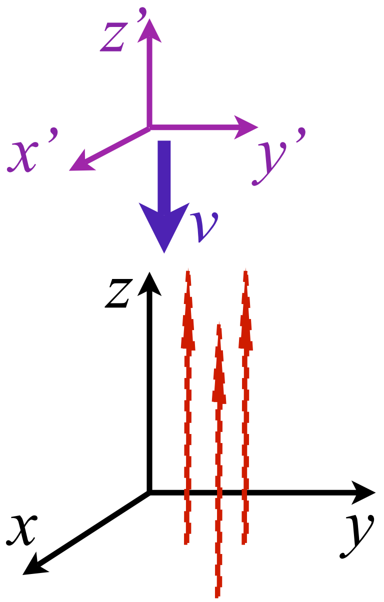
graviton



return angle = 180º
spin=2

propagation speed $= c \equiv 1 \quad \Rightarrow \quad$ rest mass $= 0$

Behavior of $\mathbf{h}_{\mathbf{jk}}^{\mathbf{GW}}$ Under boosts in z direction



$$t - z = \mathcal{D}(t' - z')$$

$$\mathcal{D} = \sqrt{\frac{1+v}{1-v}}$$

GW Field

$$h_{jk}^{\rm GW},\ h_+,\ h_\times$$

transform as
scalar fields

$$h'^{\rm GW}_{jk}(t' - z')$$
$$= h_{jk}^{\rm GW}(t - z)$$
$$= h_{jk}^{\rm GW}\left(\mathcal{D}(t' - z')\right)$$

Riemann amplified

$$R'_{jtkt} = \frac{1}{2}\ddot{h}'^{\rm GW}_{jk}$$
$$= \mathcal{D}^2 R_{jtkt}$$

boost weight 2

EM waves in
Transverse Lorenz gauge

$A_j^{\rm T}$ is transverse:

$$A_x^{\rm T}(t - z),\ A_y^{\rm T}(t - z)$$

transform as
scalar fields

$$A'^{\rm T}_j(t' - z')$$
$$= A_j^{\rm T}\left(\mathcal{D}(t' - z')\right)$$

Electric field amplified

$$E'_j = -\dot{A}_j^{\rm T}$$
$$= -\mathcal{D}E_j$$

boost weight 1

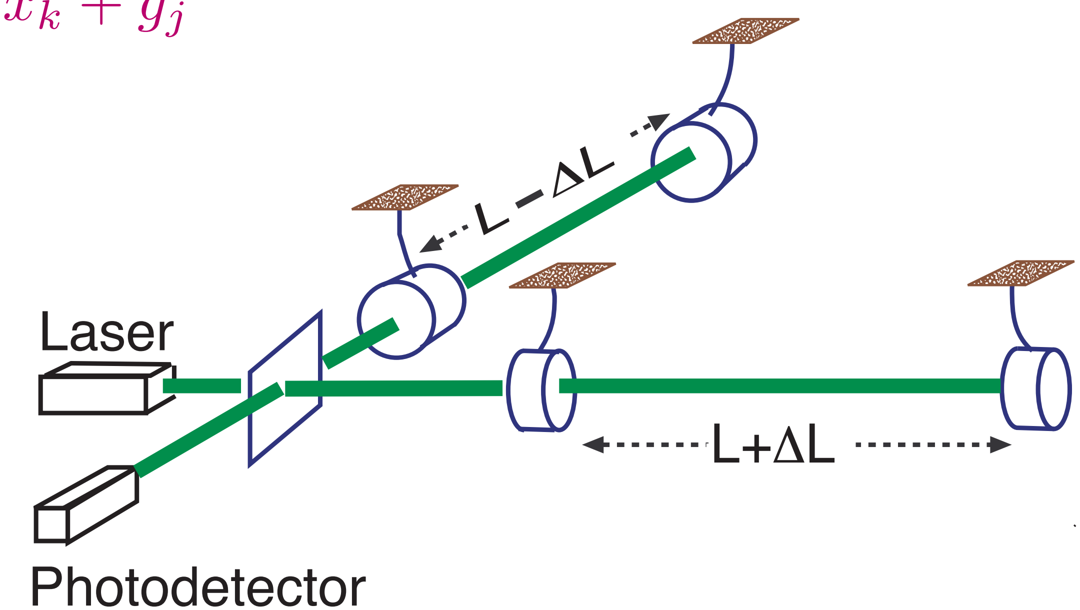
GWs as Seen in Laboratory on Earth

- **Proper Reference Frame:** analog of local Lorentz frame

$$ds^2 = -(1 + 2\mathbf{g} \cdot \mathbf{x})dt^2 + dx^2 + dy^2 + dz^2$$

- **GWs unaffected by earth's gravity**
  - except for a very tiny, unimportant gravitational blue shift
- **Total gravitational force**

$$\ddot{x}_j = \frac{1}{2}\ddot{h}^{\mathrm{GW}}_{jk} x_k + g_j$$



2. GWs in Linearized Approximation to General Relativity

Metric Perturbation, Lorenz Gauge, Einstein Field Equation

- **Metric:** $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

  Flat

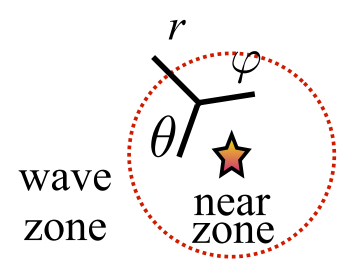
  Grav'l Field

- **Field theory in flat spacetime** $h_{\mu\nu}$ analogous to A_μ
- **Gauge freedom (ripple coordinates)** $x^\mu_{\rm new} = x^\mu_{\rm old} - \xi^\mu$

  $$h^{\rm new}_{\mu\nu} = h^{\rm old}_{\mu\nu} + \frac{\partial \xi_\mu}{\partial x^\nu} + \frac{\partial \xi_\nu}{\partial x^\mu} \text{ analogous to } A^{\rm new}_\mu = A^{\rm old}_\mu + \frac{\partial \phi}{\partial x_\mu}$$

- **Lorenz gauge** $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h^\alpha{}_\alpha \eta_{\mu\nu}$

  $$\frac{\partial \bar{h}^{\mu\nu}}{\partial x^\nu} = 0 \text{ analogous to } \frac{\partial A^\nu}{\partial x^\nu} = 0$$



- **Einstein field equation in Lorenz gauge**

  $$\Box \bar{h}^{\mu\nu} \equiv \eta^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\beta} \bar{h}^{\mu\nu} = -16\pi G T^{\mu\nu} \text{ analogous to } \Box A^\mu = -4\pi J^\mu$$

- **Gravitational-wave field**

  In wave zone, gauge change with $\Box \xi_\alpha = 0$ (analogous to $\Box \phi = 0$) $\rightarrow$

  Project out TT piece; get GW field: $h^{\rm new}_{tt} = h^{\rm new}_{jt} = 0,\ h^{\rm new}_{jk} = (h^{\rm old}_{jk})^{\rm TT} = h^{\rm GW}_{jk}$

  where $(h^{\rm old}_{\theta\varphi})^{\rm TT} = h^{\rm old}_{\theta\varphi} = h_\times,\ (h^{\rm old}_{\theta\theta})^{\rm TT} = h^{\rm old}_{\theta\theta} - \frac{1}{2}(h_{\theta\theta} + h_{\varphi\varphi}) = \frac{1}{2}(h_{\theta\theta} - h_{\varphi\varphi}) = h_+,$

  analogous to obtaining Transverse Lorenz gauge by projecting: $A^{\rm T}_j = (A^{\rm old}_j)^{\rm T}$

3. Gravitational Wave Generation

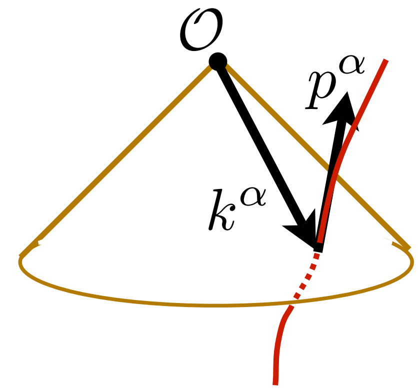
Example: Linearized, Point Particles in Lorenz Gauge

- **Electromagnetic**

$$\Box A^\alpha = -4\pi J^\alpha \Rightarrow \text{ at } \mathcal{O},\ A^\alpha = \frac{q\,p^\alpha}{k_\mu p^\mu}$$

in rest frame of particle, reduces to $A^t = \dfrac{q}{r}$

In wave zone $E_j = -(\dot{A}_j)^{\mathrm{T}}$ (Liénard-Wiechart)



- **Gravitational**

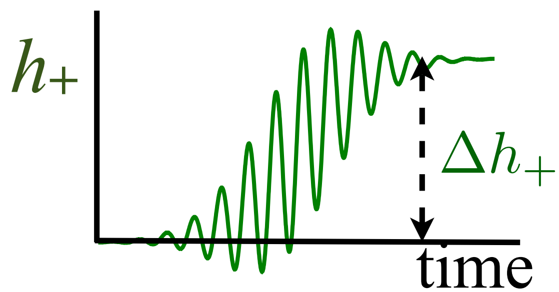
$$\Box \bar{h}^{\alpha\beta} = -16\pi G T^{\alpha\beta} \Rightarrow \text{ at } \mathcal{O},\ \bar{h}^{\alpha\beta} = G\frac{p^\alpha p^\beta}{k_\mu p^\mu}$$

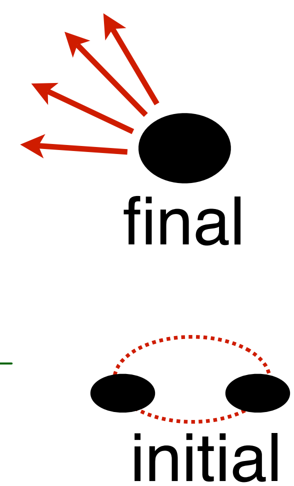
in rest frame of particle, reduces to $\bar{h}^{tt} = \dfrac{4Gm}{r}$

In wave zone $h^{\mathrm{GW}}_{jk} = (\bar{h}_{jk})^{\mathrm{TT}} = G\left(\dfrac{p^j p^k}{k_\mu p^\mu}\right)^{\mathrm{TT}}$

- **Gravitational-Wave Memory**

$$\Delta h^{\mathrm{GW}}_{jk} = G\left(\Delta \sum_A \frac{4\,p^j_A p^k_A}{k_{A\,\mu} p^\mu_A}\right)^{\mathrm{TT}}$$

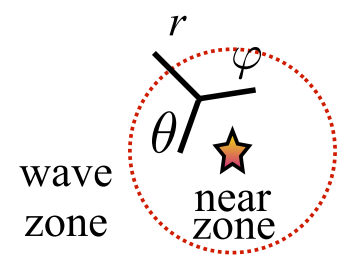




Slow-Motion GW Sources

Slow Motion: speeds << $c = 1$; wavelength = λ >> (source size) = L

examples: me waving arms; pulsar (spinning neutron star); binary made from two black holes



- **Weak-field, near zone:** ***Newtonian Potential***

$$\Phi = -G\frac{m}{r} \;\&\; G\frac{\text{mass dipole}}{r^2} \;\&\; G\frac{\text{mass quadrupole}}{r^3} \;\&\; \ldots$$

- **Wave zone:** $h_{jk}^{\rm GW} \sim \frac{1}{r}$ [by energy conservation] and dimensionless $\Rightarrow$

$$h_{jk}^{\rm GW} \sim G\frac{m}{r} \;\&\; G\frac{\partial(\text{mass dipole})/\partial t}{r} \;\&\; G\frac{\partial^2(\text{mass quadrupole})/\partial t^2}{r} \;\&\; \ldots$$

momentum; cannot oscillate

mass; cannot oscillate

canonical field theory ⇒ radiation field carried by quanta with spin s has multipoles confined to $\ell \geq s$

Mass quadrupole dominates

$$h_{jk}^{\rm GW} = 2G\left(\frac{\ddot{\mathcal{I}}_{jk}}{r}\right)^{\rm TT} \quad \text{for Newtonian source } \mathcal{I}_{jk} = \int \rho\left(x^j x^k - \frac{1}{3}r^2\right)d^3x$$

Common Textbook Derivation

1. Linearized Approximation to General Relativity (set G=1)

$$\Box \bar{h}^{jk} = -16\pi \bar{h}^{jk} \;\Rightarrow\; \bar{h}^{jk}(t,\mathbf{x}) = 4\int \frac{T^{jk}(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$$

$$\text{slow motion} \;\Rightarrow\; \bar{h}^{jk}(t,\mathbf{x}) = \frac{4\int T^{jk}(t-r,|\mathbf{x}')d^3x'}{r}$$

2. Conservation of 4-momentum $T^{\alpha\beta}{}_{,\beta} = 0 \;\Rightarrow$

$$2T^{jk} = (T^{tt}x^j x^k)_{,tt} - (T^{ab}x^j x^k)_{,ab} - 2(T^{aj}x^k + T^{ak}x^j)_{,a}$$

3. Insert 2 into 1; integral of divergence vanishes

$$\bar{h}^{jk}(t,\mathbf{x}) = \frac{2\ddot{I}^{jk}(t-r)}{r}, \text{ where } I^{jk} = \int T^{00}x^j x^k d^3x$$

4. Take transverse traceless part

$$\bar{h}^{\rm GW}_{jk}(t,\mathbf{x}) = \frac{2\left(\ddot{\mathcal{I}}_{jk}(t-r)\right)^{\rm TT}}{r}, \text{ where } \mathcal{I}^{jk} = \int T^{00}(x^j x^k - r^2\delta^{jk})d^3x$$

PROBLEM: Derivation Not valid when self gravity influences source's dynamics!!

Derivation via Propagation from Weak-Field Near Zone to Wave Zone

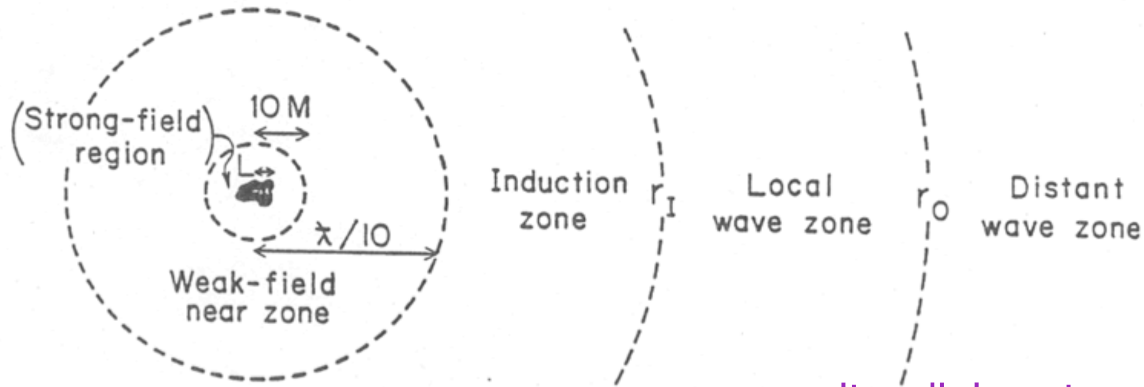
- **Weak-gravity regions: Linearized approximation to GR**

$$\Box h^{\alpha\beta} = 0$$

$$\bar{h}^{\alpha\beta}{}_{,\beta} = 0$$

i.e. $\bar{h}^{tt}{}_{,t} = -\bar{h}^{tj}{}_{,j}$

$\bar{h}^{tj}{}_{,t} = -\bar{h}^{jk}{}_{,k}$



In weak-field near zone (wfnz) $\Phi = -\frac{M}{r} - \frac{3\mathcal{I}_{jk}n^j n^k}{2r^3}$, $\bar{h}^{tt} = -4\Phi$

unit radial vector

- **Quadrupolar solution in Induction Zone:**

$$\bar{h}^{tt} = 2\left[\frac{1}{r}\mathcal{I}_{jk}(t-r)\right]_{,jk} \simeq \frac{6}{r^3}\mathcal{I}_{jk}n^j n^k \text{ in wfnz}, \quad \simeq \frac{2}{r}\ddot{I}_{jk}n^j n^k \text{ in lwz}$$

pure gauge

$$\bar{h}^{tj} = 2\left[\frac{1}{r}\dot{\mathcal{I}}_{jk}(t-r)\right]_{,k} \simeq \frac{-2}{r^2}\dot{\mathcal{I}}_{jk}n^k \text{ in wfnz}, \quad \simeq \frac{-2}{r^2}\ddot{\mathcal{I}}_{jk}(t-r)n^l \text{ in lwz}$$

tiny

pure gauge

$\bar{h}^{jk} = \frac{2}{r}\ddot{\mathcal{I}}_{jk}(t-r)$ Take TT part to get GW field in wfnz: $\bar{h}^{\rm GW}_{jk}(t,\mathbf{x}) = \frac{2\left(\ddot{\mathcal{I}}_{jk}(t-r)\right)^{\rm TT}}{r}$

h_{jk}^{GW} Order of Magnitude

- **Source parameters:**

mass ~ M, size ~ L, rate of quadrupolar oscillations ~ ω, distance ~ r, internal kinetic energy of quadrupolar oscillations ~ $E_{kin} \sim M(\omega L)^2$

- **GW strength:**

$$h_{jk}^{GW} = 2G\frac{\ddot{\mathcal{I}}_{jk}}{r} \sim G\frac{\omega^2(ML^2)}{r} \sim G\frac{E_{kin}/c^2}{r}$$

$\sim \Phi$ produced by kinetic energy of shape changes.

$$h_{jk}^{GW} \sim h_+ \sim h_\times \sim 10^{-21}\left(\frac{E_{kin}}{M_\odot c^2}\right)\left(\frac{100\text{Mpc}}{r}\right)$$

$$100\text{Mpc} = 300 \text{ million light years} \sim \frac{1}{30}(\text{Hubble distance})$$

Slow-Motion Sources: Higher-Order Corrections

- **Source Dynamics: Post-Newtonian Expansion**

  $$\text{in } v/c \sim \sqrt{\Phi/c^2} \sim \sqrt{P/\rho c^2}$$

- **GW Field: Higher-Order Moments (octopole, ...)**
  - computed in same manner as quadrupolar waves: by analyzing the transition from weak-field near zone, through induction zone, to local wave zone
  - actually Two families of moments (like electric and magnetic) -
    - moments of mass distribution, moments of angular-momentum distribution
  - Use symmetric, trace-free (STF) tensors to describe the moments and the GW field ... (19th century approach; Great Computational Power)

STF Tensors [an aside]

- **Multipole moments of Newtonian gravitational potential**
  - Spherical-harmonic description: $\Phi \sim \sum_{m=-\ell}^{+\ell} \frac{\mathcal{M}_{\ell m} Y_{\ell m}(\theta\phi)}{r^{\ell+1}}$

    $\mathcal{M}_{\ell m}$ has $2\ell+1$ components: $m = -\ell, \ell+1, ..., +\ell$
  - STF description: $\Phi \sim \frac{\mathcal{I}_{a_1 a_2 \ldots a_\ell} n^{a_1} n^{a_2} \ldots n^{a_\ell}}{r^{\ell+1}}$

    $\mathcal{I}_{a_1 a_2 \ldots a_\ell}$ has $2\ell+1$ independent components

- **Multipolar Expansion of gravitational-wave field**

$$h_{jk}^{\rm GW} = \left\{ \sum_{\ell=2}^{\infty} \frac{4}{\ell!} \frac{\partial^\ell}{\partial t^\ell} \frac{\mathcal{I}_{jka_1\ldots a_{\ell-2}}(t-r)}{r} n^{a_1} \ldots n^{a_{\ell-2}} \right\}^{\rm TT}$$

$$+ \left\{ \sum_{\ell=2}^{\infty} \frac{8\ell}{(\ell+1)!} \epsilon_{pq(j} \frac{\partial^\ell}{\partial t^\ell} \frac{\mathcal{S}_{k)pa_1\ldots a_{\ell-1}}(t-r)}{r} n^q n^{a_1} \ldots n^{a_{\ell-1}} \right\}^{\rm TT}$$

mass moments

angular momentum moments
current moments

  - Indices carry directional, multipolar and tensor information, all at once

Strong-Gravity $(GM/c^2L \sim 1)$, Fast-Motion $(v \sim c)$ Sources

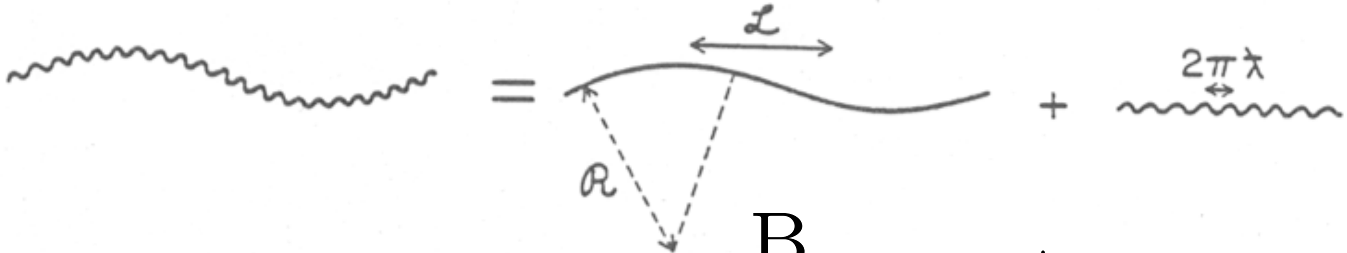
- **Most important examples [next week]**
  - Black-hole binaries: late inspiral, collision, merger, ringdown
  - Black-hole / neutron-star binaries: late inspiral, tidal disruption and swallowing
  - neutron star / neutron-star binaries: late inspiral, collision and merger
  - supernovae
- **These are the strongest and most interesting of all sources**
- **Slow-motion approximation fails**
- **Only way to compute waves:** ***Numerical Relativity***

4. Gravitational Waves in curved spacetime; geometric optics; GW energy

GWs Propagating Through Curved Spacetime (distant wave zone)

- **Definition of gravitational wave: the rapidly varying part of the metric and of the curvature**

$$\lambdabar = \lambda/2\pi \ll \mathcal{L} = (\text{lengthscale on which background metric varies}) \lesssim \mathcal{R}$$



$$g_{\alpha\beta} = g^{\rm B}_{\alpha\beta} + h_{\alpha\beta}$$

$g^{\rm B}_{\alpha\beta} \equiv \langle g_{\alpha\beta} \rangle$; $h_{\alpha\beta} \equiv g_{\alpha\beta} - \langle g_{\alpha\beta} \rangle$

Same definition used for waves in plasmas, fluids, solids

- **In local Lorentz frame of background: GW theory same as in flat spacetime (above)**

$\Box h_{\alpha\beta} = 0$ in vacuum. Same propagation equation as for EM waves: $\Box A_\alpha = 0$

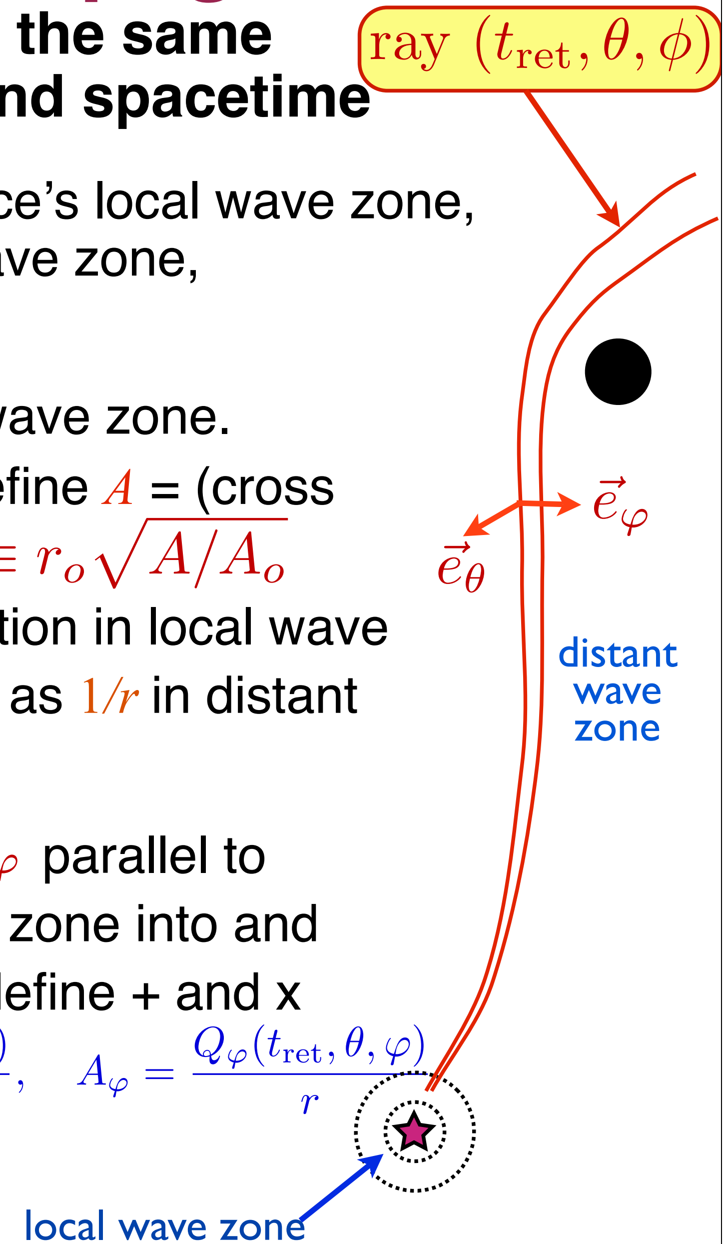
⇒ GWs exhibit same geometric-optics behavior as EM waves

Geometric-Optics Propagation

- **GWs and EM waves propagate along the same rays: null geodesics in the background spacetime**

  - Label each ray by its direction (θ, φ) in source's local wave zone, and the retarded time it has in the local wave zone,
    $t_{\rm ret} \equiv (t-r)_{\rm local\ wave\ zone}$
  - Wave's amplitude dies out as $1/r$ in local wave zone. Along the the ray, in distant wave zone, define A = (cross sectional area of a bundle of rays, and $r \equiv r_o \sqrt{A/A_o}$ where r_o and A_o are values at some location in local wave zone. Then amplitude continues to die out as $1/r$ in distant wave zone.
  - Transport the unit basis vectors $\vec{e}_\theta$ and $\vec{e}_\varphi$ parallel to themselves along the ray, from local wave zone into and through distant wave zone. Use them to define + and x
  - Then in distant wave zone: $A_\theta = \frac{Q_\theta(t_{\rm ret},\theta,\varphi)}{r}, \quad A_\varphi = \frac{Q_\varphi(t_{\rm ret},\theta,\varphi)}{r}$

$$h_+ = \frac{Q_+(t_{\rm ret},\theta,\varphi)}{r}, \quad h_\times = \frac{Q_\times(t_{\rm ret},\theta,\varphi)}{r}$$

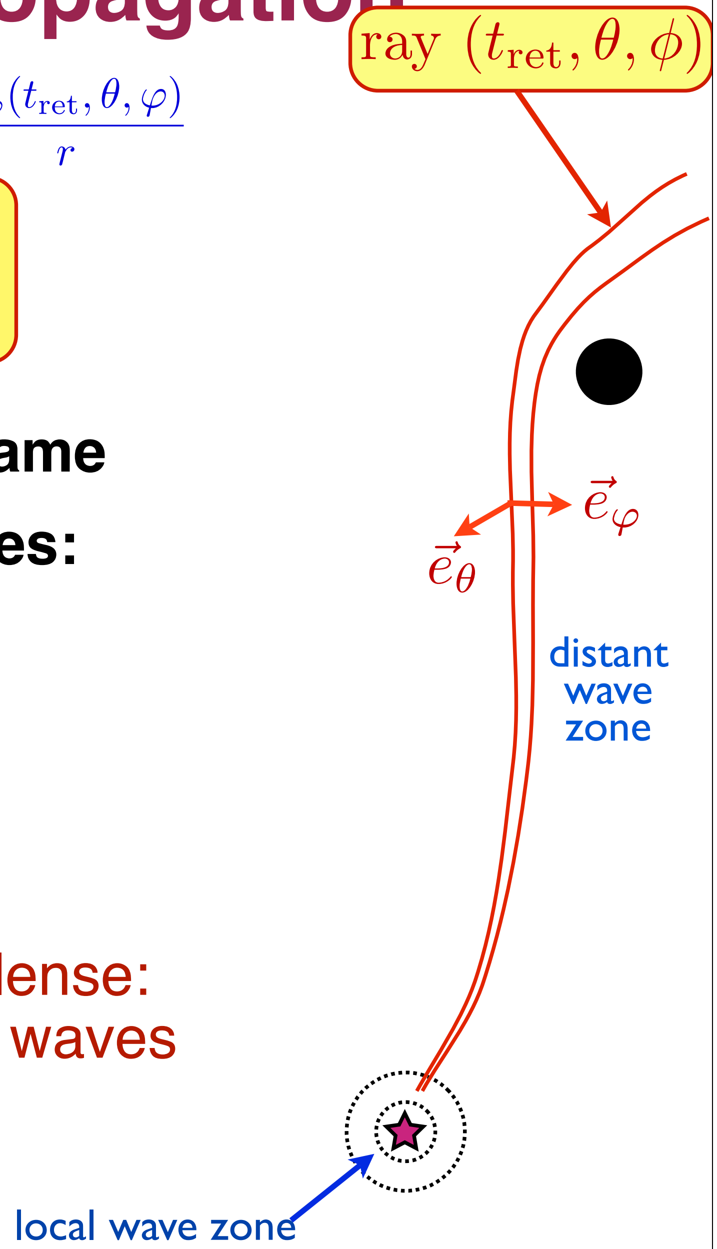


Geometric-Optics Propagation

- **Form of waves:** $A_\theta = \frac{Q_\theta(t_{\rm ret}, \theta, \varphi)}{r}, \quad A_\varphi = \frac{Q_\varphi(t_{\rm ret}, \theta, \varphi)}{r}$

$$h_+ = \frac{Q_+(t_{\rm ret}, \theta, \varphi)}{r}, \quad h_\times = \frac{Q_\times(t_{\rm ret}, \theta, \varphi)}{r}$$

- **⇒ GWs experience identically the same geometric-optics effects as EM waves:**
  - gravitational redshift,
  - cosmological redshift,
  - gravitational lensing, ...
    - ▸ and at the focus of a gravitational lense: the same diffraction effects as EM waves



Energy and Momentum in GWs

- **Einstein's general relativity field equations say**

$$G_{\alpha\beta} = 8\pi \; G \; T_{\alpha\beta}$$

- **Break metric into background plus GW:** $g_{\alpha\beta} = g^B_{\alpha\beta} + h_{\alpha\beta}$

- **Expand Einstein tensor in powers of** $h_{\alpha\beta}$

$$G_{\alpha\beta} = G^B_{\alpha\beta} + G^{(1)}_{\alpha\beta} + G^{(2)}_{\alpha\beta}$$

- **Average over a few wavelengths to get quantities that vary on background scale $\mathcal{L}$, not wavelength scale $\lambda\!\!\!^{-}$**

$$\langle G_{\alpha\beta} \rangle = G^B_{\alpha\beta} + \langle G^{(2)}_{\alpha\beta} \rangle = 8\pi \langle T_{\alpha\beta} \rangle$$

- **Rearrange:** $G^B_{\alpha\beta} = 8\pi(\langle T_{\alpha\beta}\rangle + T^{\rm GW}_{\alpha\beta})$ where $T^{\rm GW}_{\alpha\beta} \equiv -\frac{\langle G^{(2)}_{\alpha\beta}\rangle}{8\pi}$

- **Evaluate the average:** $$T^{\rm GW}_{\alpha\beta} = \frac{1}{16\pi}\langle h_{+,\alpha} h_{+,\beta} + h_{\times,\alpha} h_{\times,\beta} \rangle$$

- **In Local Lorentz Frame**

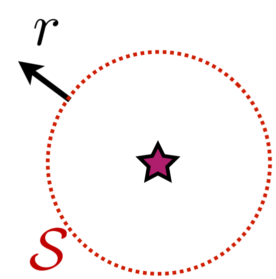
$$T^{tt}_{\rm GW} = T^{tz}_{\rm GW} = T^{zz}_{\rm GW} = \frac{1}{16\pi}\langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle \; .$$

Energy and Momentum Conservation

- **Einstein's field equations** $G^B_{\alpha\beta} = 8\pi(\langle T_{\alpha\beta}\rangle + T^{\rm GW}_{\alpha\beta})$ **guarantee energy and momentum conservation, e.g.**

- **Source loses mass (energy) at a rate**

$$\frac{dM}{dt} = -\oint_{\mathcal{S}} T^{tr}_{\rm GW} dA = -\frac{1}{16\pi}\oint_{\mathcal{S}} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle dA$$



- **Source loses linear momentum at a rate**

$$\frac{dp^j}{dt} = -\oint_{\mathcal{S}} T^{jr}_{\rm GW} dA = -\frac{1}{16\pi}\oint_{\mathcal{S}} (\vec{e}_j \cdot \vec{e}_r)\langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle dA$$

- **Angular momentum is a little more delicate**

5. Interaction of Gravitational Waves with matter and EM fields

Plane GW Traveling Through Homogeneous Matter

- **Fluid:**
  - GW shears the fluid, (rate of shear) = $\sigma_{jk} = \frac{1}{2}\dot{h}^{\rm GW}_{jk}$
  - no resistance to shear, so no action back on wave
  - Viscosity $\eta \sim \rho v s = \text{(density)(mean speed of particles)(mean free path)}$
    produces stress $T_{jk} = -2\eta\sigma_{jk} = -\eta\dot{h}^{\rm GW}_{jk}$ **NOTE:** s must be $< \lambda\!\!\!^{-}$
  - Linearized Einstein field equation: $\Box h^{\rm GW}_{jk} = -16\pi(T_{jk})^{\rm TT} = 16\pi\eta\dot{h}^{\rm GW}_{jk}$
  - Wave attenuates: $h^{\rm GW}_{jk} \sim \exp(-z/\ell_{\rm att})$ where $\ell_{\rm att} = \frac{1}{8\pi\eta} = \frac{1}{8\pi\rho v s}$
  - Fluid's density curves spacetime (background Einstein equations)
    $\frac{1}{\mathcal{R}^2} \sim G^B_{00} = 8\pi\rho$
  - Therefore $\ell_{\rm att} \sim \frac{\mathcal{R}^2}{vs} = \mathcal{R}\frac{\mathcal{R}}{s}\frac{c}{v} \gtrsim \mathcal{R}\frac{\mathcal{R}}{\lambda\!\!\!^{-}}\frac{c}{v} \gg \mathcal{R}$

The viscous attenuation length is always far larger than the background radius of curvature. Attenuation is never significant!

Plane GW Traveling Through Homogeneous Matter

- **Elastic Medium:**
  - GW shears the medium, (rate of shear) = $\sigma_{jk} = \frac{1}{2}\dot{h}_{jk}^{\rm GW}$, (shear)=$\Sigma_{jk} = \frac{1}{2}h_{jk}^{\rm GW}$
  - Medium resists with stress $T_{jk} = -2\mu\Sigma_{jk} - 2\eta\sigma_{jk} = -\mu h_{jk}^{\rm GW} - \eta\dot{h}_{jk}^{\rm GW}$
  - Einstein equation becomes $\Box h_{jk}^{\rm GW} = -16\pi(T_{jk})^{\rm TT} = 16\pi(\mu h_{jk}^{\rm GW} + \eta\dot{h}_{jk}^{\rm GW})$
  - Insert $h_{jk}^{\rm GW} \propto \exp(-i\omega t + ikz)$. Obtain dispersion relation

    $\omega^2 - k^2 = 16\pi(\mu - i\omega\eta);$ i.e. $\omega = k(1 + 8\pi\lambdabar^2\mu) - i8\pi\eta,$ where $\lambdabar = 1/k$
  - Same attenuation length as for fluid: $\ell_{\rm att} = \frac{1}{8\pi\eta} \gg \mathcal{R}$
  - Phase and group velocities (dispersion):

    $v_{\rm phase} = \frac{\omega}{k} = 1 + 8\pi\lambdabar^2\mu, \quad v_{\rm group} = \frac{d\omega}{dk} = 1 - 8\pi\lambdabar^2\mu$
  - Dispersion length (one radian phase slippage) $\ell = \frac{\lambdabar}{\delta v_{\rm phase}} = \frac{1}{8\pi\lambdabar\mu} \gtrsim \frac{\mathcal{R}^2}{\lambdabar} \gg \mathcal{R}$

The dispersion length is always far larger than the background radius of curvature. Dispersion is never significant!

GW Scattering

- **Strongest scattering medium is a swarm of black holes: hole mass M, number density of holes n**
  - Scattering cross section $\sigma \lesssim M^2$
  - Graviton mean free path for scattering

$$\ell = \frac{1}{n\sigma} \gtrsim \frac{1}{nM^2} = \frac{1}{\rho M} \sim \frac{\mathcal{R}^2}{M} \gg \mathcal{R}$$

The scattering mean free path is always far larger than the background radius of curvature. Scattering is never significant!

Interaction with an Electric or Magnetic Field

- **Consider a plane EM wave propagating through a DC magnetic field** $\mathrm{B}_{\mathrm{wave}} = B_o \sin[\omega(t-z)]\mathbf{e_y}, \quad \mathbf{B}_{\mathrm{DC}} = B_{\mathrm{DC}}\mathbf{e_y}$
- **Beating produces a TT stress** $T_{xx} = -T_{yy} = \frac{B_o B_{\mathrm{DC}}}{4\pi}$
- **TT stress resonantly generates a GW** $\Box h_{jk}^{\mathrm{GW}} = -16\pi (T_{jk})^{\mathrm{TT}}$

$$h_+ = h_{xx}^{\mathrm{GW}} = -h_{yy}^{\mathrm{GW}} = \frac{2B_{\mathrm{DC}}B_o}{\omega} z \cos[\omega(t-z)]$$ The "Gertsenshtein effect"

- **Ratio of GW energy to EM wave energy:**

$$\frac{T_{\mathrm{GW}}^{tt}}{T_{\mathrm{EMwave}}^{tt}} = \frac{\langle \dot{h}_+^2 \rangle / 16\pi}{B_o^2/8\pi} = B_{DC}^2 z^2 = \frac{z^2}{\mathcal{R}^2}$$

The lengthscale for significant conversion of EM wave energy into GW energy is equal to the radius of curvature of spacetime produced by the catalyzing DC magnetic field.

- **The lengthscale for the inverse process is the same**

There can never be significant conversion in the astrophysical universe.

Conclusion

- **Gravitational Waves propagate through the astrophysical universe without significant attenuation, scattering, dispersion, or conversion into EM waves**
- **Next Friday: Astrophysical and Cosmological Sources of Gravitational Waves, and the Information they Carry**
  - slides will be available Thursday night at http://www.cco.caltech.edu/~kip/LorentzLectures/