TOPICS IN QUANTUM FIELD THEORIES

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Content

1 Green functions

Green functions play an important role in Quantum Field Theory. They occur in the perturbative expansion of the S-matrix from which one can derive cross sections and decay widths of particles. They also appear in Statistical Mechanics and Condensed Matter where they represent the correlation functions from which one can derive quantities like susceptibility, magnetisation or specific heat. Since we are mainly interested in scalar fields we will derive the Green function together with other distributions for that case only. The free scalar quantum field operator denoted by $\Phi(x)$ can be expressed in the creation and annihilation operators $a^\dagger(\vec{p})$ and $a(\vec{p})$ as follows

$$\Phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_p}} \Big(a(\vec{p})\, e^{-i\,p\cdot x} + a^\dagger(\vec{p})\, e^{i\,p\cdot x} \Big) \tag{1.1}$$

with $p^0 = E_p = \sqrt{|\vec{p}|^2 + m^2}$ and

$$\Big[a(\vec{p}), a^\dagger(\vec{p}\,')\Big] = \delta^{(3)}(\vec{p} - \vec{p}\,') \qquad \Big[a(\vec{p}), a(\vec{p}\,')\Big] = 0 \qquad \Big[a^\dagger(\vec{p}), a^\dagger(\vec{p}\,')\Big] = 0 \tag{1.2}$$

The scalar field satisfies the Klein-Gordon equation given by

$$\Big(\Box + m^2\Big)\, \Phi(x) = 0 \tag{1.3}$$

From Eq. (1.1) one can derive various distributions like the causal commutator

$$i\,\Delta(x-y) \equiv \Big[\Phi(x), \Phi(y)\Big] \tag{1.4}$$

or the Green function

$$i\,\Delta_F(x-y) \equiv \langle 0|T\Big(\Phi(x)\,\Phi(y)\Big)|0\rangle \tag{1.5}$$

where the time ordered product for Boson fields is defined by

$$T\Big(\Phi(x)\,\Phi(y)\Big) \equiv \theta(x_0 - y_0)\, \Phi(x)\,\Phi(y) + \theta(y_0 - x_0)\, \Phi(y)\,\Phi(x) \tag{1.6}$$

For convenience one can also define other distributions like

$$\begin{aligned} \Delta_+(x-y) &\equiv \langle 0|\Phi(x)\,\Phi(y)|0\rangle \\ \Delta_-(x-y) &\equiv \langle 0|\Phi(y)\,\Phi(x)|0\rangle \\ \Delta_1(x-y) &\equiv \langle 0|\Big\{\Phi(x), \Phi(y)\Big\}|0\rangle \end{aligned} \tag{1.7}$$

Let us first derive an expression for the causal commutator in Eq. (1.4) Insertion of the free fields into the commutator yields

$$i\,\Delta(x-y) =$$

$$\frac{1}{(2\pi)^3}\int d^3\vec{p}\int d^3\vec{p}'\,\frac{1}{\sqrt{4E_pE_{p'}}}\,\left(e^{i\,(p'\cdot y-p\cdot x)}\left[a(\vec{p}),a^\dagger(\vec{p}')\right]+e^{i\,(p\cdot x-p'\cdot y)}\left[a^\dagger(\vec{p}),a(\vec{p}')\right]\right)$$

$$\frac{1}{(2\pi)^3}\int d^3\vec{p}\int d^3\vec{p}'\,\frac{1}{\sqrt{4E_pE_{p'}}}\,\delta^{(3)}(\vec{p}-\vec{p}')\,\left(e^{i\,(p'\cdot y-p\cdot x)}-e^{i\,(p\cdot x-p'\cdot y)}\right)$$

$$\frac{1}{(2\pi)^3}\int d^3\vec{p}\,\frac{1}{2E_p}\,\left(e^{i\,p\cdot(y-x)}-e^{i\,p\cdot(x-y)}\right) \tag{1.8}$$

Note that $\vec{p}' = \vec{p} \rightarrow E_{p'} = E_p$. Replace in the last exponent $\vec{p} \rightarrow -\vec{p}$

$$i\,\Delta(x-y) = -i\,\frac{1}{(2\pi)^3}\int\frac{d^3\vec{p}}{E_p}\,e^{i\,p\cdot(x-y)}\,\sin p^0\,(x_0-y_0) \tag{1.9}$$

The latter we can write in a Lorentz invariant way

$$\begin{aligned} i\,\Delta(x-y) &= \frac{1}{(2\pi)^3}\int\frac{d^3\vec{p}}{2E_p}\,e^{i\,\vec{p}\cdot(\vec{x}-\vec{y})}\,\left(e^{-i\,p^0\,(x_0-y_0)}-e^{-i\,(-p^0)\,(x_0-y_0)}\right) \\ &= \frac{1}{(2\pi)^3}\int d^4p\,\delta(p^2-m^2)\,\epsilon(p^0)\,e^{-i\,p\cdot(x-y)} \end{aligned} \tag{1.10}$$

with

$$\delta(p^2-m^2) = \frac{1}{2E_p}\left(\delta(p^0-E_p)+\delta(p^0+E_p)\right) \qquad \epsilon(p^0) = \theta(p^0)-\theta(-p^0) \tag{1.11}$$

Therefore we can split up $\delta(p^2-m^2)$ into a positive and negative energy part. From Eq. (1.9) we can derive that for $x_0 = y_0$ the causal commutator $\Delta(x-y)$ vanishes. For those values $x-y$ is spacelike. Because of the Lorentz invariance of $\Delta(x-y)$ (see Eq. (1.10)) the latter vanishes for all spacelike distances. Further we can derive

$$\frac{\partial}{\partial y_0}\,i\,\Delta(x-y)|_{y_0=x_0} = i\,\delta^{(3)}(\vec{x}-\vec{y}) \tag{1.12}$$

Hence we obtain the canonical commutation relation

$$\begin{aligned} \left[\Phi(\vec{x},x_0),\Pi(\vec{y},x_0)\right] &= \frac{\partial}{\partial y_0}\,\left[\Phi(\vec{x},x_0),\Phi(\vec{y},y_0)\right]\Big|_{y_0=x_0} = \frac{\partial}{\partial y_0}\,i\,\Delta(x-y)|_{y_0=x_0} \\ &= i\,\delta^{(3)}(\vec{x}-\vec{y}) \end{aligned} \tag{1.13}$$

where $\Pi(\vec{y}, y_0)$ is the canonical momentum operator given by $\frac{\partial}{\partial y_0}\,\Phi(\vec{y}, y_0)$. From Eq. (1.10) it also follows that the causal commutator satisfies the Klein Gordon equation

$$\left(\Box_x + m^2\right)\, i\,\Delta(x-y) = 0 \tag{1.14}$$

Similar expressions as the one presented in Eq. (1.10) can be given for the distributions in Eq. (1.7)

$$\begin{aligned}
\Delta_+(x-y) &= \frac{1}{(2\pi)^3}\int d^4p\,\delta(p^2-m^2)\,\theta(p^0)\,e^{-i\,p\cdot(x-y)} \\
\Delta_-(x-y) &= \frac{1}{(2\pi)^3}\int d^4p\,\delta(p^2-m^2)\,\theta(-p^0)\,e^{-i\,p\cdot(x-y)} \\
\Delta_1(x-y) &= \frac{1}{(2\pi)^3}\int d^4p\,\delta(p^2-m^2)\,e^{-i\,p\cdot(x-y)}
\end{aligned} \tag{1.15}$$

All expressions above satisfy the Klein Gordon equation as given in Eq. (1.14). Finally we compute this equation when it acts on the time-ordered product in Eq. (1.5). First we look at the effect on this product (see Eq. (1.6)) when it is differentiated twice w.r.t. the time coordinate x_0. Differentiating once w.r.t. x_0 yields

$$\begin{aligned}
&\frac{\partial}{\partial x_0}\left(\theta(x_0-y_0)\,\Phi(x)\,\Phi(y) + \theta(y_0-x_0)\,\Phi(y)\,\Phi(x)\right) = \\
&\delta(x_0-y_0)\left[\Phi(x),\Phi(y)\right] + \left(\theta(x_0-y_0)\,\frac{\partial\Phi(x)}{\partial x_0}\,\Phi(y) + \theta(y_0-x_0)\,\Phi(y)\,\frac{\partial\Phi(x)}{\partial x_0}\right) =
\end{aligned} \tag{1.16}$$

The first term represented by the commutator equals zero because of Eqs. (1.4) and (1.9). Differentiating the expression above again w.r.t. x_0 gives

$$\begin{aligned}
&\frac{\partial}{\partial x_0}\left(\theta(x_0-y_0)\,\frac{\partial\Phi(x)}{\partial x_0}\,\Phi(y) + \theta(y_0-x_0)\,\Phi(y)\,\frac{\partial\Phi(x)}{\partial x_0}\right) = \\
&\delta(x_0-y_0)\left[\Pi(x),\Phi(y)\right] + \left(\theta(x_0-y_0)\,\frac{\partial^2\Phi(x)}{\partial^2 x_0}\,\Phi(y) + \theta(y_0-x_0)\,\Phi(y)\,\frac{\partial^2\Phi(x)}{\partial^2 x_0}\right)
\end{aligned} \tag{1.17}$$

Using the equal time commutator in Eq. (1.13) we obtain

$$\begin{aligned}
&\Box_x\Big(\theta(x_0-y_0)\,\Phi(x)\,\Phi(y) + \theta(y_0-x_0)\,\Phi(y)\,\Phi(x)\Big) = -i\,\delta^{(4)}(x-y) \\
&+\Big(\theta(x_0-y_0)\,\Box_x\Phi(x)\,\Phi(y) + \theta(y_0-x_0)\,\Phi(y)\,\Box_x\Phi(x)\Big)
\end{aligned} \tag{1.18}$$

Using the equation of motion in Eq. (1.3) and sandwiching the expression between the vacuum state we finally obtain

$$\left(\Box_x + m^2\right)\, i\,\Delta_F(x-y) = -i\,\delta^{(4)}(x-y) \tag{1.19}$$

Because of the δ-function on the right-hand side $\Delta_F(x-y)$ becomes a Green's function. The equation above admits the following solution

$$i\,\Delta_F(x-y) = i \lim_{\epsilon \to 0} \int \frac{d^4p}{(2\pi)^4}\, e^{-i\,p\cdot(x-y)}\, \frac{1}{p^2 - m^2 + i\,\epsilon} \qquad \epsilon \geq 0 \tag{1.20}$$

The ϵ occurs in order to regularize the pole at $p^2 = m^2$. Very often one omits the limit symbol which is implicitly understood. The sign of ϵ is determined in such a way that it reproduces the answer which is obtained by substituting the free field in Eq. (1.5). Until know we have considered distributions containing the product of two field only which are therefore called two-point functions. This can be generalised to products involving more fields like

$$\langle 0|T\Big(\Phi(x_1)\,\Phi(x_2)\cdots\Phi(x_n)\Big)|0\rangle \quad \text{or} \quad \langle 0|\Phi(x_1)\,\Phi(x_2)\cdots\Phi(x_n)|0\rangle \quad \text{etc.} \tag{1.21}$$

Let us take for example the four-point function

$$\langle 0|\Phi(x_1)\,\Phi(x_2)\,\Phi(x_3)\,\Phi(x_4)|0\rangle \tag{1.22}$$

Insertion of the expression for the free fields yields

$$\prod_{j=1}^{4} \int \frac{d^3\vec{p}_j}{(2\pi)^{3/2}}\, \frac{1}{2E_{p_j}}\, \langle 0|a(\vec{p}_j)\, e^{-i\,p_j\cdot x_j} + a^\dagger(\vec{p}_j)\, e^{i\,p_j\cdot x_j}|0\rangle \tag{1.23}$$

To the expression above only the following combination of creation and annihilation operators contribute

$$\langle 0|a(\vec{p}_1)\, a(\vec{p}_2)\, a^\dagger(\vec{p}_3)\, a^\dagger(\vec{p}_4)|0\rangle \qquad \langle 0|a(\vec{p}_1)\, a^\dagger(\vec{p}_2)\, a(\vec{p}_3)\, a^\dagger(\vec{p}_4)|0\rangle \tag{1.24}$$

Using the commutation relations in Eq. (1.2) we get

$$\begin{aligned}
\langle 0|a(\vec{p}_1)\, a(\vec{p}_2)\, a^\dagger(\vec{p}_3)\, a^\dagger(\vec{p}_4)|0\rangle &= \delta^{(3)}(\vec{p}_1 - \vec{p}_3)\,\delta^{(3)}(\vec{p}_2 - \vec{p}_4) \\
&\quad + \delta^{(3)}(\vec{p}_2 - \vec{p}_3)\,\delta^{(3)}(\vec{p}_1 - \vec{p}_4) \\
\langle 0|a(\vec{p}_1)\, a^\dagger(\vec{p}_2)\, a(\vec{p}_3)\, a^\dagger(\vec{p}_4)|0\rangle &= \delta^{(3)}(\vec{p}_1 - \vec{p}_2)\,\delta^{(3)}(\vec{p}_3 - \vec{p}_4)
\end{aligned} \tag{1.25}$$

Insertion of the expressions in the equation above into Eq. (1.23), we obtain after having integrated over the δ-functions

$$\langle 0|\Phi(x_1)\,\Phi(x_2)\,\Phi(x_3)\,\Phi(x_4)|0\rangle =$$

$$\int \frac{d^3\vec{p}_1}{(2\pi)^3}\,\frac{1}{2E_{p_1}}\,e^{-i\,p_1\cdot(x_1-x_4)}\int \frac{d^3\vec{p}_2}{(2\pi)^3}\,\frac{1}{2E_{p_2}}\,e^{-i\,p_2\cdot(x_2-x_3)}$$

$$+\int \frac{d^3\vec{p}_1}{(2\pi)^3}\,\frac{1}{2E_{p_1}}\,e^{-i\,p_1\cdot(x_1-x_3)}\int \frac{d^3\vec{p}_2}{(2\pi)^3}\,\frac{1}{2E_{p_2}}\,e^{-i\,p_2\cdot(x_2-x_4)}$$

$$+\int \frac{d^3\vec{p}_1}{(2\pi)^3}\,\frac{1}{2E_{p_1}}\,e^{-i\,p_1\cdot(x_1-x_2)}\int \frac{d^3\vec{p}_3}{(2\pi)^3}\,\frac{1}{2E_{p_3}}\,e^{-i\,p_3\cdot(x_3-x_4)} \tag{1.26}$$

Since

$$\Delta_+(x-y) = \int \frac{d^3\vec{p}}{(2\pi)^3}\,\frac{1}{2E_p}\,e^{-i\,p\cdot(x-y)} \tag{1.27}$$

the four-point function becomes equal to

$$\langle 0|\Phi(x_1)\,\Phi(x_2)\,\Phi(x_3)\,\Phi(x_4)|0\rangle = \Delta_+(x_1-x_2)\,\Delta_+(x_3-x_4)$$

$$+\Delta_+(x_1-x_3)\,\Delta_+(x_2-x_4)+\Delta_+(x_1-x_4)\,\Delta_+(x_2-x_3) \tag{1.28}$$

This expression can be generalised as follows (Wick's Theorem)

$$\langle 0|\Phi(x_1)\,\Phi(x_2)\cdots\Phi(x_{2n})|0\rangle =$$

$$\sum_{P(i_1\,i_2\cdots i_{2n})} \Delta_+(x_{i_1}-x_{i_2})\,\Delta_+(x_{i_3}-x_{i_4})\cdots\Delta_+(x_{i_{2n-1}}-x_{i_{2n}}) \tag{1.29}$$

with the condition that in $\Delta_+(x_{i_k}-x_{i_l})$ one has $x_k > x_l$. It is clear that in free field theory only even point functions lead to non-zero values. This is because the number of creation operators have to be equal to the number of annihilation operators. The same expression can be also written for the time ordered product using the relation

$$i\,\Delta_F(x-y) = \theta(x_0-y_0)\,\Delta_+(x-y)+\theta(y_0-x_0)\,\Delta_-(x-y) \tag{1.30}$$

In this case one has

$$\langle 0|T\Big(\Phi(x_1)\,\Phi(x_2)\cdots\Phi(x_{2n})\Big)|0\rangle =$$

$$\sum_{P(i_1\,i_2\cdots i_{2n})} \Delta_F(x_{i_1}-x_{i_2})\,\Delta_F(x_{i_3}-x_{i_4})\cdots\Delta_F(x_{i_{2n-1}}-x_{i_{2n}}) \tag{1.31}$$

The proof of Wick's theorem can e.g. be found in [1]. In the next section we can show that Eq. (1.31) can be also derived in the path integral formalism.

2 Path integrals for scalar fields

There are various ways to derive the path integral formalism. Here we follow the approach in [2] where the Path Integral is represented as a generalisation of Gaussian integrals to infinite dimension (for other approaches see [3], [4], [5]).

2.1 Gaussian real and complex integrals for finite degrees of freedom

In one dimension the Gaussian integral becomes

$$I_1 = \int_{-\infty}^{\infty} dy \, \exp\left(-\frac{1}{2}\, a\, y^2\right) = \sqrt{\frac{2\pi}{a}} \quad \text{for} \quad a > 0 \tag{2.1}$$

This expression can be generalised to higher dimensions by introducing a real symmetric matrix A which is positive definite. These properties of A imply that all eigenvalues of A are real and positive definite. Let A be an $n \times n$ matrix then the Gaussian integral becomes equal to

$$\begin{aligned} I_n &= \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_n \, \exp\left(-\frac{1}{2}\, Y^T\, A\, Y\right) = (2\pi)^{n/2}\, (\det A)^{-1/2} \\ &\text{with} \quad Y^T = (y_1, y_2, \cdots y_n) \end{aligned} \tag{2.2}$$

The result above can be easily proven by bringing A on the diagonal form

$$\begin{aligned} &A^D = S\, A\, S^{-1} \\ &A^T = A \quad \rightarrow \quad S^{-1} = S^T \\ &\det A = \det A^D \quad \rightarrow \quad |\det S| = 1 \end{aligned} \tag{2.3}$$

Further we have $Y' = S\, Y$ or $Y = S^{-1}\, Y'$ so that

$$\begin{aligned} I_n &= |\det S| \int_{-\infty}^{\infty} dy'_1 \cdots \int_{-\infty}^{\infty} dy'_n \, \exp\left(-\frac{1}{2}\, Y'^T\, A^D\, Y'\right) = \\ &\int_{-\infty}^{\infty} dy'_1 \cdots \int_{-\infty}^{\infty} dy'_n \, \exp\left(-\frac{1}{2} \sum_{i=1}^{n} \lambda_i\, (y'_i)^2\right) = \\ &(2\pi)^{n/2} \prod_{i=1}^{n} \frac{1}{\sqrt{\lambda_i}} = (2\pi)^{n/2} \left(\prod_{i=1}^{n} \lambda_i\right)^{-1/2} = (2\pi)^{n/2}\, (\det A)^{-1/2} \end{aligned} \tag{2.4}$$

where λ_i are the eigenvalues of the matrix A. The righthand side of the expression above can be also written as.

$$I_n = (2\pi)^{n/2}\,\exp\left(-\frac{1}{2}\,\mathbf{Tr}(\ln A)\right) \tag{2.5}$$

because of

$$\det A = \exp\left(\ln(\det A)\right) = \exp\left(\mathbf{Tr}(\ln A)\right) \tag{2.6}$$

which is easy to proof if the matrix A takes the diagonal form.

$$\begin{aligned} \det A &= \det A^D = \exp\left(\ln(\det A^D)\right) = \exp\left(\ln\left(\prod_{i=1}^{n}\lambda_i\right)\right) = \exp\left(\sum_{i=1}^{n}\ln\lambda_i\right) = \\ & \exp\left(\mathbf{Tr}(\ln A^D)\right) = \exp\left(\mathbf{Tr}(\ln A)\right) \end{aligned} \tag{2.7}$$

The last step is left as a proof for the reader.

Let us now consider the Gaussian integral

$$F(b) = \int_{-\infty}^{\infty} dy\,\exp\left(-\frac{1}{2}\,a\,y^2 + b\,y\right) = \sqrt{\frac{2\pi}{a}}\,\exp\left(\frac{b^2}{2a}\right) \tag{2.8}$$

The generalisation of the above integral to n dimensions is

$$\begin{aligned} F(\rho) &= \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_n\,\exp\left(-\frac{1}{2}\,Y^T A\,Y + \rho^T\,Y\right) = \\ & (2\pi)^{n/2}\,\exp\left(-\frac{1}{2}\,\mathbf{Tr}(\ln A)\right)\,\exp\left(\frac{1}{2}\,\rho^T\,A^{-1}\,\rho\right) \\ & \text{with} \quad \rho^T = (\rho_1, \rho_2, \cdots \rho_n) \end{aligned} \tag{2.9}$$

where A has the same properties as mentioned below Eq. (2.1).
Proof:
Since $\rho^T\,Y = Y^T\,\rho = \vec{\rho}\cdot\vec{Y}$ is a scalar we have

$$Y^T A\,Y - 2\,\rho^T\,Y = (Y - A^{-1}\,\rho)^T\,A\,(Y - A^{-1}\,\rho) - \rho^T\,A^{-1}\,\rho \tag{2.10}$$

Therefore

$$\begin{aligned} F(\rho) &= \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_n\,\exp\left(-\frac{1}{2}\,(Y - A^{-1}\,\rho)^T\,A\,(Y - A^{-1}\,\rho)\right. \\ & \left.+\frac{1}{2}\,\rho^T\,A^{-1}\,\rho\right) \end{aligned} \tag{2.11}$$

With $Y' = Y - A^{-1}\,\rho$ we can rewrite the integral above and get

$$\begin{aligned}
F(\rho) &= \int_{-\infty}^{\infty} dy'_1 \cdots \int_{-\infty}^{\infty} dy'_n \,\exp\left(-\frac{1}{2}\,Y'^T\,A\,Y' + \frac{1}{2}\,\rho^T\,A^{-1}\,\rho\right) = \\
&(2\pi)^{n/2}\,(\det A)^{-1/2}\,\exp\left(\frac{1}{2}\,\rho^T\,A^{-1}\,\rho\right) = \\
&(2\pi)^{n/2}\,\exp\left(-\frac{1}{2}\,\mathbf{Tr}(\ln A)\right)\,\exp\left(\frac{1}{2}\,\rho^T\,A^{-1}\,\rho\right)
\end{aligned} \tag{2.12}$$

From $F(\rho)$ we can derive the Wick theorem mentioned in the previous section below Eq. (1.28). First we express this function into components

$$\begin{aligned}
F(\rho) &= \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_n \,\exp\left(-\frac{1}{2}\,y_m\,A_{mn}\,y_n + \rho_n\,y_n\right) = \\
&(2\pi)^{n/2}\,\exp\left(-\frac{1}{2}\,\mathbf{Tr}(\ln A)\right)\,\exp\left(\frac{1}{2}\,\rho_i\,(A^{-1})_{ij}\,\rho_j\right)
\end{aligned} \tag{2.13}$$

and differentiate this expression w.r.t. $\rho = (\rho_{m_1}, \cdots, \rho_{m_p})$ and put $\rho = 0$ i.e.

$$\begin{aligned}
\left.\frac{d^p\,F(\rho)}{d\,\rho_{m_1}\cdots\rho_{m_p}}\right|_{\rho=0} &= \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_n\,y_{m_1}\cdots y_{m_p}\,\exp\left(-\frac{1}{2}\,y_m\,A_{mn}\,y_n\right) = \\
&(2\pi)^{n/2}\,\exp\left(-\frac{1}{2}\,\mathbf{Tr}(\ln A)\right) \sum_{P(m_1\,m_2\cdots m_p)} \Big\{(A^{-1})_{m_1 m_2}\,(A^{-1})_{m_3 m_4} \\
&\cdots (A^{-1})_{m_{p-1} m_p}\Big\}
\end{aligned} \tag{2.14}$$

Notice that p has to be even otherwise the integral is equal to zero. Notice that the factor 1/2 coming from $1/2\,\rho_i\,(A^{-1})_{ij}\,\rho_j$ vanishes because the term $1/2\,\rho_j\,(A^{-1})_{ji}\,\rho_i$ appears too and A is symmetric in i and j. For example take $p = 4$ then one obtains for the sum of permutations

$$(A^{-1})_{12}\,(A^{-1})_{34} + (A^{-1})_{13}\,(A^{-1})_{24} + (A^{-1})_{14}\,(A^{-1})_{23} \tag{2.15}$$

Compare the expression above with Eq. (1.28).

In the above we have considered Gaussian integrals over real integration variables and real symmetric matrices. This can be extended to complex integration variables and hermitian matrices. The latter has real eigenvalues which all have to be positive. For complex variables the integral in Eq. (2.2) becomes

$$\begin{aligned}
I_n &= \int dz_1 \int dz_1^* \cdots \int dz_n \int dz_n^* \,\exp\left(-Z^\dagger\,A\,Z\right) = (2\pi)^n\,(\det A)^{-1} = \\
&(2\pi)^n\,\exp\left(-\mathbf{Tr}(\ln A)\right)
\end{aligned} \tag{2.16}$$

The integral above can be computed by bringing again A on the diagonal form

$$
\begin{aligned}
& A^D = S\,A\,S^{-1} \\
& A^\dagger = A \quad \to \quad S^{-1} = S^\dagger \\
& \det A = \det A^D \quad \to \quad |\det S| = 1
\end{aligned} \tag{2.17}
$$

so that the complex vector Z is changed into $Z' = S\,Z$ or $Z = S^\dagger\,Z'$ Therefore Eq. (2.16) can be written as

$$
\begin{aligned}
I_n \;&=\; \int dz'_1 \int dz'^*_1 \cdots \int dz'_n \int dz'^*_n \, \exp\left(-Z'^\dagger\,A^D\,Z'\right) = \\
& 2^n \int d\Re z'_1 \int d\Im z'_1 \cdots \int d\Re z'_n \int d\Im z'_n \, \exp\left(-Z'^\dagger\,A^D\,Z'\right)
\end{aligned} \tag{2.18}
$$

with

$$
z_i = \Re z_i + \Im z_i \qquad \int dz_i \int dz^*_i = 2\int_{-\infty}^{\infty} d\Re z_i \int_{-\infty}^{\infty} d\Im z_i \tag{2.19}
$$

Using

$$
\begin{aligned}
& Z'^\dagger\,A^D\,Z' = \sum_{i=1}^{n} \lambda_i \left\{(\Re z'_i)^2 + (\Im z'_i)^2\right\} \\
& \int_{-\infty}^{\infty} d\Re z'_i \, \exp\left(-\lambda_i\,(\Re z'_i)^2\right) = \int_{-\infty}^{\infty} d\Im z'_i \, \exp\left(-\lambda_i\,(\Re z'_i)^2\right)) = \sqrt{\frac{\pi}{\lambda_i}}
\end{aligned} \tag{2.20}
$$

we obtain

$$
I_n = (2\pi)^n \left(\prod_{i=1}^{n} \lambda_i\right)^{-1} = (2\pi)^n\,(\det A)^{-1} \tag{2.21}
$$

Further we can define a function $F(\rho)$ like in Eq. (2.9) where now ρ is a complex vector. The Wick theorem can be derived in the same way as is done for real variables and matrices.

2.2 Functional differentiation and integration

Rule of differentiation

$$
\frac{\delta f(x)}{\delta f(y)} = \delta(x-y) \qquad \frac{\delta f(x)}{\delta g(y)} = 0 \quad \text{when} \quad f \neq g \tag{2.22}
$$

Definition functional

$$
F[f] = \int dx\,V(f(x)) \tag{2.23}
$$

Functional derivative

$$\frac{\delta F[f]}{\delta f(y)} = \int dx \, \frac{\delta f(x)}{\delta f(y)} \, \frac{\partial V(f(x))}{\partial f(x)} = \frac{\partial V(f(y))}{\partial f(y)} \tag{2.24}$$

If x and y are n-dimensional vectors we have

$$\frac{\delta f(x)}{\delta f(y)} = \delta^{(n)}(x-y) = \delta(x_1 - y_1) \cdots \delta(x_n - y_n) \tag{2.25}$$

and

$$F[f] = \int d^n x \, V(f(x)) \tag{2.26}$$

Functional integration:
As we have seen in the previous section the n-dimensional integral over real variables reads

$$(2\pi)^{-n/2} \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_n \, \exp\left(-\frac{1}{2}\, y_i \, A_{ij} \, y_j\right) = \exp\left(-\frac{1}{2}\, \mathbf{Tr}(\ln A)\right) \tag{2.27}$$

where n denotes a finite number of degrees of freedom. For an infinite number of degrees of freedom $n \to \infty$ which implies that i and j in the expression above becomes continuous we have the following replacements

$$(2\pi)^{-n/2} \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_n \quad \to \quad \int \mathcal{D}\, Y(x)$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} y_i \, A_{ij} \, y_j \quad \to \quad \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \, Y(y) \, A(y,x) \, Y(x)$$

$$\mathbf{Tr} A = \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij} \, A_{ij} \quad \to \quad \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \, \delta(y-x) \, A(y,x) \tag{2.28}$$

Then the finite dimensional integral in Eq. (2.27) becomes a Gaussian functional integral

$$\int \mathcal{D}\, Y(x) \, \exp\left(-\frac{1}{2} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \, Y(y) \, A(y,x) \, Y(x)\right) \equiv \exp\left(-\frac{1}{2}\, \mathbf{Tr}(\ln A)\right) \tag{2.29}$$

<u>Example</u>

$$A(y,x) = \left(\frac{\partial^2}{dy \, dx} + r\right) \delta(y-x) \quad \text{with} \quad \delta(y-x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \, e^{i\,p\,(y-x)}$$

$$\to \quad A(y,x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \, e^{i\,p\,(y-x)} \, (p^2 + r)$$

$$\to \quad \mathbf{Tr}(\ln A) = \int_{-\infty}^{\infty} dx \ln\left(\int_{-\infty}^{\infty} \frac{dp}{2\pi} \, (p^2 + r)\right) \quad \text{(divergent)} \tag{2.30}$$

We can also define in analogy to Eq. (2.9) the generating functional

$$F[\rho] = \int \mathcal{D}\, Y(x) \, \exp\left(-\frac{1}{2} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx\, Y(y)\, A(y,x)\, Y(x) + \int_{-\infty}^{\infty} dx\, \rho(x)\, Y(x)\right)$$

$$\equiv \exp\left(-\frac{1}{2}\, \mathbf{Tr}(\ln A)\right) \exp\left(\frac{1}{2} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx\, \rho(y)\, A^{-1}(y,x)\, \rho(x)\right) \qquad (2.31)$$

In our example above we get the Green function

$$G(y,x) = A^{-1}(y,x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, e^{i\,p\,(y-x)}\, \frac{1}{p^2+r} \qquad (2.32)$$

Proof:

$$A^{-1}(y,z)\, A(z,x) \equiv \int_{-\infty}^{\infty} dz\, A^{-1}(y,z)\, A(z,x) =$$

$$\int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} \frac{dq}{2\pi} \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, e^{i\,q\,(y-z)}\, e^{i\,p\,(z-x)}\, \frac{p^2+r}{q^2+r} =$$

$$\int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, \delta(q-p)\, \frac{p^2+r}{q^2+r}\, e^{i\,(q\,y - p\,x)} = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, e^{i\,p\,(y-x)} = \delta(y-x) \qquad (2.33)$$

By differentiation p times w.r.t. the source $\rho(x_i)$ one obtains the Green function

$$G(x_1, \cdots, x_p) = \left.\frac{\delta^p\, F[\rho]}{\delta\rho(x_1)\cdots\rho(x_p)}\right|_{\rho(x_i)=0} =$$

$$\int \mathcal{D}\, Y(x)\, Y(x_1)\cdots Y(x_p)\, \exp\left(-\frac{1}{2} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx\, Y(y)\, A(y,x)\, Y(x)\right) =$$

$$\exp\left(-\frac{1}{2}\, \mathbf{Tr}(\ln A)\right) \sum_{P(x_1\cdots x_p)} \left\{A^{-1}(x_1,x_2)\, A^{-1}(x_3,x_4)\cdots A^{-1}(x_{p-1},x_p)\right\} \qquad (2.34)$$

For complex functions $Z(x)$ we can derive analogous expressions. In this case the functional integral reads

$$\int \mathcal{D}\, Z(x) \int \mathcal{D}\, Z^*(x)\, \exp\left(-\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx\, Z^*(y)\, A(y,x)\, Z(x)\right) =$$

$$\exp\left(-\mathbf{Tr}(\ln A)\right) \qquad (2.35)$$

Here we want to emphasize that the derivation of the path integral is far from rigorous. Better derivations can be found in some text books like [3], [4], [5]. Note that the path integral, when it exists at all, can be only defined in Euclidian dimensions and not in Minkowski space. That there are problems in finding the correct measure of the path integral is discussed in section 6 of [4].

2.3 Generating functionals for a scalar field theory

The functional integrals can be generalised to n-dimensions by replacing $dx \to d^n x$. Taking $n = 4$ the generating functional integral for a scalar field theory is given by

$$W[J] = N \int \mathcal{D}\,\phi(x)\,\exp\left\{ i \int_{-\infty}^{\infty} d^4x \left(\mathcal{L}(\phi(x)\,\partial_\mu \phi(x)) + J(x)\,\phi(x) \right) \right\} \tag{2.36}$$

where $\mathcal{L}$ denotes the Lagrangian density and $J(x)$ is an external source. Further N is a normalisation factor chosen in such a way that $W[0] = 1$. Hence

$$N = \left[\int \mathcal{D}\,\phi(x)\,\exp\left\{ i \int_{-\infty}^{\infty} d^4x\, \mathcal{L}(\phi(x)\,\partial_\mu \phi(x)) \right\} \right]^{-1} \tag{2.37}$$

Notice that the argument of the exponent is not real since d^4x is a volume element in Minkowski space. Therefore the integral is not Gaussian anymore. Usually one performs a Wick rotation in Minkowski space i.e. $x_0 = i\,x_4$ so that $d^4x = i\,d^4x_E$ with $x_E = (x_1, x_2, x_3, x_4)$. The consequence of this Wick rotation is that all Green's functions become real and the term $i\,\epsilon$ is then missing in the denominator of Eq. (1.20). Another way to make the integral Gaussian is to multiply the exponent by a Gaussian factor so that the path integral in Eq. (2.36) becomes

$$W[J] = N \int \mathcal{D}\,\phi(x)\,\exp\left\{ i \int_{-\infty}^{\infty} d^4x \left(\mathcal{L}(\phi(x)\,\partial_\mu \phi(x)) + J(x)\,\phi(x) + i\,\frac{\epsilon}{2}\,\phi(x)^2 \right) \right\} \tag{2.38}$$

which accounts for the extra term $i\,\epsilon$. The disconnected Green's function defined in Eq. (1.21), denoted by

$$G^{(n)}(x_1, \cdots, x_n) \equiv \langle 0 | T\Big(\Phi(x_1) \cdots \Phi(x_n) \Big) | 0 \rangle\,, \tag{2.39}$$

is now obtained from

$$i^n\, G^{(n)}(x_1, \cdots, x_n) = \left. \frac{\delta^n\, W[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J(x_i)=0} \tag{2.40}$$

In this way the Green function can be interpreted as the coefficients in the Taylor expansion of the generating functional around $J = 0$ i.e.

$$W[J] = \sum_{n=0}^{n} \frac{i^n}{n!} \int d^4x_1 \cdots \int d^4x_n\, G^{(n)}(x_1, \cdots, x_n)\, J(x_1) \cdots J(x_n) \tag{2.41}$$

Notice that $\Phi(x)$ denotes a quantum field operator which works in a Hilbert space consisting of spin zero particles whereas $\phi(x)$ is a classical field. The path integral represents an integration over all classical field configurations which leads to the same Green's functions as those obtained by the operator formalism as we will show below.
Example: Free scalar field theory

Here the Lagrangian density is given by:

$$\mathcal{L} = \frac{1}{2}\,\partial_\nu\phi(x)\,\partial^\nu\phi(x) - \frac{m^2}{2}\,\phi^2(x) \tag{2.42}$$

Using Eq. (2.38) we obtain

$$\begin{aligned} W[J] &= N\int \mathcal{D}\,\phi(x)\,\exp\Big\{i\int_{-\infty}^{\infty} d^4x\,\Big(\frac{1}{2}\,\partial_\nu\phi(x)\,\partial^\nu\phi(x) - \frac{1}{2}\,(m^2 - i\epsilon)\,\phi^2(x) \\ &\qquad + J(x)\,\phi(x)\Big)\Big\} \end{aligned} \tag{2.43}$$

After partial integration w.r.t. ∂_ν this expression becomes equal to

$$\begin{aligned} W[J] &= N\int \mathcal{D}\,\phi(x)\,\exp\Big\{i\int_{-\infty}^{\infty} d^4x\,\Big(-\frac{1}{2}\,\phi(x)\Big(\Box + m^2 - i\,\epsilon\Big)\phi(x) \\ &\qquad + J(x)\,\phi(x)\Big)\Big\} \end{aligned} \tag{2.44}$$

We now proceed in the same way as in the case of one-dimensional systems shown in the previous section. First define

$$A(y,x) = i\,\Big(\Box_x + m^2 - i\,\epsilon\Big)\,\delta^{(4)}(y-x) = -i\int \frac{d^4p}{(2\pi)^4}\,e^{-i\,p\cdot(y-x)}\,\Big(p^2 - m^2 + i\,\epsilon\Big) \tag{2.45}$$

The generating functional in Eq. (2.44) equals

$$\begin{aligned} W[J] &= N\int \mathcal{D}\,\phi(x)\,\exp\Big\{-\frac{1}{2}\int_{-\infty}^{\infty} d^4y\int_{-\infty}^{\infty} d^4x\,\phi(y)\,A(y,x)\,\phi(x) \\ &\qquad + i\int d^4x\,J(x)\,\phi(x)\Big)\Big\} \end{aligned} \tag{2.46}$$

which becomes equal to

$$W[J] = N\,\exp\Big(-\frac{1}{2}\,\mathbf{Tr}(\ln A)\Big)\,\exp\Big\{-\frac{1}{2}\int_{-\infty}^{\infty} d^4y\int_{-\infty}^{\infty} d^4x\,J(y)\,A^{-1}(y,x)\,J(x)\Big\} \tag{2.47}$$

with

$$\begin{aligned} A^{-1}(y,x) &= i\,\Delta_F(y-x) = \int \frac{d^4p}{(2\pi)^4}\,e^{-i\,p\cdot(y-x)}\,\frac{i}{p^2 - m^2 + i\,\epsilon} \\ \mathbf{Tr}(\ln A) &= \int d^4x\,\ln\Big\{-i\int \frac{d^4p}{(2\pi)^4}\,\Big(p^2 - m^2 + i\,\epsilon\Big)\Big\} \end{aligned} \tag{2.48}$$

where A^{-1} represents the Green function (Feynman propagator) which is derived in Eq. (1.20) using canonical quantum field theory. Choosing

$$N = \exp\Big(\frac{1}{2}\,\mathbf{Tr}(\ln A)\Big) \tag{2.49}$$

so that $W[0] = 1$ the generating functional equals

$$W[J] = \exp\left\{-\frac{i}{2}\int_{-\infty}^{\infty} d^4y \int_{-\infty}^{\infty} d^4x\, J(y)\,\Delta_F(y-x)\,J(x)\right\} \tag{2.50}$$

The Green's function in free field theory can now be derived by differentiation w.r.t. the sources (see Eq. (2.40))

$$\frac{\delta W[J]}{\delta J(x_1)} = -\frac{i}{2}\left\{\int d^4x\,\Delta_F(x_1-x)\,J(x) + \int d^4y\,J(y)\,\Delta_F(y-x_1)\right\} W[J] \tag{2.51}$$

Using

$$\frac{\delta J(y)}{\delta J(x)} = \delta^{(4)}(y-x) \qquad \Delta_F(x-y) = \Delta_F(y-x) \tag{2.52}$$

Eq. (2.51) can be rewritten as

$$\frac{\delta W[J]}{\delta J(x_1)} = -i\left\{\int d^4x\,\Delta_F(x_1-x)\,J(x)\right\} W[J] \tag{2.53}$$

so that the one-points Green's function reads

$$i\,G^{(1)}(x_1) = \left.\frac{\delta W[J]}{\delta J(x_1)}\right|_{J(x_1)=0} = 0 \tag{2.54}$$

Proceeding in the same way we get after differentiating the generating functional twice w.r.t. the source $J(x_i)$

$$\begin{aligned}\frac{\delta^2 W[J]}{\delta J(x_1)\,\delta J(x_2)} &= \Big\{-i\,\Delta_F(x_1-x_2) + (-i)^2\int d^4x\,\Delta_F(x_1-x)\,J(x) \\ &\quad\times\int d^4y\,\Delta_F(x_2-y)\,J(y)\Big\}\,W[J]\end{aligned} \tag{2.55}$$

from which follows the two-point Green's function

$$i^2\,G^{(2)}(x_1,x_2) = \left.\frac{\delta^2 W[J]}{\delta J(x_1)\,\delta J(x_2)}\right|_{J(x_i)=0} = -i\,\Delta_F(x_1-x_2) \tag{2.56}$$

$$\begin{aligned}&\frac{\delta^3 W[J]}{\delta J(x_1)\,\delta J(x_2)\,\delta J(x_3)} = \Big\{(-i)^2\,\Delta_F(x_1-x_2)\int d^4z\,\Delta_F(x_3-z)\,J(z) \\ &+(-i)^2\,\Delta_F(x_1-x_3)\int d^4y\,\Delta_F(x_2-y)\,J(y) + (-i)^2\,\Delta_F(x_2-x_3) \\ &\times\int d^4x\,\Delta_F(x_1-x)\,J(x) + (-i)^3\int d^4x\,\Delta_F(x_1-x)\,J(x)\int d^4y\,\Delta_F(x_2-y)\,J(y) \\ &\times\int d^4z\,\Delta_F(x_3-z)\,J(z)\Big\}\,W[J]\end{aligned} \tag{2.57}$$

so that the three-points Green's function reads

$$i^3\, G^{(3)}(x_1, x_2, x_3) = \left. \frac{\delta^3\, W[J]}{\delta\, J(x_1)\, \delta\, J(x_2)\, \delta\, J(x_3)} \right|_{J(x_i)=0} = 0 \tag{2.58}$$

$$\frac{\delta^4\, W[J]}{\delta\, J(x_1)\, \delta\, J(x_2)\, \delta\, J(x_3)\, \delta\, J(x_4)} = \Big\{ (-i)^2\, \Delta_F(x_1 - x_2)\, \Delta_F(x_3 - x_4)$$

$$+(-i)^3\, \Delta_F(x_1 - x_2) \int d^4z\, \Delta_F(x_3 - z)\, J(z) \int d^4v\, \Delta_F(x_4 - v)\, J(v)$$

$$+(-i)^2\, \Delta_F(x_1 - x_3)\, \Delta_F(x_2 - x_4) + (-i)^3\, \Delta_F(x_1 - x_3) \int d^4y\, \Delta_F(x_2 - y)\, J(y)$$

$$\times \int d^4v\, \Delta_F(x_4 - v)\, J(v) + (-i)^2\, \Delta_F(x_1 - x_4)\, \Delta_F(x_2 - x_3) + (-i)^3\, \Delta_F(x_2 - x_3)$$

$$\times \int d^4x\, \Delta_F(x_1 - x)\, J(x) \int d^4v\, \Delta_F(x_4 - v)\, J(v) + (-i)^3\, \Delta_F(x_1 - x_4)$$

$$\times \int d^4y\, \Delta_F(x_2 - y)\, J(y) \int d^4z\, \Delta_F(x_3 - z)\, J(z) + (-i)^3\, \Delta_F(x_2 - x_4)$$

$$\times \int d^4x\, \Delta_F(x_1 - x)\, J(x) \int d^4z\, \Delta_F(x_3 - z)\, J(z) + (-i)^3\, \Delta_F(x_3 - x_4)$$

$$\times \int d^4x\, \Delta_F(x_1 - x)\, J(x) \int d^4y\, \Delta_F(x_2 - y)\, J(y) + (-i)^4 \int d^4x\, \Delta_F(x_1 - x)\, J(x)$$

$$\times \int d^4y\, \Delta_F(x_2 - y)\, J(y) \int d^4x\, \Delta_F(x_3 - z)\, J(z) \int d^4v\, \Delta_F(x_4 - v)\, J(v) \Big\}\, W[J] \tag{2.59}$$

Hence the four-points Green's function equals

$$i^4\, G^{(4)}(x_1, x_2, x_3, x_4) = \left. \frac{\delta^4\, W[J]}{\delta\, J(x_1)\, \delta\, J(x_2)\, \delta\, J(x_3)\, \delta\, J(x_4)} \right|_{J(x_i)=0} =$$

$$i\, \Delta_F(x_1 - x_2)\, i\, \Delta_F(x_3 - x_4) + i\, \Delta_F(x_1 - x_3)\, i\, \Delta_F(x_2 - x_4)$$

$$+i\, \Delta_F(x_1 - x_4)\, i\, \Delta_F(x_2 - x_3) \tag{2.60}$$

The Feynman graphs for the disconnected Green's functions in Eqs. (2.56) and (2.60) can be found in the first column of Fig. 1. Comparing the expression above with Eqs. (1.28) and (1.31) we conclude that the path integral method leads to the same expressions for the Green's functions as the operator formalism in Chapter 1. We can formulate the Wick theorem as follows

	$G(x_1, \cdots, x_n)$			$G_c(x_1, \cdots, x_n)$
$n = 2$		$x_1 \text{------} x_2$		$x_1 \text{------} x_2$
$n = 4$	$x_1 \text{------} x_2$ $x_3 \text{------} x_4$	$x_1 \text{------} x_3$ $x_2 \text{------} x_4$	$x_1 \text{------} x_4$ $x_2 \text{------} x_3$	

Figure 1: Disconnected and connected Green's function for $n \leq 4$ in a free scalar field theory

$$G^{(n)}(x_1, \cdots, x_n) = \frac{(i)^{n/2}}{2^{n/2}} \sum_{P(i_1, i_2 \cdots i_n)} \Delta_F(x_{i_1} - x_{i_2})\, \Delta_F(x_{i_3} - x_{i_4}) \cdots \Delta_F(x_{i_{n-1}} - x_{i_n}) \tag{2.61}$$

for n is even. The factor $2^{-n/2}$ disappears when using $\Delta_F(x_{i_k} - x_{i_l}) = \Delta_F(x_{i_l} - x_{i_k})$. For n is odd we have

$$G^{(n)}(x_1, \cdots, x_n) = 0 \tag{2.62}$$

Besides the Green functions above which are in general disconnected we also can compute the connected Green's function. The corresponding generating functional $X[J]$ is defined by

$$W[J] = \exp\{i\, X[J]\} \quad \text{or} \quad i\, X[J] = \ln W[J] \tag{2.63}$$

The connected Green's function can be obtained from the Taylor expansion

$$i\, X[J] = \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots \int d^4x_n\, G_c^{(n)}(x_1, \cdots, x_n)\, J(x_1) \cdots J(x_n) \tag{2.64}$$

with

$$i^n\, G_c^{(n)}(x_1, \cdots, x_n) = \left. \frac{\delta^n\, i\, X[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J(x_i)=0} \tag{2.65}$$

In the case of the free scalar field (see Eq. (2.42)) we get from Eq. (2.50)

$$i\, X[J] = \ln W[J] = -\frac{i}{2} \int d^4y \int d^4x\, J(y)\, \Delta_F(y - x)\, J(x) \tag{2.66}$$

From Eq. (2.65) it follows

$$i^2\, G_c^{(2)}(x_1,x_2) = \left.\frac{\delta^2\, i\, X[J]}{\delta J(x_1)\,\delta J(x_2)}\right|_{J(x_i)=0} = -i\,\Delta_F(x_1-x_2) = i^2\, G^{(2)}(x_1,x_2)$$

$$i^n\, G_c^{(n)}(x_1,\cdots,x_n) = 0 \quad \text{for} \quad n>2 \tag{2.67}$$

The Feynman graphs for the connected Green's functions in Eq. (2.67) can be found in the second column of Fig. 1. Connected Green's functions are very important because they determine the S-matrix in scattering processes. Further we can define the classical field as follows

$$\phi(x) = \frac{\delta X[J]}{\delta J(x)} \tag{2.68}$$

so that

$$\begin{aligned} G^{(1)}(x) &= \langle 0|\Phi(x)|0\rangle_J = -i\,\frac{\delta W[J]}{\delta J(x)} = -i\,W[J]\,\frac{\delta \ln W[J]}{\delta J(x)} = W[J]\,\frac{\delta X[J]}{\delta J(x)} = \\ & W[J]\,\phi(x) \end{aligned} \tag{2.69}$$

Since $W[J] = \langle 0|0\rangle_J$ the classical field equals

$$\phi(x) = \frac{\langle 0|\Phi(x)|0\rangle_J}{\langle 0|0\rangle_J} = \frac{\int \mathcal{D}\phi(x)\ \phi(x)\ \exp\left\{i\int d^4x\left(\mathcal{L}+J\,\phi\right)\right\}}{\int \mathcal{D}\phi(x)\ \exp\left\{i\int d^4x\left(\mathcal{L}+J\,\phi\right)\right\}} \tag{2.70}$$

For the free field we have

$$\phi(x) = \frac{\delta X[J]}{\delta J(x)} = -\int d^4y\,\Delta_F(x-y)\,J(y) \tag{2.71}$$

which implies the classical equation of motion at the presence of a source

$$\left(\Box_x + m^2\right)\phi(x) = J(x) \tag{2.72}$$

Another important quantity is the effective action which is defined by the Legendre transformation

$$\Gamma[\phi] = X[J] - \int d^4z\,J(z)\,\phi(z) \tag{2.73}$$

Hence it follows

$$\frac{\delta\,\Gamma[\phi]}{\delta\,J(x)} = \frac{\delta\,X[J]}{\delta\,J(x)} - \int d^4z\,\delta^{(4)}(z-x)\,\phi(z) = 0$$

$$\frac{\delta\,\Gamma[\phi]}{\delta\,\phi(x)} = \frac{\delta\,X[J]}{\delta\,\phi(x)} - \int d^4z\,\delta^{(4)}(z-x)\,J(z) = -J(x) \tag{2.74}$$

Notice that $X[J]$ only depends on the source J. In the case of a free field theory $\Gamma[\phi]$ can be exactly computed. In the case of an interacting field theory we will limit ourselves to perturbation theory in the coupling constant or to a loop expansion. In the former case we can write using Eqs. (2.66) and (2.71)

$$\Gamma[\phi] = \frac{1}{2} \int d^4y \int d^4x \, J(y) \, \Delta_F(y-x) \, J(x) \tag{2.75}$$

Using Eq. (2.72) we can rewrite the action above in

$$\Gamma[\phi] = \frac{1}{2} \int d^4y \int d^4x \left(\Box_y + m^2\right)\phi(y) \, \Delta_F(y-x) \left(\Box_x + m^2\right)\phi(x) \tag{2.76}$$

After partial integration w.r.t. y we get

$$\begin{aligned}
\Gamma[\phi] &= \frac{1}{2} \int d^4y \int d^4x \, \phi(y) \left(\Box_y + m^2\right) \Delta_F(y-x) \left(\Box_x + m^2\right) \phi(x) = \\
& -\frac{1}{2} \int d^4y \int d^4x \, \phi(y) \, \delta^{(4)}(y-x) \left(\Box_x + m^2\right) \phi(x) = \\
& -\frac{1}{2} \int d^4x \, \phi(x) \left(\Box_x + m^2\right) \phi(x)
\end{aligned} \tag{2.77}$$

Since $\Box = \partial_\nu \, \partial^\nu$ we can perform a partial integration again and obtain the classical action of the free field theory

$$\Gamma[\phi] = \frac{1}{2} \int d^4x \left(\partial_\nu \, \phi(x) \, \partial^\nu \, \phi(x) - m^2 \, \phi^2(x)\right) \equiv \int d^4x \, \mathcal{L}^{\rm free} \tag{2.78}$$

That is the reason why we call $\Gamma[\phi]$ an action. The one-particle irreducible (OPI) Green's function $\Gamma^{(n)}$ follows from the Taylor series

$$\Gamma[\phi] = \sum_{i=1}^{n} \frac{i^n}{n!} \int d^4x_1 \cdots \int d^4x_n \, \Gamma^{(n)}(x_1, \cdots, x_n) \, \phi(x_1) \cdots \phi(x_n) \tag{2.79}$$

with

$$i^n \, \Gamma^{(n)}(x_1, \cdots, x_n) = \left. \frac{\delta^n \, \Gamma[\phi]}{\delta\phi(x_1) \cdots \delta \, \phi(x_n)} \right|_{\phi(x_i)=0} \tag{2.80}$$

For an interacting field theory, OPI means that the Feynman graphs involved in the calculation of $\Gamma^{(n)}$ are all connected which follows from Eq. (2.73) since $X[J]$ leads to connected graphs. Furthermore they cannot be made disconnected by cutting a single internal line. These graphs also have no external lines. This will be illustrated if we treat an interacting field theory in the next section. For a free field theory this means that only the two-point function becomes unequal to zero. In the latter case we obtain from Eq. (2.77)

$$\Gamma[\phi] = -\frac{1}{2}\int d^4y \int d^4x\, \phi(y)\left(\Box_x + m^2\right)\delta^{(4)}(y-x)\,\phi(x) =$$

$$\frac{i^2}{2}\int d^4y \int d^4x\, \phi(y)\,\Gamma^{(2)}(y,x)\,\phi(x) \quad \rightarrow \quad \Gamma^{(2)}(y,x) = \left(\Box_x + m^2\right)\delta^{(4)}(y-x) \tag{2.81}$$

We can also find a representation of all Green's functions in momentum space via a Fourier transform. In the case of the OPI Green's function this becomes

$$\tilde{\Gamma}^{(n)}(p_1,\cdots,p_n)\,(2\pi)^4\,\delta\left(\sum_{i=1}^n p_i\right) = \int d^4x_1\cdots\int d^4x_n\,\exp\left\{i\sum_{j=1}^n p_j\cdot x_j\right\}$$

$$\Gamma^{(n)}(x_1,\cdots,x_n) \tag{2.82}$$

The two-point Green's function $\Gamma^{(2)}(y,x)$ in Eq. (2.81) can be written as

$$\Gamma^{(2)}(y,x) = -\left(\Box_y + m^2\right)\Delta_F(y-x)\left(\Box_x + m^2\right) \tag{2.83}$$

The last operator works on the lefthand part. The Fourier transform reads

$$\tilde{\Gamma}^{(2)}(p,-p) = -\left(-p^2+m^2\right)\frac{1}{p^2-m^2+i\,\epsilon}\left(-p^2+m^2\right) = -p^2+m^2$$

$$\rightarrow \quad \tilde{\Gamma}^{(2)}(p,-p) = -\left(p^2-m^2\right) = -\tilde{\Delta}_F^{-1}(p) \tag{2.84}$$

We can now expand $\Gamma^{(n)}(p_1,\cdots,p_n)$ around the momenta $p_1, p_2, \cdots p_n$. In the case of massive fields we are allowed to put $p_i = 0$. Further we can make the following expansion for $\Gamma[\phi]$

$$\Gamma[\phi] = \int d^4x\left\{-V(\phi) + \frac{1}{2}\,Z(\phi)\,\partial_\nu\,\phi\,\partial^\nu\,\phi + \text{higher order derivatives in } \phi\right\} \tag{2.85}$$

Note that $V(\phi)$ and $Z(\phi)$ are no functionals. The function $V(\phi)$ is called the effective potential which besides the classical potential also includes all quantum corrections. The same holds for $Z(\phi)$ which is the generalisation of the field renormalization constant. We can now write

$$J(x) = -\frac{\delta\,\Gamma[\phi]}{\delta\,\phi(x)} = -\int d^4z\,\frac{\delta\,\phi(z)}{\delta\,\phi(x)}\left\{-\frac{\partial\,V(\phi)}{\partial\,\phi(z)} + \frac{1}{2}\frac{\partial\,Z(\phi)}{\partial\,\phi(z)}\,\partial_\nu\,\phi(z)\,\partial^\nu\,\phi(z) + \cdots\right\} =$$

$$\frac{\partial\,V(\phi)}{\partial\,\phi(x)} - \frac{1}{2}\frac{\partial\,Z(\phi)}{\partial\,\phi(x)}\,\partial_\nu\,\phi(x)\,\partial^\nu\,\phi(x) \tag{2.86}$$

We can simplify the above expression by assuming that$\phi(x)$ is independent of x so that $\partial_\nu \phi(x) = 0$. In this case we have

$$J = \left. \frac{\partial V(\phi)}{\partial \phi(x)} \right|_{\phi(x)=const.} \tag{2.87}$$

From Eq. (2.70) it follows that for $J = 0$ where $\langle 0|0\rangle = 1$ the vacuum expectation value for the quantum field $\Phi(x)$ is given by the classical field $\phi(x)$ i.e.

$$\langle 0|\Phi(x)|0\rangle = \phi(x) = const. \tag{2.88}$$

However from Eq. (2.87) it follows that space-time independent fields have to satisfy the condition

$$\left. \frac{\partial V(\phi)}{\partial \phi(x)} \right|_{\phi(x)=const.} = 0 \tag{2.89}$$

Hence the vacuum expectation value is a stationary point of $V(\phi)$. One can show that this is a minimum. In this case we can interpret the potential as the expectation value of the energy density given by langle $0|\mathcal{H}(x)|0\rangle$ where $\mathcal{H}(x)$ is the Hamiltonian density provided condition (2.88) holds. In the case that $\phi(x)$ depends on x the vacuum expectation value is obtained from the condition

$$\frac{\delta \Gamma[\phi]}{\delta \phi(x)} = 0 \tag{2.90}$$

Let us now assume that $\phi(x) = \phi = const.$. For a free field theory with

$$V(\phi) = \frac{1}{2} m^2 \phi^2 \tag{2.91}$$

this implies

$$\frac{\partial V(\phi)}{\partial \phi} = m^2 \phi = 0 \quad \rightarrow \quad \phi = 0 \tag{2.92}$$

Hence $\langle 0|\Phi(x)|0\rangle = 0$ is the vacuum expectation value of a free field. We also can express $V(\phi)$ and $Z(\phi)$ into the OPI Green's functions. Here we will limit ourselves to $V(\phi)$ only. From Eq. (2.82) it follows

$$\begin{aligned} \Gamma(x_1, \cdots, x_n) &= \frac{1}{(2\pi)^{4(n-1)}} \int d^4p_1 \cdots \int d^4p_n \, \delta^{(4)}\left(\sum_{i=1}^{n} p_i\right) \exp\left\{ i \sum_{j=1}^{n} p_j \cdot x_j \right\} \\ &\quad \times \tilde{\Gamma}^{(n)}(p_1, \cdots, p_n) \end{aligned} \tag{2.93}$$

Expand $\Gamma^{(n)}(p_1, \cdots, p_n)$ around $p_i = 0$ (only possible when $m \neq 0$)

$$\Gamma^{(n)}(p_1, \cdots, p_n) = \Gamma^{(n)}(0, 0, \cdots, 0) + \text{derivatives w.r.t } \mathrm{p_i} \text{ at } \mathrm{p_i} = 0 \tag{2.94}$$

The generating functional becomes equal to

$$\Gamma[\phi] = \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots \int d^4x_n \frac{1}{(2\pi)^{4(n-1)}} \int d^4p_1 \cdots \int d^4p_n \, \delta^{(4)} \left(\sum_{i=1}^{n} p_i \right)$$

$$\times \exp\left\{ i \sum_{j=1}^{n} p_j \cdot x_j \right\} \left[\tilde{\Gamma}^{(n)}(0,0,\cdots,0) + \cdots \right] \phi(x_1) \cdots \phi(x_n) =$$

$$\sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots \int d^4x_n \frac{1}{(2\pi)^{4(n-1)}} \int d^4p_1 \cdots \int d^4p_n$$

$$\times \exp\left\{ i \sum_{j=1}^{n-1} p_j \cdot (x_j - x_n) \right\} \left[\tilde{\Gamma}^{(n)}(0,0,\cdots,0) + \cdots \right] \phi(x_1) \cdots \phi(x_n) =$$

$$\sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots \int d^4x_n \, \delta^{(4)}(x_1 - x_n) \cdots \delta^{(4)}(x_{n-1} - x_n) \left[\tilde{\Gamma}^{(n)}(0,0,\cdots,0) \right.$$

$$\left. + \cdots \right] \phi(x_1) \cdots \phi(x_n) = \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_n \, \tilde{\Gamma}^{(n)}(0,0,\cdots,0) \, \phi^n(x_n) + \cdots =$$

$$- \int d^4x_n \, V(\phi) + \cdots \tag{2.95}$$

Hence the effective potential reads

$$V(\phi) = - \sum_{n=1}^{\infty} \frac{i^n}{n!} \tilde{\Gamma}^{(n)}(0,0,\cdots,0) \, \phi^n \tag{2.96}$$

For a free field theory we have (see Eq. (2.84))

$$\tilde{\Gamma}^{(2)}(p. - p) = -p^2 + m^2 \qquad \Gamma^{(n)}(p_1, \cdots p_n) = 0 \quad \text{for} \quad n \neq 2 \tag{2.97}$$

From which follows the free field potential in Eq. (2.91)

2.4 Green's functions in an interacting scalar field theory

Let us take for example ϕ^4-theory. In this theory the Lagrangian density is given by

$$\mathcal{L}(\phi, \partial_\mu \phi) = \mathcal{L}_0 + \mathcal{L}_I$$

$$\text{with} \quad \mathcal{L}_0 = \frac{1}{2} \partial^\mu \phi \, \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 \qquad \mathcal{L}_I = -\frac{\lambda}{4!} \phi^4 \tag{2.98}$$

The generating functional for the disconnected Green's functions are given by

$$W[J] = N' \int \mathcal{D}\phi \, \exp\left\{ i \int d^4x \left(\mathcal{L}(\phi, \partial_\mu \phi) + J \phi \right) \right\} \tag{2.99}$$

where we have omitted for convenience the exponential damping factor in Eq. (2.38). Further N' is a normalisation factor which will be chosen in such a way that $W[0] = 1$. For $\mathcal{L} = \mathcal{L}_0$ we get the generating functional $W_0[J]$ for the free scalar field theory given in Eq. (2.50). One can now show that

$$W[J] = N' \, \exp\left\{ i \int d^4x \, \mathcal{L}_I \left(-i \, \frac{\delta}{\delta \, J(x)} \right) \right\} \, W_0[J] \tag{2.100}$$

First write the exponent in Eq. (2.99) in the form

$$\begin{aligned} &\exp\left\{ i \int d^4x \Big(\mathcal{L}(\phi, \partial_\mu \phi) + J\,\phi \Big) \right\} = \\ &\exp\left\{ i \int d^4x \, \mathcal{L}_I(\phi) \right\} \, \exp\left\{ i \int d^4x \, \mathcal{L}_0 + J\,\phi \right\} \end{aligned} \tag{2.101}$$

Now, since

$$i\,\phi(x) \, \exp\left\{ i \int d^4x \, \mathcal{L}_0(\phi) + J\,\phi \right\} = \frac{\delta}{\delta \, J(x)} \, \exp\left\{ i \int d^4z \, \mathcal{L}_0 + J\,\phi \right\} \tag{2.102}$$

it follows that

$$\begin{aligned} \int d^4x \, \mathcal{L}_I(\phi) \, \exp\left\{ i \int d^4z \, \mathcal{L}_0 + J\,\phi \right\} &= \int d^4x \, \mathcal{L}_I \left(-i \, \frac{\delta}{\delta \, J(x)} \right) \\ &\quad \times \exp\left\{ i \int d^4z \, \mathcal{L}_0 + J\,\phi \right\} \end{aligned} \tag{2.103}$$

Since the operator:

$$\int d^4x \, \mathcal{L}_I \left(-i \, \frac{\delta}{\delta \, J(x)} \right) \tag{2.104}$$

is independent of $\phi(x)$ it can be taken outside the functional integral in Eq. (2.99). Let us expand $W[J]$ up to second order in the coupling constant λ

$$W[J] = N' \, W_0[J] \left\{ 1 + (-i\,\lambda)\, \omega_1[J] + (-i\,\lambda)^2 \, \omega_2[J] + \cdots \right\} \tag{2.105}$$

with

$$\omega_1[J] = \frac{1}{4!} \, W_0[J]^{-1} \int d^4x \, \frac{\delta^4 \, W_0[J]}{\delta \, J(x)^4} \tag{2.106}$$

Differentiation of $W_0[J]$ w.r.t. $J(x)$ yields

$$\frac{\delta \, W_0[J]}{\delta \, J(x_1)} = \Big\{ -i \int d^4y_1 \, \Delta_F(x_1 - y_1) \, J(y_1) \Big\} \, W_0[J] \tag{2.107}$$

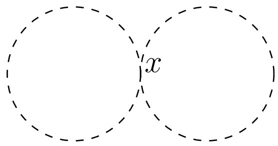

Figure 2: vacuum bubble occurring in $W[0]$ and N'

$$\frac{\delta^2 W_0[J]}{\delta J(x_1)^2} = \Big\{ -i\,\Delta_F(0) + (-i)^2 \int d^4y_1 \int d^4y_2\, \Delta_F(x_1-y_1)\,\Delta_F(x_1-y_2)\,J(y_1) \times J(y_2)\Big\}\, W_0[J] \quad (2.108)$$

$$\frac{\delta^3 W_0[J]}{\delta J(x_1)^3} = \Big\{3\,(-i)^2\,\Delta_F(0) \int d^4y_1\,\Delta_F(x_1-y_1)\,J(y_1) + (-i)^3 \int d^4y_1 \int d^4y_2 \int d^4y_3 \times \Delta_F(x_1-y_1)\,\Delta_F(x_1-y_2)\,\Delta_F(x_1-y_3)\,J(y_1)\,J(y_2)\,J(y_3)\Big\}\, W_0[J] \quad (2.109)$$

$$\frac{\delta^4 W_0[J]}{\delta J(x_1)^4} = \Big\{3\,(-i)^2\,\Delta_F^2(0) + 6\,(-i)^3\,\Delta_F(0) \int d^4y_1 \int d^4y_2\,\Delta_F(x_1-y_1)\,\Delta_F(x_1-y_2) \times J(y_1)\,J(y_2) + (-i)^4 \int d^4y_1 \int d^4y_2 \int d^4y_3 \int d^4y_4\,\Delta_F(x_1-y_1) \times \Delta_F(x_1-y_2)\,\Delta_F(x_1-y_3)\,\Delta_F(x_1-y_4)\,J(y_1)\,J(y_2)\,J(y_3)\,J(y_4)\Big\}\, W_0[J] \quad (2.110)$$

The disconnected Green function are obtained from Eq. (2.40). Up to order λ the first term in Eq. (2.110) contributes to the zero-point Green's function

$$G^{(0)} = W[0] = N'\Big\{1 - i\,\lambda\omega_1[0] - \lambda^2\,\omega_2[0]\Big\} = N'\Big(1 + \frac{i\lambda}{8} \int d^4x\Delta_F^2(0) + \cdots\Big) \quad (2.111)$$

The last term represents the vacuum bubble in Fig. 2. Since we The expression for this graph contains the function $\Delta_F(0)$ (see Eq. (2.48)) which is kwadratically divergent. If we want to normalise $W[J]$ in such a way that $W[0] = 1$ we have to choose

$$N' = 1 + i\,\lambda\,\omega_1[0] + \lambda^2\Big(\omega_2[0] - \omega_1^2[0]\Big) = 1 - i\,\frac{\lambda}{4} \int d^4x\,\Delta_F^2(0) + \cdots \quad (2.112)$$

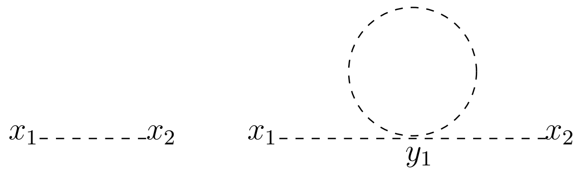

Figure 3: Graphs contributing to $G^{(2)}(x_1, x_2)$ up to order λ

After this choice we get

$$G^{(0)} = 1 \quad \text{for all } \lambda \tag{2.113}$$

Up to order λ the two-point function is given by (see Fig. 3)

$$G^{(2)}(x_1, x_2) = G_0^{(2)}(x_1, x_2) - \frac{\lambda}{2}\,\Delta_F(0) \int d^4y_1\, \Delta_F(y_1 - x_1)\, \Delta_F(y_1 - x_2) \tag{2.114}$$

The second graph in Fig. 3) is called a tadpole graph. Like the bubble graph contribution it is also characterised by the function $\Delta_F(0)$. as we have discussed above. and the four point function becomes (see Fig. 4)

$$\begin{aligned}
&G^{(4)}(x_1, x_2, x_3, x_4) = G_0^{(4)}(x_1, x_2, x_3, x_4) - \frac{\lambda}{2}\, G_0^{(2)}(x_1, x_2)\, \Delta_F(0) \int d^4y_1\, \Delta_F(y_1 - x_3) \\
&\Delta_F(y_1 - x_4) - \frac{\lambda}{2}\, G_0^{(2)}(x_1, x_3)\, \Delta_F(0) \int d^4y_1\, \Delta_F(y_1 - x_2)\, \Delta_F(y_1 - x_4) \\
&-\frac{\lambda}{2}\, G_0^{(2)}(x_1, x_4)\, \Delta_F(0) \int d^4y_1\, \Delta_F(y_1 - x_2)\, \Delta_F(y_1 - x_3) - \frac{\lambda}{2}\, G_0^{(2)}(x_2, x_3)\, \Delta_F(0) \\
&\times \int d^4y_1\, \Delta_F(y_1 - x_1)\, \Delta_F(y_1 - x_4) - \frac{\lambda}{2}\, G_0^{(2)}(x_2, x_4)\, \Delta_F(0) \int d^4y_1\, \Delta_F(y_1 - x_1) \\
&\times \Delta_F(y_1 - x_3) - \frac{\lambda}{2}\, G_0^{(2)}(x_3, x_4)\, \Delta_F(0) \int d^4y_1\, \Delta_F(y_1 - x_1)\, \Delta_F(y_1 - x_2) \\
&-i\,\lambda \int d^4y_1\, \Delta_F(y_1 - x_1)\, \Delta_F(y_1 - x_2)\, \Delta_F(y_1 - x_3)\, \Delta_F(y_1 - x_4)
\end{aligned} \tag{2.115}$$

Here $G_0^{(n)}(x_1, \cdots, x_n)$ denotes the Green's function in free field theory (see Eqs. (2.56), (2.60)). Further we have

$$\begin{aligned}
&G^{(n)}(x_1, \cdots, x_n) = 0 \quad \text{for all}\, \lambda \quad \text{if n is odd} \\
&G^{(n)}(x_1, \cdots, x_n) = G_0^{(n)}(x_1, \cdots, x_n) \quad \text{up to order}\, \lambda \quad \text{for n} > 4
\end{aligned} \tag{2.116}$$

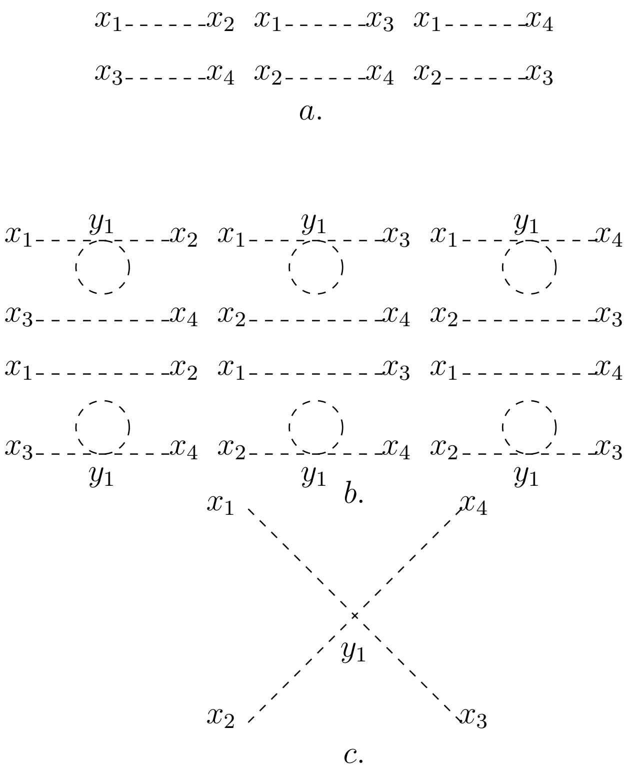

Figure 4: Graphs contributing to $G^{(4)}(x_1, x_2, x_3, x_4)$ up to order λ

From (2.65) we obtain the connected Green's function. Using Eq. (2.105) we get

$$i\,X[J] = \ln N' + \ln W_0[J] + (-i\,\lambda)\,\omega_1[J] + (-i\,\lambda)^2\left(\omega_2[J] - \frac{1}{2}\,\omega_1^2[J]\right) + \cdots \quad (2.117)$$

which up to order λ equals

$$\begin{aligned} i\,X[J] &= \ln N' - \frac{i}{2}\int d^4y_1 \int d^4y_2\, J(y_1)\,\Delta_F(y_1-y_2)\,J(y_2) - i\,\lambda \int d^4z \Big\{ -\frac{1}{8}\,\Delta_F^2(0) \\ &\quad + \frac{i}{4}\,\Delta_F(0)\int d^4y_1 \int d^4y_2\,\Delta_F(z-y_1)\,\Delta_F(z-y_2)\,J(y_1)\,J(y_2) \\ &\quad + \frac{1}{4!}\int d^4y_1 \int d^4y_2 \int d^4y_3 \int d^4y_4\,\Delta_F(z-y_1)\,\Delta_F(z-y_2)\,\Delta_F(z-y_3) \\ &\quad \times \Delta_F(z-y_4)\,J(y_1)\,J(y_2)\,J(y_3)\,J(y_4) \Big\} \qquad (2.118) \end{aligned}$$

After differentiation w.r.t. the sources (see Eq. (2.65)) the connected Green's functions are given by

$$G_c^{(0)} = \ln N'$$

$$G_c^{(2)}(x_1, x_2) = i\,\Delta_F(x_1 - x_2) - \frac{\lambda}{2}\,\Delta_F(0) \int d^4z\,\Delta_F(z - x_1)\,\Delta_F(z - x_2)$$

$$G_c^{(4)}(x_1, x_2, x_3, x_4) = -i\,\lambda \int d^4z\,\Delta_F(z - x_1)\,\Delta_F(z - x_2)\,\Delta_F(z - x_3)$$

$$\times\Delta_F(z - x_4) \tag{2.119}$$

Here $G_c^{(2)}$ and $G_c^{(4)}$ are represented by the graphs in Fig. 3 and Fig. 4c respectively. First notice that because of differentiation w.r.t. the source all vacuum bubble contributions represented by N' disappear in $G_c^{(n)}$ for $n \geq 2$. Choosing the normalisation constant N' in Eq. (2.112) we get $G_c^{(0)} = 0$. Furthermore the disconnected graphs showing up in $G^{(4)}$ (see Figs. 4a,b have all disappeared in $G_c^{(4)}$. However the tadpole graphs are still present see $G_c^{(2)}$ in Fig. 3. This feature also persists in higher order. As an illustration we have presented in order λ^2 all graphs contributing to the connected Green's function in Fig. 5 (two-point function) and Fig 6 (four-point function). In this order the six-point function shows up for the first time (see Fig. 7). All Green's functions and their corresponding Feynman graphs can be represented in momentum space where the external lines stand for the propagators. Finally we will determine the OPI Green function $\Gamma^{(n)}(x_1, \cdots, x_n)$ up to order λ with the corresponding Feynman graphs. First we have to determine the classical field from Eq. (2.68). From Eq. (2.117) we infer

$$\begin{aligned}\phi(x) \;&=\; \frac{\delta\,X[J]}{\delta\,J(x)} = -\int d^4y_1\,\Delta_F(x - y_1)\,J(y_1) - \lambda \int d^4z \left\{ \frac{i}{2}\,\Delta_F(0) \int d^4y_1\,\Delta_F(z - x) \right.\\ &\times\Delta_F(z - y_1)\,J(y_1) + \frac{\lambda}{6} \int d^4y_1 \int d^4y_2 \int d^4y_3\,\Delta_F(z - x)\,\Delta_F(z - y_1)\\ &\left.\times\Delta_F(z - y_2)\,\Delta_F(z - y_3)\,J(y_1)\,J(y_2)\,J(y_3) \right\} + \cdots \end{aligned} \tag{2.120}$$

Up to zeroth order we have the solution given in Eq. (2.72). The first order term is determined via iteration

$$J(x) = \left(\Box_x + m^2\right)\phi(x) + \lambda\,J^{(1)}(x) \tag{2.121}$$

Substitution in Eq. (2.119) yields

$$\phi(x) \;=\; -\int d^4y_1\,\Delta_F(x - y_1) \left\{ \left(\Box_{y_1} + m^2\right)\phi(y_1) + \lambda\,J^{(1)}(y_1) \right\} - \lambda \int d^4z \left\{ \frac{i}{2}\,\Delta_F(0) \right.$$

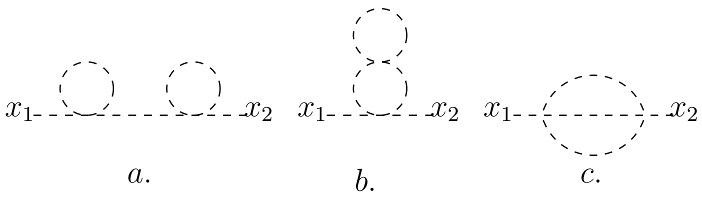

Figure 5: Graphs contributing to $G_c^{(2)}(x_1, x_2)$ in order λ^2

$$\times \int d^4y_1\, \Delta_F(z-x)\, \Delta_F(z-y_1)\, J(y_1) + \frac{\lambda}{6} \int d^4y_1 \int d^4y_2 \int d^4y_3\, \Delta_F(z-x)$$

$$\Delta_F(z-y_1)\Delta_F(z-y_2)\, \Delta_F(z-y_3)\, J(y_1)\, J(y_2)\, J(y_3) \Big\} \tag{2.122}$$

After partial integration w.r.t. y_1 we get

$$\begin{aligned} \phi(x) &= \phi(x) + \lambda \Big\{ - \int d^4y_1\, \Delta_F(x-y_1)\, J^{(1)}(y_1) + \frac{i}{2}\, \Delta_F(0) \int d^4y_1\, \Delta_F(x-y_1)\, \phi(y_1) \\ & + \frac{1}{6} \int d^4y_1\, \Delta_F(x-y_1)\, \phi^3(y_1) \Big\} \end{aligned} \tag{2.123}$$

From which follows

$$J^{(1)}(x) = \frac{i}{2}\, \Delta_F(0)\, \phi(y_1) + \frac{1}{6}\, \phi^3(y_1) \tag{2.124}$$

Hence up to order λ we get

$$J(x) = \left(\Box_x + m^2 \right) \phi(x) + \lambda \left\{ \frac{i}{2}\, \Delta_F(0)\, \phi(x) + \frac{1}{6}\, \phi^3(x) \right\} \tag{2.125}$$

The effective action in Eq. (2.73) which generates the OPI Green's functions is obtained from Eqs. (2.117) and (2.119)

$$\begin{aligned} \Gamma[\phi] &= X[J] - \int d^4x\, J(x)\, \phi(x) = -i \ln N' + \frac{1}{2} \int d^4y_1 \int d^4y_2\, \Delta_F(y_1 - y_2) \\ & \times J(y_1)\, J(y_2) + \lambda \int d^4z \Big\{ \frac{1}{8}\, \Delta_F^2(0) + \frac{i}{4}\, \Delta_F(0) \int d^4y_1 \int d^4y_2\, \Delta_F(z-y_1) \\ & \times \Delta_F(z-y_2)\, J(y_1)\, J(y_2) + \frac{1}{8} \int d^4y_1 \int d^4y_2 \int d^4y_3\, \Delta_F(z-y_1)\, \Delta_F(z-y_2) \\ & \times \Delta_F(z-y_3)\, \Delta_F(z-y_4)\, J(y_1)\, J(y_2)\, J(y_3)\, J(y_4) \Big\} \end{aligned} \tag{2.126}$$

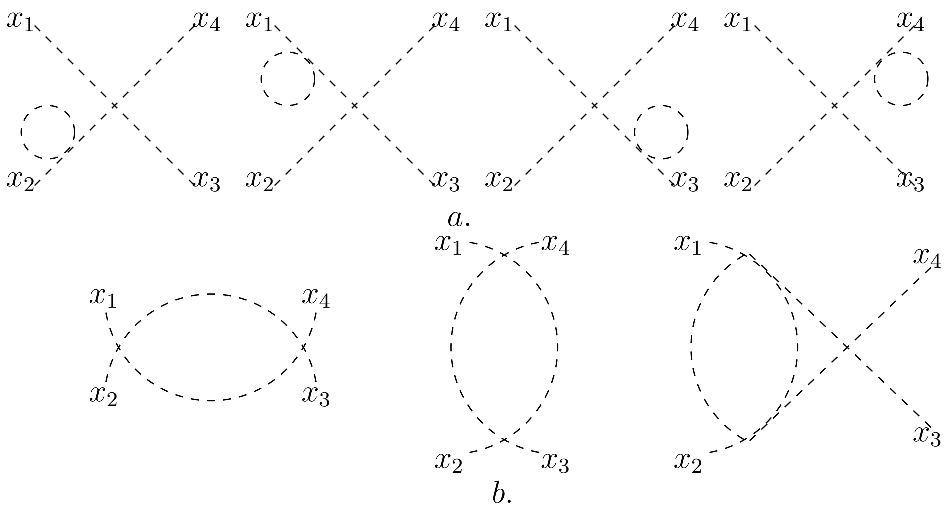

Figure 6: Graphs contributing to $G_c^{(4)}(x_1, x_2, x_3, x_4)$ in order λ^2

Substitution of $J(x)$ (see Eq. (2.124)) in the expression above yields

$$\begin{aligned}
\Gamma[\phi] &= -i \ln N' + \frac{1}{2} \int d^4y_1 \int d^4y_2 \, \Delta_F(y_1 - y_2) \left(\Box_{y_1} + m^2\right) \phi(y_1) \left(\Box_{y_2} + m^2\right) \phi(y_2) \\
&\quad + \lambda \Bigg[\int d^4z \Bigg\{ \frac{1}{8} \, \Delta_F^2(0) + \frac{i}{4} \, \Delta_F(0) \int d^4y_1 \int d^4y_2 \, \Delta_F(z - y_1) \, \Delta_F(z - y_2) \\
&\quad \times \left(\Box_{y_1} + m^2\right) \phi(y_1) \left(\Box_{y_2} + m^2\right) \phi(y_2) + \frac{1}{8} \int d^4y_1 \int d^4y_2 \int d^4y_3 \int d^4y_4 \\
&\quad \times \Delta_F(z - y_1) \, \Delta_F(z - y_2) \, \Delta_F(z - y_3) \, \Delta_F(z - y_4) \left(\Box_{y_1} + m^2\right) \phi(y_1) \\
&\quad \times \left(\Box_{y_2} + m^2\right) \phi(y_2) \left(\Box_{y_3} + m^2\right) \phi(y_3) \left(\Box_{y_4} + m^2\right) \phi(y_4) \Bigg\} + \int d^4y_1 \int d^4y_2 \\
&\quad \times \Delta_F(y_1 - y_2) \left(\frac{i}{2} \, \Delta_F(0) \, \phi(y_1) + \frac{1}{6} \phi^3(y_1) \right) \left(\Box_{y_2} + m^2\right) \phi(y_2) \Bigg]
\end{aligned} \tag{2.127}$$

After partial integration w.r.t. y_i

$$\begin{aligned}
\Gamma[\phi] &= -i \ln N' - \frac{1}{2} \int d^4y_1 \, \phi(y_1) \left(\Box_{y_1} + m^2\right) \phi(y_1) + \lambda \int d^4z \Bigg\{ \frac{1}{8} \, \Delta_F^2(0) \\
&\quad - \frac{i}{4} \, \Delta_F(0) \, \phi^2(z) - \frac{1}{4!} \, \phi^4(z) \Bigg\}
\end{aligned} \tag{2.128}$$

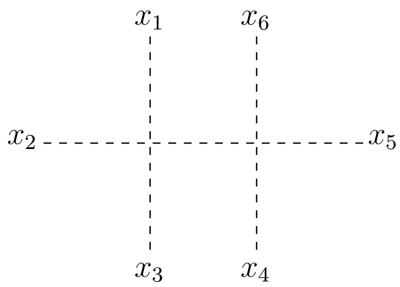

Figure 7: Graph contributing to $G_c^{(6)}(x_1, \cdots, x_6)$ in order λ^2

Choosing the normalisation constant N' as in Eq. (2.112) we get

$$\begin{aligned}\Gamma[\phi] &= \int d^4z \Big\{ \mathcal{L}_{classic}\Big(\phi(z), \partial_\mu\phi(z)\Big) - i\,\frac{\lambda}{4}\,\Delta_F(0)\,\phi^2(z) \Big\} \\ &\equiv \int d^4z \Big\{ \mathcal{L}_{eff}\Big(\phi(z), \partial_\mu\phi(z)\Big) \end{aligned} \tag{2.129}$$

where the indices *classic* and *eff* refer to the classical and effective Lagrangian density respectively. The latter represents the lowest order Lagrangian density as defined in Eq. (2.98) plus all quantum corrections which here are computed up to order λ and given by

$$-i\,\frac{\lambda}{4}\,\Delta_F(0) \int d^4z\,\phi^2(z) \tag{2.130}$$

The OPI Green's functions are given by Eq. (2.80). The result up to order λ is

$$\begin{aligned} &\Gamma^{(2)}(x_1, x_2) = \Big(\Box_{x_1} + m^2\Big)\,\delta^{(4)}(x_1 - x_2) + i\,\frac{\lambda}{2}\,\Delta_F(0)\,\delta^{(4)}(x_1 - x_2) \\ &\Gamma^{(4)}(x_1, x_2, x_3, x_4) = -\lambda\,\delta^{(4)}(x_1 - x_2)\,\delta^{(4)}(x_1 - x_3)\,\delta^{(4)}(x_1 - x_4) \\ &\Gamma^{(4)}(x_1, x_2, x_3, x_4, x_5, x_6) = 0 \end{aligned} \tag{2.131}$$

In first order the corresponding graphs are presented in Fig. 3 (two-point function) and Fig. 4c (four-point function). Notice that the graphs in Fig. 4 a,b which occur in $G_c^{(4)}(x_1, x_2, x_3, x_4)$ are one particle reducible so that they do not contribute to $\Gamma^{(4)}(x_1, x_2, x_3, x_4)$. In the momentum representation Eq. (2.130) reads

$$\begin{aligned} &i\,\tilde{\Gamma}^{(2)}(p_1, -p_1) = -i\,(p_1^2 - m^2) + \frac{i\,\lambda}{2} \int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2 - m^2 + i\,\epsilon} \\ &i\,\tilde{\Gamma}^{(4)}(p_1, p_2, p_3, -p_1 - p_2 - p_3) = -i\,\lambda \end{aligned} \tag{2.132}$$

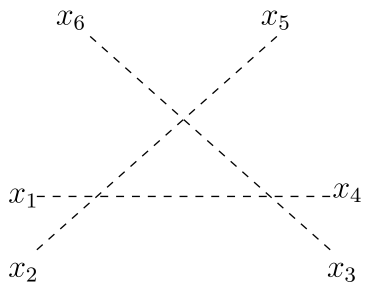

Figure 8: Graph contributing to $\Gamma^{(6)}(x_1, \cdots, x_6)$ in order λ^3

Like in the case of the connected Green's function we have also shown the graphs contributing to the OPI Green's function when the calculation is carried out up to order λ^2. They are presented in Fig. 5b,c and 6b for the two- and four point functions respectively. Notice that the six-point function in Fig. 7 is one particle reducible and does not show up in $\Gamma(x_1, \cdots x_6)$. This is a particular example of a general property which states that in higher order in λ there are no OPI tree diagrams for $n > 4$. Loop graphs are still possible which show up in order λ^3 or higher. Although the graphs for the connected and OPI Green's functions are similar, the expressions for these functions are different. In fact the OPI formulae can be obtained from the connected ones by multiplying the latter by the inverse propagators corresponding to the external lines. Finally we give the Feynman rules for a scalar field theory with a ϕ^4 interaction which hold for connected Green's functions, OPI Green's function and for the scattering amplitude M_F. They are given by

1. Draw for a given order in the coupling constant λ all topologically inequivalent connected diagrams.

2. Each internal line is represented by a propagator

$$- - - - - - \qquad\qquad i\,\Delta_F(p) = \frac{i}{p^2 - m^2 + i\,\epsilon} \tag{2.133}$$

3. For every vertex in Fig. 4c we have to assign the factor

$$-i\,\lambda\,\delta^{(4)}(p_1 + p_2 + p_3 + p_4) \tag{2.134}$$

4. Integrate over each internal momentum k which has the measure

$$\frac{d^4k}{(2\pi)^4} \tag{2.135}$$

5 One gets a factorial factor $1/n!$ if n lines start in one vertex and end in another vertex.

In this way one obtains an expression for

$$M_{fi}\,(2\pi)^4\,\delta^{(4)}\left(\sum_{j=1}^{n} p_j\right) \tag{2.136}$$

3 Ultraviolet singularities and renormalization

3.1 Origin of ultraviolet singularities

We have seen in the previous chapter that vacuum bubbles and tadpole diagrams yield infinite integrals. This is due to the fact that distributions like $\Delta_\pm(x)$ or $\Delta_F(x)$ are not defined at $x = 0$ in other words they are singular at that space-time point. However also diagrams like the ones shown in Fig. 1 diverge in spite of the fact that $x \neq y$ in Fig. 9. For historical reasons this phenomena is called ultraviolet catastrophe. This is because these graphs contribute to physical quantities like decay widths and cross sections but are still not defined in the mathematical sense. We will investigate the origin of these divergences before the solution is given. The graph in Fig 9 yields

$$\tilde{\Gamma}^{(4)}(p_1, \cdots, p_4)\,(2\pi)^4\,\delta^{(4)}\left(\sum_{j=1}^{4} p_j\right) = \frac{1}{2}\,(-i\,\lambda)^2 \int d^4x \int d^4y\, e^{-i\,y\cdot(p_3+p_4)-i\,x\cdot(p_1+p_2)} \Big[i\,\Delta_F(x-y)\Big]^2 \tag{3.1}$$

After changing variables $z = x - y$ we can perform the integral over x

$$\tilde{\Gamma}^{(4)}(p_1, \cdots, p_4)\,(2\pi)^4\,\delta^{(4)}\left(\sum_{j=1}^{4} p_j\right) = \frac{1}{2}\,(-i\,\lambda)^2\,(2\pi)^4\,\delta^{(4)}\left(\sum_{j=1}^{4} p_j\right) \int d^4z\, e^{-i\,z\cdot(p_1+p_2)} \Big[i\,\Delta_F(z)\Big]^2 \tag{3.2}$$

This integral diverges at $z = 0$ as we will show below. The Feynman propagator Eq. (2.48) in space time coordinates reads (for the derivation see chapter 15 in [8])

$$i\,\Delta_F(z) = -\frac{i}{4\pi}\,\delta(z^2) + \frac{i\,m}{8\pi\sqrt{z^2}}\,\theta(z^2)\left\{J_1(m\sqrt{z^2}) - i\,N_1(m\sqrt{z^2})\right\} + \frac{m}{4\pi^2\sqrt{z^2}}\,\theta(-z^2)\,K_1(m\sqrt{z^2}) \tag{3.3}$$

where $J_1(x)$, $N_1(x)$ and $K_1(x)$ are the first order Bessel, Neumann and Hankel functions respectively. The most singular part of the above expression is independent of m. Taking the limit $x \to 0$ we have

$$J_1(x) \to -\frac{1}{2}\,x$$

$$N_1(x) \to \frac{2}{\pi\,x}$$

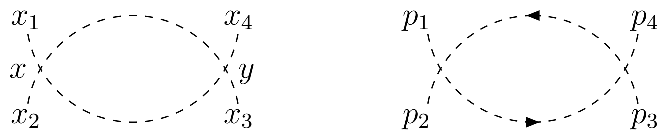

Figure 9: Graph contributing to $\Gamma^{(4)}(x_1, \cdots, x_4)$ and $\tilde{\Gamma}^{(4)}(p_1, \cdots, p_4)$

$$K_1(x) \quad \to \quad -\frac{1}{x} \tag{3.4}$$

In this limit the Feynman propagator becomes

$$i\,\Delta_F(z) = -\frac{i}{4\pi}\,\delta(z^2) - \frac{1}{4\pi^2}\,\frac{1}{z^2} = -\frac{1}{4\pi^2}\,\frac{1}{z^2 - i\,\epsilon} \tag{3.5}$$

The latter form can be derived directly from Eq. (2.48) by putting $m = 0$. Hence the most singular part of Fey-man propagator is mass independent. Substitution of Eq. (3.5) into Eq. (3.2) gives

$$\tilde{\Gamma}^{(4)}(p_1, \cdots, p_4) = -\frac{\lambda^2}{2}\,\frac{1}{(2\pi)^4}\int d^4z\, e^{-i\,z\cdot(p_1+p_2)}\,\frac{1}{(z^2 - i\,\epsilon)^2} \tag{3.6}$$

This integral is logarithmic divergent at $z = 0$. In order to show this perform a Wick rotation $z_0 = i\,z_4$ so that $z^2 = -Z_E^2$ (Z_E is an Euclidian vector). Further we have $d^4z = i\,d^4Z_E = i\,2\,\pi^2\,z_E^3\,dZ_E$. The $i\,\epsilon$ in the numerator can now be omitted and we get

$$\tilde{\Gamma}^{(4)}(p_1, \cdots, p_4) = \lambda^2\,\frac{1}{(16\pi^2)}\int_0^\infty \frac{dz_E}{z_E}\, e^{z_4\,(E_1+E_2)+i\,\vec{z}\cdot(\vec{p}_1+\vec{p}_2)} \tag{3.7}$$

the above form really shows the divergence at $z = 0$. The divergence at $z_E = \infty$ is artificial because we have approximated the propagator near $z = \infty$ by neglecting all the masses. We will perform the same analysis of the four point function in Fig. 9 but now in momentum space. Here the function reads

$$\tilde{\Gamma}^{(4)}(p_1, \cdots, p_4)\,(2\pi)^4\,\delta^{(4)}\left(\sum_{j=1}^4 p_j\right) = \frac{1}{2}\,\lambda^2\,(2\pi)^4\,\delta^{(4)}\left(\sum_{j=1}^4 p_j\right)$$

$$\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2 - m^2 + i\,\epsilon}\,\frac{1}{(k + p_1 + p_2)^2 - m^2 + i\,\epsilon} \tag{3.8}$$

We perform again a Wick rotation $k_0 = i\,k_4$, $p_{j,0} = i\,p_{j,4}$ ($j = 1 \cdots 4$) so that the four point-function becomes

$$\tilde{\Gamma}^{(4)}(p_1, \cdots, p_4) = i\,\frac{\lambda^2}{2}\int \frac{d^4k_E}{(2\pi)^4}\,\frac{1}{k_E^2 + m^2}\,\frac{1}{(k_E + p_{1,E} + p_{2,E})^2 + m^2} \tag{3.9}$$

Approximate the integrand for $k_E \to \infty$ then the integral looks like

$$-i\frac{\lambda^2}{16\pi^2}\int_0^\infty \frac{dk_E}{k_E} \tag{3.10}$$

which is logarithmic divergent at $k_E = \infty$. The divergence at $k_E = 0$ is again artificial because we neglected masses. If there would be no masses then we also have an infrared divergences at $k_E = 0$. From the observations above we infer the following. The ultraviolet divergence at $z = 0$ (short distance) corresponds with $k_E = \infty$ (large momentum) whereas the infrared divergence at $z = \infty$ (long distance) corresponds with $k_E = 0$ (small momentum) in the loop integral. Concluding our findings above we have shown that the ultraviolet divergences can be traced back to the multiplication of two distributions $\Delta_F(z)$ in Eq. (3.2) at the same space time leading to an undefined integral (compare $\delta(x)\,\delta(x)$).

3.2 Regularization and renormalisation

Since we want to compute the loop graphs we have to find a method to regularise these divergences. The only problem is that this procedure is arbitrary. This arbitrariness is reflected by the introduction of a mass scale in the integral as we will show below. Among the methods to regularise the integrals we adopt n-dimensional regularization. It boils down to the reformulation of the field theory in n dimensions. The consequence is that the coupling constant acquires a dimension which is the scale mentioned above. Let us first determine the dimension of all quantities.

$$\begin{aligned}
&\dim m = \mu \quad \to \quad \dim x = \mu^{-1} \qquad \dim d^n x = \mu^{-n} \\
&\dim S[\phi] = \int d^n x\,\mathcal{L} = 1 \quad \to \quad \dim \mathcal{L} = \mu^n \\
&\dim \partial_\mu = \mu \\
&\dim \mu^2\,\phi^2(x) = \mu^n \quad \to \quad \dim \phi(x) = \mu^{(n-2)/2} \\
&\dim \lambda_n\,\phi^4(x) = \mu^n \quad \to \quad \dim \lambda_n = \mu^{4-n} \qquad \lambda_n = \lambda\,\mu^{4-n}
\end{aligned} \tag{3.11}$$

Because the Feynman propagator for massive theories is much more simple in momentum space than in space-time space (compare Eq. (2.48) with Eq. (3.3)) we will carry out all integrals in momentum space. The Feynman rules in Eqs. (2.134) and (2.135) have to be modified as follows

3. For every vertex in Fig. 4c we have to assign the factor

$$-i\,\lambda\,\delta^{(4)}(p_1+p_2+p_3+p_4) \quad \to \quad -i\,\lambda_n\,\delta^{(n)}(p_1+p_2+p_3+p_4) \tag{3.12}$$

4. Integrate over each internal momentum k which has the measure

$$\frac{d^4k}{(2\pi)^4} \quad \rightarrow \quad \frac{d^nk}{(2\pi)^4} \tag{3.13}$$

which together with the other rules mentioned in 1, 2, 5 at the end of the previous section determines the scattering amplitude

$$M_{fi}\,(2\pi)^n\,\delta^{(n)}\left(\sum_{j=1}^{n} p_j\right) \tag{3.14}$$

As an example we compute the scattering amplitude M_{fi} for the reaction

$$p_1 + p_2 \quad \rightarrow \quad p_3 + p_4 \tag{3.15}$$

where the incoming scalar particles indicated by the momenta p_1 and p_2 belong to the initial state $|i\rangle$ and the outgoing particles, indicated by the momenta p_3 and p_4 belong to the initial state $\langle f|$. Up to order λ^2 the scattering amplitude M_{fi}, represented by the graphs in Figs. 4c, 6b, is given by (see Eq. (3.8))

$$M_{fi} = -i\,\lambda_n + J(s) + J(t) + J(u) \tag{3.16}$$

with

$$J(s) = \frac{\lambda_n^2}{2}\int \frac{d^nk}{(2\pi)^n}\,\frac{1}{k^2 - m^2 + i\,\epsilon}\,\frac{1}{(k + p_1 + p_2)^2 - m^2 + i\,\epsilon} \tag{3.17}$$

If $n < 4$ this integral is defined. We will now calculate this expression by introducing Feynman parameters. This trick reads

$$\frac{1}{a^ib^j} = \frac{\Gamma(i+j)}{\Gamma(i)\,\Gamma(i)}\int_0^1 dx\,\frac{x^{i-1}\,(1-x)^{j-1}}{[a\,x + b\,(1-x)]^{i+j}} \tag{3.18}$$

The Gamma function is defined as

$$\Gamma(x)\int_0^\infty dt\,t^{x-1}\,e^{-t} \qquad \Gamma(x+1) = x\,\Gamma(x)$$

$$\Gamma(n+1) = n! \quad \text{with} \quad n \in \mathcal{N} \tag{3.19}$$

Further $\Gamma(x)$ has poles for $x = -n$ with $n \geq 0$. Using the Feynman parameter trick the expression in Eq. (3.17) equals

$$J(s) = \frac{\lambda_n^2}{2}\int \frac{d^nk}{(2\pi)^n}\int_0^1 dx\,\frac{1}{[k^2 - 2\,x\,k\cdot(p_1+p_2) + x.(p_1+p_2)^2 - m^2 + i\,\epsilon]^2} \tag{3.20}$$

Define

$$k' = k - x(p_1 + p_2) \qquad s = (p_1 + p_2)^2 \tag{3.21}$$

$$J(s) = \frac{\lambda_n^2}{2} \int \frac{d^n k'}{(2\pi)^n} \int_0^1 dx \, \frac{1}{[k'^2 + x\,(1-x)\,s - m^2 + i\,\epsilon]^2} \tag{3.22}$$

If $a = m^- x\,(1-x)\,s > 0$ we can drop $i\,\epsilon$. Let us calculate integrals of the type

$$I_1 = \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - a} \quad \text{with} \quad a > 0 \tag{3.23}$$

Perform a Wick rotation: $k_0 = i\,k_{E,n}$ so that $d^n k = i\,d^n k_E$ and $k^2 = -k_E^2$

$$I_1 = -i \int \frac{d^n k}{(2\pi)^n} \frac{1}{k_E^2 + a} \quad \text{with} \quad d^n k_E = \frac{2\,\pi^{n/2}}{\Gamma(n/2)}\, k_E^{n-1}\, dk_E \tag{3.24}$$

$$I_1 = -i\, \frac{2\,\pi^{n/2}}{(2\pi)^n\, \Gamma(n/2)} \int_0^\infty dk_E\, \frac{k_E^{n-1}}{k_E^2 + a} \tag{3.25}$$

Substitute $k_E = \sqrt{ax}$ $(a > 0)$

$$I_1 = -i\, \frac{\pi^{n/2}}{(2\pi)^n\, \Gamma(n/2)}\, a^{n/2-1} \int_0^\infty dx\, x^{n/2-1}\, (1+x)^{-1} \tag{3.26}$$

Substitute $x = y/(1-y)$

$$I_1 = -i\, \frac{\pi^{n/2}}{(2\pi)^n\, \Gamma(n/2)}\, a^{n/2-1} \int_0^1 dy\, y^{n/2-1}\, (1-y)^{-n/2} = -i\, \frac{\pi^{n/2}}{(2\pi)^n}\, \Gamma(1-n/2)\, a^{n/2-1} \tag{3.27}$$

Differentiate the expression above $l-1$ times w.r.t. a we get

$$I_l = \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 - a)^l} = (-1)^l\, i\, \frac{\pi^{n/2}}{(2\pi)^n} \frac{\Gamma(l - n/2)}{\Gamma(l)}\, a^{n/2-l} \tag{3.28}$$

For a scalar field theory the following integrals are not necessary.

$$\int \frac{d^n k}{(2\pi)^n} \frac{k_\mu}{(k^2 - a)^l} = 0$$

$$\int \frac{d^n k}{(2\pi)^n} \frac{k_\mu}{(k^2 + 2\,k\cdot p - a)^l} = (-1)^{l+1}\, i\, \frac{\pi^{n/2}}{(2\pi)^n} \frac{\Gamma(l - n/2)}{\Gamma(l)}\, (a + p^2)^{n/2-l}\, p_\mu$$

$$\int \frac{d^n k}{(2\pi)^n} \frac{k_\mu\, k_\nu}{(k^2 + 2\,k\cdot p - a)^l} = (-1)^l\, i\, \frac{\pi^{n/2}}{(2\pi)^n} \frac{\Gamma(l - n/2 - 1)}{\Gamma(l)}$$

$$\times \Big\{ p_\mu\, p_\nu\, (l - n/2 - 1) - \frac{1}{2}\, g_{\mu\nu}\, (a + p^2) \Big\}\, (a + p^2)^{n/2-l} \tag{3.29}$$

Watch $g^{\mu\nu}\, g\mu\nu = n$. For the case $a = m^2 - x\,(1-x)\,s < 0$ we have to include the effect of $i\,\epsilon$. The integral I_1 in Eq. (3.25) becomes

$$I_1 = -i\,\frac{2\,\pi^{n/2}}{(2\pi)^n\,\Gamma(n/2)}\int_0^\infty dk_E\,\frac{k_E^{n-1}}{k_E^2 + a - i\,\epsilon} =$$

$$-i\,\frac{2\,\pi^{n/2}}{(2\pi)^n\,\Gamma(n/2)}\left\{P\int_0^\infty dk_E\,\frac{k_E^{n-1}}{k_E^2+a} + i\,\pi\int_0^\infty dk_E\,k_E^{n-1}\,\delta(k_E^2+a)\right\} \quad (3.30)$$

The first integral we already evaluated. The second integral becomes equal to

$$\int_0^\infty dk_E\,k_E^{n-1}\,\delta(k_E^2+a) = \frac{1}{2}\int_0^\infty dk_E^2\,(k_E^2)^{n/2-1}\,\delta(k_E^2+a) = \frac{1}{2}\,(-a)^{n/2-1} \quad (3.31)$$

Hence for general a we can write

$$I_1 = -i\,\frac{\pi^{n/2}}{(2\pi)^n}\left\{\Gamma(1-n/2)\,|a|^{n/2-1} + i\,\frac{\pi}{\Gamma(n/2)}\;(-a)^{n/2-1}\Big|_{a<0}\right\} \quad (3.32)$$

Differentiation w.r.t. a yields

$$I_l = i\,\frac{\pi^{n/2}}{(2\pi)^n}\,(-1)^l\left\{\Gamma(l-n/2)\,|a|^{n/2-l} + i\,\frac{\pi}{\Gamma(n/2-l)}\;(-a)^{n/2-l}\Big|_{a<0}\right\} \quad (3.33)$$

Let us first calculate $J(s)$ in Eq. (3.22) for $s<0$ $(a>0)$

$$J_I(s) = i\,\frac{\lambda_n^2}{2}\,\frac{\pi^{n/2}}{(2\pi)^n}\,\Gamma(2-n/2)\int_0^1 dx\,(m^2 - x\,(1-x)\,s)^{n/2-2} \quad (3.34)$$

Since we are interested in the limit $n \to 4$ we expand de Gamma function around $n-4$

$$\begin{aligned}\Gamma(1+x) &= \Gamma(1) + x\,\Gamma'(1) + \frac{x^2}{2}\,\Gamma''(1) + \cdots = 1 - \gamma_E + \frac{1}{2}\left(\gamma_E^2 + \frac{\pi^2}{6}\right)x^2 + \cdots = \\ & e^{-\gamma_E\,x}\left(1 + \frac{\pi^2}{6}\,x^2\right) \end{aligned} \quad (3.35)$$

where $\gamma_E = 0.5772156649\cdots$ is the Euler's constant. Define $\omega = n-4$ then $\lambda_n = \lambda^{-\omega}$ and the integral equals

$$J_I(s) = i\,\frac{\lambda^2}{32\pi^2}\,\mu^{-\omega}\left[-\frac{2}{\omega} - \gamma_E + \ln 4\pi - \int_0^1 dx\,\ln\left(\frac{m^2 - x\,(1-x)\,s}{\mu^2}\right)\right] \quad (3.36)$$

The calculation of the last integral is straightforward and we obtain

$$\begin{aligned} J_I(s) &= i\,\frac{\lambda^2}{32\pi^2}\,\mu^{-\omega}\left[-\frac{2}{\omega} - \gamma_E + \ln 4\pi - \ln\frac{m^2}{\mu^2} + 2\right. \\ &\left. + \sqrt{1-4m^2/s}\,\ln\left(\frac{\sqrt{1-4m^2/s}-1}{\sqrt{1-4m^2/s}+1}\right)\right] \end{aligned} \quad (3.37)$$

For $0 < s < 4m^2$ we have to analytically continuate $J(s)$

$$\sqrt{1 - 4m^2/s} = i\sqrt{4m^2/s - 1} \tag{3.38}$$

$$\begin{aligned} J_{II}(s) &= i\,\frac{\lambda^2}{32\pi^2}\,\mu^{-\omega}\Big[-\frac{2}{\omega} - \gamma_E + \ln 4\pi - \ln\frac{m^2}{\mu^2} + 2 \\ & -2\sqrt{4m^2/s - 1}\,\arctan\left(\frac{1}{\sqrt{4m^2/s - 1}}\right)\Big] \quad J_I(o) = J_{II}(0) \end{aligned} \tag{3.39}$$

For $s > 4\,m^2$ (III) staan twee opties open
a) We zetten $J_{II}(s)$ analytisch voort naar $s > 4\,m^2$. Dit betekent dat we de $i\,\epsilon$ in $J(s)$ nu mee moeten nemen

$$-2\sqrt{\frac{4\,(m^2 - i\,\epsilon)}{s} - 1}\,\arctan\left(\frac{1}{\sqrt{\frac{4\,(m^2 - i\,\epsilon)}{s} - 1}}\right) =$$

$$\sqrt{1 - 4\,m^2/s}\,\ln\left(\frac{1 - \sqrt{1 - 4\,m^2/s}}{1 + \sqrt{1 - 4\,m^2/s}}\right) - \pi\,i\,\theta(s - 4\,m^2)\sqrt{1 - 4\,m^2/s} \tag{3.40}$$

b) We nemen de tweede term mee in formula (3.33)

$$\begin{aligned} &\frac{1}{32\pi}\int_0^1 dx\,\theta(x\,(1-x)\,s - m^2) = \frac{1}{32\pi}\theta(s - 4\,m^2)\int_{x_1}^{x_2} = \\ &\frac{i\,\mu^{-2\,\omega}\,\lambda^2}{32\pi^2}\Big[- i\,\pi\int_{x_1}^{x_2} dx\,\theta(s - 4\,m^2)\Big] = \frac{i\,\mu^{-2\,\omega}\,\lambda^2}{32\pi^2}\Big[- i\,\pi\,\theta(s - 4\,m^2) \\ &\sqrt{1 - 4\,m^2/s}\Big] \end{aligned} \tag{3.41}$$

$$\begin{aligned} &J_{III}(s) = \frac{i\,\mu^{-\omega}\,\lambda^2}{32\pi^2}\Big[-\frac{2}{\omega} - \gamma_E + \ln 4\,pi - \ln\frac{m^2}{\mu^2} + 2 + \sqrt{1 - 4\,m^2/s} \\ &\ln\left(\frac{1 - \sqrt{1 - 4\,m^2/s}}{1 + \sqrt{1 - 4\,m^2/s}}\right) - \pi\,i\,\theta(s - 4\,m^2)\sqrt{1 - 4\,m^2/s}\Big] \end{aligned} \tag{3.42}$$

Resuming

1. We have found a way to define a Feynman integral. This procedure is called regularization.

2. It Appears that the Feynman integral an analytical function is that has cuts.

3. In the Feynman integral is a term that diverges for $n \to 4\ (\omega)$. This term cannot contribute to physical quantitities like effective cross sections.

Let us take for example de effective cross section of the proces $p_1 + p_2 \to p_3 + p_4$ calculated in higher order in λ. We take definition 1 for the S-matrix because here no tadpole occur. In lowest order in λ we have for M_{fi} the diagram in Fig. 4c. This diagram is enhanced in order λ^2 by the the diagrams in Fig. 6b. Definitie: Mandelstam variables

$$s = (p_1 + p_2)^2 \qquad t = (p_1 - p_3)^3 \qquad u = (p_1 - p_4)^2 \qquad s + t + u = 4\, m^2 \tag{3.43}$$

$$|\vec{p}_1| = |\vec{p}_2| = |\vec{p}_3| = |\vec{p}_4| = \frac{1}{2}\, sqrts - 4\, m^2 \tag{3.44}$$

In the fysical region

$$\begin{aligned} & s = 4\, (|\vec{p}_1|^2 + m^2) \geq 4\, m^2 \qquad t = -2\, |\vec{p}_1|^2\, (1 - \cos\theta) \leq 0 \\ & u = -2\, |\vec{p}_1|^2\, (1 + \cos\theta) \leq 0 \end{aligned} \tag{3.45}$$

$$\begin{aligned} M_{fi} \quad &= \quad -i\, \lambda_n \left[1 + \frac{3\, \lambda_n}{32\pi^2} \left\{ \frac{2}{\omega} + \gamma_E - \ln 4\pi + \ln \frac{m^2}{\mu^2} - \frac{1}{3}\, F_{III}(s) - \frac{1}{3}\, F_I(t) \right.\right. \\ & \left.\left. -\frac{1}{3}\, F_I(u) \right\} \right] \end{aligned} \tag{3.46}$$

Watch that $F_{III}(s)$ has a real as well as an imaginairy part because $s \geq 4\, m^2$. However $F_I(t)$ and $F_I(u)$ have no imaginairy parts since $t < 0$ and $u < 0$.

$$\begin{aligned} |M_{fi}|^2 \quad &= \quad \lambda_n^2 \left[1 + \frac{3\, \lambda_n}{16\, \pi^2} \left\{ \frac{2}{\omega} + \gamma_E - \ln 4\, \pi + \ln \frac{m^2}{\mu^2} - \frac{1}{3}\, \Re F_{III}(s) - \frac{1}{3}\, F_I(t) \right.\right. \\ & \left.\left. -\frac{1}{3}\, F_I(u) \right\} \right] \end{aligned} \tag{3.47}$$

Therefore the cross section becomes

$$\begin{aligned} d\,\sigma \quad &= \quad \frac{1}{4}\, \frac{1}{\sqrt{(p_1.p_2)^2 - m_1^2\, m_2^2}} \int \frac{d^4 p_3}{(2\pi)^3}\, \theta(p_{3,0})\, \delta(p_3^2 - m_3^2) \int \frac{d^3 p_4}{(2\pi)^3}\, \theta(p_{4,0})\, \delta(p_4^2 - m_4^2) \\ & (2\pi)^4\, \delta^{(4)}(p_1 + p_2 - p_3 - p_4)\, |M_{fi}|^2 \end{aligned} \tag{3.48}$$

$$\begin{aligned} \frac{d\,\sigma}{d\,\Omega} \quad &= \quad \frac{\lambda_n^2}{64\pi^2 s} \left[1 + \frac{3\lambda_n}{16\pi^2} \left\{ \frac{2}{\omega} + \gamma_E - \ln 4\pi + \ln \frac{m^2}{\mu^2} - \frac{1}{3}\, \Re F_{III}(s) - \frac{1}{3}\, F_I(t) \right.\right. \\ & \left.\left. -\frac{1}{3}\, F_I(u) \right\} \right] \end{aligned} \tag{3.49}$$

If we assume that this effective cross section describes the experimental data can λ be no physical parameter because $2/\omega \to \infty$ when $n \to 4$. In order to observe a physical coupling constant we have to define a renormalized coupling constant. This is called renormalisation of the coupling constant. This is done as follows Define the renormalized coupling constant $\lambda_R(i)$ by

$$\lambda_n = \lambda_R(i)\left[1 - \frac{3\lambda_r(i)}{32\pi^2}\left\{\frac{2}{\omega} + C_i\right\} + O(\lambda_R^2(i))\right] \tag{3.50}$$

The renormalized cross section becomes now

$$\begin{aligned}\frac{d\,\sigma^R}{d\,\Omega} &= \frac{\lambda_R^2(i)}{64\pi^2 s}\left[1 + \frac{3\lambda_R(i)}{16\pi^2}\left\{\gamma_E - \ln 4\pi - C_i + \ln\frac{m^2}{\mu^2} - \frac{1}{3}\Big(\Re F_{III}(s) + F_I(t)\right.\right.\\ &\left.\left.+F_I(u)\Big)\right\} + O(\lambda_R^2(i))\right]\Big\}\end{aligned} \tag{3.51}$$

The arbitrary constant C_i indicates that the perturbation series for $d\,\sigma^R$ is renormalization dependent. This dependence is indicated by the parameter i. However nature is not sensitive to the renormalisation choice. Remind that the coupling constant is not given by the theory. We have to determine it experimentally. The cross section is symmetric in t and u or equivalently $\cos\theta \Leftrightarrow -\cos\theta$. If we fit the cross section to the data then we get a whole list of values for $\lambda_R(i)$ that depends on C_i. We have to watch the following.

a. The C_i cannot be chosen so that $\lambda_R(i)$ becomes too large (condition $\lambda_R(i) \ll 1$) otherwise the perturbation serie diverges.

b. It can happen that the form of the curve disagrees with the measured values. In this case one has to take higher powers in $\lambda_R(i)$ into account.

c. It is also possible that the form never agrees with the experimental data. Higher order terms cannot bring agreement. One has to take $\lambda_R(i)$ so large that one can get a conflict with a.. The conclusion is then that the theorie is wrong.

d. However even if the cross section is in agreement with the data we are not there. We have to show that for any arbitrary other process the data can be accurately predicted with the same renormalisation scheme i.

If we use the same renormalization scheme as in the previous process then we can use the the value for the coupling constant $\lambda_R(i)$ in the new process. In this we can give an absolute prediction for the latter process.

Remark

We can the constant C_i absorb in μ^2 by defining $\mu^2 = \mu_i^2\,\exp(-C_i)$. In the literature one sees instead of $\lambda_R(i)$ the coupling constant $\lambda_R(\mu_i^2)$ or simply $\lambda_R(\mu^2)$ (μ^2 is a continuous parameter). We can now renormalize in various ways

$$\begin{aligned} C_i &= \ln \frac{\mu_i^2}{\mu^2} \\ \lambda_n &= \lambda_R(\mu_i^2) \left[1 - \frac{3\lambda_R(\mu_i^2)}{32\pi^2} \left\{ \frac{2}{\omega} + \ln \frac{\mu_i^2}{\mu^2} \right\} \right] \\ &= \lambda_R(\mu_j^2) \left[1 - \frac{3\lambda_R(\mu_j^2)}{32\pi^2} \left\{ \frac{2}{\omega} + \ln \frac{\mu_j^2}{\mu^2} \right\} \right] \end{aligned} \tag{3.52}$$

so that we can express

$$\lambda_R(\mu_j^2) = \frac{\lambda_R(\mu_i^2)}{1 - \frac{3\lambda_R(\mu_i^2)}{32\pi^2} \ln \frac{\mu_j^2}{\mu_i^2}} \tag{3.53}$$

Take μ_i fixed and vary μ_j. Let $\mu_j^2 \to 0$ then $\lambda_R(\mu_j^2) \to 0$. The zero is IR stabil point. Besides coupling constant renormalization there exists also mass renormalization. Finally we want to make another remark. Perturbation theory will not finally determine coupling constant renormalization. This is achieved by non-perturbative arguments for instance lattice calculations.

$$\lambda = \lambda_R(\mu)\, Z_\lambda \left(\frac{\Lambda}{\mu} \right) \tag{3.54}$$

Λ is a cut off or an inverse lattice distance in lattice theories. Suppose λ is finite

$$\lim_{\Lambda \to \infty} Z_\lambda = \text{finite} \;\; \lambda_R \neq 0 \qquad\qquad \lim_{\Lambda \to \infty} Z_\lambda = \text{infinite} \;\; \lambda_R = 0 \tag{3.55}$$

In perturbation theory is λ undetermined.

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