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Topics from 20^{th} century physics.
An introductory course for students in mathematics

II. CLASSICAL MECHANICS

1. HISTORICAL REMARKS

1.1. *Aristotelian physics*

Classical mechanics describes the motion of bodies under the influence of forces. Superficial observation leads to the impression that physical objects are normally in rest and start moving only when forces are acting on them. Their velocities seem moreover to increase when the forces increase. Aristotle, the Greek philospher who gave the first all-encompassing picture of the physical world based on empirical observation instead of pure speculation, followed this train of thought. One may – somewhat anachronistically – formulate his basic dynamical law of motion as

$$F = mv,$$

i.e. the *velocity* of a moving body is proportional to the force acting on it. One should add that to understand what happened when an object was thrown or was allowed to fall freely, it was necessary to devise special explanations, none of which seem to us very convincing now. The ideas of Aristotle dominated physics in the western world and in the world of Islam from classical antiquity until the end of the middle ages.

1.2. *Galileo and Newton*

If one realizes the importance of friction in the motion of bodies and observes situations in which friction is negligible – think of a stone moving on a surface of ice, a different picture emerges: an object which moves with constant velocity will persist in this motion when left alone. The consequences of such observations were first clearly understood by Galileo. He was led to the general *principle of inertia*, which we may formulate as:

A physical body which is free, i.e. on which no forces act, is either in rest or moves in a straight line with constant velocity.

This principle has remained the basis of classical mechanics. Galileo, who made many other important contributions to the new post-Aristotelian physics that arose in the sixteenth and seventeenth century, can be seen as the first representative of the method that led to the great successses of modern natural science: the combination of careful empirical observation with the use of precise mathematical models.

Another new and important insight that Galilei helped to establish was that the laws of physics are the same for events on earth and in the heavens. This made in a certain sense astronomy and in particular planetary motion into a part of mechanics, which greatly stimulated its further development. The heliocentric picture of the solar system had been put forward already by Copernicus. (Galileo was a strong defender of it, with very unpleasant consequences for him personally, as is well-known.) Thinking – more or less – in terms of Copernicus' model and using the precise numerical data on planetary positions, collected by Brahe in years of observation, Kepler was able to establish that the planets move in ellipses with the sun in one of the focal points. This set the stage for the fundamental work of Newton. Starting from the principle of inertia he developed mechanics as a complete mathematical theory for the description of the motion of physical bodies under the influence of forces, with as central dynamical law the formula

$$F = am,$$

stating that instead of the velocity the *acceleration*, i.e. the second derivative of the position with respect to time, should be proportional to the force, developing in the process differential and integral calculus as the necessary mathematical apparatus for this. Introducing a universal gravitational force between two arbitrary massive bodies, proportional to the product of the masses of the two bodies and inversely proportional to the square of their distance, he was able to obtain – as a first application of his general ideas – a precise description of the motion of the moon around the earth, essentially in terms of Kepler's laws of planetary motion.

Newton's mechanics was further developed mathematically during the eighteenth and nineteenth century, by mathematicians as Laplace, Lagrange, Hamilton and Poincaré, however with no changes in its basic laws. It remains today a lively subject of mathematical research, particularly as celestial mechanics, with many interesting unsolved problems. Its modern formulation is geometrical, in terms of vector fields on differential manifolds, in particular so-called symplectic manifolds. Nevertheless, as a part of physics, 'classical mechanics' is essentially complete, a theory belonging to nineteenth century physics. The reason classical mechanics is discussed here in some detail is that it is necessary for the understanding of much of twentieth century physics, in particular quantum mechanics, one of the two main topics of this course.

It should also be remarked that classical mechanics, like much of the physics from the end of the nineteenth century, still describes many of the physical phenomena around us with very high precision. It fails however in situations where velocities comparable with the velocity of light are involved, or in situations at the level of atoms. In the first case Newton's theory has to be replaced by Einstein's theory of relativity and in the second case classical mechanics is superseded by quantum mechanics.

2. NEWTONIAN CLASSICAL MECHANICS

2.1. *Newton's equations for a system of point particles*

Classical mechanics as it is taught nowadays to physics students is essentially Newton's mechanics, with some further developments that will be discussed in the next chapters. Consider the typical situation of a system of N point particles with masses $m_1, \ldots, m_N$, described by cartesian coordinates $\vec{r}_1, \ldots, \vec{r}_N$, $\vec{r}_j = (x_j, y_j, z_j)$, $j = 1, \ldots, N$. We assume that there are forces acting on the particles which come from a potential, i.e. the force $\vec{F}_j$ on the j^{th} particle is equal to

$$\vec{F}_j(\vec{r}_1,\ldots,\vec{r}_N) = -\frac{\vec{\partial}}{\partial \vec{r}_j} V(\vec{r}_1,\ldots,\vec{r}_N),$$

with $\frac{\vec{\partial}}{\partial \vec{r}_j}$ denoting the triple of partial differentiations $(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}, \frac{\partial}{\partial z_j})$, for $j = 1,\ldots,N$, and $V(\vec{r}_1,\ldots,\vec{r}_N)$ a given real function on R^{3N}, the potential energy of the system. The time evolution of the system is described by the N vector-valued functions $\vec{r}_j(t)$, which are solutions of *Newton's equations*, in this case the system of coupled second order ordinary differential equations

$$m_j \frac{d^2\vec{r}_j(t)}{dt^2} = -\frac{\vec{\partial}}{\partial \vec{r}_j} V(\vec{r}_1(t),\ldots,\vec{r}_N(t)),$$

for $j = 1,\ldots,N$. Such a classical mechanical system is *deterministic*: if we mean by the state of the system at time $t = t_1$ the $2N$ positions and velocity vectors $\vec{r}_j(t_1)$ and $\vec{v}_j(t_1) = \frac{d}{dt}\vec{r}_j(t_1)$, for $j = 1,\ldots,N$, then the state of the system at $t = t_1$ completely determines the state at a later time $t = t_2 > t_1$, because one can, for a sufficiently smooth potential function V, prove the existence of a unique solution for each given set of initial conditions $\vec{r}_j(t_1)$ and $\vec{v}_j(t_1)$. This does of course not mean that such a solution can always be found explicitly. For a system of two particles with masses m_1 and m_2, interacting through a potential

$$V(\vec{r}_1,\vec{r}_2) = -g\,\frac{m_1 m_2}{|\vec{r}_2 - \vec{r}_1|},$$

with g a constant, Newton's equations can be solved in closed form. This is of course the problem of the sun and a planet attracting each other by gravitation; the periodic solutions are Keppler's elliptic planetary orbits. For a similar system consisting of three bodies Newton's equations can not be solved; the solutions can be approximated to arbitrary precision.

2.2. *Newton's equations as a system of first order equations*

By using the velocities $\vec{v}_j(t)$ as independent functions, Newton's equations can be written as a system of $2N$ vector-valued or $6N$ real-valued first order equations. Write the position variables as $x_1,\ldots,x_n$ instead of $\vec{r}_1,\ldots,\vec{r}_N$, and the velocities as $x_{n+1},\ldots,x_{2n}$ instead of $\vec{v}_1,\ldots,\vec{v}_n$, with $n = 3N$. Newton's equations then take the form

$$\frac{d}{dt}\,x_j(t) = X_j(x_1(t),\ldots,x_{2n}(t)),$$

for $j = 1,\ldots,2n$, with

$$X_j(x_1,\ldots,x_{2n}) = x_{n+j},$$

for $j = 1,\ldots,n$,

$$X_j(x_1,\ldots,x_{2n}) = -\frac{\partial}{\partial x_{j-n}}V(x_1,\ldots,x_n),$$

and for $j = n+1, \ldots, 2n$. This is the standard form of a system of first order ordinary differential equations.

2.3. *A more intrinsic mathematical formulation*

A system of ordinary differential equations in the above form, with smooth functions F^j, has a mathematical formulation in terms of differential geometry. See Appendix A (*Manifolds*). Let $\mathcal{M}$ be an m-dimensional C^∞ manifold and X a vector field on $\mathcal{M}$. Such a vector field is an assignment of a tangent vector X_p to every point p of $\mathcal{M}$. An *integral curve* is a curve, i.e. a map γ from an open interval I in R into $\mathcal{M}$, such that it is tangent to the vector field in each point $\gamma(t)$, for t in I. (We assume in this context all maps, vector fields, etc., to be smooth, i.e. infinitely differentiable.) A general theorem about vector fields states that there is a unique integral curve through each p of $\mathcal{M}$, for some interval I. These curves together form a *flow* on $\mathcal{M}$. If one can take for this I the real line R, the vector field is called *complete*. In terms of local coordinates $x^1, \ldots, x^m$ the vector field can be written, as $X = X^1 \frac{\partial}{\partial x^1} + \ldots + X^m \frac{\partial}{\partial x^m}$; with the components X^j functions of these coordinates. A curve γ is represented by m functions $x^j(t)$; it is an integral curve of X if these functions are solutions of the differential equations

$$\frac{dx^j(t)}{dt} = X^j(x^1(t), \ldots, x^m(t)),$$

for $j = 1, \ldots, m$. Summarizing, a pair $(\mathcal{M}, X)$, with $\mathcal{M}$ a differential manifold and X a vector field on $\mathcal{M}$, may be called a *general dynamical system*. It incorporates the idea of a standard system of first order ordinary differential equations; these are obtained explicitly by choosing local coordinates on $\mathcal{M}$.

The equations of Newtonian mechanics in this form represent a special case. The manifold $\mathcal{M}$ is of even dimension $m = 2n$, and is the tangent bundle manifold of an underlying n-dimensional manifold of 'positions' $\mathcal{Q}$, with the vector field X satisfying conditions corresponding with this fact.

3. THE LAGRANGIAN AND HAMILTONIAN FORMULATION OF CLASSICAL MECHANICS

3.1. *Lagrangian variational problems*

In the Lagrangian formulation Newtonian mechanics is treated as a particular example of a class of variational problems, i.e. problems in which a certain function, or a set of functions, is determined by finding the extremum of a given functional. Let $\mathcal{U}$ be an open set in R^n and let L be a given real-valued function on $\mathcal{U} \times R^n$. Consider curves in $\mathcal{U}$, i.e. functions γ from R into $\mathcal{U}$. To keep things simple all functions are supposed to be C^∞. We denote the n coordinates on U by $q^1, \ldots, q^n$ and those on $\mathcal{U} \times R^n$ by $q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n$. Consider a fixed finite interval $[t_1, t_2]$ on the real line. A curve γ, a set of functions $q^1(t), \ldots, q^n(t)$, determines an integral

$$I_{t_1,t_2}(\gamma) = \int_{t_1}^{t_2} L(q^1(t), \ldots, q^n(t), \frac{d}{dt}q^1(t), \ldots, \frac{d}{dt}q^n(t))\, dt.$$

This integral is a *functional* on the space of curves γ parametrized by t from the interval $[t_1, t_2]$.

The variational problem defined by this set-up is to find the curve or curves for which the integral is *extremal* with respect to variations which are arbitrary except that they leave the end points fixed. A curve γ is extremal in this sense if the integral is constant up to first order under each 1-parameter deformation $\gamma \mapsto \gamma_\varepsilon$ of the form

$$q^j(t) \mapsto q^j(t) + \varepsilon\, \eta^j(t),$$

for arbitrary (smooth) functions $\eta^j(t)$ with $\eta^j(t_1) = \eta^j(t_2) = 0$, for $j = 1, \ldots, n$, or

$$\frac{d}{d\varepsilon} I_{t_1,t_2}(\gamma_\varepsilon) = 0$$

in $\varepsilon = 0$ for such deformations. One can show that this condition leads to a system of second order differential equation for the extremal functions $q^j(t)$, the *Euler-Lagrange equations*

$$\frac{\partial L}{\partial q^j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^j}\right) = 0,$$

for $j = 1, \ldots, n$.

In physics and in some of the variational calculus literature one employs a symbolic notation for 'variations'. Written in this notation the statement above says that the requirement $\delta I_{t_1,t_2} = 0$ under variations δq^j with $\delta q^j(t_1) = \delta q^j(t_2) = 0$ implies the Euler-Lagrange equations.

3.2. *Newton's equations as variational equations*

Consider a system of N point particles, as before. Let $\mathcal{U} = R^n$, with $n = 3N$. The coordinates $q^1, \ldots, q^n$ are the position variables $x_1, y_1, z_1, \ldots, x_N, y_N, z_N$. Define the function L as

$$L(q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n) = \sum_{j=1}^{n} \frac{1}{2} m_j (\dot{q}^j)^2 - V(q^1, \ldots, q^n),$$

the *kinetic energy* minus the *potential energy* of the system, if we interprete the $\dot{q}^j$ as the components of the velocities of the particles. In this context the function L is called the *Lagrangian function*, or *Lagrangian* of the system and the integral $I_{t_1,t_2}(\gamma)$ is called the *action*. It is an easy exercise to show that the Euler-Lagrange equations for this L reduce to Newton's equations. This means that the time evolution from t_1 to t_2 is described precisely by those curves γ for which the action is extremal, in fact *minimal* in this case.

Writing Newton's equations in Lagrangian form in this manner does of course not add anything to their contents, but has nevetheless great advantages:

a. There is no need to restrict oneself to cartesian coordinates; the formulas hold for arbitrary curvilinear coordinates.

b. The Lagrangian formulation is very useful in situations where there are constrains on the system, for instance when the particles are restricted in their motion to a lower dimensional surface.

c. Symmetries and their consequences such as conserved quantities can be easily read of from the Lagrangian.

3.3. *A more intrinsic formulation*

Suppose that the space of possible positions of the particles is not just R^n, but a more general smooth n-dimensional manifold $\mathcal{Q}$. The space of all tangent vectors on $\mathcal{Q}$, the tangent bundle over $\mathcal{Q}$, is a $2n$-dimensional manifold, which may be denoted as $T(\mathcal{Q})$. Let $\mathcal{U}$ be an coordinate neighbourhood on $\mathcal{Q}$, with coordinates $q^1, \ldots, q^n$, and $q^1, \ldots, q^n, \dot{q}^1, \dot{q}^1, \ldots, \dot{q}^n$ the corresponding coordinates on $T(\mathcal{U})$, the part of $T(\mathcal{Q})$ above $\mathcal{U}$. The Lagrangian $L(q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)$ is the local coordinate expression of a function on $T(\mathcal{Q})$. The further development of Lagrangian variational systems in this intrinsic differential geometric picture would lead to *jet bundles*, objects that can be seen as 'higher order tangent bundles'. In this course Lagrangian classical mechanics is only used as an intermediate step to the Hamiltonian form of classical mechanics. This will be useful for the discussion of quantum mechanics in Part III of this course. It should however be remarked that Lagrangian mechanics plays a role in the formulation of quantum mechanics in terms of *Feynman's path integral*, an interesting topic that will not be touched upon in this course.

3.4. *From Lagrangian to Hamiltonian classical mechanics.*

In this section we finally obtain the Hamilton formalism of classical mechanics which is useful as background for our presentation of quantum mechanics. We start from a Lagrangian variational system formulated in terms of local coordinate coordinate expressions, i.e. with coordinates $q^1, \ldots, q^n$, not necessarily cartesian, with the associated velocities $\dot{q}^1, \ldots, \dot{q}^n$ and with a given Lagrange function $L(q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)$. Using L we introduce new variables, the *canonically conjugated momenta* $p_1, \ldots, p_n$, as

$$p_j = \frac{\partial}{\partial \dot{q}^j} L,$$

for $j = 1, \ldots, n$. We assume that this transformation, which is called the *Legendre transformation*, can be inverted, i.e. that the velocities $\dot{q}^j$ can be written as functions of $q^1, \ldots, q^n$ and the new momenta $p_1, \ldots, p_n$. If this is possible, the Lagrangian L is called *regular* or *nondegenerate*. If L is *singular* or *degenerate* the simple road to a Hamiltonian formalism breaks down at this point. More complicated procedures can be found to overcome the problem of degenerateness of L, but this will not be discussed here.

Define next the *Hamiltonian* as

$$H = \sum_{j=1}^{n} p_j \dot{q}_j - L.$$

Note that this H should be seen as a function of $q^1, \ldots, q^n$ and $p_1, \ldots, p_n$. The time evolution, given in the Lagrangian formulation by functions $q^1(t), \ldots, q^n(t)$, $\dot{q}^1(t), \ldots, \dot{q}^n(t)$, satisfying the system of n Euler-Lagrange equations, is now given by functions $q^1(t), \ldots, q^n(t)$ and $p_1(t), \ldots, p_n(t)$, which are solutions of $2n$ first order equations involving H, namely

$$\frac{dp_j}{dt} = -\frac{\partial H}{\partial q^j} \qquad \frac{dq^j}{dt} = \frac{\partial H}{\partial p_j},$$

for $j = 1, \ldots, n$. The equivalence of these equations – called, not surprisingly *Hamilton's equations* – with the Euler-Lagrange equations is not hard to prove. These equations can be written in a more uniform manner as

$$\frac{dp_j}{dt} = \{H, p_j\} \qquad \frac{dq^j}{dt} = \{H, q^j\},$$

for $j = 1, \ldots, n$, with the *Poisson bracket* $\{\cdot, \cdot\}$ defined for an arbitrary pair of functions f and g of the variables $p_1, \ldots, p_n, q^1, \ldots, q^n$ as

$$\{f, g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q^j} - \frac{\partial g}{\partial p_j} \frac{\partial f}{\partial q^j} \right).$$

The $2n$-dimensional space of the variables $p_1, \ldots, p_n, q^1, \ldots, q^n$ is called the *phase space* of the system. The time evolution of the system, a *flow* in this space, has the property that it leaves the integration measure $dp_1 \ldots dp_n dq^1 \ldots dq^n$ invariant. This fact is known as *Liouville's theorem*.

In a more intrinsic picture phase space turns out to be the cotangent bundle manifold $T^*(\mathcal{Q})$ of the position manifold $\mathcal{Q}$ discussed in 3.3. The Legendre transformation

$$p_j = \frac{\partial L}{\partial \dot{q}^j}$$

is the representation in local coordinates of a map from the tangent bundle $T(\mathcal{Q})$ to the cotangent bundle $T^*(\mathcal{Q})$, consisting of maps from the tangent spaces $T_p(\mathcal{Q})$ in all points p of $\mathcal{Q}$ onto the corresponding cotangent spaces $T_p^*(\mathcal{Q})$. This map transforms the motion of the system from a flow on $T(\mathcal{Q})$ into a flow on $T^*(\mathcal{Q})$.

Hamiltonian classical mechanics in this picture is a special case of a more general class of dynamical systems. As this is important for understanding the relation between classical and quantum mechanics, it will be discussed in some detail in the next chapter.

4. GENERAL HAMILTONIAN DYNAMICAL SYSTEMS

4.1. *Symplectic manifolds*

The intrinsic differential geometric formulation of Hamiltonian classical mechanics, mentioned at the end of 3.4, leads to what me may be called 'general Hamiltonian dynamical systems', with *symplectic manifold* as the basic mathematical notion.

A 2-form ω on the C^∞-manifold $\mathcal{M}$ is called a *symplectic form* if it is closed and nondegenerate, i.e. if it satisfies $d\omega = 0$ and is such that $\omega(X,Y) = 0$, for all vector fields Y, implies $X = 0$. One can show that the requirement of nondegeneratess forces $\mathcal{M}$ to have even dimension. A *symplectic manifold* is a manifold provided with a symplectic form.

A symplectic form ω on $\mathcal{M}$ gives a map which assigns to each function f in $C^\infty(\mathcal{M})$ a vector field X_f, as the solution of the equation

$$\omega(X_f, Y) + (df)(Y) = 0,$$

for all vector fields Y in $V(\mathcal{M})$.

To see this, observe first that the form ω gives for each point p in $\mathcal{M}$ a bilinear map $T_p(\mathcal{M}) \times T_p(M) \to R$, or equivalently, a linear map $T_p(\mathcal{M}) \to T_p^*(\mathcal{M})$, which is invertible because ω is nondegenerate. This means that we can define a map $\phi : V(\mathcal{M}) \to \Omega^1(\mathcal{M})$, $X \mapsto \phi_X$, by keeping in the expression $\omega(X,Y)$ the vector field X fixed and letting Y run through $V\mathcal{M})$, i.e. as $\phi_X(Y) = \omega(X,Y)$, for all Y in $V(\mathcal{M})$. This map is linear, even $C^\infty(\mathcal{M})$-linear, and is also invertible. Using this, we define $X_f = -(\phi^{-1} \circ d)f$. This is equivalent to $\phi_{X_f} = -df$. Applying this to an arbitrary vector field Y gives $\phi_{X_f}(Y) = -(df)(Y)$. According to the definition of the map ϕ the left-hand side of this is $\omega(X_f, Y)$. The map $f \mapsto X_f$ is neither injective nor surjective. We have $X_f = 0$ iff f is a constant function. The vector fields in the image are called the *Hamiltonian vector fields* on $\mathcal{M}$.

It should be finally remarked that a symplectic manifold has a very useful canonical $2n$-form. It is simply defined as

$$\theta_\omega = \frac{1}{n!}(-1)^{\frac{1}{2}n(n-1)} \underbrace{\omega \wedge \ldots \wedge \omega}_{n\, times}.$$

The $2n$-form θ_ω is a *volume form*, i.e. it is everywhere nonzero. This implies that $\mathcal{M}$ is orientable.

4.2. *The Poisson bracket*

Let M be a symplectic manifold with symplectic form ω. Define a bilinear map

$$\{\cdot,\cdot\} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(M)$$

as

$$\{f, g\} = \omega(X_f, X_g),$$

for all f and g in $C^\infty(\mathcal{M})$. This is the *Poisson bracket* of the symplectic manifold $\mathcal{M}$. It has the following properties:

$$\begin{aligned}
&X_{\{f,g\}} = [X_f, X_g] \\
&\{f, gh\} = \{f,g\}h + \{f,h\}g \\
&\{f,g\} = -\{g,f\} \\
&\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0,
\end{aligned}$$

for all f, g and h in $C^\infty(M)$. The fourth property is called the *Jacobi identity*. Together with the third property, antisymmetry, it means that the Poisson bracket makes the associative algebra of functions $C^\infty(\mathcal{M})$ into a *Lie algebra*. See for this notion Appendix D (*Lie groups and Lie algebras*). The first property then says that the map $f \mapsto X_f$ is a Lie algebra homomorphism from $C^\infty(\mathcal{M})$ into $V(\mathcal{M})$, relating the Poisson bracket with the commutator of vector fields.

To prove the first statement we start from $\omega([X_f, Y_g], Z)$. Adding and substracting terms gives

$$\begin{aligned}
&\omega([X_f, Y_g], Z) = \\
&= -X_f(\omega(X_g, Z)) - X_g(\omega(Z, X_f)) - Z(\omega(X_f, X_g)) + \\
&+\omega([X_f, X_g], Z) + \omega([X_g, Z], X_f) + \omega([Z, X_f], X_g) + \\
&+ X_f(\omega(X_g, Z)) - Z(\omega(X_g, X_f)) - \omega(X_g, [X_f, Z]) + \\
&+X_f(\omega(X_f, Z)) - Z(\omega(X_g, X_f)) - \omega(X_g, [X_f, Z]) + \\
&-X_g(\omega(X_f, Z)) + Z(\omega(X_f, X_g)) + \omega(X_f, [X_g, Z])
\end{aligned}$$

The second and third line together are just $-(d\omega)(X_f, X_g, Z)$, which vanishes because we have $d\omega = 0$. The fourth line can be rewritten, by using the formula that connects functions and their associated Hamiltonian vector fields, as

$$\begin{aligned}
&X_f(\omega(X_g, Z)) - Z(\omega(X_g, X_f)) + \omega(X_g([X_f, Z]) = \\
&= X_f(-(dg)(Z)) - Z(-(dg)(X_f)) + (dg)([X_f, Z]) = \\
&-X_f(Z(g)) + Z(X_f(g)) + [X_f, Z](g) = 0.
\end{aligned}$$

The fifth line vanishes in the same way. We are left with the first line which give us therefore

$$\begin{aligned}
&\omega([X_f, X_g], Z) = Z(\omega(X_g, X_f) = -Z(\{f,g\}) = \\
&= -(d\{f,g\})(Z) = \omega(X_{\{f,g\}}, Z).
\end{aligned}$$

This holds for all Z in $V(\mathcal{M})$; the nondegeneracy of ω implies then finally $X_{\{f,g\}} = [X_f, X_g]$, which was to be proved. To prove the second property of $\{\cdot,\cdot\}$ we write

$$\begin{aligned}
&X_{fg} = -(\phi^{-1} \circ d)(fg) = -\phi^{-1}(d(fg) = \\
&= -\phi^{-1}((df)g + f\,dg) = -g\phi^{-1}(df) - f\phi^{-1}(dg) = \\
&= fX_g + gX_f.
\end{aligned}$$

Using this – and the $C^\infty(\mathcal{M})$-bilinearity of ω – we obtain

$$\begin{aligned}\{f, gh\} &= \omega(X_f, X_{gh}) = \omega(X_f, gX_h + hX_g) = \\ &= g\,\omega(X_f, X_h) + h\,\omega(X_f, X_g) = \{f, g\}h + \{f, h\}g,\end{aligned}$$

for all f, g and h in $C^\infty(\mathcal{M})$. The third property, the antisymmetry of $\{\cdot,\cdot\}$, follows immediately from the antisymmetry of ω as a 2-form. For the fourth property, the Jacobi identity, we use $d\omega = 0$ together with the first property. We write

$$\begin{aligned}&0 = (d\omega)(X_f, X_g, X_h) = \\ &= X_f(\omega(X_g, X_h)) + X_g(\omega(X_h, X_f)) + X_h(\omega(X_f, X_g)) + \\ &-\omega([X_f, X_g], X_h) - \omega([X_g, X_h], X_f) - \omega([X_h, X_f], X_g).\end{aligned}$$

The first term in the second line can be written as

$$\begin{aligned}&X_f(\omega(X_g, X_h)) = X_f(\{g, h\}) = (d\{g, h\})(X_f) = \\ &= -\omega(X_{\{g,h\}}, X_f) = -\{\{g, h\}, f\} = \{f, \{g, h\}\},\end{aligned}$$

and the first term in the third line as

$$\begin{aligned}&-\omega([X_f, X_g], X_h) = -\omega(X_{\{f,g\}}, X_h) = \\ &= -\{\{f, g\}, h\} = \{h, \{f, g\}\},\end{aligned}$$

with similar results for the cyclically permuted terms. This gives

$$0 = 2(\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}),$$

which is the Jacobi identity, up to the trivial factor 2.

4.3. *Hamiltonian dynamical systems*

A dynamical system is a pair $(\mathcal{M}, X)$, with $\mathcal{M}$ a manifold and X a vector field on $\mathcal{M}$. X defines the equation for a flow on $\mathcal{M}$, as was briefly discussed in 2.3. On a symplectic manifold there is a special class of vector fields, the Hamiltonian vector fields, which come from functions in a mannner described in the first section of this chapter. This suggests the following obvious and simple definition:

A *Hamiltonian dynamical system* is a symplectic manifold $(\mathcal{M}, \omega)$ provided with a function H in $C^\infty(\mathcal{M})$.

The function H is called the *Hamiltonian function* of the system. It gives a Hamiltonian vector field X_H, which in turn gives the differential equation for a flow on $\mathcal{M}$. This 'Hamiltonian flow' has many interesting properties. It leaves the symplectic form ω, and therefore also the volume form θ_ω invariant. The same is true for the Poisson bracket $\{\cdot,\cdot\}$ which plays an important role in the stydy of the properties of the system.

4.4. *Darboux coordinates*

Explicit calculations in differential geometry are often facilitated by the use of local coordinates. For symplectic manifolds there exist special coordinates in which the symplectic form becomes particularly simple. This is stated by the *Darboux theorem* which we give here without proof:

A $2n$-dimensional symplectic manifold possesses a system of coordinate neighbourhoods, each with coordinates $q^1, \ldots, q^n, p_1, \ldots, p_n$, such that the symplectic form ω is locally represented by the expression $\omega = \sum_{j=1}^{n} dp_j \wedge dq^j$.

In such *Darboux coordinates* the map $f \mapsto X_f$ assigns to a function f a Hamiltonian vector field X_f represented locally by the differential operator

$$X_f = \sum_{j=1}^{n} \frac{\partial f}{\partial p_j} \frac{\partial}{\partial q^j} - \sum_{j=1}^{n} \frac{\partial f}{\partial q^j} \frac{\partial}{\partial p_j}.$$

For a Hamiltonian dynamical system this means that X_H, the vector field defined by the Hamiltonian function, has the components $\frac{\partial H}{\partial p_1}, \ldots, \frac{\partial H}{\partial p_n}$ with respect to the coordinates $q_1, \ldots, q_n$, and $-\frac{\partial H}{\partial q^1}, \ldots, -\frac{\partial H}{\partial q^n}$ with respect to the $p_1, \ldots, p_n$. An integral curve $\gamma(t)$ is represented by functions $q^1(t), \ldots, q^n(t), p_1(t) \ldots, p_n(t)$ which are solutions of the differential equations

$$\frac{dq^j(t)}{dt} = \frac{\partial H}{\partial p_j} \qquad \frac{dp_j(t)}{dt} = -\frac{\partial H}{\partial q^j},$$

for $j = 1, \ldots, n$. The Poisson bracket of two functions f and g takes the local form

$$\{f, g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q^j} - \frac{\partial g}{\partial p_j} \frac{\partial f}{\partial q_j} \right).$$

This can be used to write the differential equations for $p_j(t)$ and $q^j(t)$ as

$$\frac{dp_j}{dt} = \{H, p_j\} \qquad \frac{dq^j(t)}{dt} = \{H, q^j\}.$$

The canonical volume form becomes $\theta_\omega = dp_1 \wedge \ldots \wedge dp_n \wedge dq^1 \wedge \ldots \wedge dq^n$, which gives, with an appropriate choice of orientation, the integration measure $dp_1 \ldots d_n \, dq^1 \ldots dq^n$.

4.5. *Hamilton classical mechanics*

The local formulas for the description of a general Hamiltonian dynamical system given in the preceding section are precisely the formulas for classical mechanics in its Hamiltonian form that were given in 3.4. This means that, locally at least, a general Hamiltonian dynamical system is classical Hamiltonian mechanics. Globally it is a more general notion.

Let us go back to a Newtonian classical mechanical system of N point particles. The positions of these particles at any moment in time can be thought of as points of an n-dimensional manifold $\mathcal{Q}$, with $n = 3N$. The Langrangian description of such a system is formulated in terms of a set of first order ordinary differential equations defined on the tangent bundle $T(\mathcal{Q})$, a $2n$-dimensional manifold associated with $\mathcal{Q}$. The transition to the Hamiltonian description involves the introduction of canonical momenta as $p_j = \frac{\partial L}{\partial \dot{q}^j}$. This is a local formula which defines the Legendre transform, a global map from the tangent bundle onto the cotangent bundle $T^*(\mathcal{Q})$, again a $2n$-dimensional manifold. It transforms the dynamics of the system as a flow on the tangent bundle into an equivalent flow on the cotangent bundle.

The cotangent bundle $T^*(\mathcal{Q})$ of an arbitrary manifold $\mathcal{Q}$ has a natural symplectic structure given by a symplectic form ω. This form can be defined intrinsically. In terms of local coordinates $q^1, \ldots, q^n$, the original coordinates on $\mathcal{Q}$, and $p_1, \ldots, p_n$, the additional coordinates of the cotangent bundle $T^*(\mathcal{Q})$, it is just $\omega = \sum_{j=1}^n dp_j \wedge dq^j$. A general Hamiltonian dynamical system is defined on a symplectic manifold. The cotangent bundle $T^*(\mathcal{Q})$ of the manifold $\mathcal{Q}$ of positions of a classical mechanical system is in a natural way a symplectic manifold, but not every symplectic manifold is a cotangent bundle of some underlying manifold.