Problems

1. The function $f_p(x)$, which appears in the scalar field, is given by

$$f_p(x) = \frac{c}{(2\pi\hbar)^{3/2}} \sqrt{\frac{\hbar}{2E}}\, e^{-i\,px/\hbar}$$

   Derive the following relations (see lectures)

$$(f_p, f_{p'}) = \delta^{(3)}(\vec{p} - \vec{p}') \quad (f_p^*, f_{p'}^*) = -\delta^{(3)}(\vec{p} - \vec{p}')$$

$$(f_p^*, f_{p'}) = (f_p, f_{p'}^*) = 0$$

2. Derive the following properties of the Dirac matrices

   a.

$$\gamma_\mu \not{a}\, \gamma^\mu = -2\not{a} \quad \gamma_\mu \not{a} \not{b}\, \gamma^\mu = 4a \cdot b \gamma_\mu \not{a} \not{b} \not{c}\, \gamma^\mu = -2 \not{c} \not{b} \not{a}$$

   b.

$$Tr \not{a} \not{b} = 4\, a \cdot b$$

$$Tr \not{a} \not{b} \not{c} \not{d} = 4\Big[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)\Big]$$

   c.

$$Tr \not{a}_1 \not{a}_2 \cdots \not{a}_n = 0 \quad n \text{ is odd}$$

$$Tr \not{a}_1 \not{a}_2 \cdots \not{a}_n = (a_1 \cdot a_2) Sp \not{a}_3 \cdots \not{a}_n - (a_1 \cdot a_3) Tr \not{a}_2 \not{a}_4 \cdots \not{a}_n$$

$$+ (a_1 \cdot a_4) Sp \not{a}_2 \not{a}_3 \not{a}_5 \cdots \not{a}_n + \cdots + (a_1 \cdot a_n) Sp \not{a}_2 \not{a}_3 \cdots \not{a}_{n-1}$$

$$n \text{ is even}$$

3. Show the following relations

$$\bar{u}(\vec{p}, s)\, u(\vec{p}, s') = \delta_{ss'} \qquad \bar{v}(\vec{p}, s)\, v(\vec{p}, s') = -\delta_{ss'}$$

$$\bar{u}(\vec{p}, s)\, v(\vec{p}, s') = 0 \qquad u(-\vec{p}, s)^\dagger\, v(\vec{p}, s') = 0$$

$$\bar{v}(\vec{p}, s)\, u(\vec{p}, s') = 0 \qquad v(-\vec{p}, s)^\dagger\, u(\vec{p}, s') = 0$$

$$\bar{u}(\vec{p}, s) = u(\vec{p}, s)^\dagger\, \gamma^0 = u(-\vec{p}, s)^\dagger \qquad \bar{v}(\vec{p}, s) = v(\vec{p}, s)^\dagger\, \gamma^0 = -v(-\vec{p}, s)^\dagger \quad (1)$$

4. One can derive a number of properties for the Dirac spinors.

   a. Show that the Dirac spinors $u(\vec{p}, s)$ and $v(\vec{p}, s)$ are eigen vectors of

   $$h = \frac{\vec{\Sigma} \cdot \vec{p}}{|\vec{p}|} \quad \text{met} \quad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & \emptyset \\ \emptyset & \vec{\sigma} \end{pmatrix}$$

   provided for the quantization axis holds $\vec{n} = \vec{p}/|\vec{p}|$.

   b. Show also that these Dirac spinors are eigen states of $\gamma_5 \not{W}$ provided $W \cdot p = 0$ (W is a four vector), $\vec{n}$ is allowed to be arbitrary. Hint: use the property that the Dirac spinors are eigenstates of $\not{p}$ and show that

   $$[\gamma_5 \not{W}, \not{p}] = 0$$

   c. Show that

   $$[\gamma_5 \not{W}, h] = 0 \quad \text{mits} \quad \vec{W} \parallel \vec{p}$$

   d. For which value of W commute $\not{p}$, h and $\gamma_5 \not{W}$ with each other provided we have the additional requirement that $W^2 = -1$. Why does $W^2 < 0$ always hold?

5. Prove the following idendity (Gordon-decomposition)

$$\bar{u}(\vec{p}', s')\, \gamma^\mu\, u(\vec{p}, s) = \bar{u}(\vec{p}', s') \left[\frac{(p' + p)^\mu}{2mc} + \frac{i\sigma^{\mu\nu}(p' - p)_\nu}{2mc} \right] u(\vec{p}, s)$$

6. Compute the following inner product

$$(\psi_{i,p,s}, \psi_{j,p',s'}) = \int d^3\vec{x}\, \psi^\dagger_{i,p,s}(x) \psi_{j,p',s'}(x) \quad i, j = 1, 2$$

   where $\psi_{1,p,s}$ and $\psi_{2,p,s}$ are the solutions of Dirac equation which represents the fermion and the anti-fermion respectively.

7. The Maxwell-equations can be written in a Lorentz-covariant manner.

a. Show that the Maxwell-equations

$$\vec{\nabla} \cdot \vec{E} = \rho \qquad \vec{\nabla} \times \vec{B} - \frac{\partial\, \vec{E}}{c\partial\, t} = \frac{\vec{J}}{c}$$

follow from $\quad \partial^\mu F_{\mu\nu} = \dfrac{J_\nu}{c}$

b. Show that the Maxwell-equations

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} + \frac{\partial\, \vec{B}}{c\partial\, t} = 0$$

follow from the Bianchi-identity given by

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$$

8. The electromagnetic field $\vec{A}(x)$ in the radiation gauge is given by

$$\vec{A}(x) = \sum_\alpha \int d^3\vec{k} \left[a(\vec{k},\alpha)\, \vec{f}_k^{(\alpha)} + a^*(\vec{k},\alpha)\, \vec{f}_k^{(\alpha)*}\right]$$

with $\quad f_k^{(\alpha)}(x) = \dfrac{1}{(2\pi)^{3/2}}\, c\, \dfrac{\hbar}{\sqrt{2\omega}}\, \epsilon^{(\alpha)}(\vec{k})\, e^{-i\,kx}$

Show that for the inner product holds

$$(f_k^{(\alpha)}, f_{k'}^{(\alpha')}) = \delta_{\alpha\alpha'}\, \delta^{(3)}(\vec{k} - \vec{k}')$$

$$(f_k^{(\alpha)*}, f_{k'}^{(\alpha')*}) = -\delta_{\alpha\alpha'}\, \delta^{(3)}(\vec{k} - \vec{k}')$$

$$(f_k^{(\alpha)*}, f_{k'}^{(\alpha')}) = (f_k^{(\alpha)}, f_{k'}^{(\alpha')*}) = 0$$

9. Show that the four momentum of the fields below can be expressed in their Fourier coefficients as follows

scalar field

$$P^\nu = \frac{1}{2} \int d^3\vec{p}\, p^\nu \left[a^*(\vec{p})\, a(\vec{p}) + a(\vec{p})\, a^*(\vec{p})\right]$$

Dirac field

$$P^\nu = \sum_s \int d^3\vec{p}\, p^\nu \left[b^*(\vec{p},s)\, b(\vec{p},s) - d(\vec{p},s)\, d^*(\vec{p},s) \right]$$

E.M. field

$$P^\nu = \frac{1}{2} \sum_\alpha \int d^3\vec{k}\, \hbar k^\nu \left[a^*(\vec{k},\alpha)\, a(\vec{k},\alpha) + a(\vec{k},\alpha)\, a^*(\vec{k},\alpha) \right]$$

10. The permutation group S_3 contains the following mappings

$$\begin{aligned} 1 &\rightarrow i \\ 2 &\rightarrow j \\ 3 &\rightarrow k \\ i,j,k &= 1,2,3 \quad i \neq j \quad j \neq k \quad k \neq i \end{aligned}$$

The number of mappings is $3! = 6$ elements. The multiplication law reads

$$\left\{ \begin{array}{ccc} 1 & \rightarrow & i \\ 2 & \rightarrow & j \\ 3 & \rightarrow & k \end{array} \right\} \times \left\{ \begin{array}{ccc} i & \rightarrow & l \\ j & \rightarrow & m \\ k & \rightarrow & n \end{array} \right\} = \left\{ \begin{array}{ccc} 1 & \rightarrow & l \\ 2 & \rightarrow & m \\ 3 & \rightarrow & n \end{array} \right\}$$

Example of a multiplication

$$\left\{ \begin{array}{ccc} 1 & \rightarrow & 2 \\ 2 & \rightarrow & 3 \\ 3 & \rightarrow & 1 \end{array} \right\} \times \left\{ \begin{array}{ccc} 1 & \rightarrow & 1 \\ 2 & \rightarrow & 3 \\ 3 & \rightarrow & 2 \end{array} \right\} = \left\{ \begin{array}{ccc} 1 & \rightarrow & 3 \\ 2 & \rightarrow & 2 \\ 3 & \rightarrow & 1 \end{array} \right\}$$

Show by means of a multiplication table that this set constitutes a non-Abelian group. What are the subgroups.

11. We have the following Lagrange density which describes the interaction between spin 1/2 particles and spin 0 particles (Yukawa-theory)

$$\begin{aligned} \mathcal{L}(x) &= i\, c\, \hbar\, \bar{\psi}(x)\, \gamma^\mu\, \partial_\mu\, \psi(x) + \frac{1}{2}\, \partial^\mu\, \phi(x)\, \partial_\mu\, \phi(x) - mc^2\, \bar{\psi}(x)\, \psi(x) \\ &\quad - \frac{\mu^2 c^2}{2\hbar^2}\, \phi^2(x) - i\, \sqrt{\hbar c}\, g\, \bar{\psi}(x)\, \gamma_5\, \psi(x)\, \phi(x) - \frac{\lambda}{4!}\, \phi^4(x) \end{aligned}$$

a. Derive the equation of motions for the fields $\psi(x)$, $\bar{\psi}(x)$ and $\phi(x)$.

b. Express the stress energy momentum tensor, which is given by

$$T^{\mu\nu}(x) = \sum_r \frac{\partial \mathcal{L}(x)}{\partial \partial_\mu \phi_r(x)} \partial^\nu \phi_r(x) - g^{\mu\nu} \mathcal{L}(x)$$

in the fields above.

c. Show by using Euler Lagrange equations that $\partial_\mu T^{\mu\nu} = 0$.

12. Show that for the scalar field the Noether charge can be expressed in the Fourier-coefficients as follows.

$$Q_a = (T_a)_{kl} \int d^3\vec{p} \left[a_{+,k}(\vec{p})\, a^*_{+,l}(\vec{p}) - a^*_{-,k}(\vec{p})\, a_{-,l}(\vec{p})\right]$$

Do the same for the Dirac field.

$$Q_a = (T_a)_{kl} \sum_s \int d^3\vec{p} \left[b^*_k(\vec{p}, s)\, b_l(\vec{p}, s) + d_k(\vec{p}, s)\, d^*_l(\vec{p}, s)\right]$$

13. Prove that the relation between the structure constants where f_{abc} is totally antisymmetric in a, b and c

$$f_{abe} f_{edc} + f_{aec} f_{edb} = f_{ade} f_{ebc}$$

follows from the Jacobi identity.

$$[[T_a, T_b], T_c] + [[T_b, T_c], T_a] + [[T_c, T_a], T_b] = 0$$

14 T_a are the elements of the semi-simpel Lie-algebra $\mathcal{L}(G)$ which satisfy

$$[T_a, T_b] = i\, f_{abc}\, T_c$$

For the groups $SO(3)$ and $SU(2)$ hold that $f_{abc} = \epsilon_{abc}$. Write for these groups the fundamental and the adjoint representation for T_a.

15. Given is the L:agrange density for the free Dirac field

$$\mathcal{L}\Big(\psi(x), \partial_\lambda \psi(x)\Big) = i\, c\, \hbar\, \bar{\psi}(x)\, \gamma^\mu\, \partial_\mu \psi(x) - mc^2\, \bar{\psi}(x)\, \psi(x)$$

The chiral $U(1)$ transformation of the Dirac field is given by

$$\psi(x) \quad \rightarrow \quad \psi'(x) = e^{i\,\alpha(x)\gamma_5}\, \psi(x)$$

a. Show that the Lagrange density above is not invariant under chiral transformationis even when $\alpha(x)$ is indepent of x (global transformations).

b. Calculate the Noether current $j^5_\mu(x)$ and determine $\partial^\mu j^5_\mu(x)$.

c. Wenn is the Lagrange density global invariant under the gauge transformation above?

16. The Lagrange densities for the interaction of the E.M. veld $A_\mu(x)$ with the scalar field $\phi(x)$ and the Dirac field $\psi(x)$ respectively are given by

$$\begin{aligned} \mathcal{L}^B(x) &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D^\mu \phi)^* \; D_\mu \phi - \frac{m^2 c^2}{\hbar^2} \, \phi^* \phi \\ \mathcal{L}^F(x) &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \, c \, \hbar \, \bar{\psi} \, \gamma^\mu \, D_\mu \psi - mc^2 \, \bar{\psi} \, \psi \\ \text{with} \qquad & D_\mu = \partial_\mu + \frac{ie}{\hbar c} \, A_\mu \end{aligned}$$

as covariant derivative

a. Derive the equations of motion for all fields.

b. Calculate the stress energy momentum tensor $T^{\mu\nu}$ and show that $\partial_\mu T^{\mu\nu} = 0$.

17. For an arbitrary quantum field operator $\hat{\varphi}_r(x)$ with spin r is given that

$$[\hat{P}^\mu, \hat{\varphi}_r(x)] = -i \, \hbar \, \partial^\mu \hat{\varphi}_r(x) \tag{2}$$

Show that the relation above holds for the scalar field and the Dirac field. Use the $\hat{P}^\mu$ which are derived for these fields.

a. skalaire veld

$$\hat{P}^0 = \frac{1}{c} \int d^3\vec{x} \, \hat{\mathcal{H}} \qquad \hat{P}^k = \frac{1}{c} \int d^3\vec{x} \, \hat{\Pi} \, \partial^k \hat{\phi} \tag{3}$$

b. Dirac field

$$\hat{P}^\mu = \int d^3\vec{x} \, \hat{\Pi} \, \partial^\mu \hat{\psi} \tag{4}$$

$$: \hat{P}^\mu := \int d^3\vec{p}\; p^\mu \,\hat{N}(\vec{p}) \tag{5}$$

18. Using the expression for the E.M. veld $\vec{\hat{A}}$ one can determine fields $\vec{\hat{E}}$ and $\vec{\hat{B}}$.

    a. Calculate the following commutators

    $$[\hat{E}_i(\vec{x},t), \hat{E}_j(\vec{x}',t)] \quad [\hat{E}_i(\vec{x},t), \hat{B}_j(\vec{x}',t)] \quad [\hat{B}_i(\vec{x},t), \hat{B}_j(\vec{x}',t)]$$

    b. Determine the expectation values

    $$\langle 0|\vec{\hat{E}}(\vec{x},t)|0\rangle \qquad \langle 0|\vec{\hat{E}}(\vec{x},t)\cdot\vec{\hat{E}}(\vec{x},t)|0\rangle$$

    and show that the latter diverges

    c. To lift the divergence one defines the average field operator $\vec{\hat{E}}(b,t)$ as follows

    $$\vec{\hat{E}}(b,t) = \frac{1}{b^3}\int d^3\vec{x}\, e^{-\vec{x}\cdot\vec{x}/2b^2}\,\vec{\hat{E}}(\vec{x},t)$$

    Calculate the expectation value

    $$\langle 0|\vec{\hat{E}}(b,t)\cdot\vec{\hat{E}}(b,t)|0\rangle$$

    en laat zien dat deze eindig is

19. Consider the following scattering process $1{+}2 \;\rightarrow\; 3{+}4$ where $m_1 = m_2$ and $m_3 = m_4$ (m_i is the mass of particle i). The law of conservation of energy and momentum (four momentum) reads: $p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu$

    a. Define de Mandelstam variabels s, t and u

    $$s = (p_1+p_2)^2 \qquad t = (p_1-p_3)^2 \qquad u = (p_1-p_4)^2$$

    Show that

    $$s + t + u = 2\,m_1^2 + 2\,m_3^2$$

b. Express in the center of mass system the energy E_i and the momentum $|\vec{p}_i|$ for each particle i in the variables s, t en u.

c. The laboratory system is the system where one of the incoming particles is at rest. Choose particle 2 to be at rest. Express in the laboratory system the energies E_i and the momenta $|\vec{p}_i|$ for each particle i in the variables s, t en u.

d. Express the scattering angles in b en c, given by $\cos\theta_{cm}$ and $cos\theta_{13,lab}$, $cos\theta_{14,lab}$ respectively, in the variables s, t en u.

20. The causal commutator in its Lorentz invariant form is given by

$$i\,\Delta(x-x') = \frac{1}{(2\pi)^3}\int d^4p\,\delta(p^2-\mu^2)\,\epsilon(p^0)\,e^{i\,p\,(x-x')}$$

Show that

$$i\,\Delta(x-x') = \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{2E_p}\left\{e^{-i\,p^0\,(x_0'-x_0)} - e^{-i\,p^0\,(x_0-x_0')}\right\}e^{i\,\vec{p}\cdot(\vec{x}-\vec{x}')}$$

Show that it is impossible to transform a four vector with $p^0 > 0$ into a four vector with $p^0 < 0$ and vice versa.

21. A scalar current density is given by

$$J(x) =: \phi^2(x) :$$

Show that

a.

$$\begin{aligned} J(x)\,J(y) &= 2\,\Delta_+^2(x-y)\,1 + 4\,\Delta_+(x-y)\ :\phi(x)\,\phi(y): \\ &\quad + :\phi^2(x)\,\phi^2(y): \end{aligned}$$

b.

$$\Big[J(x), J(y)\Big] = 2\,i\,\Delta(x-y)\,\Delta_1(x-y)\,1 + 4\,i\,\Delta(x-y)\ :\phi(x)\,\phi(y):$$

c.

$$T\Big((x)\,J(y)\Big) = 2\,(i\,\Delta_F(x-y))^2\,1 + 4\,i\,\Delta(x-y)\;:\phi(x)\,\phi(y): \\ + :\phi^2(x)\,\phi^2(y):$$

Hint:express $:\phi(x)$ in creation and annihilation operators and calculate a..

22. Calculate the matrix element $S_{fi}^{(1)} = \langle f|U^{(1)}(\infty,-\infty)|i\rangle$ for ϕ^4 theory.

23. Prove the relations

$$\frac{1}{a\,b} = \int_0^1 dx\,\frac{1}{[a\,x + b\,(1-x)]^2}$$

$$\frac{1}{a^i\,b^j} = \frac{\Gamma(i+j)}{\Gamma(i)\,\Gamma(j)}\int_0^1 dx\,\frac{1}{[a\,x+b\,(1-x)]^{i+j}}$$

24. Show that for arbitrary l and $a > 0$ the follow holds

$$\int \frac{d^nk}{(2\pi)^n}\,\frac{1}{[k^2-a]^l} = (-1)^l\,\frac{i\,\pi^{n/2}}{(2\pi)^n}\,\frac{\Gamma(l-n/2)}{\Gamma(l)}\,a^{n/2-l}$$

$$\int \frac{d^nk}{(2\pi)^n}\,\frac{k_\mu}{[k^2-a]^l} = 0$$

$$\int \frac{d^nk}{(2\pi)^n}\,\frac{k_\mu}{[k^2+2\,k\cdot p-a]^l} = (-1)^l\,\frac{i\,\pi^{n/2}}{(2\pi)^n}\,\frac{\Gamma(l-n/2)}{\Gamma(l)}$$

$$\times(a+p^2)^{n/2-l}\,(-p_\mu)$$

$$\int \frac{d^nk}{(2\pi)^n}\,\frac{k_\mu\,k_\nu}{[k^2+2\,k\cdot p-a]^l} = (-1)^l\,\frac{i\,\pi^{n/2}}{(2\pi)^n}\,\frac{\Gamma(l-n/2-1)}{\Gamma(l)}$$

$$\times\Big\{p_\mu\,p_\nu\,(l-n/2-1) - \frac{1}{2}\,g_{\mu\nu}\,(a+p^2)\Big\}\,(a+p^2)^{n/2-l}$$

25. In the case that we allow for tadpoles in ϕ^4 theory the two point Green's connected function up to first order in λ is given by

$$G_{1c}(x_1, x_2) = G_0(x_1, x_2) + \frac{i\,\lambda}{2} \int d^4x\, \Delta_F(x_1 - x)\, \Delta_F(x_2 - x)\, \Delta_F(0)$$

   a. Write this Green's function in the momentum representation in other words calculate the Fourier transform of G_{1c}.

   b. Carry out the integral over the inner momentum by using n-dimensional regularisation.

   c. Render the Green's function finite by mass renormalisation i.e. replace the bare mass μ by

$$m^2 \quad \rightarrow \quad m_R^2 \left(1 + \frac{a}{\omega}\right) \qquad \omega = n - 4$$

26. Prove the following identities in n dimensions if $\{\gamma_\mu, \gamma_\nu\} = 2\, g_{\mu\nu}\, 1$ and $\gamma_\mu\, \gamma^\mu = \delta^\mu_\mu\, 1 = n\, 1$

$$\gamma_\mu\, \gamma_\lambda\, \gamma^\mu = (2 - n)\, \gamma_\lambda \qquad \gamma_\mu\, \gamma_\alpha\, \gamma_\beta\, \gamma^\mu = 4\, g_{\alpha\beta}\, 1 + (n - 4)\, \gamma_\alpha\, \gamma_\beta$$

$$\gamma_\mu\, \gamma_\alpha\, \gamma_\beta\, \gamma_\lambda\, \gamma^\mu = -2\, \gamma_\lambda\, \gamma_\beta\, \gamma_\alpha - (n - 4)\, \gamma_\alpha\, \gamma_\beta\, \gamma_\lambda$$

27. Verify that the Feynman integral $J_I(s)$ ($s \leq 0$) with

$$J_I(s) = \frac{\lambda_n^2}{2} \int \frac{d^n k}{(2\pi)^n} \frac{1}{[k^2 - m^2]\,[(k + p_1 + p_2)^2 - m^2]}$$

is given by

$$\begin{aligned} J_I(s) &= \frac{i\,\lambda^2}{32\,\pi^2} \left[-\frac{2}{\omega} - \gamma_E + \ln 4\pi - \ln \frac{m^2}{\mu^2} + 2 \right. \\ &\quad \left. + \sqrt{1 - \frac{4\,m^2}{s}}\, \ln \left(\frac{\sqrt{1 - 4\,m^2/s} - 1}{\sqrt{1 - 4\,m^2/s} + 1} \right) \right] \end{aligned}$$

with $\omega = n - 4$.