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Topics from 20th century physics.
An introductory course for students in mathematics

III. QUANTUM THEORY: CHAPTERS 10 - 13

– APPLYING SYMMETRY PRINCIPLES

1. TRANSLATION SYMMETRY AND LINEAR MOMENTUM

1.1. *Translation symmetry for a single particle*

The translations in space over vectors $\vec{a}$ form a 3-dimensional abelian Lie group. It has an obvious unitary representation in the Hilbert space $\mathcal{H} = L^2(R^3, d\vec{x})$ of a single particle, given as

$$(U(\vec{a})\psi)(\vec{x}) = \psi(\vec{x} - \vec{a}),$$

for all $\vec{a}$ in R^3. The Hamiltonian of this system has a kinetic energy term

$$H_0 = \frac{\vec{P}^2}{2m} = -\frac{\hbar^2}{2m}\Delta,$$

and possibly a potential energy term V consisting of the operator of multiplication by a potential function $V(\vec{x})$. The unitary operators $U(\vec{a})$ commute with H_0 because taking second derivatives and translation over $\vec{a}$ commutes. The $U(\vec{a})$ do not commute with V because $V(\vec{x})\psi(\vec{x} - \vec{a}) \neq V(\vec{x} - \vec{a})\psi(\vec{x} - \vec{a})$.

Taylor expansion gives

$$\begin{aligned}(U(\vec{a})\psi)(\vec{x}) &= \psi(\vec{x} - \vec{a}) = \\ &= \psi(\vec{x}) + \sum_{j=1}^{3}(-a_j)\frac{\partial}{\partial x_j}\psi(\vec{x}) + \frac{1}{2}\sum_{j,k=1}^{3}(-a_j)(-a_k)\frac{\partial^2}{\partial x_j \partial x_k}\psi(\vec{x}) + \ldots\end{aligned}$$

This suggests that $U(\vec{a})$ can be written as

$$U(\vec{a}) = e^{\vec{a}\cdot\vec{P}'},$$

with the three commuting antihermitian operators

$$P'_j = -\frac{\partial}{\partial x_j},$$

representing the three basis elements of the 3-dimensional abelian Lie algbra of the group of translations. Instead of the antihermitic P'_j we can use the *selfadjoint* generators

$$P_j = i\hbar P'_j = \frac{\hbar}{i}\frac{\partial}{\partial x_j},$$

which as observables we already know as the components of linear momentum. This gives

$$U(\vec{a}) = e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{P}}.$$

Conclusion: The group of translations in space is a symmetry group for a single *free* particle. The conserved quantities associated with this symmetry are the three components of *linear momentum*. A single particle in a potential has no translation symmetry.

10.2. *Translation symmetry for a system of two particles*

Consider a system of two different particles, with masses m_1 and m_2. Its Hilbert space of states is $\mathcal{H} = L^2(R^6, d\vec{x}d\vec{y})$. The group of 3-dimensional translations has a unitary representation in this $\mathcal{H}$ given by

$$(U(\vec{a})\psi)(\vec{x},\vec{y}) = \psi(\vec{x}-\vec{a},\vec{y}-\vec{a}),$$

for all $\vec{a}$ in R^3. The Hamiltonian of this system contains a kinetic enery term

$$H_0 = \frac{\vec{P}_x^2}{2m_1} + \frac{\vec{P}_y^2}{2m_2} = -\frac{\hbar^2}{2m_1}\Delta_x - \frac{\hbar^2}{2m_2}\Delta_y,$$

which clearly commutes with all $U(\vec{a})$. An interaction potential of the form $V(\vec{x},\vec{y}) = V(\vec{x}-\vec{y})$, i.e. depending only on the difference of the positions, also commutes with the $U(\vec{a})$. However, an additional potential $V_1(\vec{x}) + V_1(\vec{y})$, associated with an external force, does *not* commute with the $U(\vec{a})$.

Taylor expansion of $(U(\vec{a})\psi)(\vec{x},\vec{y})$ gives in this case

$$\begin{aligned}
&(U(\vec{a})\psi)(\vec{x},\vec{y}) = \psi(\vec{x}-\vec{a},\vec{y}-\vec{a}) = \\
&= \psi(\vec{x},\vec{y}) + \sum_{j=1}^{3}(-a_j)\left(\frac{\partial}{\partial x_j} + \frac{\partial}{\partial y_j}\right)\psi(\vec{x},\vec{y}) + \\
&+ \sum_{j,k=1}^{3}(-a_j)(-a_k)\left(\frac{\partial^2}{\partial x_j\partial x_k} + 2\frac{\partial^2}{\partial x_j\partial y_k} + \frac{\partial^2}{\partial y_j\partial y_k}\right)\psi(\vec{x},\vec{y}) + \ldots,
\end{aligned}$$

which suggest that the $U(\vec{a})$ can be written as

$$U(\vec{a}) = e^{\vec{a}\cdot\vec{P}'},$$

with the three commuting antihermitian operators P'_j in this case defined as

$$P'_j = -\frac{\partial}{\partial x_j} - \frac{\partial}{\partial y_j},$$

or as

$$U(\vec{a}) = e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{P}},$$

with three commuting selfadjoint generators

$$P_j = i\hbar P'_j = \frac{\hbar}{i}\frac{\partial}{\partial x_j} + \frac{\hbar}{i}\frac{\partial}{\partial y_j} = (P_x)_j + (P_y)_j,$$

the three components of the total linear momentum of the system.

Conclusion: A system of two particles, interacting through a potential depending only on the difference $\vec{x}-\vec{y}$, has translation symmetry. The conserved quantities associated with this symmetry are the three components of the *total* *linear momentum* of the system. Adding a potential connected with external forces breaks this symmetry.

Remark: The examples of this section and the preceding one are meant as very simple demonstrations of the relation between symmetries and conserved quantities. The fact that the momentum $\vec{P}$ of a free particle is a conserved quantity follows of course immediately from the form of the Hamiltonian; it does not need the context of Lie groups. The example of a system of two particles is only slightly less trivial as a group theory exercise. However, the general principle involved in the relation between translation symmetry and conservation of momentum is very important in systems where one does not have an obvious suitable notion of total momentum, such as classical and quantum field theory. In such cases it is used to *define* it.

11. ROTATION SYMMETRY AND ANGULAR MOMENTUM

11.1. *The rotation group $SO(3)$ and its Lie algebra $so(3)$*

$SO(3)$, the group of (continuous) rotations in 3-dimensional Euclidean space, consists of all 3×3 orthogonal matrices with determinant equal to 1. Consider the 1-parameter subgroup of matrices

$$D_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

the rotations over an angle θ around the x_1-axis. One has

$$\left(\frac{d}{d\theta} D_1(\theta)\right)_{\theta=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = d_1,$$

which implies that $D_1(\theta)$ can be written as $D_1(\theta) = e^{\theta d_1}$. Similarly, the subgroups of rotations around the x_2- and x_3-axis can be written as $D_2(\theta) = e^{\theta d_2}$ and $D_3(\theta) = e^{\theta d_3}$, with

$$d_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

respectively

$$d_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrices $D_j(\theta)$ describe what are called *infinitesimal rotations* around the x_j-axes. They form a basis for the vector space of all antisymmetric 3×3 matrices, the Lie algebra $so(3)$ of $SO(3)$, and satisfy the commutation relations

$$[d_1, d_2] = d_3, \quad [d_2, d_3] = d_1, \quad [d_3, d_1] = d_2.$$

11.2. *Unitary representations in the single particle Hilbert space*

The group of spatial rotations is represented in $\mathcal{H} = L^2(R^3, d\vec{x})$, the single particle state space, by unitary operators as

$$(U(D)\psi)(\vec{x}) = \psi(D^{-1}\vec{x}),$$

for all D in $SO(3)$ and with $(D^{-1}\vec{x})_j = \sum_{k=1}^{3}(D^{-1})_{jk}x_k$. According to the general scheme discussed in Appendix D, Section 5.3, the 1-parameter groups of orthogonal matrices $D_j(\theta) = e^{\theta d_j}$ are represented by the operators

$$U(D_j(\theta)) = U(e^{\theta d_j}) = e^{\theta\pi(d_j)},$$

with the antihermitian operators $\pi(d_j)$ representing the Lie algebra elements d_j, with the same commutation relations as the d_j, i.e. $[\pi(d_1), \pi(d_2)] = \pi([d_1, d_2]) = \pi(d_3)$, etc.

Using the Taylor expansion of $(U(D_j(\theta))\psi)(\vec{x})$ one finds the generators $\pi(d_j)$ as differential operators in $\mathcal{H} = L^2(R^3, d\vec{x})$. For $j = 1$ one has

$$\begin{aligned}(U(D_1(\theta))\psi)(\vec{x}) &= \psi(D_1^{-1}(\theta)\vec{x}) = \\ &= \psi(e^{-\theta d_1}\vec{x}) = \psi(\vec{x} - \theta d_1\vec{x} + \ldots).\end{aligned}$$

After substitution of the explicit form of the matrix d_1 this becomes

$$\begin{aligned}&\psi(x_1,\ x_2 + \theta x_3 + \ldots,\ x_3 - \theta x_2 + \ldots) = \\ &= \psi(\vec{x}) + \theta\left(x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3}\right)\psi(\vec{x}) + \ldots\end{aligned}$$

Comparison with

$$U(D_1(\theta)) = e^{\theta\pi(d_1)} = 1 + \theta\pi(d_1) + \ldots$$

gives finally

$$\pi(d_1) = x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3} = -\frac{i}{\hbar}(Q_2P_3 - Q_3P_2) = -\frac{i}{\hbar}L_1,$$

with L_1 the selfadjoint operator for the x_1 component of the quantum mechanical angular momentum, introduced and discussed in 7.1. For $j = 1$ and $j = 3$ the same procedure gives

$$\pi(d_2) = x_1\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_1} = -\frac{i}{\hbar}(Q_3P_1 - Q_1P_3) = -\frac{i}{\hbar}L_2,$$

respectively

$$\pi(d_3) = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} = -\frac{i}{\hbar}(Q_1P_2 - Q_2P_1) = -\frac{i}{\hbar}L_3.$$

The commutation relations for the L_j,

$$[L_1, L_2] = i\hbar L_3, \quad [L_2, L_3] = i\hbar L_1, \quad [L_3, L_1] = i\hbar L_2,$$

can be calculated using these explicit expressions, but they follow here immediately from the group theoretic relation

$$[\pi(d_j), \pi(d_k)] = \pi([d_j, d_k]),$$

for $j, k = 1, 2, 3$. We have for the unitary operators

$$U(D_j(\theta)) = e^{\theta\pi(d_j)} = e^{-\frac{i}{\hbar}\theta L_j}.$$

11.3. *Rotation symmetry for a single particle*

The operators $U(D)$, representing the group $SO(3)$ in $\mathcal{H} = L^2(R^3, d\vec{x})$ commute with the Hamiltonian $H = H_0 + V$ of a particle moving in a rotation invariant potential, i.e. a potential of the form $V(|\vec{x}|)$. (Note that the simplest way to check that the $U(D)$ commute with the kinetic energy part of the Hamiltonian H_0 is to look at the Fourier transformed picture in which H_0 acts on the Fourier transformed functions $\hat{\psi}(\vec{p})$ as multiplication by a factor $-(1/\hbar^2)|\vec{p}|^2$ and $U(D)$ changes $\hat{\psi}(\vec{p})$ into $\hat{\psi}(D^{-1}\vec{p})$.

Conclusion: $SO(3)$, the group of rotations in space, is a symmetry group for the system of a single particle moving in a centrally symmetric potential $V(|\vec{x}|)$. The conserved quantities associated with this symmetry are the three components of *angular momentum*.

To understand the physical properties of angular momentum in this quantum mechanical system one has to study in the first place the spectrum of the operators L_j. Note that because the L_j do not commute one cannot look for a common system of eigenvectors; the L_j have to be discussed separately. The L_j can be transformed into each other by unitary operators $U(D)$. One can for instance easily derive that $e^{\frac{i}{\hbar}\frac{\pi}{2}L_1} L_2 \, e^{-\frac{i}{\hbar}\frac{\pi}{2}L_1}$ is equal to L_3. This implies that the three L_j have the same spectrum. To find this spectrum, say that of L_3, it is of no help that it is a simple expression in operators for position and momentum, operators of which we know the spectrum. In fact the spectrum of L_3 turns out to be discrete, contrary to that of the position and momentum operators.

To find the spectrum of L_3 we introduce spherical coordinates r, ν and θ, by the formulas

$$\begin{aligned} x_1 &= r \sin\nu \, \cos\theta \\ x_2 &= r \sin\nu \, \sin\theta \\ x_3 &= r \cos\nu \end{aligned}$$

Writing the wave functions ψ as functions of these coordinates, the inner product of two wave functions ψ_1 and ψ_2 becomes

$$(\psi_1, \psi_2) = \int_0^\infty \int_0^{2\pi} \int_0^\pi \overline{\psi_1(r, \nu, \theta)} \psi_2(r, \nu, \theta) r^2 \sin\nu \, d\nu dr d\theta.$$

It is clear that the operators $U(D)$ act only on the angle variables ν and θ; the 1-parameter group $U(D_3(\theta))$ acts only on θ. We have

$$(U(D_3(\theta')\psi)(r,\nu,\theta) = \left(e^{-\frac{i}{\hbar}\theta' L_3}\psi\right)(r,\nu,\theta) = \psi(r,\nu,\theta-\theta'),$$

which implies

$$L_3\psi = \frac{\hbar}{i}\frac{\partial\psi}{\partial\theta}.$$

The general solution of the eigenvalue/eigenvector equation

$$\frac{\hbar}{i}\frac{\partial\psi}{\partial\theta} = m_3\psi$$

has the form

$$\psi_{m_3}(r,\nu,\theta) = \phi(r,\nu)\, e^{im_3\theta},$$

for a real eigenvalue $m_3\hbar$ and an arbitrary function $\phi(r,\nu)$ for which the integral $\int_0^\infty\int_0^{2\pi}|\phi(r,\nu)|^2 r^2 \sin\nu\, dr d\nu$ is convergent. The 1-parameter group of unitary operators $e^{-\frac{i}{\hbar}\theta' L_3}$, representing the rotations along the x_3-axis over an angle θ', act on these eigenstates as

$$\left(e^{-\frac{i}{\hbar}\theta' L_3}\psi_{m_3}\right)(r,\nu,\theta) = \psi_{m_3}(r,\nu,\theta-\theta') = e^{-im_3\theta'}\psi_{m_3}(r,\nu,\theta).$$

Because a rotation over an angle θ' is the same as a rotation over $\theta'+2\pi$, the number m_3 has to be one of the discrete values $m_3 = 0, \pm1, \pm2, \ldots$.

This result can be slightly generalized to the statement that the component of the angular momentum in any given direction is 'quantized' in the sense that it can only assume the discrete values $m_3\hbar$, $m_3 \in Z$. The eigenvalues clearly have infinite degeneration, due to the presence of the functions $\phi(r,\nu)$. To discuss this and other physically interesting features of the situation one needs to know more about the way the representation $\{U(D)\}_{D\in SO(3)}$ can be written in terms of irreducible representations. Mathematical material for this and for a further study of eigenstates, etc., will be supplied in the next section. This will be useful for our treatment of the *hydrogen atom* in Chapter 12 and for the notion of *spin* in Chapter 13.

11.4. *The irreducible unitary representations of* $so(3)$ and $S)(3)$

The group $SO(3)$ is compact. It is known that the irreducible unitary representations of compact Lie groups, together with the corresponding Lie algebra representations, are all finite dimensional.

Assume that we have an irreducible unitary representation of the Lie algebra $so(3)$ in a finite dimensionial Hilbert space $\mathcal{K}$. Such a representation is generated by three selfadjoint operators A_1, A_2 and A_3 in $\mathcal{K}$, satisfying the commutation relations

$$[A_1,A_2] = iA_3, \quad [A_2,A_3] = iA_1, \quad [A_3,A_1] = iA_2.$$

Define the operators $A^2 = (A_1)^2 + (A_2)^2 + (A_3)^2$ and $A_\pm = A_1 \pm iA_2$. The operator A^2 is positive and commutes with the three A_j, as one easily verifies. *Schur's lemma* states that in a finite dimensional space a linear operator that commutes with all the members of an irreducible system of operators is necessarily a multiple of the unit operator. Therefore $A^2\psi = \lambda\psi$, for a fixed positive constant λ and for all ψ in $\mathcal{K}$. For the $A_\pm$ one has $A_-^* = A_+$ and the commutation relations

$$[A_+, A_-] = 2A_3, \quad [A_3, A_\pm] = \pm A_\pm.$$

Write $A_+A_- = (A_1+iA_2)(A_1-iA_2) = (A_1)^2+(A_2)^2-i[A_1,A_2] = A^2-(A_3)^2+A_3$, etc., and a similar expression for A_-A_+. This gives the expressions

$$\begin{aligned} A_+A_- &= A^2 - A_3(A_3 - 1) \\ A_-A_+ &= A^2 - A_3(A_3 + 1) \end{aligned}$$

LEMMA 1: Let ϕ_α be an eigenvector of A_3 with eigenvalue α. For $\alpha(\alpha+1) \neq \lambda$ one has $A_+\phi_\alpha \neq 0$, $A_+\phi_\alpha$ an eigenvector of A_3 with eigenvalue $\alpha + 1$, and $A_-A_+\phi_\alpha = (\lambda - \alpha(\alpha+1))\phi_\alpha$. For $\alpha(\alpha+1) = 0$ one has $A_+\phi_\alpha = 0$

Proof: Follows easily from the property $A_+^* = A_-$ and two identities:
1. $A_3(A_+\phi_\alpha) = ([A_3, A_+] + A_+A_3)\phi_\alpha = (A_+ + A_+A_3)\phi_\alpha = (\alpha+1)A_+\phi_\alpha$.
2. $A_-A_+\phi_\alpha = (A^2 - A_3(A_3+1))\phi_\alpha = (\lambda - \alpha(\alpha+1))\phi_\alpha$.

LEMMA 2: Let ϕ_α be an eigenvector of A_3 with eigenvalue α. For $\alpha(\alpha-1) \neq \lambda$ one has $A_-\phi_\alpha \neq 0$, $A_-\phi_\alpha$ an eigenvector of A_3 with eigenvalue $\alpha - 1$, and $A_+A_-\phi_\alpha = (\lambda - \alpha(\alpha-1))\phi_\alpha$. For $\alpha(\alpha-1) = 0$ one has $A_-\phi_\alpha = 0$

Proof: Follows easily from the property $A_+^* = A_-$ and two identities:
1. $A_3(A_-\phi_\alpha) = ([A_3, A_-] + A_-A_3)\phi_\alpha = (-A_- + A_-A_3)\phi_\alpha = (\alpha-1)A_-\phi_\alpha$.
2. $A_+A_-\phi_\alpha = (A^2 - A_3(A_3-1))\phi_\alpha = (\lambda - \alpha(\alpha-1))\phi_\alpha$.

The operator A_3 has at least one eigenvector ϕ_α with eigenvalue α. Because of the two lemmas there is a sequence of eigenvectors of A_3

$$\ldots,\ (A_-)^2\phi_\alpha,\ A_-\phi_\alpha,\ \phi_\alpha,\ A_+\phi_\alpha,\ (A_+)^2\phi_\alpha,\ \ldots$$

with eigenvalues $\ldots, \alpha-2, \alpha-1,\ \alpha,\ \alpha+1,\ \alpha+2, \ldots$. Because $\mathcal{K}$ is finite dimensional this sequence is finite; it starts with an eigenvector ϕ_{α_1} with the eigenvalue α_1 and ends with an eigenvector ϕ_{α_2} with eigenvalue α_2, with the difference $\alpha_2 - \alpha_1$ a nonnegative integer p. One has $A_-\phi_{\alpha_1} = 0$ and $A_+\phi_{\alpha_2} = 0$, which implies $\alpha_1(\alpha_1 - 1) = \lambda$ and $\alpha_2(\alpha_2+1) = \lambda$. Combining these two relations with $\alpha_2 - \alpha_1 = p$ and $p \geq 0$ leads to $\alpha_1 = -\frac{1}{2}p$, $\alpha_2 = +\frac{1}{2}p$ and $\lambda = \frac{1}{2}p(\frac{1}{2}p+1)$. The linear subspace spanned by the sequence of $p+1$ eigenvectors of A_3, say $\phi_{\alpha_1}, \ldots, \phi_{\alpha_2}$, is invariant under the action of A_-, A_+ and A_3, and therefore under A_1, A_2 and A_3. These three operators were assumed to generate an irreducible representation of $so(3)$, so the invariant subspace must be $\mathcal{K}$ itself.

Introducing $j = \frac{1}{2}p$, the result obtained in this manner can be stated as follows: the irreducible representation that we assumed to be given is characterized by a number j taking one of the values $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$. The representation space $\mathcal{K}$ is $(2j+1)$-dimensional, spanned by eigenvectors ϕ_m of A_3 with eigenvalues $m = -j,\ -j+1,\ \ldots,\ +j-1,\ +j$. The operator A^2 multiplies all vectors in $\mathcal{K}$ by the constant $j(j+1)$.

The vectors ϕ_m are orthogonal but not normalized. We are free to choose one eigenvector, say the one with the lowest eigenvalue $m = -j$, to be of unit lenght. We call it ϕ_{-j}. The length of the next eigenvector is then given by

$$\begin{aligned} ||A_+\phi_{-j}||^2 &= (A_+\phi_{-j}, A_+\phi_{-j}) = (\phi_{-j}, A_-A_+\phi_{-j}) = \\ &= (\phi_{-j}, (A^2 - A_3(A_3+1))\phi_{-j}) = 2l(\phi_{-j}, \phi_{-j}) = 2j. \end{aligned}$$

We choose therefore the second normalized eigenvector as

$$\phi_{-j+1} = (2j)^{-1/2} A_+ \phi_{-j}.$$

Continuing this process we obtain an orthonormal basis of $2j+1$ eigenvectors of A_3, denoted as

$$\phi_m = C_m (A_+)^{m+l} \phi_{-j},$$

for $m = -j,\ -j+1,\ \ldots,\ +j$. The normalization constants C_m can be calculated; they are determined up to a phase factor which we fix by requiring that the C_m are positive. One next calculates the matrix elements of the operators A_- and A_+ with respect to the ϕ_m as

$$\begin{aligned} (\phi_m, A_-\phi_{m'}) &= (j(j+1) - m(m+1))^{1/2}\ \delta_{m+1,m'} \\ (\phi_m, A_+\phi_{m'}) &= (j(j+1) - m'(m'+1))^{1/2}\ \delta_{m,m'+1}. \end{aligned}$$

From this we obtain the matrix elements of A_1 and A_2. This means that from our initial assumption that an irreducible representation of $so(3)$ was given, we have finally derived the explicit matrix form of such a representation, in fact the explicit form of all possible finite dimensional irreducible representations of $so(3)$.

The irreducible representations of the *Lie algebra* $so(3)$ are characterized by a number j, integer or half-integer, i.e. taking the values $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$. Can all these representations be exponentiated to representations of the *group* $SO(3)$? The answer follows immediately from the form of the representation of the 1-parameter subgroup $D_3(\theta) = e^{\theta d_3}$, as defined in 11.1. One should have

$$U(e^{\theta d_3})\phi_m = e^{-\frac{i}{\hbar}\theta L_3}\phi_m = e^{-i\theta A_3}\phi_m = e^{-i\theta m}\phi_m.$$

Rotation over an angle θ is the same as rotation over an angle $\theta + 2\pi$. This excludes the half integer values of m, and consequently the half-integer values of j. We can state therefore the following general result for the representations of $SO(3)$:

An irreducible unitary representation of the group $SO(3)$, necessarily finite dimensional, is characterized by an integer usually denoted as l, which takes one of the values $l = 0, 1, 2, 3, \ldots$. The dimension of such a representation is $2l + 1$; it is spanned by vectors ϕ_{m_3}, eigenvectors of L_3 with eigenvalues $\hbar m_3$, for $m_3 = -l,\ -l+1,\ \ldots,\ +l$.

11.5. *The irreducible unitary representations of $su(2)$ and $SU(2)$*

Consider the Lie algebra $su(2)$, which consists of all 2×2 antihermitian matrices. The three matrices

$$\hat{d}_1 = \begin{pmatrix} 0 & -\frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}, \quad \hat{d}_2 = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad \hat{d}_3 = \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix}$$

form a basis of $su(2)$. Their commutation relations are

$$[\hat{d}_1, \hat{d}_2] = \hat{d}_3, \quad [\hat{d}_2, \hat{d}_3] = \hat{d}_1, \quad [\hat{d}_3, \hat{d}_1] = \hat{d}_2.$$

This shows that $su(2)$ is isomorphic to $so(3)$. The representations of $so(3)$ that we have found can therefore also be seen as representations of $su(2)$. It is not hard to check that as such they can be exponentiated to representations of the group $SU(2)$ for *all* values of $j = 0, \frac{1}{2}, 2, \frac{3}{2}, 3, \ldots$. The map $su(2) \to so(3)$ is the Lie algebra isomorphism associated with a two-to-one Lie group homomorphism from $SU(2)$ onto $SO(3)$. This homomorphism can be worked out explicitly, but this will not be done here. A general element of $SU(3)$ can be written as

$$\begin{pmatrix} z_1 & -z_2 \\ \overline{z_2} & \overline{z_1} \end{pmatrix},$$

with z_1 and z_2 two complex numbers satisfying $|z_1|^2 + |z_2|^2 = 1$. Writing $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, one realizes that this means that $SU(2)$ as a manifold is diffeomorphic to the 3-sphere S^3, and is therefore simply connected. $SU(2)$ is in fact the unique simply connected covering group of $SO(3)$.