QUANTUM FIELD THEORY

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Recommended literature

1 A. Bailin and A. Love, "Introduction to Gauge Field Theory", Adam Hilger, ISBN 0-85274-817-5.

2 L. Ryder, "Quantum Field Theory", Cambridge Univ. Press, ISBN 0 521 33859 X.

3 R.J. Rivers, "Path Integral Methods in Quantum Field Theory", Cambridge Univ. Press, ISBN 0 521 25979 7.

4 "Quantum Field Theory, A Modern Introduction", M. Kaku, Oxford University Press, ISBN 0-19-509158-2 (pbk). hoofdstukken 1 t/m 4.

5 Ta-Pei Cheng and Ling-Fong Li, "Gauge theory of Elementary Particle Physics", Oxford Univ. Press, ISBN 0-19-851956-7.

6 D.J. Amit, "Field Theory, The Renormalization Group and Critical Phenomena", World Scientific, ISBN 9971-966-10-7.

7 M. Le Bellac, "Quantum and Statistical Field Theory", Clarendon Press Oxford, ISBN 0-19-853929-0.

1 Relativistic classical field theory

1.1 The principle of special relativity

In the classical theory t is shown that the speed of light is not the same in all inertial systems i.e. $c' \neq c$ ($c' = c + v_0$ follows from $x' = x + v_0 t$). Further Galilei transformations do not leave the Maxwell equations, which describe electromagnetism, covariant (Larmoor, H.A. Lorentz 1904, H. Poincaré 1905).
However the Michelson-Morley experiment (1887) reveals that the speed of light in all inertial systems is the same and finite. From this we conclude that we need another symmetry group. It turns out that the Maxwell equations and the invariance of the speed of light are maintained if we assume the Lorentz group as the symmetry group of nature. (or her extension the Poincaré group). This follows from the Principle of Special Relativity as formulated by A. Einstein in 1905. This states

1 Relativity principle: The laws of nature are the same in all inertial systems. (analogue of the classical relativity).

2 Principle of the invariant speed of light: The speed of light is the same in all inertial systems en and it is the largest speed at all.

A light signal is emitted from the origin in the coordinate system S (ct, x, y, z) and S' $(c't', x', y', z')$. The origins coincide at $t = t' = 0$. The radius of the spherical wave is given by $|\vec{x}| = ct$ en $|\vec{x}'| = c't'$ respectively. According to Einstein is $c' = c$. Next we have the equation

$$c^2t^2 - |\vec{x}|^2 = c^2t'^2 - |\vec{x}'|^2 = 0 \tag{1.1}$$

The above equation can be rewritten using the metric tensor. For that purpose we introduce a four dimensional space, the Minkowski space. This unifies the three dimensional space with time. Choose a basis in Minkowski space which is given by

$$e_0, e_1, e_2, e_3 \quad \rightarrow \quad e_\mu \quad \mu = 0, 1, 2, 3 \tag{1.2}$$

where e_0 the time vector. A four vector (denoted without an arrow) can then be denoted as $x = x^\mu \, e_\mu$ (watch the index). This can be also written in coordinate form $x = (x^0, x^1, x^2, x^3) = (ct, \vec{x})$. From mathematics it is known that each vector space V of dimension n has a dual vector space V^* with the same dimension. The latter consists of all linear functions (mappings) $\{f\}$ for which holds

$$f: \quad V \quad \rightarrow \quad \Re \quad \text{met} \quad x \in V \quad f \in V^* \quad f(x) \in \Re \tag{1.3}$$

Here $\Re$ is the set of real numbers. The basis in V^* is denoted by

$$e^0, e^1, e^2, e^3 \quad \rightarrow \quad e^\mu \quad \mu = 0, 1, 2, 3$$

$$\text{with} \quad e^\mu \, e_\nu = \delta^\mu_\nu \tag{1.4}$$

Here δ represents the Kronecker delta tensor in four dimensions with the properties

$$\delta^\mu_\nu = 1 \quad \mu = \nu \qquad \delta^\mu_\nu = 0 \quad \mu \neq \nu \tag{1.5}$$

If $x \in V^*$ then it can be written as $x = x_\mu \, e^\mu$ or $x = (x_0, x_1, x_2, x_3)$. In the formalism above e_μ and e^μ are called covariant and contra variant basis vectors respectively whereas x_μ en x^μ are the covariant and contra variant components.
Remind: $\mu\ is\ lowerindex \ \rightarrow\ \mu\ is\ co-variant$, $\mu\ is\ upperindex\ \rightarrow \mu\ is\ contra-variant$. We introduce a metric tensor g with components $g_{\mu\nu}$. It has the properties

$$g_{00} = 1 \quad g_{ij} = -1 \quad i = j = 1, 2, 3 \qquad g_{\mu\nu} = 0 \quad \mu \neq \nu$$

$$\begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array} \tag{1.6}$$

Notice that 0 is the time index and i, j are the spatial indices. The Minkowski metric is indefinite this is in contrast to the Euclidian metric which is represented by the Kronecker delta tensor $\vec{\vec{\delta}}$. That is why g does not define a norm in contrast to $\vec{\vec{\delta}}$. Equation (1.1) can now be cast in the form

$$c^2t^2 - |\vec{x}|^2 = c^2t'^2 - |\vec{x}'|^2 = 0 = g_{\mu\nu} \, x^\mu \, x^\nu \tag{1.7}$$

Besides $g_{\mu\nu}$ which is a covariant tensor we have also a contra variant tensor $g^{\mu\nu}$ with exactly the same properties as in (1.6).

$$g_{\mu\nu} \, g^{\nu\lambda} = \delta^\lambda_\mu \tag{1.8}$$

$$g_{\mu\nu} \, g^{\nu\lambda} = \delta^\lambda_\mu \tag{1.9}$$

Further we have the relations

$$e_\mu \cdot e_\nu = g_{\mu\nu} \qquad e^\mu \cdot e^\nu = g^{\mu\nu}$$

$$e_\mu = g_{\mu\nu} \, e^\nu \qquad e^\mu = g^{\mu\nu} \, e_\nu$$

$$x_\mu = g_{\mu\nu} \, x^\nu \qquad x^\mu = g^{\mu\nu} \, x_\nu$$

$$x \cdot y = x_\mu y_\nu e^\mu e^\nu = x_\mu y_\nu g^{\mu\nu} = g^{\mu\nu} g_{\mu\lambda} x^\lambda y_\nu = \delta^\nu_\lambda x^\lambda y_\nu$$

$$x \cdot y = x^\mu y^\nu e_\mu e_\nu = x^\mu y^\nu g_{\mu\nu} = g_{\mu\nu} g^{\mu\lambda} x_\lambda y^\nu = \delta^\lambda_\nu x_\lambda y^\nu$$

$$x \cdot y = x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3 \qquad x \cdot y = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3$$

$$x \cdot y = x_0 y^0 + x_1 y^1 + x_2 y^2 + x_3 y^3 = x^0 y_0 + x^1 y_1 + x^2 y_2 + x^3 y_3$$

$$x \cdot y = x_0 y_0 - \vec{x} \cdot \vec{y} = x_0 y_0 - x_i y_i = x^0 y^0 - x^i y^i = x^0 y_0 + x^i y_i = x_0 y^0 + x_i y^i \tag{1.10}$$

Let A be a transformation which connects the components x and the basis vectors e of systems S and S' then the transformations read

$$e'^{\mu} = A^{\mu}_{\nu}\, e^{\nu} \qquad x'_{\mu} = (A^{-1})^{\nu}_{\mu}\, x_{\nu}$$

$$e'_{\mu} = (A^{-1})^{\nu}_{\mu}\, e_{\nu} \qquad x'^{\mu} = A^{\mu}_{\nu}\, x^{\nu}$$

$$(A^T)^{\mu}_{\nu} \equiv A^{\nu}_{\mu} \qquad A^{\mu}_{\nu}\,(A^{-1})^{\nu}_{\lambda} = \delta^{\mu}_{\lambda} \qquad A = \begin{pmatrix} A^0_0 & A^1_0 & A^2_0 & A^3_0 \\ A^0_1 & A^1_1 & A^2_1 & A^3_1 \\ A^0_2 & A^1_2 & A^2_2 & A^3_2 \\ A^0_3 & A^1_3 & A^2_3 & A^3_3 \end{pmatrix} \tag{1.11}$$

1.2 The real scalar (spin zero) field

A relativistic classical field, notation $\varphi_r(\vec{x}, t) \equiv \varphi_r(x)$, is a function defined on space and time. This field satisfies a covariant differential equation. Further $\varphi_r(x)$ is a representation of a certain symmetry group where r is the representation index. In these lecture the symmetry group is given by the Lorentz group or Poincaré group. A classical field can be considered as an object which describes a system of infinite degrees of freedom. In classical mechanics can a system with N particles be described by n independent generalised coordinates $q_i(t)$ $(i = 1, 2, \cdots, n)$. If one takes the index i continuous i.e. $q_i(t) \rightarrow q(i, t)$ is a function of i and t. When we take for i the spatial coordinates $(x_1, x_2, x_3) = (x, y, z)$ then $q(\vec{x}, t)$ becomes a classical field.

The scalar free field satisfies the Klein-Gordon equation

$$\left(\Box + \frac{m^2 c^2}{\hbar^2} \right) \phi(x) = 0 \quad \text{met} \quad \Box \equiv \partial_{\mu} \partial^{\mu} \tag{1.12}$$

The symbol $\Box$ is called the d'Alembertian. This is a scalar quantity under the Lorentz group. It can be written as

$$\Box = \frac{\partial^2}{c^2 \partial\, t^2} - \sum_{i=1}^{3} \frac{\partial^2}{\partial\, x_i^2} \qquad \vec{\nabla} \cdot \vec{\nabla} \equiv \Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial\, x_i^2} \tag{1.13}$$

Here Δ is the Laplacian. Further is $\hbar = h/2\pi$ where h is Planck's constant. This is introduced here on the classical level because later the field will be quantised. This field is also called the Klein-Gordon field. The most general solution of the Klein-Gordon equation looks like

$$\phi(x) = \int d^3\vec{p} \left[a(\vec{p})\, f_p(x) + a^*(\vec{p})\, f_p^*(x) \right] \tag{1.14}$$

met

$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right) f_p(x) = 0 \qquad \left(\Box + \frac{m^2c^2}{\hbar^2}\right) f_p^*(x) = 0 \tag{1.15}$$

The solutions are

$$f_p(x) = A\, e^{-i\, px/\hbar} \qquad f_p^*(x) = A\, e^{i\, px/\hbar}$$

$$p = (\frac{E}{c}, \vec{p}) \qquad E = \sqrt{m^2c^4 + |\vec{p}|^2c^2} \qquad \frac{px}{\hbar} = \frac{1}{\hbar}(E\, t - \vec{p}\cdot\vec{x}) \tag{1.16}$$

We have written the field $\phi(x)$ as a Fourier expansion in the plane wave functions f_p, f_p^*. The quantities $a(\vec{p})$, $a^*(\vec{p})$ are the Fourier coefficients. A is a real normalisation constant which has still to be determined. This depends on the convention which has to be introduced. Define the inner product

$$(f, g) = \frac{i}{\hbar c}\int d^3\vec{x}\, f^*(x)\, \overleftrightarrow{\partial_0}\, g(x)$$

$$f^*(x)\, \overleftrightarrow{\partial_\mu}\, g(x) = f^*(x)(\partial_\mu g(x)) - (\partial_\mu f^*(x))g(x) \tag{1.17}$$

Notice that this product does not lead to a norm. Calculate $(f_p, f_{p'})$

$$\begin{aligned}
(f_p, f_{p'}) &= i\frac{A^2}{\hbar c}\int d^3\vec{x}\, e^{-i\,\vec{p}\cdot\vec{x}/\hbar}\, e^{i\,\vec{p}'\cdot\vec{x}/\hbar}\Big(e^{i\,Et/\hbar}\frac{\partial}{c\partial\, t}e^{-i\,E't/\hbar} - e^{-i\,E't/\hbar}\frac{\partial}{c\partial\, t}e^{i\,Et/\hbar}\Big) \\
&= A^2\, e^{i\,(E-E')t/\hbar}\frac{E+E'}{(\hbar c)^2}\int d^3\vec{x}\, e^{-i\,(\vec{p}-\vec{p}')\cdot\vec{x}/\hbar} \\
&= A^2\, e^{i\,(E-E')t/\hbar}\frac{E+E'}{(\hbar c)^2}\,(2\pi)^3\,\delta^{(3)}\left(\frac{(\vec{p}-\vec{p}')}{\hbar}\right) \\
&= A^2\,\frac{2E}{(\hbar c)^2}\,(2\pi\hbar)^3\,\delta^{(3)}(\vec{p}-\vec{p}')
\end{aligned} \tag{1.18}$$

Notice that from $\vec{p} = \vec{p}'$ follows $E = E'$. The properties of the delta function are

$$\delta^{(n)}(x) = \frac{1}{(2\pi)^n}\int d^n k\, e^{\pm i\, kx} \qquad n = 1, 2, \cdots$$

$$\delta^{(n)}(x) = \delta(x_1)\delta(x_2)\cdots\delta(x_n)$$

$$\delta^{(n)}(ax) = \frac{1}{a^n}\delta^{(n)}(x) \tag{1.19}$$

Choose A such that $(f_p, f_{p'}) = \delta^{(3)}(\vec{p} - \vec{p}')$. One obtains

$$A = \frac{1}{(2\pi\hbar)^{3/2}} \frac{\hbar c}{\sqrt{2E}} \tag{1.20}$$

Further one can show

$$(f_p^*, f_{p'}^*) = -\delta^{(3)}(\vec{p} - \vec{p}') \qquad (f_p^*, f_{p'}) = (f_p, f_{p'}^*) = 0$$

$$a(\vec{p}) = (f_p, \phi) \qquad a^*(\vec{p}) = -(f_p^*, \phi) \tag{1.21}$$

Notice that (f, g) does not define a norm as is the case in non relativistic relativistic quantum mechanics. In the latter case each wave function has to satisfy

$$|\psi|^2 = (\psi, \psi) \geq 0 \tag{1.22}$$

Here the norm is definite positive! In field theory this is not the case e.g. $(f_p^*, f_{p'}^*) < 0$. Hence $\phi(x)$ is field and no wave function as one originally thought. After the normalisation choice we can write for the Klein-Gordon field

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi\hbar)^{3/2}} \frac{\hbar c}{\sqrt{2E}} \Big[a(\vec{p})\, e^{-i\,p\cdot x/\hbar} + a^*(\vec{p})\, e^{i\,p\cdot x/\hbar}\Big] \tag{1.23}$$

Watch that in the most field theory books is written $\hbar = c = 1$.

1.3 The complex scalar (spin zero) field

This field is defined by

$$\phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\,\phi_2(x)) \tag{1.24}$$

where $\phi_1(x)$ and $\phi_2(x)$ are real fields which satisfy Klein-Gordon equations

$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\phi(x) = 0 \qquad \left(\Box + \frac{m^2c^2}{\hbar^2}\right)\phi^*(x) = 0 \tag{1.25}$$

Hence follow the solutions

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi\hbar)^{3/2}} \frac{\hbar c}{\sqrt{2E}} \Big[a_+(\vec{p})\, e^{-i\,p\cdot x/\hbar} + a_-^*(\vec{p})\, e^{i\,p\cdot x/\hbar}\Big]$$

$$\phi^*(x) = \int \frac{d^3\vec{p}}{(2\pi\hbar)^{3/2}} \frac{\hbar c}{\sqrt{2E}} \Big[a_-(\vec{p})\, e^{-i\,p\cdot x/\hbar} + a_+^*(\vec{p})\, e^{i\,p\cdot x/\hbar}\Big] \tag{1.26}$$

From the definition of $\phi(x)$ the next relations follow

$$a_+(\vec{p}) = \frac{1}{\sqrt{2}}(a_1(\vec{p}) + i\,a_2(\vec{p})) \qquad a_-^*(\vec{p}) = \frac{1}{\sqrt{2}}(a_1^*(\vec{p}) + i\,a_2^*(\vec{p}))$$

$$a_+^*(\vec{p}) = \frac{1}{\sqrt{2}}(a_1^*(\vec{p}) - i\,a_2^*(\vec{p})) \qquad a_-(\vec{p}) = \frac{1}{\sqrt{2}}(a_1(\vec{p}) - i\,a_2(\vec{p})) \tag{1.27}$$

where $a_i(\vec{p})$, $a_i^*(\vec{p})$ are the Fourier coefficients of $\phi_i(x)$.

1.4 The complex Dirac (spin half) field

To describe spin half field we follow a procedure which is also applied in non relativistic quantum mechanics to describe spin half wave functions. To that purpose we introduce spinors and the Pauli matrices.

$$\sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.28}$$

with the properties

$$\begin{aligned} &[\sigma_k, \sigma_l] = 2\, i\, \epsilon_{klm}\, \sigma_m \qquad \{\sigma_k, \sigma_l\} = 2\, \delta_{kl} \qquad \sigma_k^\dagger = \sigma_k \\ &\Rightarrow \quad \sigma_k\, \sigma_l = \delta_{kl} + i\, \epsilon_{klm}\, \sigma_m \end{aligned} \tag{1.29}$$

The Pauli spinors are

$$\chi^{(1/2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi^{(-1/2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \sigma_z\, \chi^{(s)} = 2\, s\, \chi^{(s)} \tag{1.30}$$

Here $\chi^{(s)}$ is a eigen state of σ_z. In the above representation the quantisation axis is chosen to be in the z direction. If an arbitrary direction $\vec{n}$ is chosen with $|\vec{n} = 1$ then the spinors are eigen states of $\vec{\sigma} \cdot \vec{n}$

$$\begin{aligned} &\vec{\sigma} \cdot \vec{n}\, \chi_n^{(s)} = 2\, s\, \chi_n^{(s)} \quad \text{met} \quad \vec{n} = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta) \\ &\vec{\sigma} \cdot \vec{n} = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix} \\ &\chi_n^{(1/2)} = \begin{pmatrix} e^{-i\varphi/2} \cos\theta/2 \\ e^{i\varphi/2} \sin\theta/2 \end{pmatrix} \qquad \chi_n^{(-1/2)} = \begin{pmatrix} -e^{-i\varphi/2} \sin\theta/2 \\ e^{i\varphi/2} \cos\theta/2 \end{pmatrix} \end{aligned} \tag{1.31}$$

From (1.29) follows

$$\vec{\sigma} \cdot \vec{a}\, \vec{\sigma} \cdot \vec{b} = \vec{a} \cdot \vec{b}\, \mathbf{1} + i\, \vec{\sigma} \cdot (\vec{a} \times \vec{b}) \qquad \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \rightarrow \quad (\vec{\sigma} \cdot \vec{a})^2 = |\vec{a}|^2\, \mathbf{1} \tag{1.32}$$

Using the above we can write the Klein-Gordon equation as follows

$$\mathbf{1} \left(-\frac{\partial^2}{\partial\, (x^0)^2} + \vec{\nabla} \cdot \vec{\nabla} \right) \phi(x) = \left(-\mathbf{1}\, \frac{\partial^2}{\partial\, (x^0)^2} + (\vec{\sigma} \cdot \vec{\nabla})^2 \right) \phi(x) = \mathbf{1}\, \frac{m^2 c^2}{\hbar^2}\, \phi(x) \tag{1.33}$$

or

$$\left(i\hbar\, \mathbf{1}\, \frac{\partial}{\partial\, x^0} + i\, \hbar\, \vec{\sigma} \cdot \vec{\nabla} \right) \left(i\hbar\, \mathbf{1}\, \frac{\partial}{\partial\, x^0} - i\, \hbar\, \vec{\sigma} \cdot \vec{\nabla} \right) \phi(x) = \mathbf{1}\, m^2 c^2\, \phi(x) \tag{1.34}$$

Call $\phi(x) \equiv \phi_L(x)$ then we can define

$$\phi_R(x) = \frac{1}{mc}\left(i\,\hbar\,\mathbf{1}\,\frac{\partial}{\partial\, x^0} - i\,\hbar\,\vec{\sigma}\cdot\vec{\nabla}\right)\phi_L(x) \tag{1.35}$$

where $\phi_R(x)$ and $\phi_L(x)$ are two component vectors. We get two equations

$$i\,\hbar\left(\mathbf{1}\,\frac{\partial}{\partial\, x^0} - \vec{\sigma}\cdot\vec{\nabla}\right)\phi_L(x) = \mathbf{1}\,mc\,\phi_R(x)$$

$$i\,\hbar\left(\mathbf{1}\,\frac{\partial}{\partial\, x^0} + \vec{\sigma}\cdot\vec{\nabla}\right)\phi_R(x) = \mathbf{1}\,mc\,\phi_L(x) \tag{1.36}$$

Take the sum and difference of the equations in (1.36)

$$i\,\hbar\left(\mathbf{1}\,\frac{\partial}{\partial\, x^0}\Big(\phi_R(x)+\phi_L(x)\Big) + \vec{\sigma}\cdot\vec{\nabla}\Big(\phi_R(x)-\phi_L(x)\Big)\right) = \mathbf{1}\,mc\Big(\phi_R(x)+\phi_L(x)\Big)$$

$$-i\,\hbar\left(\mathbf{1}\,\frac{\partial}{\partial\, x^0}\Big(\phi_R(x)-\phi_L(x)\Big) + \vec{\sigma}\cdot\vec{\nabla}\Big(\phi_R(x)+\phi_L(x)\Big)\right) = \mathbf{1}\,mc\Big(\phi_R(x)-\phi_L(x)\Big) \tag{1.37}$$

We introduce a four component spinor which is called Dirac spinor. It is given by

$$\psi(x) = \begin{pmatrix} \psi_A(x) \\ \psi_B(x) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_R(x)+\phi_L(x) \\ \phi_R(x)-\phi_L(x) \end{pmatrix} = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}$$

$$\phi_R(x)+\phi_L(x) = \sqrt{2}\begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \qquad \phi_R(x)-\phi_L(x) = \sqrt{2}\begin{pmatrix} \psi_3(x) \\ \psi_4(x) \end{pmatrix} \tag{1.38}$$

One can rewrite the equation in (1.37)

$$i\,\hbar\begin{pmatrix} \mathbf{1}\frac{\partial}{\partial\, x^0} & \vec{\sigma}\cdot\vec{\nabla} \\ -\vec{\sigma}\cdot\vec{\nabla} & -\mathbf{1}\frac{\partial}{\partial\, x^0} \end{pmatrix}\begin{pmatrix} \psi_A(x) \\ \psi_B(x) \end{pmatrix} = mc\begin{pmatrix} \psi_A(x) \\ \psi_B(x) \end{pmatrix} \tag{1.39}$$

Define the four by four γ-matrices (Dirac-representation)

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \emptyset \\ \emptyset & -\mathbf{1} \end{pmatrix} \qquad \gamma^k = \begin{pmatrix} \emptyset & \sigma_k \\ -\sigma_k & \emptyset \end{pmatrix} \quad k=1,2,3 \quad \text{of} \quad k=x,y,z \tag{1.40}$$

Further we have the conventions

$$\vec{\nabla} = (\frac{\partial}{\partial\, x},\frac{\partial}{\partial\, y},\frac{\partial}{\partial\, z}) = (\frac{\partial}{\partial\, x^1},\frac{\partial}{\partial\, x^2},\frac{\partial}{\partial\, x^3}) = (\partial_1,\partial_2,\partial_3) \qquad \partial_0 = \frac{\partial}{\partial\, x^0} \tag{1.41}$$

In this notation equation (1.39) looks like

$$i\hbar \begin{pmatrix} \mathbf{1}\partial_0 & \sigma_k\partial_k \\ -\sigma_k\partial_k & -\mathbf{1}\partial_0 \end{pmatrix} \begin{pmatrix} \psi_A(x) \\ \psi_B(x) \end{pmatrix} = mc \begin{pmatrix} \psi_A(x) \\ \psi_B(x) \end{pmatrix} \tag{1.42}$$

The equation above can be written in a shorter way

$$(i\hbar\,\gamma^\mu\,\partial_\mu - mc)\;\psi(x) = 0 \qquad \text{Dirac-vergelijking} \tag{1.43}$$

Because $\phi_R(x)$, $\phi_L(x)$ satisfy the Klein-Gordon equation (1.1), $\psi(x)$ satisfies this equation too. From this one can derive a general relation between the γ-matrices

$$\begin{aligned} &\hbar^2\,\gamma^\nu\,\gamma^\mu\,\partial_\nu\,\partial_\mu\psi(x) = -i\,\hbar\,mc\,\gamma^\nu\,\partial_\nu\,\psi(x) = -m^2c^2\,\psi(x) \\ &\hbar^2\,\gamma^\mu\,\gamma^\nu\,\partial_\mu\,\partial_\nu\psi(x) = -m^2c^2\,\psi(x) \end{aligned} \tag{1.44}$$

Addition gives

$$\begin{aligned} &\hbar^2\left(\gamma^\mu\,\gamma^\nu + \gamma^\nu\,\gamma^\mu\right)\partial_\mu\,\partial_\nu\psi(x) = -2\,m^2c^2\psi(x) = 2\,\hbar^2\,\Box\psi(x) = \\ &2\,\hbar^2\,g^{\mu\nu}\,\partial_\mu\,\partial_\nu\psi(x) \quad \Rightarrow \quad \{\gamma^\mu,\gamma^\nu\} = 2\,g^{\mu\nu}\,1 \end{aligned} \tag{1.45}$$

The four by four unit matrix 1 is often omitted in the literature but it is present!! The anti-commutator $\{\}$ is defined as

$$\{a,b\} = a\,b + b\,a \tag{1.46}$$

Further one can derive the following relations

$$\begin{aligned} &(\gamma^0)^\dagger = \gamma^0 \qquad (\gamma^i)^\dagger = -\gamma^i \qquad (\gamma^0)^2 = 1 \qquad (\gamma^i)^2 = -1 \\ &Sp\,\gamma^\mu = 0 \qquad Sp\,1 = 4 \qquad \gamma_\mu = g_{\mu\nu}\gamma^\nu \quad \rightarrow \quad \gamma_0 = \gamma^0 \quad \gamma_i = -\gamma^i \end{aligned} \tag{1.47}$$

where 1 is a four by four unit matrix which very often is omitted. We introduce the following notation

$$\not{a} = a_\mu\gamma^\mu \tag{1.48}$$

We can now derive the following relations

$$\begin{aligned} &\gamma_\mu\,\not{a}\,\gamma^\mu = -2\,\not{a} \qquad \gamma_\mu\,\not{a}\,\not{b}\,\gamma^\mu = 4\,a\cdot b \qquad \gamma_\mu, \not{a}\,\not{b}\,\not{c}\,\gamma^\mu = -2\not{c}\,\not{b}\,\not{a} \\ &Sp\,\not{a}\,\not{b} = 4a\cdot b \qquad Sp\,\not{a}_1\,\not{a}_2\cdots\not{a}_n = 0 \quad \text{if } n \text{ is odd} \\ &Sp\,\not{a}\,\not{b}\,\not{c}\,\not{d} = 4[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)] \end{aligned} \tag{1.49}$$

Definition adjoint Dirac spinor

$$\bar{\psi}(x) = \psi^\dagger(x)\,\gamma^0 \qquad \psi^\dagger(x) = (\psi^T(x))^*$$
$$\psi^T(x) = (\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x))$$
$$\psi^\dagger(x) = (\psi_1^*(x), \psi_2^*(x), \psi_3^*(x), \psi_4^*(x))$$
$$\psi^\dagger(x)\,\gamma^0 = (\psi_1^*(x), \psi_2^*(x), -\psi_3^*(x), -\psi_4^*(x)) \tag{1.50}$$

The adjoint spinor satisfies the Dirac equation. This follows from hermitian conjugation

$$\Big(i\,\hbar\,\gamma^0\,\partial_0 + i\,\hbar\,\gamma^k\,\partial_k - mc\Big)\;\psi(x) = 0 \quad \rightarrow \quad \psi^\dagger(x)\left(-i\,\hbar\,\gamma^0\,\overleftarrow{\partial_0} + i\,\hbar\,\gamma^k\,\overleftarrow{\partial_k} - mc\right) = 0$$

$$\psi^\dagger(x)\left(-i\,\hbar\,\gamma^0\,\overleftarrow{\partial_0} + i\,\hbar\,\gamma^k\,\overleftarrow{\partial_k} - mc\right)\,\gamma^0 = 0$$

$$\rightarrow \quad \bar{\psi}(x)\left(-i\,\hbar\,\gamma^0\,\overleftarrow{\partial_0} - i\,\hbar\,\gamma^k\,\overleftarrow{\partial_k} - mc\right) = 0 \tag{1.51}$$

or

$$\bar{\psi}(x)\left(i\,\hbar\,\gamma^\mu\,\overleftarrow{\partial_\mu} + mc\right) = 0 \tag{1.52}$$

Theorem: Each four by four matrix can be expressed in the complete set of matrices Γ_k which contains 16 elements. They are

$$\Gamma_S = 1 \qquad \Gamma_V^\mu = \gamma^\mu \qquad \Gamma_T^{\mu\nu} = \sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$
$$\Gamma_P = \gamma_5 = \gamma^5 \equiv i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 = \begin{pmatrix} \emptyset & \mathbf{1} \\ \mathbf{1} & \emptyset \end{pmatrix} \qquad \Gamma_A^\mu = \gamma^5\,\gamma^\mu \tag{1.53}$$

Further holds

$$\{\gamma^\mu, \gamma^5\} = 0 \qquad Sp\,\gamma_5\,\gamma_\mu\,\gamma_\nu\,\gamma_\lambda\,\gamma_\sigma = 4\,i\,\epsilon_{\mu\nu\lambda\sigma} \tag{1.54}$$

We have to find the solutions of the Dirac equation. There are two solutions i.e. a positive energy solution and a negative energy solution

1 Positive energy solution

$$\psi_1(x) = \begin{pmatrix} \psi_{1A}(x) \\ \psi_{1B}(x) \end{pmatrix} = \sum_s \int d^3\vec{p}\,A \begin{pmatrix} u_{1A}(\vec{p}, s) \\ u_{1B}(\vec{p}, s) \end{pmatrix} e^{-i\,px/\hbar} \qquad s = \pm\frac{1}{2} \tag{1.55}$$

where A is a normalisation factor which depends on the energy. Substitute this solution in the Dirac equation

$$\left(\frac{E}{c}\gamma^0 - \vec{\gamma}\cdot\vec{p} - mc\right)\begin{pmatrix} u_{1A}(\vec{p},s) \\ u_{1B}(\vec{p},s) \end{pmatrix} = 0$$

$$\begin{pmatrix} \mathbf{1}\frac{E}{c} & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -\mathbf{1}\frac{E}{c} \end{pmatrix}\begin{pmatrix} u_{1A}(\vec{p},s) \\ u_{1B}(\vec{p},s) \end{pmatrix} = mc\begin{pmatrix} u_{1A}(\vec{p},s) \\ u_{1B}(\vec{p},s) \end{pmatrix} \tag{1.56}$$

The solution is

$$u_{1B}(\vec{p},s) = \frac{c}{E+mc^2}\vec{\sigma}\cdot\vec{p}\,u_{1A}(\vec{p},s) \quad \text{met} \quad u_{1A}(\vec{p},s) = \chi_n^{(s)} \quad \text{Pauli-spinor} \tag{1.57}$$

The solution

$$u_{1A}(\vec{p},s) = \frac{c}{E-mc^2}\vec{\sigma}\cdot\vec{p}\,u_{1B}(\vec{p},s) \tag{1.58}$$

can be dropped because the limit $E \to mc^2$ or $\vec{p} \to 0$ does exist.

2 Negative energy solution

$$\psi_2(x) = \begin{pmatrix} \psi_{2A}(x) \\ \psi_{2B}(x) \end{pmatrix} = \sum_s \int d^3\vec{p}\,A\begin{pmatrix} u_{2A}(\vec{p},s) \\ u_{2B}(\vec{p},s) \end{pmatrix} e^{i\,px/\hbar} \qquad s = \pm\frac{1}{2} \tag{1.59}$$

Substitute this solution in the Dirac equation

$$\left(-\frac{E}{c}\gamma^0 + \vec{\gamma}\cdot\vec{p} - mc\right)\begin{pmatrix} u_{2A}(\vec{p},s) \\ u_{2B}(\vec{p},s) \end{pmatrix} = 0$$

$$\begin{pmatrix} -\mathbf{1}\frac{E}{c} & \vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & \mathbf{1}\frac{E}{c} \end{pmatrix}\begin{pmatrix} u_{2A}(\vec{p},s) \\ u_{2B}(\vec{p},s) \end{pmatrix} = mc\begin{pmatrix} u_{2A}(\vec{p},s) \\ u_{2B}(\vec{p},s) \end{pmatrix} \tag{1.60}$$

The solution is

$$u_{2A}(\vec{p},s) = \frac{c}{E+mc^2}\vec{\sigma}\cdot\vec{p}\,u_{2B}(\vec{p},s) \quad \text{met} \quad u_{2B}(\vec{p},s) = \chi_n^{(-s)} \quad \text{Pauli-spinor} \tag{1.61}$$

The $-s$ which occurs in the Pauli spinor is a convention which becomes clear when we quantise the Dirac field. The solution

$$u_{2B}(\vec{p},s) = \frac{c}{E-mc^2}\vec{\sigma}\cdot\vec{p}\,u_{2A}(\vec{p},s) \tag{1.62}$$

can be dropped because the limit $E \to mc^2$ or $\vec{p} \to 0$ does not exist.

From the above we can derive the following solutions. The first Dirac spinor corresponds to the fermion

$$u(\vec{p},s) = \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \chi_n^{(s)} \\ \frac{c\,\vec{\sigma}\cdot\vec{p}}{E+mc^2}\chi_n^{(s)} \end{pmatrix} \tag{1.63}$$

The second Dirac spinor belongs to the anti-fermion anti-fermion

$$v(\vec{p},s) = \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \frac{c\,\vec{\sigma}\cdot\vec{p}}{E+mc^2}\chi_n^{(-s)} \\ \chi_n^{(-s)} \end{pmatrix} \tag{1.64}$$

The square roots are chosen in such a way that Dirac spinors satisfy the following orthogonality relations

$$u(\vec{p},s)^\dagger\, u(\vec{p},s') = \frac{E}{mc^2}\delta_{ss'} \qquad v(\vec{p},s)^\dagger\, v(\vec{p},s') = \frac{E}{mc^2}\delta_{ss'} \tag{1.65}$$

However in the literature other conventions are also accepted

$$\sqrt{\frac{E+mc^2}{2mc^2}} \quad \to \quad = \sqrt{\frac{E+mc^2}{2E}} \tag{1.66}$$

In this case we have the orthonormal relations

$$u(\vec{p},s)^\dagger\, u(\vec{p},s') = \delta_{ss'} \qquad v(\vec{p},s)^\dagger\, v(\vec{p},s') = \delta_{ss'} \tag{1.67}$$

The advantage of this convention is that we can immediately take the limit $m \to 0$. In these lectures the convention (1.63)- (1.65) is assumed. There are other relations. They can be derived from (1.63) and (1.64)

$$\begin{aligned}
&\bar{u}(\vec{p},s)\, u(\vec{p},s') = \delta_{ss'} \qquad \bar{v}(\vec{p},s)\, v(\vec{p},s') = -\delta_{ss'} \\
&\bar{u}(\vec{p},s)\, v(\vec{p},s') = 0 \qquad u(-\vec{p},s)^\dagger\, v(\vec{p},s') = 0 \\
&\bar{v}(\vec{p},s)\, u(\vec{p},s') = 0 \qquad v(-\vec{p},s)^\dagger\, u(\vec{p},s') = 0 \\
&\bar{u}(\vec{p},s) = u(\vec{p},s)^\dagger\, \gamma^0 = u(-\vec{p},s)^\dagger \qquad \bar{v}(\vec{p},s) = v(\vec{p},s)^\dagger\, \gamma^0 = -v(-\vec{p},s)^\dagger
\end{aligned} \tag{1.68}$$

The Dirac equation for spinors read

$$\begin{aligned}
&(\not{p} - mc)u(\vec{p},s) = 0 \qquad \bar{u}(\vec{p},s)(\not{p} - mc) = 0 \\
&(\not{p} + mc)v(\vec{p},s) = 0 \qquad \bar{v}(\vec{p},s)(\not{p} + mc) = 0
\end{aligned} \tag{1.69}$$

Further one can define the projection operators

$$\begin{aligned}
&\Lambda_+ = \sum_s u(\vec{p},s)\, \bar{u}(\vec{p},s) \qquad \Lambda_- = -\sum_s v(\vec{p},s)\, \bar{v}(\vec{p},s) \\
&\Lambda_+\, u(\vec{p},s) = u(\vec{p},s) \qquad \Lambda_+\, v(\vec{p},s) = 0 \\
&\Lambda_-\, u(\vec{p},s) = 0 \qquad \Lambda_-\, v(\vec{p},s) = v(\vec{p},s)
\end{aligned} \tag{1.70}$$

From which follows

$$\Lambda_+ = \frac{\not{p} + mc}{2mc} \qquad \Lambda_- = \frac{-\not{p} + mc}{2mc} \tag{1.71}$$

The most general solution of the Dirac equation follows from (1.57) and (1.61)

$$\psi(x) = \sum_s \int d^3\vec{p}[b(\vec{p},s)\,\psi_{1,\vec{p},s}(x) + d^*(\vec{p},s)\,\psi_{2,\vec{p},s}(x)]$$

$$\psi_{1,\vec{p},s}(x) = A\,u(\vec{p},s)\,e^{-i\,px/\hbar} \qquad \psi_{2,\vec{p},s}(x) = A\,v(\vec{p},s)\,e^{i\,px/\hbar} \tag{1.72}$$

Here $b(\vec{p},s)$ and $d(\vec{p},s)$ are complex Fourier coefficients. We can define again a inner product

$$(\psi_{i,\vec{p},s}, \psi_{j,\vec{p}',s'}) = \int d^3\vec{x}\,\psi^\dagger_{i,\vec{p},s}(x)\,\psi_{j,\vec{p}',s'}(x) = \delta_{ij}\,\delta^{(3)}(\vec{p}-\vec{p}')\,\delta_{ss'} \tag{1.73}$$

Hence it follows

$$A = \frac{1}{(2\pi\hbar)^{3/2}}\sqrt{\frac{mc^2}{E}} \tag{1.74}$$

Notice that the inner product above defines a real norm in contrast to the inner product for scalar fields in (1.17). The classical Dirac field gets the following notation

$$\psi(x) = \sum_s \int \frac{d^3\vec{p}}{(2\pi\hbar)^{3/2}}\sqrt{\frac{mc^2}{E}}[b(\vec{p},s)\,u(\vec{p},s)\,e^{-i\,px/\hbar} + d^*(\vec{p},s)\,v(\vec{p},s)\,e^{i\,px/\hbar}] \tag{1.75}$$

From which follows

$$\psi^\dagger(x) = \sum_s \int \frac{d^3\vec{p}}{(2\pi\hbar)^{3/2}}\sqrt{\frac{mc^2}{E}}[b^*(\vec{p},s)\,u^\dagger(\vec{p},s)\,e^{i\,px/\hbar} + d(\vec{p},s)\,v^\dagger(\vec{p},s)\,e^{-i\,px/\hbar}]$$

$$\bar{\psi}(x) = \sum_s \int \frac{d^3\vec{p}}{(2\pi\hbar)^{3/2}}\sqrt{\frac{mc^2}{E}}[b^*(\vec{p},s)\,\bar{u}(\vec{p},s)\,e^{i\,px/\hbar} + d(\vec{p},s)\,\bar{v}(\vec{p},s)\,e^{-i\,px/\hbar}] \tag{1.76}$$

Remarks

$$\vec{n} = \frac{\vec{p}}{|\vec{p}|} \quad \rightarrow \quad \chi_n^{(s)} = \chi_p^{(s)} \quad \text{met} \quad \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\chi_p^{(s)} = 2s\,\chi_p^{(s)} \tag{1.77}$$

In this case the Dirac spinors are eigen vectors of

$$\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|} \quad \text{with} \quad \vec{\Sigma} = (\Sigma_1, \Sigma_2, \Sigma_3) \qquad \Sigma_k = \begin{pmatrix} \sigma_k & \emptyset \\ \emptyset & \sigma_k \end{pmatrix}$$

$$\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}u(\vec{p},s) = 2s\,u(\vec{p},s) \qquad \frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}v(\vec{p},s) = -2s\,v(\vec{p},s) \qquad s = \pm\frac{1}{2} \tag{1.78}$$

The quantity $h = 2s$ is called the helicity.

1.5 The electromagnetic (massless spin one) field

This field is given by Maxwell's equations

$$\vec{\nabla}\cdot\vec{E}=\rho \qquad \vec{\nabla}\times\vec{B}-\frac{\partial\,\vec{E}}{c\partial\,t}=\frac{\vec{J}}{c}$$

$$\vec{\nabla}\cdot\vec{B}=0 \qquad \vec{\nabla}\times\vec{E}+\frac{\partial\,\vec{B}}{c\partial\,t}=0 \tag{1.79}$$

where the electrostatic field and the magnetic field (magnetic induction) are given by $\vec{E}$ and $\vec{B}$ respectively. Furthermore ρ and $\vec{J}$ represent the charge distribution and the current density. One can now express $\vec{E}$ and $\vec{B}$ in the electrostatic potential ϕ and the magnetic potential $\vec{A}$

$$\vec{E}=-\vec{\nabla}\phi-\frac{\partial\,\vec{A}}{c\partial\,t} \qquad \vec{B}=\vec{\nabla}\times\vec{A} \tag{1.80}$$

We will now represent the Maxwell equations in a Lorentz covariant form. Define the electromagnetic field tensor (second rank tensor)

$$F_{\mu\nu}(x)=\partial_\mu\,A_\nu-\partial_\nu\,A_\mu \qquad F_{\mu\nu}=-F_{\nu\mu}$$

$$A(x)=(\phi(x),\vec{A}(x))=(A_0(x),A_1(x),A_2(x),A_3(x)) \tag{1.81}$$

Using

$$\partial=(\partial_0,\vec{\partial})=(\partial_0,\partial_1,\partial_2,\partial_3)=\left(\frac{\partial}{c\partial\,t},\frac{\partial}{\partial\,x^1},\frac{\partial}{\partial\,x^2},\frac{\partial}{\partial\,x^3},\right)=$$

$$\left(\frac{\partial}{c\partial\,t},-\frac{\partial}{\partial\,x_1},-\frac{\partial}{\partial\,x_2},-\frac{\partial}{\partial\,x_3},\right)=\left(\frac{\partial}{c\partial\,t},-\vec{\nabla}\right) \tag{1.82}$$

we can derive

$$E_i=F_{i0} \quad\rightarrow\quad \vec{E}=-\vec{\nabla}A_0-\frac{\partial\,\vec{A}}{c\partial\,t}$$

$$B_i=-\frac{1}{2}\epsilon_{ijk}\,F_{jk} \qquad F_{ij}=-\epsilon_{ijk}\,B_k \quad\rightarrow\quad \vec{B}=\vec{\nabla}\times\vec{A}$$

$$B_1=-F_{23} \qquad B_2=-F_{31} \qquad B_3=-F_{12} \tag{1.83}$$

The current density (four vector) is given by

$$J=(c\,\rho,\vec{J})=(J_0,J_1,J_2,J_3) \tag{1.84}$$

where ρ represents the charge density. One can now write Maxwell equations in a relativistic covariant manner

$$\partial^\mu F_{\mu\nu} = \frac{J_\nu}{c} \quad \text{of} \quad \Box A_\nu - \partial_\nu \partial^\mu A_\mu = \frac{J_\nu}{c}$$

$$\vec{\nabla} \cdot \vec{E} = \rho \qquad \vec{\nabla} \times \vec{B} - \frac{\partial\, \vec{E}}{c\partial\, t} = \frac{\vec{J}}{c} \tag{1.85}$$

The other two equations in (1.79) follow from the Bianchi identity

$$\partial_\mu\, F_{\nu\lambda} + \partial_\nu\, F_{\lambda\mu} + \partial_\lambda\, F_{\mu\nu} = 0$$

$$\Rightarrow \quad \vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} + \frac{\partial\, \vec{B}}{c\partial\, t} = 0 \tag{1.86}$$

Further we have current conservation

$$\partial^\mu \partial^\nu F_{\mu\nu} = \frac{1}{c} \partial^\nu J_\nu = 0 \tag{1.87}$$

because $F_{\mu\nu}$ is antisymmetric. therefore J_ν is a conserved current. The solution for A_μ is determined up to a gauge-transformation in other words if A_μ is a solution then is the gauge transformed potential $A'_\mu = A_\mu + \partial_\mu \Lambda$ also a solution

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x) \quad \rightarrow \quad F'_{\mu\nu} = \partial_\mu\, A'_\nu - \partial_\nu\, A'_\mu$$

$$= F_{\mu\nu} + \partial_\mu \partial_\nu \Lambda(x) - \partial_\nu \partial_\mu \Lambda(x) = F_{\mu\nu} \tag{1.88}$$

Hence it follows that $\vec{E}$ and $\vec{B}$ are independent of the choice for Λ. this has to be for $\vec{E}$ and $\vec{B}$ are observables. To find an expression for A_μ we have to make a choice for Λ. This is called a gauge choice. Let us limit ourselves to the free electromagnetic field which implies $J_\mu = 0$

$$\Box A_\nu - \partial_\nu \partial^\mu A_\mu = 0 \tag{1.89}$$

The equation above becomes the Klein-Gordon equation when we choose

1 Lorentz gauge

$$\partial^\mu A_\mu = 0 \tag{1.90}$$

Suppose $\partial^\mu A_\mu \neq 0$, choose a Λ for which holds

$$\partial^\mu A'_\mu = 0 \quad \rightarrow \partial^\mu A_\mu = -\Box \Lambda \tag{1.91}$$

such that

$$\Box A'_\nu = 0. \tag{1.92}$$

The Lorentz gauge is a Lorentz covariant gauge.

2 radiation gauge

$$\phi = 0 \qquad \vec{\nabla}\cdot\vec{A} = 0 \quad \rightarrow \quad \partial^\mu A_\mu = 0 \tag{1.93}$$

The disadvantage of this gauge choice is that it is not Lorentz covariant. For example in a system S it appears that $\phi = 0$ and $\vec{\nabla}\cdot\vec{A} = 0$. One can now make a Lorentz transformation to a system S' for which holds $\phi' \neq 0$ and $\phi' \neq 0$ en $\vec{\nabla}'\cdot\vec{A}' \neq 0$. However $\partial'^\mu A'_\mu = 0$ is still valid in all Lorentz frames. The advantage of the radiation gauge is that it only contains the physical components of the radiation field so that for the time being we shall choose the radiation gauge.

In the radiation gauge $\vec{A}$ satisfies the Klein-Gordon equation $\Box\vec{A} = 0$. The solution is

$$\vec{A}(x) = \sum_\alpha \int \frac{d^3\,\vec{k}}{(2\pi)^{3/2}}\, c\sqrt{\frac{\hbar}{2\omega}}\Big[a(\vec{k},\alpha)\,\vec{\epsilon}^{(\alpha)}(\vec{k})\,e^{-i\,kx} + a^*(\vec{k},\alpha)\,\vec{\epsilon}^{(\alpha)*}(\vec{k})\,e^{i\,kx}\Big] \tag{1.94}$$

with $k^2 = 0$.

$$\vec{\nabla}\cdot\vec{A} = 0 \quad \rightarrow \quad \vec{k}\cdot\vec{\epsilon}^{(\alpha)}(\vec{k}) = 0 \tag{1.95}$$

Here $\vec{\epsilon}^{(\alpha)}(\vec{k})$ represents a polarisation vector. From the condition above it follows that there are two independent polarisations α for the electromagnetic field. These physical polarisation vectors are perpendicular on the direction of propagation (transversality requirement). Notice that the Klein-Gordon equation for the electromagnetic field has no mass term hence $k^2 = 0$. If we compare with the real scalar field equation (1.12) we have made the following substitutions

$$\begin{aligned} &\vec{p} \quad \rightarrow \quad \hbar\vec{k} \qquad E \quad \rightarrow \quad \hbar\omega \qquad a(\vec{p}) \quad \rightarrow \quad \hbar^{-3/2}\,a(\vec{k},\alpha) \\ &k = \left(\frac{\omega}{c},\vec{k}\right) \qquad k^2 = 0 \quad \rightarrow \quad \omega = c|\vec{k}| \end{aligned} \tag{1.96}$$

We can also write equation (1.94)

$$\vec{A}(x) = \sum_\alpha \int d^3\vec{k}\,\Big[a(\vec{k},\alpha)\,\vec{f}_k^{(\alpha)}(x) + a^*(\vec{k},\alpha)\,\vec{f}_k^{(\alpha)*}(x)\Big]$$

$$\text{with} \quad \vec{f}_k^{(\alpha)}(x) = \frac{c}{(2\pi)^{3/2}}\sqrt{\frac{\hbar}{2\omega}}\,\vec{\epsilon}^{(\alpha)}(\vec{k})\,e^{-i\,kx} \tag{1.97}$$

From the requirement:

$$\left(f_k^{(\alpha)}, f_{k'}^{(\alpha')}\right) = \delta_{\alpha\alpha'}\,\delta^{(3)}(\vec{k}-\vec{k}') \tag{1.98}$$

where

$$\left(f_k^{(\alpha)}, f_{k'}^{(\alpha')}\right) = \frac{i}{\hbar c}\int d^3\vec{x}\,f_k^{(\alpha)*}(x)\,\overleftrightarrow{\partial_0}\,f_{k'}^{(\alpha')}(x) \tag{1.99}$$

follows that the polarisation vectors are mutually perpendicular.

$$\epsilon^{(\alpha)*}(\vec{k}) \cdot \vec{\epsilon}^{(\alpha')}(\vec{k}) = \delta_{\alpha\alpha'} \tag{1.100}$$

Proof:

$$\left(f_k^{(\alpha)}, f_{k'}^{(\alpha')}\right) = \frac{i}{(2\pi)^3} \frac{c}{2\sqrt{\omega\omega'}} \int d^3\vec{x}\, \epsilon^{(\alpha)*}(\vec{k}) \cdot \vec{\epsilon}^{(\alpha')}(\vec{k}')\, e^{-i\,\vec{k}\cdot\vec{x}}\, e^{i\,\vec{k}'\cdot\vec{x}} \left\{ e^{i\,\omega t} \frac{\partial}{c\partial\ t} e^{-i\,\omega' t} \right.$$

$$\left. - e^{-i\,\omega' t} \frac{\partial}{c\partial\ t} e^{i\,\omega t} \right\} = \frac{1}{(2\pi)^3} \frac{c}{2\sqrt{\omega\omega'}} \frac{\omega + \omega'}{c}\, \epsilon^{(\alpha)*}(\vec{k}) \cdot \vec{\epsilon}^{(\alpha')}(\vec{k}') \int d^3\vec{x}\, e^{i\,(\vec{k}'-\vec{k})\cdot\vec{x}}\, e^{i\,(\omega-\omega')t}$$

$$= \frac{\omega + \omega'}{2\sqrt{\omega\omega'}}\, \epsilon^{(\alpha)*}(\vec{k}) \cdot \vec{\epsilon}^{(\alpha')}(\vec{k}')\, \delta^{(3)}(\vec{k} - \vec{k}') = \epsilon^{(\alpha)*}(\vec{k}) \cdot \vec{\epsilon}^{(\alpha')}(\vec{k}')\, \delta^{(3)}(\vec{k} - \vec{k}')$$

$$= \delta_{\alpha\alpha'} \delta^{(3)}(\vec{k} - \vec{k}') \tag{1.101}$$

Watch that from $\vec{k} = \vec{k}'$ follows $\omega = \omega'$. The Fourier coefficients can be expressed in the inner product

$$a(\vec{k}, \alpha) = \left(\vec{f}_k^{(\alpha)}, \vec{A}\right) \qquad a^*(\vec{k}, \alpha) = -\left(\vec{f}_k^{(\alpha)*}, \vec{A}\right) \tag{1.102}$$

Further two different polarisation vectors are chosen in the literature

1 Linear polarisation vectors

$$\begin{aligned}
\vec{k} &= |\vec{k}|(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \\
\epsilon^{(1)}(\vec{k}) &= (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) \\
\epsilon^{(2)}(\vec{k}) &= (-\sin\varphi, \cos\varphi, 0)
\end{aligned} \tag{1.103}$$

so that $\vec{\epsilon}^{(\alpha)}$ is real. Further we have the properties

$$\begin{aligned}
&\vec{\epsilon}^{(1)}(\vec{k}) \times \vec{\epsilon}^{(2)}(\vec{k}) = \hat{\vec{k}} \quad \text{met} \quad \hat{\vec{k}} = \frac{\vec{k}}{|\vec{k}|} \\
&\vec{\epsilon}^{(1)}(-\vec{k}) = -\vec{\epsilon}^{(1)}(\vec{k}) \qquad \vec{\epsilon}^{(2)}(-\vec{k}) = \vec{\epsilon}^{(2)}(\vec{k}) \\
&\rightarrow \quad \vec{\epsilon}^{(1)}(-\vec{k}) \times \vec{\epsilon}^{(2)}(-\vec{k}) = -\hat{\vec{k}}
\end{aligned} \tag{1.104}$$

From the above it appears that $\vec{\epsilon}^{(1)}$, $\vec{\epsilon}^{(2)}$ and $\vec{k}$ form a right handed system. If we sum over the polarisations then we get

$$\sum_{\alpha=1}^{2} \epsilon_i^{(\alpha)}(\vec{k})\, \epsilon_j^{(\alpha)}(\vec{k}) = T_{ij}(\vec{k}) = \delta_{ij} - \frac{k_i\, k_j}{|\vec{k}|^2} \tag{1.105}$$

This follows from

$$T_{ij}(\vec{k}) = a\,\delta_{ij} + b\,\frac{k_i\,k_j}{|\vec{k}|^2} \quad \text{met} \quad k_i\,T_{ij}(\vec{k}) = T_{ij}(\vec{k})\,k_j = 0 \quad T_{ii} = 2 \tag{1.106}$$

2 Circular polarisations vectors
We can now choose a left handed $\alpha = L$ and a right handed $\alpha = R$ polarisation. They are given by

$$\vec{\epsilon}^{(R)}(\vec{k}) = -\frac{\vec{\epsilon}^{(1)}(\vec{k}) + i\,\vec{\epsilon}^{(2)}(\vec{k})}{\sqrt{2}} \qquad \vec{\epsilon}^{(L)}(\vec{k}) = \frac{\vec{\epsilon}^{(1)}(\vec{k}) - i\,\vec{\epsilon}^{(2)}(\vec{k})}{\sqrt{2}} \tag{1.107}$$

in other words the polarisation vectors are complex. The Fourier coefficients become

$$a(\vec{k}, R) = -\frac{a(\vec{k},1) - i\,a(\vec{k},2)}{\sqrt{2}} \qquad a(\vec{k}, L) = \frac{a(\vec{k},1) + i\,a(\vec{k},2)}{\sqrt{2}} \tag{1.108}$$

The properties of these polarisation vectors are

$$\vec{\epsilon}^{(R)*}(\vec{k}) = -\vec{\epsilon}^{(L)}(\vec{k}) \qquad \vec{\epsilon}^{(R)}(-\vec{k}) = \vec{\epsilon}^{(L)}(\vec{k}) \qquad \vec{\epsilon}^{(L)}(-\vec{k}) = \vec{\epsilon}^{(R)}(\vec{k})$$

$$\vec{\epsilon}^{(\alpha)*}(\vec{k}) \cdot \vec{\epsilon}^{(\alpha')}(\vec{k}) = \delta_{\alpha\alpha'} \quad \rightarrow \quad \vec{\epsilon}^{(\alpha)}(\vec{k}) \cdot \vec{\epsilon}^{(\alpha)}(\vec{k}) = 0 \quad \alpha = R, L \tag{1.109}$$

Finally one can derive from equation (1.107)

$$\sum_{\alpha=1}^{2} \epsilon_i^{(\alpha)}(\vec{k})\,\epsilon_j^{(\alpha)*}(\vec{k}) = \delta_{ij} - \frac{k_i\,k_j}{|\vec{k}|^2} \tag{1.110}$$

One can now determine from $\vec{A}$ in (1.94) the field $\vec{E}$ and $\vec{B}$. From (1.80) with $\phi = 0$ it follows

$$\vec{E}(x) = i\sum_\alpha \int \frac{d^3\vec{k}}{(2\pi)^{3/2}}\,\omega\sqrt{\frac{\hbar}{2\omega}}\Big[a(\vec{k},\alpha)\,\vec{\epsilon}^{(\alpha)}(\vec{k})\,e^{-i\,kx} - a^*(\vec{k},\alpha)\,\vec{\epsilon}^{(\alpha)*}(\vec{k})\,e^{i\,kx}\Big]$$

$$\vec{B}(x) = i\,c\sum_\alpha \int \frac{d^3\vec{k}}{(2\pi)^{3/2}}\sqrt{\frac{\hbar}{2\omega}}\,\vec{k}\times\Big[a(\vec{k},\alpha)\,\vec{\epsilon}^{(\alpha)}(\vec{k})\,e^{-i\,kx} - a^*(\vec{k},\alpha)\,\vec{\epsilon}^{(\alpha)*}(\vec{k})\,e^{i\,kx}\Big] \tag{1.111}$$

In stead of the radiation gauge one could also choose the Lorentz gauge This has the disadvantage that the electromagnetic field A_μ has unphysical unphysical polarisations. The field becomes

$$A_\mu(x) = \sum_\alpha \int \frac{d^3\vec{k}}{(2\pi)^{3/2}}\,c\sqrt{\frac{\hbar}{2\omega}}\Big[a(\vec{k},\alpha)\,\epsilon_\mu^{(\alpha)}(\vec{k})\,e^{-i\,kx} + a^*(\vec{k},\alpha)\,\epsilon_\mu^{(\alpha)*}(\vec{k})\,e^{i\,kx}\Big]$$

$$\sum_\alpha k^\mu\,\epsilon_\mu^{(\alpha)} = 0 \quad \text{en} \quad k^2 = 0 \tag{1.112}$$

We have besides the two transverse polarisation $\epsilon_\mu^{(\alpha)}$ $\alpha = 1,2$ or $\alpha = R, L$ a scalar polarisation $\epsilon_\mu^{(0)}$ and a longitudinal polarisation $\epsilon_\mu^{(3)}$. The latter two are unphysical and satisfy the following relation

$$k^\mu \, \epsilon_\mu^{(0)} = -k^\mu \, \epsilon_\mu^{(3)} \tag{1.113}$$

This follows from (1.90). For $k = (|\vec{k}|, \vec{k})$ the polarisation vectors are

$$\epsilon^{(0)} = (1, \vec{0}) \quad \epsilon^{(3)} = (0, \vec{\hat{k}}) \quad \epsilon^{(1)} = (0, \vec{\epsilon}^{(1)}) \quad \epsilon^{(2)} = (0, \vec{\epsilon}^{(2)})$$

$$\sum_\alpha \frac{\epsilon_\mu^{(\alpha)}(\vec{k}) \, \epsilon_\nu^{(\alpha)*}(\vec{k})}{\epsilon^{(\alpha)}(\vec{k}) \, \epsilon^{(\alpha)*}(\vec{k})} = g_{\mu\nu} \qquad \epsilon^{(\alpha)}(\vec{k}) \, \epsilon^{(\beta)*}(\vec{k}) = g^{\alpha\beta} \tag{1.114}$$

One can assume that the polarisation indices are Lorentz indices.

2 Symmetries in classical field theory

2.1 Lagrange en Hamilton formalism for classical fields

The Lagrangian for classical fields is defined by

$$L[\varphi_r, \dot{\varphi}_r, t] = \int d^3\vec{x}\, \mathcal{L}(\varphi_r(x), \partial_\mu \varphi_r, x_\mu) \tag{2.1}$$

This is a functional! The action is then given by

$$S[\varphi_r, \dot{\varphi}_r] = c \int_{t_1}^{t_2} dt\, L[\varphi_r, \dot{\varphi}_r, t] = \int_R d^4x\, \mathcal{L}(\varphi_r(x), \partial_\mu \varphi_r, x_\mu) \tag{2.2}$$

where $\mathcal{L}$ is the Lagrange density. In the subsequent part we omit the explicit dependence of $\mathcal{L}$ on x_μ. R is a four dimensional space time volume element such that $\int_R d^4x$ is a Lorentz-invariant.

$$\int d^4x' = \int d^4x \left|\frac{\partial\, x'_\mu}{\partial\, x_\nu}\right| = \int d^4x\, |\det\Lambda| = \int d^4x \quad \text{with} \quad x'_\mu = \Lambda_\mu^\lambda\, x_\lambda$$

$$\rightarrow \quad \frac{\partial\, x'_\mu}{\partial\, x_\nu} = \Lambda_\mu^\lambda\, \delta_\lambda^\nu = \Lambda_\mu^\nu \qquad |\Lambda_\mu^\nu| \equiv |\det\Lambda| \tag{2.3}$$

The boundary of the space R is denoted by ∂R. The upper and lower boundary of the integral $\int d^4x$ is determined by the hyper surfaces which arise by the intersections $t = t_1$ and $t = t_2$ which form a part ∂R.

Principle of the minimal action (Principle of Hamilton for fields)

The equation of motion for the field $\varphi_r(x)$ is given by that value for the field $\varphi_r(x)$ for which the action is minimal. To find the minimum we vary the field in such a way that $\varphi_r(x) = const.$ if $x \in \partial R$.

$$\varphi_r(x) \quad \rightarrow \quad \varphi_r(x) + \delta\varphi_r(x) \quad \text{with} \quad \delta\varphi_r(x) = 0 \quad \text{if} \quad x \in \partial R \tag{2.4}$$

The action varies like

$$\delta S = \int_R d^4x \left\{ \mathcal{L}(\varphi_r(x) + \delta\varphi_r(x), \partial_\lambda\varphi_r(x) + \delta\partial_\lambda\varphi_r(x)) - \mathcal{L}(\varphi_r(x), \partial_\lambda\varphi_r(x)) \right\}$$

$$= \int_R d^4x \left\{ \delta\varphi_r(x) \frac{\partial}{\partial\, \varphi_r(x)} \mathcal{L}(\varphi_r(x), \partial_\lambda\varphi_r(x)) \right.$$

$$\left. + \delta\partial_\mu\varphi_r(x) \frac{\partial}{\partial\, \partial_\mu\varphi_r(x)} \mathcal{L}(\varphi_r(x), \partial_\lambda\varphi_r(x)) \right\} \tag{2.5}$$

Using $\partial_\mu \delta\varphi_r(x) = \delta\partial_\mu\varphi_r(x)$ we can write

$$\delta S = \int_R d^4x\, \delta\varphi_r(x) \Big\{ \frac{\partial}{\partial\ \varphi_r(x)} \mathcal{L}(\varphi_r(x), \partial_\lambda\varphi_r(x)) - \partial_\mu \frac{\partial}{\partial\ \partial_\mu\varphi_r(x)} \mathcal{L}(\varphi_r(x), \partial_\lambda\varphi_r(x)) \Big\}$$

$$+ \int_R d^4x\, \partial_\mu \left\{ \delta\varphi_r(x) \frac{\partial}{\partial\ \partial_\mu\varphi_r(x)} \mathcal{L}(\varphi_r(x), \partial_\lambda\varphi_r(x)) \right\}$$

$$= \int_R d^4x\, \delta\varphi_r(x) \Big\{ \frac{\partial}{\partial\ \varphi_r(x)} \mathcal{L}(\varphi_r(x), \partial_\lambda\varphi_r(x)) - \partial_\mu \frac{\partial}{\partial\ \partial_\mu\varphi_r(x)} \mathcal{L}(\varphi_r(x), \partial_\lambda\varphi_r(x)) \Big\}$$

$$+ \int_{\partial R} d\sigma_\mu \Big\{ \delta\varphi_r(x) \frac{\partial}{\partial\ \partial_\mu\varphi_r(x)} \mathcal{L}(\varphi_r(x), \partial_\lambda\varphi_r(x)) \Big\} \tag{2.6}$$

In the last step use Gauss theorem

$$\int_R d^4x\, \partial_\mu\, V^\mu(x) = \int_{\partial R} d\sigma_\mu\, V^\mu(x) \tag{2.7}$$

which converts a four dimensional integral into a three dimensional integral $\int_{\partial R} d\sigma_\mu$. The surface integral is zero because $\delta\varphi(x) = 0$ for $x \in \partial R$. From this we derive the Euler Lagrange equation.

$$\delta S = 0 \quad \Rightarrow$$

$$\frac{\partial}{\partial\ \varphi_r(x)} \mathcal{L}(\varphi_r(x), \partial_\lambda\varphi_r(x)) = \partial_\mu \frac{\partial}{\partial\ \partial_\mu\varphi_r(x)} \mathcal{L}(\varphi_r(x), \partial_\lambda\varphi_r(x)) \tag{2.8}$$

Define the canonical momentum

$$\Pi_r(x) = \frac{\partial}{\partial\ \dot\varphi_r(x)} \mathcal{L}(\varphi_r(x), \partial_\lambda\varphi_r(x)) \quad \text{with} \quad \dot\varphi_r(x) = \frac{\partial\ \varphi_r(x)}{\partial\ t} \tag{2.9}$$

Definition Hamilton density

$$\mathcal{H}(\Pi_r(x), \varphi_r(x)) = \Pi_r(x)\, \dot\varphi_r(x) - \mathcal{L}(\varphi_r(x), \partial_\mu\varphi_r(x)) \tag{2.10}$$

The Hamilton equations read:

$$-\frac{\partial\ \mathcal{H}}{\partial\ \varphi_r} = \frac{\partial\ \mathcal{L}}{\partial\ \varphi_r} = \partial_\mu \frac{\partial\ \mathcal{L}}{\partial\ \partial_\mu\, \varphi_r} = \dot\Pi_r + \partial_k \frac{\partial\ \mathcal{L}}{\partial\ \partial_k\, \varphi_r}$$

$$\text{because} \quad \partial_0 \frac{\partial\ \mathcal{L}}{\partial\ \partial_0\, \varphi_r} = \frac{\partial}{\partial\ t} \frac{\partial\ \mathcal{L}}{\partial\ \dot\varphi_r} = \frac{\partial\ \Pi_r}{\partial\ t} = \dot\Pi_r$$

$$\frac{\partial\ \mathcal{H}}{\partial\ \Pi_r} = \dot\varphi_r \tag{2.11}$$

Applications

A <u>Scalar field</u>

The Lagrange density is given by

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2} \partial^\mu \phi \, \partial_\mu \phi - \frac{1}{2} \frac{m^2 c^2}{\hbar^2} \phi^2 \tag{2.12}$$

Application of the Euler Lagrange equation yields

$$\frac{\partial \, \mathcal{L}}{\partial \, \partial_\lambda \phi} = \frac{\partial}{\partial \, \partial_\lambda \phi} \left[\frac{1}{2} \partial^\mu \phi \, \partial_\mu \phi - \frac{1}{2} \frac{m^2 c^2}{\hbar^2} \phi^2 \right] = \frac{1}{2} \, g^{\mu\sigma} \, \delta^\lambda_\sigma \, \partial_\mu \phi$$

$$+ \frac{1}{2} \, \delta^\lambda_\mu \, \partial^\mu \phi = \partial^\lambda \phi$$

$$\frac{\partial \, \mathcal{L}}{\partial \, \phi} = - \frac{m^2 c^2}{\hbar^2} \, \phi$$

$$\Rightarrow \quad \left(\Box + \frac{m^2 c^2}{\hbar^2} \right) \phi(x) = 0 \tag{2.13}$$

The canonical momentum is

$$\Pi = \frac{1}{c} \frac{\partial \, \mathcal{L}}{\partial \, \partial_0 \phi} = \frac{1}{c} \, \partial^0 \phi = \frac{1}{c^2} \dot{\phi} \tag{2.14}$$

The Hamilton density is equal to

$$\mathcal{H} = \frac{1}{c^2} \, (\dot{\phi})^2 - \frac{1}{2c^2} \, (\dot{\phi})^2 + \frac{1}{2} \, \vec{\nabla} \phi \cdot \vec{\nabla} \phi + \frac{1}{2} \frac{m^2 c^2}{\hbar^2} \, \phi^2$$

$$= \frac{1}{2} \left(\frac{1}{c^2} \, (\dot{\phi})^2 + \vec{\nabla} \phi \cdot \vec{\nabla} \phi + \frac{m^2 c^2}{\hbar^2} \, \phi^2 \right) \tag{2.15}$$

For the complex scalar field the Lagrangian density becomes

$$\mathcal{L}(\phi, \phi^*, \partial \phi, \partial \phi^*) = \partial^\mu \phi^* \, \partial_\mu \phi - \frac{m^2 c^2}{\hbar^2} \, \phi^* \, \phi$$

$$\frac{\partial \, \mathcal{L}}{\partial \, \partial_\lambda \phi} = \partial^\lambda \phi^* \qquad \frac{\partial \, \mathcal{L}}{\partial \, \partial_\lambda \phi^*} = \partial^\lambda \phi$$

$$\frac{\partial \, \mathcal{L}}{\partial \, \phi} = - \frac{m^2 c^2}{\hbar^2} \, \phi^* \qquad \frac{\partial \, \mathcal{L}}{\partial \, \phi^*} = - \frac{m^2 c^2}{\hbar^2} \, \phi \tag{2.16}$$

From which follows

$$\left(\Box + \frac{m^2 c^2}{\hbar^2} \right) \phi(x) = 0 \qquad \left(\Box + \frac{m^2 c^2}{\hbar^2} \right) \phi^*(x) = 0$$

$$\Pi = \frac{1}{c^2}\,\dot{\phi}^* \qquad \Pi^* = \frac{1}{c^2}\,\dot{\phi}$$

$$\mathcal{H} = \frac{1}{c^2}\,\dot{\phi}^*\,\dot{\phi} + \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi + \frac{m^2 c^2}{\hbar^2}\,\phi^*\,\phi \tag{2.17}$$

B Dirac-veld

The Lagrange density is given by

$$\mathcal{L}(\psi, \bar{\psi}, \partial_\mu\psi, \partial_\mu\bar{\psi}) = i\,c\,\hbar\,\bar{\psi}\,\gamma^\mu\,\partial_\mu\psi - MC^2\,\bar{\psi}\,\psi$$

$$\frac{\partial\,\mathcal{L}}{\partial\,\partial_\mu\psi} = i\,c\,\hbar\,\bar{\psi}\,\gamma^\mu \qquad \frac{\partial\,\mathcal{L}}{\partial\,\partial_\mu\bar{\psi}} = 0$$

$$\frac{\partial\,\mathcal{L}}{\partial\,\psi} = -mc^2\,\bar{\psi} \qquad \frac{\partial\,\mathcal{L}}{\partial\,\bar{\psi}} = i\,c\,\hbar\,\gamma^\mu\,\partial_\mu\psi - mc^2\,\psi \tag{2.18}$$

The equations of motion become equal to

$$i\,\hbar\,\partial_\mu\bar{\psi}\,\gamma^\mu + mc\,\bar{\psi} = 0 \qquad i\,\hbar\,\gamma^\mu\,\partial_\mu\psi - mc\,\psi = 0 \tag{2.19}$$

The canonical momentum becomes equal to

$$\Pi = \frac{1}{c}\,\frac{\partial\,\mathcal{L}}{\partial\,\partial_0\psi} = i\,\hbar\,\bar{\psi}\,\gamma_0 = i\,\hbar\,\bar{\psi}^\dagger \qquad \Pi^\dagger = \frac{1}{c}\,\frac{\partial\,\mathcal{L}}{\partial\,\partial_0\bar{\psi}} = 0 \tag{2.20}$$

One can also derive the Dirac equation from

$$\mathcal{L}'(\psi, \bar{\psi}, \partial_\mu\psi, \partial_\mu\bar{\psi}) = -i\,c\,\hbar\,(\partial_\mu\,\bar{\psi})\,\gamma^\mu\,\psi - mc^2\,\bar{\psi}\,\psi$$

$$\Delta\mathcal{L} = \mathcal{L}' - \mathcal{L} = -i\,c\,\hbar\,\partial_\mu\Big(\bar{\psi}\,\gamma^\mu\,\psi\Big) \tag{2.21}$$

In other words the Lagrange density is determined up to a total derivative. This is a general theorem which can be derived from the action.

$$S' - S = \int_R d^4x\,\partial_\mu\,V^\mu(x) = \int_{\partial R} d\sigma_\mu\,V^\mu(x) \tag{2.22}$$

From which follows

$$S \quad \text{minimal} \quad \overset{\longleftarrow}{\longrightarrow} \quad S' \quad \text{minimal} \tag{2.23}$$

The Hamilton density becomes

$$\mathcal{H} = i\,\hbar\,\psi^\dagger\,\dot{\psi} - i\,\hbar\,\psi^\dagger\,\dot{\psi} - i\,c\,\hbar\,\bar{\psi}\,\gamma^k\,\partial_k\psi + mc^2\,\bar{\psi}\,\psi$$

$$= -i\,c\,\hbar\,\bar{\psi}\,\gamma^k\,\partial_k\psi + mc^2\,\bar{\psi}\,\psi \tag{2.24}$$

C Electromagnetic field

The Lagrange density is given by

$$\mathcal{L}(A_\mu, \partial_\nu A_\mu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.25}$$

Apply the Euler Lagrange equations

$$\frac{\partial F_{\mu\nu}}{\partial \partial_\lambda A_\sigma} = \delta^\lambda_\mu \delta^\sigma_\nu - \delta^\lambda_\nu \delta^\sigma_\mu$$

$$\frac{\partial F^{\mu\nu}}{\partial \partial_\lambda A_\sigma} = g^{\mu\alpha} g^{\nu\beta} \frac{\partial F_{\alpha\beta}}{\partial \partial_\lambda A_\sigma} = g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\lambda}$$

$$\Rightarrow \qquad \frac{\partial \mathcal{L}}{\partial \partial_\lambda A_\sigma} = -\frac{1}{4} \frac{\partial F_{\mu\nu}}{\partial \partial_\lambda A_\sigma} F^{\mu\nu} - \frac{1}{4} F^{\mu\nu} \frac{\partial F^{\mu\nu}}{\partial \partial_\lambda A_\sigma}$$

$$= -\frac{1}{4} \left(\delta^\lambda_\mu \delta^\sigma_\nu - \delta^\lambda_\nu \delta^\sigma_\mu \right) F^{\mu\nu} - \frac{1}{4} F_{\mu\nu} \left(g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\lambda} \right)$$

$$= -\frac{1}{4} F^{\lambda\sigma} + \frac{1}{4} F^{\sigma\lambda} - \frac{1}{4} F^{\lambda\sigma} + \frac{1}{4} F^{\sigma\lambda} = -F^{\lambda\sigma}$$

$$\frac{\partial \mathcal{L}}{\partial A_\sigma} = 0 \tag{2.26}$$

From this follows the equation of motion

$$\partial_\lambda F^{\lambda\sigma} = 0 \quad \text{of} \quad \Box A_\sigma - \partial_\sigma \partial_\lambda A^\lambda = 0 \tag{2.27}$$

The canonical momentum equals

$$\Pi^\sigma = \frac{1}{c} \frac{\partial \mathcal{L}}{\partial \partial_0 A_\sigma} = -\frac{1}{c} F^{0\sigma} = \frac{1}{c} F^{\sigma 0}$$

$$\rightarrow \quad \Pi^0 = 0 \qquad \Pi^k = \frac{1}{c} E^k \quad \text{Electrostatic field} \tag{2.28}$$

If the field interacts with an external source then the Lagrange density becomes

$$\mathcal{L}(A_\mu, \partial_\nu A_\mu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{e}{\hbar c} A_\mu J^\mu \tag{2.29}$$

with e as elementary unit charge. The Euler Lagrange equation yields

$$\frac{\partial \mathcal{L}}{\partial \partial_\lambda A_\sigma} = -F^{\lambda\sigma} \qquad \frac{\partial \mathcal{L}}{\partial A_\sigma} = -\frac{e}{\hbar c} J^\sigma$$

$$\rightarrow \quad \partial_\lambda F^{\lambda\sigma} = \frac{e}{\hbar c} J^\sigma \tag{2.30}$$

Expressed in the E and B fields the Lagrange density becomes

$$\mathcal{L} = -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} = \frac{1}{4}\,F_{\mu\nu}\,F^{\nu\mu} = \frac{1}{4}\,Sp(F\,F^D) = \frac{1}{2}\Big(|\vec{E}|^2 - |\vec{B}|^2\Big) \tag{2.31}$$

Hint use the notation

$$F = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \quad \rightarrow$$

$$F^D = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix} \tag{2.32}$$

The Hamilton density becomes equal to

$$\mathcal{H} = \Pi^\sigma\,\dot{A}_\sigma - \mathcal{L} = E^k \frac{\partial\;A_k}{c\partial\;t} - \frac{1}{2}\left(|\vec{E}|^2 - |\vec{B}|^2\right)$$

$$E_k = F_{k0} = \partial_k\,A_0 - \frac{\partial\;A_k}{c\partial\;t} \quad\rightarrow\quad E^k = \partial^k\,A_0 + \frac{\partial\;A_k}{c\partial\;t}$$

$$\rightarrow\quad \mathcal{H} = \frac{1}{2}\left(|\vec{E}|^2 + |\vec{B}|^2\right) - E^k\,\partial^k A_0 = \frac{1}{2}\left(|\vec{E}|^2 + |\vec{B}|^2\right) - \vec{E}\cdot\vec{\nabla}\phi$$

$$\text{with}\quad A_0 = \phi \quad \text{electrostatic potential} \tag{2.33}$$

In the radiation gauge $\phi = 0$ so that

$$\mathcal{H} = \frac{1}{2}\left(|\vec{E}|^2 + |\vec{B}|^2\right) \tag{2.34}$$

2.2 Noether theorem for classical fields

If the action S is invariant under a continuous transformation then there exists a conserved current.

This conserved current implies a conserved charge.

To prove this theorem we take all transformations infinitesimal.

$$x^\mu \quad \rightarrow x'^\mu = x^\mu + \Delta x^\mu \quad \text{with} \quad \Delta x^\mu = X_i^\mu(x)\,\Delta\omega^i(x) \tag{2.35}$$

The fields transform as

$$\varphi_r(x) \quad \rightarrow \quad \varphi'_r(x') = \varphi_r(x) + \Delta\varphi_r(x)$$

$$\partial_\mu\varphi_r(x) \quad \rightarrow \quad \partial'_\mu\varphi'_r(x') = \partial_\mu\varphi_r(x) + \Delta\,\partial_\mu\varphi_r(x)$$

$$\text{with} \quad \Delta\varphi_r(x) = \Phi_{ri}(x)\,\Delta\omega^i(x) \tag{2.36}$$

Further we define the infinitesimal transformations

$$\varphi_r(x) \quad \rightarrow \quad \varphi'_r(x) = \varphi_r(x) + \delta\varphi_r(x)$$

$$\partial_\mu\varphi_r(x) \quad \rightarrow \quad \partial_\mu\varphi'_r(x) = \partial_\mu\varphi_r(x) + \delta\,\partial_\mu\varphi_r(x) \tag{2.37}$$

Watch that the following holds

$$\delta\,\partial_\mu\varphi_r(x) = \partial_\mu\,\delta\varphi_r(x) \quad \text{but} \quad \Delta\,\partial_\mu\varphi_r(x) \neq \partial_\mu\,\Delta\varphi_r(x) \tag{2.38}$$

because δ, in contrast to Δ, does not transform x^μ. Notice that Δ transforms x as well as $\varphi_r(x)$ whereas δ only changes $\varphi_r(x)$. One can derive the following relations between both infinitesimal transformations Δ and δ.

$$\varphi'_r(x') = \varphi'_r(x^\mu + X^\mu_i(x)\Delta\omega^i(x)) = \varphi'_r(x) + X^\mu_i(x)\,\Delta\omega^i(x)\,\partial_\mu\varphi'_r(x)$$

$$= \varphi'_r(x) + X^\mu_i(x)\,\Delta\omega^i(x)\,\partial_\mu\varphi_r(x) + X^\mu_i(x)\,\Delta\omega^i(x)\,\partial_\mu\,\delta\varphi_r(x) \tag{2.39}$$

The last term can be omitted since this is of order $\Delta\,\delta$. Further we have

$$\delta\varphi_r(x) = \varphi'_r(x) - \varphi_r(x) = \varphi'_r(x') - \varphi_r(x) - X^\mu_i(x)\,\Delta\omega^i(x)\,\partial_\mu\varphi_r(x)$$

$$= \Delta\varphi_r(x) - X^\mu_i(x)\,\Delta\omega^i(x)\,\partial_\mu\varphi_r(x)$$

$$\rightarrow \quad \delta\varphi_r(x) = \Big(\Phi_{ri}(x) - X^\mu_i(x)\,\partial_\mu\varphi_r(x)\Big)\,\Delta\omega^i(x) \tag{2.40}$$

Also the following statement holds

$$\delta\,\partial_\mu\varphi_r(x) = \Delta\partial_\mu\varphi_r(x) - X^\nu_i(x)\,\Delta\omega^i(x)\,\partial_\nu\,\partial_\mu\varphi_r(x) \tag{2.41}$$

Under the transformations above the action changes into

$$\delta S = \int_{R'} d^4x'\,\mathcal{L}(\varphi'_r(x'), \partial'_\lambda\varphi'_r(x')) - \int_R d^4x\,\mathcal{L}(\varphi_r(x), \partial_\lambda\varphi_r(x)) \tag{2.42}$$

The Jacobian of the transformation is

$$\frac{\partial\ x'^\mu}{\partial\ x^\nu} = \delta^\mu_\nu + \partial_\nu\Big(X^\mu_i(x)\,\Delta\omega^i(x)\Big) \quad \rightarrow \quad d^4x' = \left|\frac{\partial\ x'^\mu}{\partial\ x^\nu}\right|\,d^4x$$

$$= \Big[1 + \partial_\mu\Big(X^\mu_i(x)\,\Delta\omega^i(x)\Big)\Big] \tag{2.43}$$

Expand $\mathcal{L}$ in $\Delta\omega$

$$\mathcal{L}(\varphi'_r(x'), \partial'_\lambda \varphi'_r(x')) = \mathcal{L}(\varphi_r(x) + \Delta\varphi_r(x), \partial_\lambda \varphi_r(x) + \Delta\, \partial_\lambda \varphi_r(x))$$

$$= \mathcal{L}(\varphi_r(x), \partial_\lambda \varphi_r(x)) + \Delta\varphi_r(x)\, \frac{\partial}{\partial\ \varphi_r(x)}\, \mathcal{L}(\varphi_r(x), \partial_\lambda \varphi_r(x))$$

$$+\Delta\, \partial_\mu \varphi_r(x) \frac{\partial}{\partial\ \partial_\mu \varphi_r(x)}\, \mathcal{L}(\varphi_r(x), \partial_\lambda \varphi_r(x)) \tag{2.44}$$

δS is then equal to

$$\delta S = \int_R d^4x \Big[\partial_\mu \Big\{ X_i^\mu(x)\, \Delta\omega^i(x) \Big\}\, \mathcal{L}(\varphi_r(x), \partial_\lambda \varphi_r(x))$$

$$+\Delta\varphi_r(x)\, \frac{\partial}{\partial\ \varphi_r(x)}\, \mathcal{L}(\varphi_r(x), \partial_\lambda \varphi_r(x))$$

$$+\Delta\, \partial_\mu \varphi_r(x) \varphi_r(x) \frac{\partial}{\partial\ \partial_\mu \varphi_r(x)}\, \mathcal{L}(\varphi_r(x), \partial_\lambda \varphi_r(x)) \Big]$$

$$= \int_R d^4x \Big[\partial_\mu \Big\{ X_i^\mu(x)\, \Delta\omega^i(x)\, \mathcal{L}(\varphi_r(x), \partial_\lambda \varphi_r(x)) \Big\}$$

$$-X_i^\mu(x)\, \Delta\omega^i(x)\, \partial_\mu \mathcal{L}(\varphi_r(x), \partial_\lambda \varphi_r(x)) + \Big\{ \delta\varphi_r(x)$$

$$+X_i^\mu(x)\, \Delta\omega^i(x)\, \partial_\mu \varphi_r(x) \Big\}\, \frac{\partial}{\partial\ \varphi_r(x)}\, \mathcal{L}(\varphi_r(x), \partial_\lambda \varphi_r(x)) + \Big\{ \delta\, \partial_\mu \varphi_r(x)$$

$$+X_i^\nu(x)\, \Delta\omega^i(x)\, \partial_\nu\, \partial_\mu \varphi_r(x) \Big\}\, \frac{\partial}{\partial\ \partial_\mu \varphi_r(x)}\, \mathcal{L}(\varphi_r(x), \partial_\lambda \varphi_r(x)) \Big] \tag{2.45}$$

Exchange in the last term μ and ν. Subsequently we have

$$\partial_\mu \mathcal{L}(\varphi_r(x), \partial_\lambda \varphi_r(x)) = \partial_\mu \varphi_r(x)\, \frac{\partial}{\partial\ \varphi_r(x)}\, \mathcal{L}(\varphi_r(x), \partial_\lambda \varphi_r(x))$$

$$+\partial_\mu\, \partial_\nu \varphi_r(x)\, \frac{\partial}{\partial\ \partial_\nu \varphi_r(x)}\, \mathcal{L}(\varphi_r(x), \partial_\lambda \varphi_r(x)) \tag{2.46}$$

Using (2.46) the second, the fourth and the last term in equation (2.45) cancel against each other and we can write

$$\delta S \;\; = \;\; \int_R d^4x \Big[\partial_\mu \Big\{ X_i^\mu(x)\, \Delta\omega^i(x)\, \mathcal{L}(\varphi_r(x), \partial_\lambda \varphi_r(x)) \Big\}$$

$$
\begin{aligned}
&+\delta\varphi_r(x)\,\frac{\partial}{\partial\;\varphi_r(x)}\,\mathcal{L}(\varphi_r(x),\partial_\lambda\varphi_r(x))\\
&+\partial_\mu\,\delta\varphi_r(x)\,\frac{\partial}{\partial\;\partial_\mu\varphi_r(x)}\,\mathcal{L}(\varphi_r(x),\partial_\lambda\varphi_r(x))\Big]\\
&=\int_R d^4x\Big[\partial_\mu\Big\{X_i^\mu(x)\,\Delta\omega^i(x)\,\mathcal{L}(\varphi_r(x),\partial_\lambda\varphi_r(x))\\
&+\delta\varphi_r(x)\,\frac{\partial}{\partial\;\partial_\mu\varphi_r(x)}\,\mathcal{L}(\varphi_r(x),\partial_\lambda\varphi_r(x))\Big\}\\
&+\delta\varphi_r(x)\Big\{\frac{\partial}{\partial\;\varphi_r(x)}\,\mathcal{L}(\varphi_r(x),\partial_\lambda\varphi_r(x))\\
&-\partial_\mu\,\frac{\partial}{\partial\;\partial_\mu\varphi_r(x)}\,\mathcal{L}(\varphi_r(x),\partial_\lambda\varphi_r(x))\Big\}\Big]
\end{aligned}
\tag{2.47}
$$

The last two terms vanish because $\varphi_r(x)$ satisfies the i Euler Lagrange equation. Substitute $\delta\varphi_r(x)$ (2.40) in the formula above then can the change in the action be written as

$$
\begin{aligned}
\delta S \;=\; -\int_R d^4x\,\partial_\mu\Big[\Big\{T_\nu^\mu(x)\,X_i^\nu(x)\\
-\Phi_{ri}(x)\,\frac{\partial}{\partial\;\partial_\mu\varphi_r(x)}\,\mathcal{L}(\varphi_r(x),\partial_\lambda\varphi_r(x))\Big\}\,\Delta\omega^i(x)\Big]
\end{aligned}
\tag{2.48}
$$

where the stress energy momentum tensor is defined as

$$
T_\nu^\mu(x)=\partial_\nu\varphi_r(x)\,\frac{\partial}{\partial\;\partial_\mu\varphi_r(x)}\,\mathcal{L}(\varphi_r(x),\partial_\lambda\varphi_r(x))-\delta_\nu^\mu\,\mathcal{L}(\varphi_r(x),\partial_\lambda\varphi_r(x)) \tag{2.49}
$$

We define now the Noether current

$$
j_i^\mu(x)=T_\nu^\mu(x)\,X_i^\nu(x)-\Phi_{ri}(x)\,\frac{\partial}{\partial\;\partial_\mu\varphi_r(x)}\,\mathcal{L}(\varphi_r(x),\partial_\lambda\varphi_r(x)) \tag{2.50}
$$

so that the change in the action can be written as

$$
\delta S=-\int_R d^4x\,\partial_\mu\Big(j_i^\mu(x)\,\Delta\omega^i(x)\Big)=-\int_{\partial R} d\sigma_\mu\,\Big(j_i^\mu(x)\,\Delta\omega^i(x)\Big) \tag{2.51}
$$

If S is left invariant under the transformation and $\Delta\omega^i(x)$ is independent of x then

$$
\delta S=-\Delta\omega^i\int_R d^4x\,\partial_\mu j_i^\mu(x)=-\Delta\omega^i\int_{\partial R} d\sigma_\mu\,j_i^\mu(x)=0 \tag{2.52}
$$

This holds for an arbitrary space R and surface ∂R bounded by $t = t_1$ and $t = t_2$. Therefore the following theorem holds

$$\partial_\mu j_i^\mu(x) = 0 \tag{2.53}$$

In other words $j_i^\mu(x)$ is a conserved current. Hence it follows that the charge defined by

$$Q_i = \frac{1}{c}\int_V d^3\vec{x}\, j_i^0(\vec{x},t) \quad \rightarrow \quad \frac{d\,Q_i}{d\,t} = 0 \tag{2.54}$$

is conserved.
Proof:

$$\begin{aligned}\frac{d\,Q_i}{d\,t} &= \frac{1}{c}\int_V d^3\vec{x}\,\partial_0 j_i^0(\vec{x},t) = \frac{1}{c}\int_V d^3\vec{x}\left(\partial_\mu j_i^\mu(\vec{x},t) - \partial_k j_i^k(\vec{x},t)\right)\\ &= \frac{1}{c}\int_V d^3\vec{x}\,\vec{\nabla}\cdot\vec{j}_i(\vec{x},t) = \frac{1}{c}\int_{\partial V} d\vec{S}\cdot\vec{j}_i(\vec{x},t) = 0\\ &\text{Gauss theorem}\end{aligned} \tag{2.55}$$

Here we have assumed that

$$\vec{j}_i(\vec{x},t) = 0 \quad \text{for} \quad x \in \partial V \tag{2.56}$$

In general one assumes $|\vec{x}| \rightarrow \infty$.
Notice: In general one can show that

$$Q_i = \frac{1}{c}\int_\sigma d\sigma_\mu\, j_i^\mu(\vec{x},t) \tag{2.57}$$

is conserved where σ_μ is a hyper surface. In the proof above we have chosen $t =$ *constant* so that $d\sigma_\mu = d^3\vec{x}\,\delta_\mu^0$.
The Noether current is not unique. One can always add two types of terms without changing the Noether charge. They are

1

$$\partial_\mu\, j_i^\mu = 0 \quad \rightarrow \quad \partial_\mu\, j_i^{'\mu} = 0 \quad \text{with} \quad j_i^{'\mu} = j_i^\mu + V_i^\mu \tag{2.58}$$

provided $V_i^\mu = 0$ using the Euler Lagrange equations to which all fields have to satisfy.

2

$$\begin{aligned}&\partial_\mu\, j_i^\mu = 0 \quad \rightarrow \quad \partial_\mu\, j_i^{'\mu} = 0 \quad \text{with} \quad j_i^{'\mu} = j_i^\mu + \partial_\nu A_i^{\mu\nu}\\ &\text{with the condition} \quad A_i^{\mu\nu} = -A_i^{\nu\mu} \quad \text{so that} \quad \partial_\mu\,\partial_\nu\, A_i^{\mu\nu} = 0\end{aligned} \tag{2.59}$$

This is given in the theory of relativity by the Poincaré group We limit ourselves to translations T_4. Take these infinitesimal $a^\mu \ll 1$.

$$
\begin{aligned}
&x^{'\mu} = x^\mu + a^\mu \quad \rightarrow \quad \Delta x^\mu = a^\mu \\
&\Delta x^\mu = X^\mu_\nu(x)\,\Delta\omega^\nu \quad \text{with} \quad \Delta\omega^\nu = a^\nu \\
&\rightarrow \quad X^\mu_\nu(x) = \delta^\mu_\nu \quad \text{or} \quad X_{\mu\nu}(x) = g_{\mu\nu} \\
&\varphi'_r(x+a) = \varphi_r(x) \quad \rightarrow \quad \Delta\varphi_r(x) = 0 \\
&\Delta\varphi_r(x) = \Phi_{r\nu}(x)\,\Delta\omega^\nu \quad \rightarrow \quad \Phi_{r\nu}(x) = 0
\end{aligned} \tag{2.60}
$$

The Noether current for translations is given by the stress energy momentum tensor (second rank tensor)

$$
T^\mu_\nu(x) = j^\mu_\nu(x) \quad \text{or} \quad T^{\mu\nu}(x) = j^{\mu\nu}(x) \quad \text{with} \quad \partial_\mu T^{\mu\nu}(x) = 0 \tag{2.61}
$$

The Noether charges are

$$
\begin{aligned}
&P^\nu = \frac{1}{c}\int d^3\vec{x}\,T^{0\nu} \qquad \frac{d\,P^\nu}{d\,t} = 0 \\
&\mathcal{H} = T^{00} \quad \text{Hamilton density}
\end{aligned} \tag{2.62}
$$

P^ν is a constant of motion and represents de four momentum of the field $\varphi_r(x)$. We study the Noether current and the Noether charge for the translations in the case of scalar, Dirac en electromagnetic field

1 Scalar veld

$$
\begin{aligned}
&T^{\mu\nu} = \partial^\mu\phi\,\partial^\nu\phi - \frac{1}{2}\,g^{\mu\nu}\left(\partial^\lambda\phi\,\partial_\lambda\phi - \frac{m^2c^2}{\hbar^2}\,\phi^2\right) \\
&\mathcal{H} = T^{00} = \frac{1}{2}\left(\frac{1}{c^2}(\dot{\phi})^2 + \vec{\nabla}\phi\cdot\vec{\nabla}\phi + \frac{m^2c^2}{\hbar^2}\,\phi^2\right) \\
&= \frac{1}{2}\left(c^2\,\Pi^2 + \vec{\nabla}\phi\cdot\vec{\nabla}\phi + \frac{m^2c^2}{\hbar^2}\,\phi^2\right) \\
&T^{0i} = \partial^0\phi\,\partial^i\phi
\end{aligned} \tag{2.63}
$$

Substitute the solution of the free field (1.23) then we can express the four momentum into its Fourier coefficients

$$
P^\nu = \frac{1}{c}\int d^3\vec{x}\,T^{0\nu} = \frac{1}{2}\int d^3\vec{p}\,p^\nu\left[a(\vec{p})\,a^*(\vec{p}) + a^*(\vec{p})\,a(\vec{p})\right] \tag{2.64}
$$

Keeping an eye on second quantisation, where the Fourier coefficients represent operators, we have assumed that the latter do not commute. For the classical field these coefficients represent complex numbers which of course commute. For the scalar field we also have

$$P^0 = \frac{1}{c} \int d^3\vec{x}\, \mathcal{H} \qquad P^k = \int d^3\vec{x}\, \Pi\, \partial^k \phi \tag{2.65}$$

Notice that for P^0 one is using the equation of motion for the scalar field.

2 Dirac-veld

$$\begin{aligned}
T^{\mu\nu} &= i\, c\, \hbar\, \bar{\psi}\, \gamma^\mu\, \partial^\nu \psi - g^{\mu\nu} \left(i\, c\, \hbar\, \bar{\psi}\, \gamma^\lambda\, \partial^\lambda \psi - mc^2\, \bar{\psi}\, \psi \right) \\
\mathcal{H} = T^{00} &= -i\, c\, \hbar\, \bar{\psi}\, \gamma^k\, \partial_k \psi + mc^2\, \bar{\psi}\, \psi \\
T^{0i} &= i\, c\, \hbar\, \bar{\psi}\, \gamma^0\, \partial^i \psi = i\, c\, \hbar\, \psi^\dagger\, \partial^i \psi
\end{aligned} \tag{2.66}$$

Because $\psi(x)$ satisfies the equation of motion (here the Dirac equation) disappears the term which is multiplied by the tensor $g^{\mu\nu}$ of $T^{\mu\nu}$.

$$\begin{aligned}
T^{\mu\nu} &= i\, c\, \hbar\, \bar{\psi}\, \gamma^\mu\, \partial^\nu \psi \quad \rightarrow \quad T^{0\nu} = i\, c\, \hbar\, \psi^\dagger\, \partial^\nu \psi \\
P^\nu &= \frac{1}{c} \int d^3\vec{x}\, T^{0\nu}(x) = i\, \hbar \int d^3\vec{x}\, \psi^\dagger(x)\, \partial^\nu \psi(x) \\
&= \int d^3\vec{x}\, \Pi(x)\, \partial^\nu \psi(x)
\end{aligned} \tag{2.67}$$

Only P^ν can be expressed in a simple way in the Fourier coefficients of the Dirac field. The results is equal to

$$P^\nu = \sum_s \int d^3\vec{p}\, p^\nu \left[b^*(\vec{p}, s)\, b(\vec{p}, s) - d(\vec{p}, s)\, d^*(\vec{p}, s) \right] \tag{2.68}$$

3 Electromagnetic field

The Noether currents are

$$\begin{aligned}
T^{\mu\nu} &= -F^{\mu\lambda}\, \partial^\nu A_\lambda - g^{\mu\nu}\, \mathcal{L} = -g_{\lambda\sigma}\, F^{\mu\lambda}\, \partial^\nu A^\sigma + g^{\mu\nu}\, \frac{1}{4}\, F_{\lambda\sigma}\, F^{\lambda\sigma} \\
T^{0\nu} &= -F^{0\lambda}\, \partial^\nu A_\lambda - g^{0\nu}\, \mathcal{L} = c\, \Pi^\lambda\, \partial^\nu A_\lambda - g^{0\nu}\, \mathcal{L} \\
T^{00} &= \mathcal{H} = \Pi^\lambda\, \dot{A}_\lambda - \mathcal{L} = \frac{1}{2} \left(|\vec{E}|^2 + |\vec{B}|^2 \right) - \vec{E} \cdot \vec{\nabla}\phi \\
T^{0i} &= c\, \Pi^\lambda\, \partial^i A_\lambda = c\, \Pi^k\, \partial^i A_k = E^k\, \partial^i A_k = -E_k\, \frac{\partial\, A_k}{\partial\, x_i} \\
&= -E_k\, \frac{\partial\, A_i}{\partial\, x_k} - (\vec{E} \times \vec{B})_i = E_k\, \frac{\partial\, A^i}{\partial\, x_k} + (\vec{E} \times \vec{B})^i
\end{aligned} \tag{2.69}$$

The Noether charges are

$$P^0 = H = \frac{1}{c}\int d^3\vec{x}\, T^{00} = \frac{1}{2c}\int d^3\vec{x}\left(|\vec{E}|^2 + |\vec{B}|^2\right)$$

E.M. energy density

$$P^i = \frac{1}{c}\int d^3\vec{x}\, T^{0i} = \frac{1}{c}\int d^3\vec{x}\,(\vec{E}\times\vec{B})^i \quad \text{Poynting-vector}$$

$$\text{or} \quad P^i = \frac{1}{c}\int d^3\vec{x}\,\Pi^\lambda\,\partial^i A_\lambda \tag{2.70}$$

Notice that

$$\int d^3\vec{x}\, E_k\,\frac{\partial\, A^i}{\partial\, x_k} = -\int d^3\vec{x}\, A^i\,\vec{\nabla}\cdot\vec{E} = 0 \quad \text{because} \quad \nabla\cdot\vec{E} = 0 \tag{2.71}$$

In the case of the radiation gauge the momentum of the field can be expressed in the Fourier coefficients as follows

$$P^\nu = \frac{1}{2}\sum_\alpha \int d^3\vec{k}\,\hbar\, k^\nu\left[a(\vec{k},\alpha)\, a^*(\vec{k},\alpha) + a^*(\vec{k},\alpha)\, a(\vec{k},\alpha)\right]$$

$$\alpha = 1,2 \quad \text{or} \quad \alpha = R, L \tag{2.72}$$

In the case of the Lorentz gauge we have also two unphysical polarisations

$$P^\nu = -\frac{1}{2}\sum_\alpha \int d^3\vec{k}\,\hbar\, k^\nu\, g^{\lambda\sigma}\left[a(\vec{k},\lambda)\, a^*(\vec{k},\sigma) + a^*(\vec{k},\lambda)\, a(\vec{k},\sigma)\right] \tag{2.73}$$

2.3 Global en Local gauge symmetries

There are two kinds of symmetries in Physics.

a External symmetries
They concern space and time. For example in special relativity these symmetries are given by the Poincaré group. For classical mechanics and non-relativistic quantum mechanics by the Galilei group.

b Internal symmetries
The concern the internal degrees of freedom. Here the space time coordinates are not changed. The consequence is that the action is not changed only but also the Lagrange density is left invariant. Examples of these symmetries are:

1 Degree of freedom: charge. Symmetry: group $U(1)$. Theory: electrodynamics.

2 Degree of freedom: isospin. Symmetry: group $SU(2)$. Theory: strong interactions.

3 Degree of freedom: colour. Symmetry: group $SU(3)$. Theory: quantum Chromodynamics.

The internal symmetry transformations, provided they are continuous, are called gauge transformations. The latter are represented by Lie groups.

Group theory

Symmetry transformations are linear transformations which change the coordinates of one or more particles into other coordinates in such a way that the physical system is left invariant. In these lectures we shall limit ourselves to group transformations.
Definition: A group G is a set of finite elements $g_1, g_2, \cdots, g_n$ and satisfies the following properties

1. Multiplication: $g_i.g_j = g_k$.
2. unit element e: $e.g_i = g_i.e = g_i$.
3. Inverse of g_i is g_i^{-1}: $g_i.g_i^{-1} = g_i^{-1}.g_i = e$.
4. Associativity: $g_i.(g_j.g_k) = (g_i.g_j).g_k$

In the special case that the following relation is satisfied

5. Commutativity: $g_i.g_j = g_j.g_i$.

G is called an Abelian group.

Infinite group:

An infinite group G has elements $g(t_1, t_2, \cdots, t_n)$ which are functions of the continuous variables $t_1, t_2, \cdots, t_n$. As infinite group we shall only consider Lie groups. The set $t_1, t_2, \cdots, t_n$ constitute a manifold. Further is $g(t_1, t_2, \cdots, t_n)$ differentiable w.r.t. $t_1, t_2, \cdots, t_n$ around $0, 0, \cdots, 0$ with $g(0, 0, \cdots, 0) \equiv e = 1$. Each element element can then be written as

$$g(t_1, t_2, \cdots, t_n) = \exp\left(\sum_{i=1}^{n} a_i\, t_i\right) \qquad a_i = \frac{d\, g}{d\, t_i}|_{(t_1,t_2,\cdots,t_n)=(0,0,\cdots,0)} \tag{2.74}$$

The elements a_i are the generators of the Lie group G. They constitute a Lie algebra of G (notation $\mathcal{L}(G)$). The dimension of the Group manifold is denoted by n. Notice the g_i and a_i can be represented by matrices but also other representations are possible.
Properties of a Lie algebra $\mathcal{L}(G)$ of G. A Lie algebra is a linear vector space in which is defined a product $[,]$ which is anticommutative

1

$$A, B \in \mathcal{L}(G) \quad \rightarrow \quad [A, B] = -[B, A] \in \mathcal{L}(G) \tag{2.75}$$

2 The elements of the Lie algebra satisfy the Jacobi identity

$$A, B, C \in \mathcal{L}(G) \quad \rightarrow \quad [[A, B], C] + [[B, C], A] + [[C, A], B] = 0 \qquad (2.76)$$

For matrices the product looks like

$$A, B \in \mathcal{L}(G) \quad \rightarrow \quad [A, B] = A\,B - B\,A \in \mathcal{L}(G) \qquad (2.77)$$

External symmetry group

An example of a symmetry group is the rotation group in three dimensions. Rotations are transformations in a vector space which leave the inner product invariant. Let us start with rotations in one plane. The Lie group is represented by $SO(2)$. For a general element $R(\theta)$ we can be write

$$R(\theta) = \exp\left(\theta\, I_{12}\right) \quad \text{with} \quad I_{12} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad (2.78)$$

Here I_{12} is the generator of $SO(2)$. From $\det R = 1$ follows Tr $I_{12} = 0$ (Tr = trace). From the above it follows

$$\begin{aligned} I_{12}^n &= E \qquad n = 0 \mod 4 \\ I_{12}^n &= I_{12} \qquad n = 1 \mod 4 \\ I_{12}^n &= -E \qquad n = 2 \mod 4 \\ I_{12}^n &= -I_{12} \qquad n = 3 \mod 4 \end{aligned} \qquad (2.79)$$

so that

$$\begin{aligned} R(\theta) &= \sum_{k=0}^{\infty} \frac{\theta^k}{k!}\, I_{12}^k = E\,\cos\theta + I_{12}\,\sin\theta \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \end{aligned} \qquad (2.80)$$

Remark: the generator one can always be found by differentiating the element of the group to θ at $\theta = 0$

$$\Rightarrow \quad I_{12} = \frac{d\ R(\theta)}{d\ \theta}\Big|_{\theta=0} \qquad (2.81)$$

The Lie algebra of $SO(2)$ (notation $\mathcal{L}(SO(2))$ contains one element only i.e. I_{12} and it is Abelian.

Orthogonal group in three dimensions $SO(3)$.

If $\det R = 1$ then $R \in SO(3)$. There are three elementary rotations in $SO(3)$ which are given by rotations around x-, y- and z-axis. [1].

$$R_x(\theta_1) = \exp(\theta_1 I_{23}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix} \qquad I_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.82}$$

$$R_y(\theta_2) = \exp(\theta_2 I_{31}) = \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \qquad I_{31} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{2.83}$$

$$R_z(\theta_3) = \exp(\theta_3 I_{12}) = \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad I_{12} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.84}$$

Notice: all rotations are right handed. Define

$$\begin{aligned} & K_1 \equiv I_{23} \qquad K_2 \equiv I_{31} \qquad K_3 \equiv I_{12} \\ \Rightarrow \quad & [K_1, K_2] = K_3 \quad \text{cyclic} \qquad K_i \in \mathcal{L}(SO(3)) \end{aligned} \tag{2.85}$$

Notice that instead of I_{23}, I_{31} and I_{12} one can also choose I_{32}, I_{13} and I_{21} provided $I_{ij} = -I_{ji}$ (There are three independent generators in $SO(3)$). The group $SO(3)$ is non-Abelian for $R(\theta_1)\,R(\theta_2) \neq R(\theta_2)\,R(\theta_1)$ (non commutative). Rotations leave quantities as kinetic energy $T = 1/2mv^2 = 1/2m\vec{v}\cdot\vec{v}$ and potentials $V(r)$ with $r = |\vec{r}|$ invariant. Newtons Law

$$\vec{F} = m\,\vec{a} = m\,\frac{d^2\vec{r}(t)}{d\,t} \tag{2.86}$$

is left covariant under rotations.

$$\vec{F}' = R\,\vec{F} \quad \vec{a}' = R\,\vec{a} \quad \rightarrow \quad \vec{F}' = m\,\vec{a}' \tag{2.87}$$

Internal symmetry group

We consider Lie groups with the following Lie algebra which is given by the generators T_i satisfying the commutation relations

$$[T_a, T_b] = i\,f_{abc}\,T_c \tag{2.88}$$

The structure constant f_{abc} are antisymmetric in a, b, c. A special representation is the adjoint representation

$$(T_a)_{bc} = -i\,f_{abc} \tag{2.89}$$

[1]For the components x, y, z has been also used the notation x_1, x_2, x_3.

Let us limit ourselves to gauge transformations on fields. They look like

$$\varphi_{r,k}(x) \quad \to \quad \varphi'_{r,k}(x) = \Big[\exp\Big(- i\,\vec{\alpha}(x)\cdot\vec{T}\Big)\Big]_{kl}\,\varphi_{r,l}(x)$$

$$\vec{\alpha}(x)\cdot\vec{T} = \alpha_i(x)\,T_i = \sum_{i=1}^{n}\alpha_i(x)\,T_i \tag{2.90}$$

Here $\alpha_i(x)$ are the n parameters of the Group manifold and are real functions of space and time. The indices k, l correspond to the internal symmetry group G whereas r refers to the external group in this case the Lorentz group $SO(3,1)$. If the symmetry group is unitary then the following holds

$$\Big[\exp\Big(- i\,\vec{\alpha}(x)\cdot\vec{T}\Big)\Big]^{\dagger} = \Big[\exp\Big(i\,\vec{\alpha}(x)\cdot\vec{T}^{\dagger}\Big)\Big] = \Big[\exp\Big(i\,\vec{\alpha}(x)\cdot\vec{T}\Big)\Big]$$

$$\Rightarrow \quad T_a^{\dagger} = T_a \quad T_a \text{ is hermitian} \quad \vec{T} = (T_1, T_2, \cdots T_n) \tag{2.91}$$

Examples

Electromagnetism

In the case of electromagnetism is the gauge group given by $U(1)$. The elements consists of a set of phase transformations given by

$$U(1) = \{e^{i\,\alpha}\} \quad \to \quad T = 1 \tag{2.92}$$

In the case of the real charge α is also multiplied by the elementary unit charge $e = 1,6.10^{-19}\,\mathrm{C}$.

Isospin

Here is the gauge group represented by $SU(2)$ which is non-Abelian. The structure constants are $f_{abc} = \epsilon_{abc}$ (Levi-Civita tensor components $a, b, c = 1, 2, 3$).

The fundamental representation is in the two dimensional space of iso-spinors. It is given by:

$$T_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad T_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.93}$$

In the adjoint representation $\Big(T_a\Big)_{mn} = -i\,\epsilon_{amn}$, which is in three dimensional space, one obtains the following matrices:

$$T_1 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad T_2 = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad T_3 = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.94}$$

One distinguishes two kinds of gauge transformations.

1 Global gauge transformations. Here $\alpha_i(x)$ are independent of x. They are also called gauge transformations of the first kind.

2 Local gauge transformations. Here $\alpha_i(x)$ is dependent on x. The are also called gauge transformations of the second kind.

Global gauge transformations

For a global gauge transformation exist a Noether current and a conserved charge. Take the gauge transformation infinitesimal

$$\varphi'_{r,k}(x) = \Big(\mathbf{1} - i\,\vec{\alpha}\cdot\vec{T}\Big)_{kl}\,\varphi_{r,l}(x) = \Big[\delta_{kl} - i\,\alpha_a\,\Big(T_a\Big)_{kl}\Big]\varphi_{r,l}(x) \tag{2.95}$$

Hence it follows from (2.35) and (2.36)

$$\begin{aligned}
&\Delta x^\mu = 0 \quad \rightarrow \quad X^\mu_k(x) = 0 \\
&\Delta\varphi_{r,k}(x) = \Phi_{ka}(x)\,\Delta\omega_a \quad \text{with} \quad \Delta\omega_a = -\alpha_a \quad \text{and} \\
&\Phi_{r,ka}(x) = i\,\Big(T_a\Big)_{kl}\,\varphi_{r,l}(x) \quad \Phi^*_{r,ka}(x) = -i\,\Big(T^*_a\Big)_{kl}\,\varphi^*_{r,l}(x)
\end{aligned} \tag{2.96}$$

Notice that the space time coordinates do not change under gauge transformations. The Noether current in (2.50) becomes equal to

$$j^\mu_a(x) = -i\,\frac{\partial\,\mathcal{L}}{\partial\,\partial_\mu\varphi_{r,k}(x)}\,\Big(T_a\Big)_{kl}\,\varphi_{r,l}(x) + i\,\frac{\partial\,\mathcal{L}}{\partial\,\partial_\mu\varphi^*_{r,k}(x)}\,\Big(T^*_a\Big)_{kl}\,\varphi^*_{r,l}(x) \tag{2.97}$$

with the Noether charge

$$Q_a = \frac{1}{\hbar c}\int d^3\vec{x}\,j^0_a(x) \quad \text{with} \quad \frac{d\,Q_a}{d\,t} = 0 \tag{2.98}$$

Examples

A Complex scalar field

$$\begin{aligned}
\phi'_k(x) &= \Big[\exp\Big(-i\,\vec{\alpha}\cdot\vec{T}\Big)\Big]_{kl}\,\phi_l(x) \\
\phi'^*_k(x) &= \Big[\exp\Big(i\,\vec{\alpha}\cdot\vec{T}^*\Big)\Big]_{kl}\,\phi^*_l(x) = \phi^*_l(x)\,\Big[\exp\Big(i\,\vec{\alpha}\cdot\vec{T}\Big)\Big]_{lk}
\end{aligned} \tag{2.99}$$

The Lagrangian density

$$\mathcal{L}^B_{mat} = \partial^\mu\phi^*_k\,\partial_\mu\phi_k - \frac{m^2c^2}{\hbar^2}\phi^*_k\,\phi_k \tag{2.100}$$

is invariant under this gauge transformation. The dependence of the fields on x is omitted for simplicity. The Noether current becomes equal to

$$j^\mu_a(x) = -i\,\frac{\partial\,\mathcal{L}^B_{mat}}{\partial\,\partial_\mu\phi_k}\Big(T_a\Big)_{kl}\,\phi_l + i\,\frac{\partial\,\mathcal{L}^B_{mat}}{\partial\,\partial_\mu\phi^*_k}\Big(T^*_a\Big)_{kl}\,\phi^*_l$$

$$= i\,\Big(\partial^\mu\phi_k\,\Big(T^*_a\Big)_{kl}\,\phi^*_l - \partial^\mu\phi^*_k\,\Big(T_a\Big)_{kl}\,\phi_l\Big)$$

$$= i\,\Big(\phi^*_l\,\Big(T_a\Big)_{lk}\,\partial^\mu\phi_k - \partial^\mu\phi^*_k\,\Big(T_a\Big)_{kl}\,\phi_l\Big)$$

$$= i\,\Big(\phi^*_k\,\Big(T_a\Big)_{kl}\,\partial^\mu\phi_l - \partial^\mu\phi^*_k\,\Big(T_a\Big)_{kl}\,\phi_l\Big) = i\,\phi^*_k\,\Big(T_a\Big)_{kl}\,\overset{\leftrightarrow}{\partial^\mu}\,\phi_l \qquad (2.101)$$

From which follows that the Noether current is real

$$j^{*\mu}_a = \quad \Big(i\,\phi^*_k\,\Big(T_a\Big)_{kl}\,\overset{\leftrightarrow}{\partial^\mu}\,\phi_l\Big)^* = -i\,\phi_k\,\Big(T^*_a\Big)_{kl}\,\overset{\leftrightarrow}{\partial^\mu}\,\phi^*_l$$

$$= -i\,\phi_k\,\Big(T_a\Big)_{lk}\,\overset{\leftrightarrow}{\partial^\mu}\,\phi^*_l = i\,\phi^*_l\,\Big(T_a\Big)_{lk}\,\overset{\leftrightarrow}{\partial^\mu}\,\phi_k = j^\mu_a$$

$$\rightarrow \quad Q^*_a = Q_a \qquad (2.102)$$

Therefore the charge is also real. If we express the field in its Fourier coefficients then the charge can be written as follows

$$Q_a = \frac{i}{\hbar}\Big(T_a\Big)_{kl}\int d^3\vec{x}\Big[\phi^*_k\,\Pi^*_l - \Pi_k\,\phi_l\Big]$$

$$Q_a = \int d^3\vec{p}\,\Big[a_{+,k}(\vec{p})\,\Big(T_a\Big)_{kl}\,a^*_{+,l}(\vec{p}) - a^*_{-,k}(\vec{p})\,\Big(T_a\Big)_{kl}\,a_{-,l}(\vec{p})\Big] \qquad (2.103)$$

B Complex Dirac field

$$\psi'_{r,k}(x) = \Big[\exp\Big(-i\,\vec{\alpha}\cdot\vec{T}\Big)\Big]_{kl}\,\psi_{r,l}(x)$$

$$\bar{\psi}'_{r,k}(x) = \Big[\exp\Big(i\,\vec{\alpha}\cdot\vec{T}^*\Big)\Big]_{kl}\,\bar{\psi}_{r,l}(x) = \bar{\psi}_{r,l}(x)\,\Big[\exp\Big(i\,\vec{\alpha}\cdot\vec{T}\Big)\Big]_{lk} \qquad (2.104)$$

The Lagrange density

$$\mathcal{L}^F_{mat} = i\,c\,\hbar\,\bar{\psi}_{r,k}\,\Big(\gamma^\mu\Big)_{rs}\,\partial_\mu\psi_{s,k} - mc^2\,\bar{\psi}_{r,k}\,\psi_{r,k} \qquad (2.105)$$

is invariant under this gauge transformation. The Noether current becomes

$$j^\mu_a(x) = -i\,\frac{\partial\,\mathcal{L}^F_{mat}}{\partial\,\partial_\mu\psi_{r,k}}\Big(T_a\Big)_{kl}\,\psi_{r,l} + i\,\frac{\partial\,\mathcal{L}^F_{mat}}{\partial\,\partial_\mu\bar{\psi}_{r,k}}\Big(T^*_a\Big)_{kl}\,\bar{\psi}_{r,l}$$

$$= c\,\hbar\,\bar{\psi}_{r,k}\,\Big(\gamma^\mu\Big)_{rs}\,\Big(T_a\Big)_{kl}\,\psi_{s,l} \qquad (2.106)$$

Hence it follows again that the Noether current is real in other words

$$j_a^{*\mu} = j_a^\mu \quad \rightarrow \quad Q_a^* = Q_a \tag{2.107}$$

The charge is also real. If we express the field in its Fourier coefficients then the charge can be written as follows

$$Q_a = -\frac{i}{\hbar}\Big(T_a\Big)_{kl}\int d^3\vec{x}\,\Pi_{r,k}\,\psi_{r,l}$$

$$Q_a = \sum_s \int d^3\vec{p}\Big[b_k^*(\vec{p},s)\Big(T_a\Big)_{kl}\,b_l(\vec{p},s) + d_k(\vec{p},s)\Big(T_a\Big)_{kl}\,d_l^*(\vec{p},s)\Big] \tag{2.108}$$

Local gauge transformations

For local gauge transformations one can show that after application of the equation of motion the Noether current can be written as $\partial_\mu T^{\mu\nu}$ where $T^{\mu\nu} = -T^{\nu\mu}$. In this case the Noether charge is equal to zero (see also the remark in (2.59)) and one has no genuine Noether current. Lagrange densities which are global invariant are very often not local gauge invariant. Examples are the Lagrange densities in (2.12) and (2.18). For simplicity we shall take the gauge transformations infinitesimal and limit ourselves to $U(1)$

$$\phi'(x) = \Big[1 - i\,\alpha(x)\Big]\,\phi(x)$$

$$\phi^{'*}(x) = \phi^*(x)\,\Big[1 + i\,\alpha(x)\Big] \tag{2.109}$$

$$\partial_\mu\,\phi'(x) = -i\,\partial_\mu\alpha(x)\,\phi(x) + \Big[1 - i\,\alpha(x)\Big]\,\partial_\mu\phi(x)$$

$$\partial_\mu\,\phi^{'*}(x) = i\,\phi^*(x)\,\partial_\mu\,\alpha(x) + \partial_\mu\phi^*(x)\,\Big[1 + i\,\alpha(x)\Big] \tag{2.110}$$

The Lagrange density of the complex scalar field in (2.16) transforms as

$$\delta\mathcal{L}^B_{mat} = \mathcal{L}^B_{mat}\Big(\phi'(x), \partial_\lambda\phi'(x)\Big) - \mathcal{L}^B_{mat}\Big(\phi(x), \partial_\lambda\phi(x)\Big) \tag{2.111}$$

with

$$\delta\mathcal{L}^B_{mat} = i\,\phi^*(x)\,\partial^\mu\,\alpha(x)\,\partial_\mu\,\phi_k(x) - i\,\partial^\mu\phi^*(x)\,\partial_\mu\,\alpha(x)\,\phi(x)$$

$$= \partial_\mu\,\alpha(x)\,j^\mu(x) \tag{2.112}$$

For the Dirac field one obtains after local gauge transformations

$$\psi_r'(x) = \Big[1 - i\,\alpha(x)\Big]\,\psi_r(x)$$

$$\bar{\psi}_r'(x) = \bar{\psi}_r(x)\,\Big[1 + i\,\alpha(x)\Big] \tag{2.113}$$

$$\partial_\mu \psi'_r(x) = -i\,\partial_\mu \alpha(x)\,\psi_r(x) + \Big[1 - i\,\alpha(x)\Big]\,\partial_\mu \psi_r(x)$$

$$\partial_\mu \bar{\psi}'_r(x) = i\,\bar{\psi}_r(x)\,\partial_\mu \alpha(x) + \partial_\mu \bar{\psi}_r(x)\,\Big[1 + i\,\alpha(x)\Big] \tag{2.114}$$

The Lagrange density for the Dirac field in (2.18) transforms as

$$\delta\mathcal{L}^F_{mat} = \hbar\,c\,\bar{\psi}_r(x)\Big(\gamma^\mu\Big)_{rs}\psi_s(x)\,\partial_\mu \alpha(x) = \partial_\mu \alpha(x)\,j^\mu(x) \tag{2.115}$$

One observes that in both cases the Lagrange densities are global ($\partial_\mu\alpha(x) = 0$) but not local gauge invariant. To make the Lagrange densities local gauge invariant we have to introduce a gauge field. Let us first remark that the gauge invariance breaking term is proportional to the Noether current which occurred at global gauge invariance. For the scalar fields the transformation proceeds as

$$j'^\mu(x) = -i\left(\partial_\mu\,\phi'^*(x)\;\phi'(x) - \phi'^*(x)\;\partial^\mu\phi'(x)\right) = j^\mu(x)$$

$$+\partial^\mu\,\alpha(x)\left(\phi^*(x)\,\phi(x) + \phi^*(x)\;\phi(x)\right) \tag{2.116}$$

$$j'^\mu(x) = j^\mu(x) + 2\,\partial^\mu\,\alpha(x)\,\phi^*(x)\,\phi(x) \tag{2.117}$$

For the Dirac field the transformation proceeds as

$$j'^\mu(x) = j^\mu(x) \tag{2.118}$$

To make the Lagrange densities in (2.105) and(2.110) local gauge invariant we introduce a new term in $\mathcal{L}_{mat}$. Because this is simpler for the Dirac field we start with the latter. Introduce the following interaction Lagrange density

$$\mathcal{L}^F_{int} = -\frac{g}{\hbar c}\,j_\mu(x)\,A^\mu(x) \tag{2.119}$$

Here g is the gauge coupling constant which indicates the strength with which the gauge field $A^\mu(x)$ couples to the current. The gauge field transforms locally as follows

$$A'^\mu(x) \quad = \quad A^\mu(x) + \frac{\hbar c}{g}\,\partial^\mu\alpha(x) \tag{2.120}$$

Under local gauge transformations transforms the interaction Lagrange density as

$$\begin{aligned}
\mathcal{L}'^F_{int} &= -\frac{g}{\hbar c}\,j'_\mu(x)\,A'^\mu(x) \\
&= \mathcal{L}^F_{int} - \partial^\mu\,\alpha(x)\,j_\mu(x) \\
&\rightarrow \;\delta\mathcal{L}^F_{int} = \mathcal{L}'^F_{int} - \mathcal{L}^F_{int} = -\partial^\mu\,\alpha(x)\,j_\mu(x)
\end{aligned} \tag{2.121}$$

and the combination

$$\mathcal{L}^F = \mathcal{L}^F_{mat} + \mathcal{L}^F_{int} \tag{2.122}$$

is therefore local gauge invariant because

$$\delta\mathcal{L}^F_{mat} = -\delta\mathcal{L}^F_{int} \tag{2.123}$$

In the case of scalar fields is the term $\mathcal{L}_{int}$ not sufficient to render the Lagrange density local gauge invariant. In this case we get after transformation

$$\mathcal{L}'_{int} = \mathcal{L}_{int} - \partial^\mu\,\alpha(x)\,j_\mu(x) - 2\,\frac{g}{\hbar c}\,\partial_\mu\alpha(x)\,\phi^*(x)\,\phi(x)\,A^\mu(x) \tag{2.124}$$

We add to $\mathcal{L}_{int}$ the next term

$$\frac{g^2}{(\hbar c)^2}\,\phi^*(x)\,\phi(x)\,A^\mu(x)\,A_\mu(x) \tag{2.125}$$

Under local gauge transformations this term becomes

$$\begin{aligned} &\frac{g^2}{(\hbar c)^2}\,\phi^{'*}(x)\,\phi'(x)\,A'^\mu(x)\,A'\mu(x) \\ &= \frac{g^2}{(\hbar c)^2}\,\phi^*(x)\,\phi(x)\,A^\mu(x)\,A_\mu(x) \\ &+2\,\frac{g}{\hbar c}\,\partial_\mu\alpha(x)\,\phi^*(x)\,\phi(x)\,A^\mu(x) \end{aligned} \tag{2.126}$$

Define for the scalar fields a new interaction Lagrange density

$$\mathcal{L}^B_{int} = -\frac{g}{\hbar c}\,j_\mu(x)\,A^\mu(x) + \frac{g^2}{(\hbar c)^2}\,\phi^*(x)\,\phi(x)\,A^\mu(x)\,A_\mu(x) \tag{2.127}$$

Under local gauge transformation this term becomes equal to

$$\mathcal{L}'^B_{int} = \mathcal{L}^B_{int} - \partial^\mu\,\alpha(x)\,j_\mu(x) \tag{2.128}$$

so that here also

$$\mathcal{L}^B = \mathcal{L}^B_{mat} + \mathcal{L}^B_{int} \tag{2.129}$$

is local gauge invariant. To complete the theory we need besides derivatives of the scalar field and the Dirac field also derivatives of the gauge field. For the gauge field this is achieved by

$$\mathcal{L}_{ijk} = -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} \tag{2.130}$$

with the gauge field tensor

$$F^{\mu\nu} = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) \tag{2.131}$$

Under local gauge transformations remains the tensor unchanged

$$F'^{\mu\nu} = F^{\mu\nu} \tag{2.132}$$

From the above it follows that $\mathcal{L}_{ijk}$ is local gauge invariant. The full Lagrange density reads:

$$\mathcal{L} = \mathcal{L}_{ijk} + \mathcal{L}_{mat} + \mathcal{L}_{int} \tag{2.133}$$

The index *mat* refers to matter fields. These are all the fields except for the gauge fields.

<u>The notion covariant derivative</u>

We have seen that under local gauge transformations $\partial_\mu\phi(x)$ and $\partial_\mu\psi(x)$ transform differently than $\phi(x)$ and $\psi(x)$ as shown in (2.104) and (2.109) respectively. Therefore $\partial_\mu\phi(x)$ and $\partial_\mu\psi(x)$ are the wrong vectors of the group. We define a new kind of derivative namely the covariant derivative:

$$D^\mu = \partial^\mu + i\,\frac{g}{\hbar c}\,A^\mu(x) \tag{2.134}$$

We now study the change in the covariant derivative under local gauge transformations. This we show for the scalar field

$$\begin{aligned} D^\mu\,\phi(x) &= \Big(\partial^\mu + i\,\frac{g}{\hbar c}\,A^\mu(x)\Big)\,\phi(x) \\ D^\mu\,\phi^*(x) &= \Big(\partial^\mu - i\,\frac{g}{\hbar c}\,A^\mu(x)\Big)\,\phi^*(x) \end{aligned} \tag{2.135}$$

Apply the local gauge transformation

$$\begin{aligned} D'^\mu\,\phi'(x) \;=\; & \Big(\partial^\mu + i\,\frac{g}{\hbar c}\,A'^\mu(x)\Big)\Big(1 - i\,\alpha(x)\,\Big)\,\phi(x) \\ &= \Big(\partial^\mu + i\,\frac{g}{\hbar c}\,\Big\{A^\mu(x) + \frac{\hbar c}{g}\,\partial^\mu\alpha(x)\Big\}\Big)\Big(1 - i\,\alpha(x)\Big)\,\phi(x) \\ &= \partial^\mu\phi(x) - i\,\partial^\mu\alpha(x)\,\phi(x) - i\,\alpha(x)\,\partial^\mu\phi(x) + i\,\frac{g}{\hbar c}\,A^\mu(x)\,\phi(x) \\ &+ i\,\partial^\mu\,\alpha(x)\,\phi(x) + \frac{g}{\hbar c}\,\alpha(x)\;A^\mu(x)\,\phi(x) \\ &= \Big(1 - i\,\alpha(x)\Big)\,\partial^\mu\phi(x) + i\,\frac{g}{\hbar c}\,A^\mu(x)\,\phi(x) + \frac{g}{\hbar c}\,\alpha(x)\,A^\mu(x)\,\phi(x) \\ &= \Big(1 - i\,\alpha(x)\Big)\,\Big(\partial^\mu + i\,\frac{g}{\hbar c}\,A^\mu(x)\Big)\,\phi(x) \\ &= \Big(1 - i\,\alpha(x)\Big)\,D^\mu\,\phi(x) \end{aligned} \tag{2.136}$$

The derivation for the Dirac field goes in an analogous way. The conclusion is that $D^\mu\phi(x)$ and $D^\mu\psi(x)$ transform in the same way as $\phi(x)$ and $\psi(x)$. They are good vectors in the representation space of the group. We can write the Lagrange density in (2.138) in a manifest local gauge invariant way. For the scalar fields this becomes

$$\mathcal{L}^B = -\frac{1}{4}\, F_{\mu\nu}\, F^{\mu\nu} + D^\mu\, \phi^*\, D_\mu\, \phi - \frac{m^2 c^2}{\hbar^2}\, \phi^*\, \phi \tag{2.137}$$

and for the Dirac field

$$\mathcal{L}^F = -\frac{1}{4}\, F_{\mu\nu}\, F^{\mu\nu} + i\, c\, \hbar\, \bar{\psi}_r \left(\gamma^\mu\right)_{rs} D_\mu\, \psi_s - mc^2\, \bar{\psi}_r\, \psi_r \tag{2.138}$$

After application of the Euler Lagrange equation one gets again the equations of motion. For the scalar fields they are

$$\partial^\mu\, F_{\mu\nu} = \frac{g}{\hbar c}\, j_\nu \qquad j_\nu = i\, \phi^*\, \overset{\leftrightarrow}{D}_\nu\, \phi$$

$$\left(D^\mu\, D_\mu + \frac{m^2 c^2}{\hbar^2}\right) \phi = 0 \qquad \phi^* \left(\, \overset{\leftarrow}{D^{*\mu}}\ \overset{\leftarrow}{D^*_\mu} + \frac{m^2 c^2}{\hbar^2}\right) = 0 \tag{2.139}$$

and for the Dirac fields

$$\partial^\mu\, F_{\mu\nu} = \frac{g}{\hbar c}\, j_\nu \qquad j_\nu = c\, \hbar\, \bar{\psi}_r \left(\gamma_\nu\right)_{rs} \psi_s$$

$$\left(i\, \hbar\, \gamma^\mu\, D_\mu - mc\right) \psi = 0 \qquad \bar{\psi} \left(i\, \hbar\, \gamma^\mu\ \overset{\leftarrow}{D^*_\mu} + mc\right) = 0 \tag{2.140}$$

3 Relativistic quantum field theory

The formalism to quantise relativistic fields is analogous to the procedure in non-relativistic quantum mechanics. In the latter we had to deal with a system of a finite degrees of freedom $\{q_i(t), p_i(t)\}$. In the former we had to deal with a system of infinite degrees of freedom given by $\{\varphi(\vec{x}, t), \Pi(\vec{x}, t)\}$.

ANALOGUE

first quantisation	second quantisation
non-relativistic quantum mechanics	relativistic quantum field theory
generalised coordinates: $q_i(t)$	generalised coordinates: $\varphi(\vec{x}, t)$
canonical momentum: $p_i(t) = \dfrac{\partial\, \mathcal{L}}{\partial\, \dot{q}_i(t)}$	canonical momentum: $\Pi(\vec{x}, t) = \dfrac{\partial\, \mathcal{L}}{\partial\, \dot{\varphi}(\vec{x}, t)}$
$H = \sum_{i=1}^{N} \dfrac{p_i^2}{2m} + V(q_1, \cdots q_n)$	$\mathcal{H} = \dfrac{1}{2}\Big[c^2\, \Pi^2(\vec{x}, t) + \vec{\nabla}\varphi(\vec{x}, t) \cdot \vec{\nabla}\varphi(\vec{x}, t) + \dfrac{m^2 c^2}{\hbar^2}\, \varphi^2(\vec{x}, t)\Big]$
$[\hat{q}_k(t), \hat{p}_l(t)] = i\, \hbar\, \delta_{kl}$	$[\hat{\varphi}(\vec{x}, t), \hat{\Pi}(\vec{x}', t)] = i\, \hbar\, \delta^{(3)}(\vec{x} - \vec{x}')$
$[\hat{q}_k(t), \hat{q}_l(t)] = 0$	$[\hat{\varphi}(\vec{x}, t), \hat{\varphi}(\vec{x}', t)] = 0$
$[\hat{p}_k(t), \hat{p}_l(t)] = 0$	$[\hat{\Pi}(\vec{x}, t), \hat{\Pi}(\vec{x}', t)] = 0$

The commutation relations for the fields are called equal time commutation (ETC) relations. Because of these relations the classical fields become quantum operators which act in a Hilbert space. This is analogous to $\hat{q}_i, \hat{p}_i$ in non-relativistic quantum mechanics. This Hilbert space consist out of particles which are characterised by their momentum and spin. The above ETC relations hold only for integer spin fields as the scalar field (spin zero) and the electromagnetic field (spin one). In the case of half numbered spin fields, like the Dirac field (spin half), the commutators are replaced by anti commutators. This procedure follows from the spin statistics theorem. From this follows that integer spin field are quantised by commutation relations and spin half numbered fields by anti commutation relations. If this does not happen the Einstein causality is violated. This principle implies that the commutator of two physical operators $A(x)$ and $B(y)$ equals zero if x and y are spacelike

$$[\hat{A}(x), \hat{B}(y)] = 0 \quad \text{if} \quad (x-y)^2 < 0 \tag{3.1}$$

An example is the electromagnetic current $\hat{j}_\mu(x)$

$$[\hat{j}_\mu(x), \hat{j}_\mu(y)] = 0 \quad \text{if} \quad (x-y)^2 < 0$$

$$\text{with} \qquad \hat{j}_\mu(x) = i : \hat{\phi}^\dagger(x) \overleftrightarrow{\partial_\mu} \hat{\phi}(x) : \quad \text{scalar field}$$

$$\hat{j}_\mu(x) = c\,\hbar \ : \bar{\hat{\psi}}(x)\, \gamma_\mu\, \hat{\psi}(x) : \quad \text{Dirac field} \tag{3.2}$$

The explanation of : follows later on.

3.1 Quantisation of the real scalar field

$$\hat{\phi}(x) = \int \frac{d^3\vec{p}}{(2\pi\hbar)^{3/2}} \frac{\hbar c}{\sqrt{2E}} \left[\hat{a}(\vec{p})\, e^{-i\,px/\hbar} + \hat{a}^\dagger(\vec{p})\, e^{i\,px/\hbar}\right] \tag{3.3}$$

Compared with the classical field we replace the complex conjugated $*$ by hermitian conjugated $\dagger$ because $\hat{a}(\vec{p})$ and $\hat{a}^\dagger(\vec{p})$ become operators by second quantisation. Notice that $\hat{\phi}(x)$ is hermitian is. The annihilation en creation operators can be expressed in the inner products (see (1.21))

$$\begin{aligned}
\hat{a}(\vec{p}) &= (f_p, \hat{\phi}) = \frac{i}{\hbar c} \int d^3\vec{x}\, f_p^*(x) \overleftrightarrow{\partial_0} \hat{\phi}(x) \\
&= \frac{i}{\hbar c} \int d^3\vec{x} \left[c\, f_p^*(x)\, \hat{\Pi}(x) - \hat{\phi}(x)\, \partial_0 f_p^*(x)\right] \\
\hat{a}^\dagger(\vec{p}) &= -(f_p^*, \hat{\phi}) = -\frac{i}{\hbar c} \int d^3\vec{x}\, f_p(x) \overleftrightarrow{\partial_0} \hat{\phi}(x) \\
&= -\frac{i}{\hbar c} \int d^3\vec{x} \left[c\, f_p(x)\, \hat{\Pi}(x) - \hat{\phi}(x)\, \partial_0 f_p(x)\right]
\end{aligned} \tag{3.4}$$

Hence it follows

$$\begin{aligned}
[\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{p}\,')] &= \frac{1}{\hbar^2 c^2} \int d^3\vec{x} \int d^3\vec{x}\,' \Big\{ - c\, f_p^*(\vec{x}, t)\, \partial_0 f_{p'}(\vec{x}\,', t)\, [\hat{\Pi}(\vec{x}, t), \hat{\phi}(\vec{x}\,', t)] \\
&\quad - c\, f_{p'}(\vec{x}\,', t)\, \partial_0 f_p^*(\vec{x}, t)\, [\hat{\phi}(\vec{x}, t), \hat{\Pi}(\vec{x}\,', t)] \Big\} \\
&= \frac{i}{\hbar c} \int d^3\vec{x}\, f_p^*(\vec{x}, t) \overleftrightarrow{\partial_0} f_{p'}(\vec{x}, t) = (f_p, f_{p'}) = \delta^{(3)}(\vec{p} - \vec{p}\,')
\end{aligned} \tag{3.5}$$

likewise one can show

$$
\begin{aligned}
[\hat{a}(\vec{p}), \hat{a}(\vec{p}')] &= -\frac{i}{\hbar c} \int d^3\vec{x}\, f_p^*(\vec{x},t) \overset{\leftrightarrow}{\partial_0} f_{p'}^*(\vec{x},t) = -(f_p, f_{p'}^*) = 0 \\
[\hat{a}^\dagger(\vec{p}), \hat{a}^\dagger(\vec{p}')] &= -\frac{i}{\hbar c} \int d^3\vec{x}\, f_p(\vec{x},t) \overset{\leftrightarrow}{\partial_0} f_{p'}(\vec{x},t) = -(f_p^*, f_{p'}) = 0 \qquad (3.6)
\end{aligned}
$$

The commutation relations above are the same as those derived for the harmonic oscillator in non-relativistic quantum mechanics. However now we have an infinite number of degrees of freedom ($\vec{p}$ is continuous). Therefore the replacement becomes equal to

$$
\begin{aligned}
&\hat{a}_i \quad \rightarrow \quad \hat{a}(\vec{p}) \\
&[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} \quad \rightarrow \quad [\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{p}')] = \delta^{(3)}(\vec{p}-\vec{p}') \qquad (3.7)
\end{aligned}
$$

By second quantisation the momentum P^μ and the angular momentum $J^{\mu\nu}$ also operators in the Hilbert space because they are expressed in field. Analogous to the harmonic oscillator for a finite degrees of freedom (non relativistic quantum mechanics) there exists a ground state with the lowest energy in the Hilbert space H. This is called the vacuum, notation $|0\rangle \in$ H. this state does not contain any particles. The nth exited state of the harmonic oscillator corresponds with the n-particle state which contains n different momenta, notation $|\vec{p}_1; \vec{p}_2; \cdots; \vec{p}_n\rangle \in$ H. Introduce the number density operator

$$
\hat{N}(\vec{p}) = \hat{a}^\dagger(\vec{p})\, \hat{a}(\vec{p}) \qquad (3.8)
$$

so that the number operator looks like

$$
\hat{N} = \int d^3\vec{p}\, \hat{N}(\vec{p}) \quad \text{met} \quad \hat{N}|\vec{p}_1; \vec{p}_2; \cdots; \vec{p}_n\rangle = n\, |\vec{p}_1; \vec{p}_1; \cdots; \vec{p}_n\rangle \qquad (3.9)
$$

From the commutation relations for the creation and the annihilation operators follows

$$
\begin{aligned}
&[\hat{N}(\vec{p}), \hat{a}(\vec{p}')] = -\delta^{(3)}(\vec{p}-\vec{p}')\, \hat{a}(\vec{p}) \qquad [\hat{N}(\vec{p}), \hat{a}^\dagger(\vec{p}')] = \delta^{(3)}(\vec{p}-\vec{p}')\, \hat{a}^\dagger(\vec{p}) \\
&\Rightarrow \quad [\hat{N}, \hat{a}(\vec{p})] = -\hat{a}(\vec{p}) \qquad [\hat{N}, \hat{a}^\dagger(\vec{p})] = \hat{a}^\dagger(\vec{p}) \qquad (3.10)
\end{aligned}
$$

One can now derive the following

$$
\begin{aligned}
&\hat{a}^\dagger(\vec{p})\, |\vec{p}_1; \vec{p}_2; \cdots; \vec{p}_n\rangle = c_+\, |\vec{p}; \vec{p}_1; \cdots; \vec{p}_n\rangle \\
&\hat{N}\, \hat{a}^\dagger(\vec{p})\, |\vec{p}_1; \vec{p}_2; \cdots; \vec{p}_n\rangle = \hat{a}^\dagger(\vec{p})\, (\hat{N} + \hat{1})\, |\vec{p}_1; \vec{p}_2; \cdots; \vec{p}_n\rangle \\
&= (n+1)\, \hat{a}^\dagger(\vec{p}) |\vec{p}_1; \vec{p}_2; \cdots; \vec{p}_n\rangle \qquad (3.11)
\end{aligned}
$$

or

$$\hat{a}^\dagger(\vec{p})\,|\vec{p}_1;\vec{p}_2;\cdots;\vec{p}_n\rangle = c_+\,|\vec{p};\vec{p}_1;\vec{p}_2;\cdots;\vec{p}_n\rangle \quad \text{met} \quad \vec{p} \neq \vec{p}_i \tag{3.12}$$

From the above it follows that $\hat{a}^\dagger(\vec{p})$ a creation operator which creates a particle with momentum $\vec{p}$ in the Hilbert space. Because of normalisation reason we shall adopt $c_+ = \sqrt{n+1}$. An n-particle state can be written as

$$|\vec{p}_1;\vec{p}_2;\cdots;\vec{p}_n\rangle = \frac{1}{\sqrt{n!}}\,\hat{a}^\dagger(\vec{p}_1)\,\hat{a}^\dagger(\vec{p}_2)\cdots\hat{a}^\dagger(\vec{p}_n)\,|0\rangle \tag{3.13}$$

$\hat{a}(\vec{p})$ is an annihilation operator which destroys a particle in the Hilbert space

$$\hat{N}\,\hat{a}(\vec{p})\,|\vec{p}_1\rangle = \hat{a}(\vec{p})\,(\hat{N}-\hat{1})\,|\vec{p}_1\rangle = 0 \quad \to \quad \hat{a}(\vec{p})\,|\vec{p}_1\rangle = c_-|0\rangle \tag{3.14}$$

Because only for the vacuum state holds that $\hat{N}\,|0\rangle = 0$. Choose $c_- = 1$ for normalisation reasons. Let $\hat{a}(\vec{p})$ act on general state

$$\hat{a}(\vec{p})\,|\vec{p}_1;\vec{p}_2;\cdots;\vec{p}_n\rangle = \frac{1}{\sqrt{n!}}\,\hat{a}(\vec{p})\hat{a}^\dagger(\vec{p}_1)\,\hat{a}^\dagger(\vec{p}_2)\cdots\hat{a}^\dagger(\vec{p}_n)\,|0\rangle$$

$$= \frac{1}{\sqrt{n}}\Big[\delta^{(3)}(\vec{p}-\vec{p}_1)|\vec{p}_2;\cdots;\vec{p}_n\rangle + \delta^{(3)}(\vec{p}-\vec{p}_2)|\vec{p}_1;\vec{p}_3;\cdots;\vec{p}_n\rangle + \cdots$$

$$+\delta^{(3)}(\vec{p}-\vec{p}_n)|\vec{p}_1;\vec{p}_2;\cdots;\vec{p}_{n-1}\rangle\Big] \tag{3.15}$$

Since the creation operators commute with each other is the state $|\vec{p}_1;\vec{p}_2;\cdots;\vec{p}_n\rangle$ symmetric under interchange of the momenta. Because of this one obtains the Bose-Einstein statistics for particles with spin zero.

$$\hat{N}(\vec{p})\,|\vec{p}_1;\vec{p}_2;\cdots;\vec{p}_n\rangle = \frac{1}{\sqrt{n!}}\,\hat{a}^\dagger(\vec{p})\,\hat{a}(\vec{p})\,\hat{a}^\dagger(\vec{p}_1)\,\hat{a}^\dagger(\vec{p}_2)\,\cdots\hat{a}^\dagger(\vec{p}_n)\,|0\rangle$$

$$= \frac{1}{\sqrt{n}}\Big\{\delta^{(3)}(\vec{p}-\vec{p}_1)\,\sqrt{n}\,|\vec{p};\vec{p}_2;\cdots;\vec{p}_n\rangle + \delta^{(3)}(\vec{p}-\vec{p}_2)\,\sqrt{n}\,|\vec{p}_1;\vec{p};\cdots;\vec{p}_n\rangle$$

$$+\cdots+\delta^{(3)}(\vec{p}-\vec{p}_n)\,\sqrt{n}\,|\vec{p}_1;\vec{p}_2;\cdots;\vec{p}\rangle\Big\}$$

$$= \Big\{\sum_{i=1}^{n}\delta^{(3)}(\vec{p}-\vec{p}_i)\Big\}\,|\vec{p}_1;\vec{p}_2;\cdots;\vec{p}_n\rangle \tag{3.16}$$

Further we have the property

$$\langle\vec{p}_1;\vec{p}_2;\cdots;\vec{p}_n| = |\vec{p}_1;\vec{p}_2;\cdots;\vec{p}_n\rangle^\dagger = \langle 0|\frac{1}{\sqrt{n!}}\,\hat{a}(\vec{p}_n)\cdots\hat{a}(\vec{p}_2)\,\hat{a}(\vec{p}_1) \tag{3.17}$$

The inner product is equal to

$$\langle \vec{p}_1; \vec{p}_2; \cdots; \vec{p}_m | \vec{p}_1'; \vec{p}_2'; \cdots; \vec{p}_n' \rangle = 0 \quad \text{for} \quad m \neq n$$

$$\langle \vec{p}_1; \vec{p}_2; \cdots; \vec{p}_n | \vec{p}_1'; \vec{p}_2'; \cdots; \vec{p}_n' \rangle = \frac{1}{n!} \sum_{P(p_1', p_2', \cdots, p_n')} \delta^{(3)}(\vec{p}_1 - \vec{p}_1')$$

$$\delta^{(3)}(\vec{p}_2 - \vec{p}_2') \cdots \delta^{(3)}(\vec{p}_n - \vec{p}_n') \quad \text{for} \quad m = n \tag{3.18}$$

$P(p_1', p_2', \cdots, p_n')$ represents the permutation of the momenta $p_1', p_2', \cdots, p_n'$. One observes that for a continuous presentation of the state vectors in Hilbert space the same problem occur as for free waves in non-relativistic quantum mechanics namely the appearance of delta functions. This has as consequence that the momenta in the multi particle states are not equal to each other. For scattering process this is not necessary. The four momentum of the quantum field can now be expressed in creation and annihilation operators. Hence from equation (2.64) follows

$$\hat{P}^\mu = \frac{1}{2} \int d^3\vec{p}\, p^\mu \left[a(\vec{p})\, a^\dagger(\vec{p}) + a^\dagger(\vec{p})\, a(\vec{p}) \right] = \int d^3\vec{p}\, p^\mu \left[\hat{N}(\vec{p}) + \frac{1}{2}\delta^{(3)}(0) \right] \tag{3.19}$$

The last term is very ugly and for the following reasons

a The distribution $\delta^{(3)}(0)$ is not defined.

b $\hat{P}^\mu |0\rangle \neq 0$. This means that the vacuum (no particles !!) has not the eigen value zero for the energy en momentum-operator.

To remedy this fault we introduce normal ordered operators. It boils down that we omit the term $\delta^{(3)}(0)$ everywhere where it appears.

Normal Ordering

Definition

$$: \hat{a}(\vec{p})\, \hat{a}^\dagger(\vec{p}) :\equiv \hat{a}^\dagger(\vec{p})\, \hat{a}(\vec{p}) \tag{3.20}$$

This operation implies that for any arbitrary product of creation and annihilation operators after normal ordering the creation operators are to left of the annihilation operators. For example

$$: \hat{a}^\dagger(\vec{p}_1)\, \hat{a}(\vec{p}_2)\, \hat{a}(\vec{p}_3)\, \hat{a}^\dagger(\vec{p}_4) := \hat{a}^\dagger(\vec{p}_1)\, \hat{a}^\dagger(\vec{p}_4)\, \hat{a}(\vec{p}_2)\, \hat{a}(\vec{p}_3) \tag{3.21}$$

Notice that the mutual order between the creation operators and between the annihilation operators does not matter. One can after normal ordering write

$$: \hat{P}^\mu := \int d^3\vec{p}\, p^\mu\, \hat{N}(\vec{p}) \quad \text{met} \quad : \hat{P}^\mu : |0\rangle = 0$$

$$: \hat{P}^\mu :\, |\vec{p}_1; \vec{p}_2; \cdots; \vec{p}_n\rangle = \left\{ \sum_{i=1}^{n} p_i^\mu \right\} |\vec{p}_1; \vec{p}_2; \cdots; \vec{p}_n\rangle \tag{3.22}$$

3.2 Quantisation van the complex scalar field

The fields are given by

$$\hat{\phi}(x) = \frac{1}{\sqrt{2}}\Big(\hat{\phi}_1(x) + i\,\hat{\phi}_2(x)\Big) \qquad \hat{\phi}^\dagger(x) = \frac{1}{\sqrt{2}}\Big(\hat{\phi}_1(x) - i\,\hat{\phi}_2(x)\Big) \tag{3.23}$$

where $\hat{\phi}_i(x)$ $(i = 1, 2)$ are hermitian (real) fields. From the latter it follows

$$\begin{aligned}
&[\hat{\phi}_k(\vec{x}, t), \hat{\Pi}_l(\vec{x}', t)] = i\,\hbar\,\delta_{kl}\,\delta^{(3)}(\vec{x} - \vec{x}') \\
&[\hat{\phi}_k(\vec{x}, t), \hat{\phi}_l(\vec{x}', t)] = 0 \\
&[\hat{\Pi}_k(\vec{x}, t), \hat{\Pi}_l(\vec{x}', t)] = 0
\end{aligned} \tag{3.24}$$

In the case of the complex scalar field read the ETC commutation relations as follows

$$\begin{aligned}
&[\hat{\phi}(\vec{x}, t), \hat{\Pi}(\vec{x}', t)] = i\,\hbar\,\delta^{(3)}(\vec{x} - \vec{x}') && [\hat{\phi}^\dagger(\vec{x}, t), \hat{\Pi}^\dagger(\vec{x}', t)] = i\,\hbar\,\delta^{(3)}(\vec{x} - \vec{x}') \\
&[\hat{\phi}(\vec{x}, t), \hat{\phi}(\vec{x}', t)] = 0 && [\hat{\phi}^\dagger(\vec{x}, t), \hat{\phi}^\dagger(\vec{x}', t)] = 0 \\
&[\hat{\phi}(\vec{x}, t), \hat{\phi}^\dagger(\vec{x}', t)] = 0 && [\hat{\Pi}(\vec{x}, t), \hat{\Pi}^\dagger(\vec{x}', t)] = 0 \\
&[\hat{\phi}(\vec{x}, t), \hat{\Pi}^\dagger(\vec{x}', t)] = 0 && [\hat{\phi}^\dagger(\vec{x}, t), \hat{\Pi}(\vec{x}', t)] = 0
\end{aligned} \tag{3.25}$$

The two quantum fields get the following representation

$$\begin{aligned}
\hat{\phi}(x) &= \int \frac{d^3\vec{p}}{(2\pi\hbar)^{3/2}}\,\frac{\hbar c}{\sqrt{2E}}\,\Big[\hat{a}_+(\vec{p})\,e^{-i\,px/\hbar} + \hat{a}_-^\dagger(\vec{p})\,e^{i\,px/\hbar}\Big] \\
\hat{\phi}^\dagger(x) &= \int \frac{d^3\vec{p}}{(2\pi\hbar)^{3/2}}\,\frac{\hbar c}{\sqrt{2E}}\,\Big[\hat{a}_-(\vec{p})\,e^{-i\,px/\hbar} + \hat{a}_+^\dagger(\vec{p})\,e^{i\,px/\hbar}\Big]
\end{aligned} \tag{3.26}$$

The creation en annihilation operators can be expresses in those of the real fields $\hat{\phi}_i$

$$\begin{aligned}
\hat{a}_+(\vec{p}) &= \frac{1}{\sqrt{2}}\Big(\hat{a}_1(\vec{p}) + i\,\hat{a}_2(\vec{p})\Big) && \hat{a}_-^\dagger(\vec{p}) = \frac{1}{\sqrt{2}}\Big(\hat{a}_1^\dagger(\vec{p}) + i\,\hat{a}_2^\dagger(\vec{p})\Big) \\
\hat{a}_+^\dagger(\vec{p}) &= \frac{1}{\sqrt{2}}\Big(\hat{a}_1^\dagger(\vec{p}) - i\,\hat{a}_2^\dagger(\vec{p})\Big) && \hat{a}_-(\vec{p}) = \frac{1}{\sqrt{2}}\Big(\hat{a}_1(\vec{p}) - i\,\hat{a}_2(\vec{p})\Big)
\end{aligned} \tag{3.27}$$

From the commutation relations

$$\begin{aligned}
&[\hat{a}_k(\vec{p}), \hat{a}_l^\dagger(\vec{p}')] = \delta_{kl}\,\delta^{(3)}(\vec{p} - \vec{p}') \\
&[\hat{a}_k(\vec{p}), \hat{a}_l(\vec{p}')] = 0 \\
&[\hat{a}_k^\dagger(\vec{p}), \hat{a}_l^\dagger(\vec{p}')] = 0
\end{aligned} \tag{3.28}$$

we infer

$$
\begin{aligned}
&[\hat{a}_\pm(\vec{p}), \hat{a}^\dagger_\pm(\vec{p}\,')] = \delta^{(3)}(\vec{p} - \vec{p}\,') && [\hat{a}_\pm(\vec{p}), \hat{a}^\dagger_\mp(\vec{p}\,')] = 0 \\
&[\hat{a}_\pm(\vec{p}), \hat{a}_\pm(\vec{p}\,')] = 0 && [\hat{a}_\pm(\vec{p}), \hat{a}_\mp(\vec{p}\,')] = 0 \\
&[\hat{a}^\dagger_\pm(\vec{p}), \hat{a}^\dagger_\pm(\vec{p}\,')] = 0 && [\hat{a}^\dagger_\pm(\vec{p}), \hat{a}^\dagger_\mp(\vec{p}\,')] = 0
\end{aligned}
\tag{3.29}
$$

The interpretation is that $\hat{a}^\dagger_+(\vec{p})$ and $\hat{a}_+(\vec{p})$ are the creation and annihilation operators of the positive charged particles whereas $\hat{a}^\dagger_-(\vec{p})$ and $\hat{a}^(_-\vec{p})$ represent the creation en annihilation operators of the negative charged particles. The Hilbert spaces of the positive and negative charges particles are denoted by $\mathrm{H}^{(+)}$ and $\mathrm{H}^{(-)}$ respectively. The state vectors are

$$
|\vec{p}_1; \vec{p}_2; \cdots; \vec{p}_m\rangle \in \mathrm{H}^{(+)} \qquad |\vec{q}_1; \vec{q}_2; \cdots; \vec{q}_n\rangle \in \mathrm{H}^{(-)} \tag{3.30}
$$

From the above one can construct the following product space $\mathrm{H} = \mathrm{H}^{(+)} \otimes \mathrm{H}^{(-)}$ with as state vector

$$
|\vec{p}_1; \vec{p}_2; \cdots; \vec{p}_m; \vec{q}_1; \vec{q}_2; \cdots; \vec{q}_n\rangle \in \mathrm{H} \tag{3.31}
$$

This can be created out of the vacuum

$$
|0\rangle = |0\rangle_+ \, |0\rangle_- \tag{3.32}
$$

via

$$
\begin{aligned}
&|\vec{p}_1; \vec{p}_2; \cdots; \vec{p}_m; \vec{q}_1; \vec{q}_2; \cdots; \vec{q}_n\rangle = \frac{1}{\sqrt{n!\, m!}} \, \hat{a}^\dagger_+(\vec{p}_1) \, \hat{a}^\dagger_+(\vec{p}_2) \cdots \hat{a}^\dagger_+(\vec{p}_m) \\
&\hat{a}^\dagger_-(\vec{q}_1) \, \hat{a}^\dagger_-(\vec{q}_2) \cdots \hat{a}^\dagger_-(\vec{q}_n) \, |0\rangle
\end{aligned}
\tag{3.33}
$$

Define again the number density operator

$$
\begin{aligned}
\hat{N}_\pm(\vec{p}) &= \hat{a}^\dagger_\pm(\vec{p}) \hat{a}_\pm(\vec{p}) \quad \text{number density operator} \\
\hat{N}_\pm &= \int d^3\vec{p} \, \hat{N}_\pm(\vec{p}) \quad \text{number operator}
\end{aligned}
\tag{3.34}
$$

We can again after normal ordering write the momentum operator $: \hat{P}^\mu :$ and the charge $: \hat{Q} :$ as

$$
\begin{aligned}
: \hat{P}^\mu &:= \int d^3\vec{p} \, p^\mu \left(\hat{N}_+(\vec{p}) + \hat{N}_-(\vec{p}) \right) \\
: \hat{Q} &:= e \int d^3\vec{p} \left(\hat{N}_+(\vec{p}) - \hat{N}_-(\vec{p}) \right)
\end{aligned}
\tag{3.35}
$$

with

$$: \hat{P}^{\mu} : \, |\vec{p}_1; \vec{p}_2; \cdots ; \vec{p}_m; \vec{q}_1; \vec{q}_2; \cdots ; \vec{q}_n\rangle = \Big\{ \sum_{i=1}^{m} p_i^{\mu} + \sum_{i=1}^{n} q_i^{\mu} \Big\} \, |\vec{p}_1; \vec{p}_2; \cdots ; \vec{p}_m;$$

$$\vec{q}_1; \vec{q}_2; \cdots ; \vec{q}_n\rangle$$

$$: \hat{Q} : \, |\vec{p}_1; \vec{p}_2; \cdots ; \vec{p}_m; \vec{q}_1; \vec{q}_2; \cdots ; \vec{q}_n\rangle = e\,(m-n)\, |\vec{p}_1; \vec{p}_2; \cdots ; \vec{p}_m;$$

$$\vec{q}_1; \vec{q}_2; \cdots ; \vec{q}_n\rangle \tag{3.36}$$

Further we have the property

$$[: \hat{Q} :, : \hat{P}^{\mu} :] = 0 \tag{3.37}$$

This commutator follows from

$$[\hat{N}_{\pm}(\vec{p}), \hat{N}_{\pm}(\vec{p})] = 0 \quad \text{en} \quad [\hat{N}_{\mp}(\vec{p}), \hat{N}_{\pm}(\vec{p})] = 0 \tag{3.38}$$

3.3 quantisation of the complex Dirac field

As mentioned in the beginning of this chapter a half integer spin field and his canonical momentum have to satisfy anti commutation relations. In the case of the Dirac field they are given by

$$\begin{aligned}
&\{\hat{\psi}_{\alpha}(\vec{x}, t), \hat{\Pi}_{\beta}(\vec{x}', t)\} = i\,\hbar\,\delta_{\alpha\beta}\,\delta^{(3)}(\vec{x} - \vec{x}') \\
&\{\hat{\psi}_{\alpha}(\vec{x}, t), \hat{\psi}_{\beta}(\vec{x}', t)\} = 0 \\
&\{\hat{\Pi}_{\alpha}(\vec{x}, t), \hat{\Pi}_{\beta}(\vec{x}', t)\} = 0 \\
&\text{met} \quad \hat{\Pi}_{\alpha}(\vec{x}, t) = i\,\hbar\,\hat{\psi}^{\dagger}_{\alpha}(\vec{x}, t) \qquad \alpha, \beta = 1, 2, 3, 4
\end{aligned} \tag{3.39}$$

The quantum fields are

$$\begin{aligned}
\hat{\psi}_{\alpha}(x) &= \sum_s \int \frac{d^3\vec{p}}{(2\pi\hbar)^{3/2}} \sqrt{\frac{mc^2}{E}} \Big[\hat{b}(\vec{p}, s)\, u_{\alpha}(\vec{p}, s)\, e^{-i\,px/\hbar} + \hat{d}^{\dagger}(\vec{p}, s)\, v_{\alpha}(\vec{p}, s)\, e^{i\,px/\hbar}\Big] \\
\hat{\psi}^{\dagger}_{\alpha}(x) &= \sum_s \int \frac{d^3\vec{p}}{(2\pi\hbar)^{3/2}} \sqrt{\frac{mc^2}{E}} \Big[\hat{b}^{\dagger}(\vec{p}, s)\, u^{\dagger}_{\alpha}(\vec{p}, s)\, e^{i\,px/\hbar} + \hat{d}(\vec{p}, s)\, v^{\dagger}_{\alpha}(\vec{p}, s)\, e^{-i\,px/\hbar}\Big]
\end{aligned} \tag{3.40}$$

An abbreviation is (see (1.72))

$$\hat{\psi}(x) = \sum_s \int d^3\vec{p}\,\Big[\hat{b}(\vec{p}, s)\,\psi_{1,\vec{p},s}(x) + \hat{d}^{\dagger}(\vec{p}, s)\,\psi_{2,\vec{p},s}(x)\Big] \tag{3.41}$$

With the definitions (see (1.72)-(1.74)) the creation and annihilation operators can be written as

$$\hat{b}(\vec{p},s) = \left(\psi_{1,\vec{p},s},\hat{\psi}\right) = \int d^3\vec{x}\,\psi^{\dagger}_{1,\vec{p},s}(x)\,\hat{\psi}(x)$$

$$\hat{b}^{\dagger}(\vec{p},s) = \left(\psi^{\dagger}_{1,\vec{p},s},\hat{\psi}^{\dagger}\right) = \frac{1}{i\hbar}\int d^3\vec{x}\,\psi_{1,\vec{p},s}(x)\,\hat{\Pi}(x)$$

$$\hat{d}(\vec{p},s) = \left(\psi^{\dagger}_{2,\vec{p},s},\hat{\psi}^{\dagger}\right) = \frac{1}{i\hbar}\int d^3\vec{x}\,\psi_{2,\vec{p},s}(x)\,\hat{\Pi}(x)$$

$$\hat{d}^{\dagger}(\vec{p},s) = \left(\psi_{2,\vec{p},s},\hat{\psi}\right) = \int d^3\vec{x}\,\psi^{\dagger}_{2,\vec{p},s}(x)\,\hat{\psi}(x) \tag{3.42}$$

Hence it follows

$$\{\hat{b}(\vec{p},s),\hat{b}^{\dagger}(\vec{p}',s')\} = \frac{1}{i\,\hbar}\int d^3\vec{x}\int d^3\vec{x}'\,\psi^{\dagger}_{1,\vec{p},s}(x)\,\psi_{1,\vec{p}',s'}(x')\,\{\hat{\psi}(\vec{x},t),\Pi(\vec{x}',t)\}$$

$$= \int d^3\vec{x}\,\psi^{\dagger}_{1,\vec{p},s}(x)\,\psi_{1,\vec{p}',s'}(x) = \delta_{ss'}\,\delta^{(3)}(\vec{p}-\vec{p}')$$

$$\{\hat{d}(\vec{p},s),\hat{d}^{\dagger}(\vec{p}',s')\} = \delta_{ss'}\,\delta^{(3)}(\vec{p}-\vec{p}')$$

$$\{\hat{b}(\vec{p},s),\hat{b}(\vec{p}',s')\} = 0 \qquad \{\hat{d}(\vec{p},s),\hat{d}(\vec{p}',s')\} = 0$$

$$\{\hat{b}^{\dagger}(\vec{p},s),\hat{b}^{\dagger}(\vec{p}',s')\} = 0 \qquad \{\hat{d}^{\dagger}(\vec{p},s),\hat{d}^{\dagger}(\vec{p}',s')\} = 0$$

$$\{\hat{b}(\vec{p},s),\hat{d}(\vec{p}',s')\} = 0 \qquad \{\hat{b}(\vec{p},s),\hat{d}^{\dagger}(\vec{p}',s')\} = 0$$

$$\{\hat{b}^{\dagger}(\vec{p},s),\hat{d}(\vec{p}',s')\} = 0 \qquad \{\hat{b}^{\dagger}(\vec{p},s),\hat{d}^{\dagger}(\vec{p}',s')\} = 0 \tag{3.43}$$

$\hat{b}^{\dagger}(\vec{p},s)$ and $\hat{b}(\vec{p},s)$ are the creation and annihilation operators of the negative charged spin half particle (e^-, μ^-,τ^- etc.).
$\hat{d}^{\dagger}(\vec{p},s)$ and $\hat{d}(\vec{p},s)$ are the creation and annihilation operators of the positive charged spin half particle (e^+, μ^+,τ^+ etc.).
The Hilbert spaces of the negative charged particles and positive charged anti-particles are given by $\mathrm{H}^{(-)}$ and $\mathrm{H}^{(+)}$ respectively. The state vectors are

$$|\vec{p}_1,s_1;\vec{p}_2,s_2;\cdots;\vec{p}_m,s_m\rangle \in \mathrm{H}^{(-)} \qquad |\vec{q}_1,t_1;\vec{q}_2,t_2;\cdots;\vec{q}_n,t_n\rangle \in \mathrm{H}^{(+)} \tag{3.44}$$

with

$$|\vec{p}_1,s_1;\vec{p}_2,s_2;\cdots;\vec{p}_m,s_m\rangle = \frac{1}{\sqrt{m!}}\,\hat{b}^{\dagger}(\vec{p}_1,s_1)\,\hat{b}^{\dagger}(\vec{p}_2,s_2)\cdots\hat{b}^{\dagger}(\vec{p}_m,s_m)|0\rangle_-$$

$$|\vec{q}_1,t_1;\vec{q}_2,t_2;\cdots;\vec{q}_n,t_n\rangle = \frac{1}{\sqrt{n!}}\,\hat{d}^{\dagger}(\vec{q}_1,t_1)\,\hat{d}^{\dagger}(\vec{q}_2,t_2)\cdots\hat{d}^{\dagger}(\vec{q}_n,t_n)|0\rangle_+ \tag{3.45}$$

Because the creation operators anti commute, the multi particle state is anti-symmetric under interchange of momenta and spins.

$$|\vec{p}_1, s_1; \cdots \vec{p}_i, s_i; \cdots \vec{p}_j, s_j; \cdots \vec{p}_m, s_m\rangle = -|\vec{p}_1, s_1; \cdots \vec{p}_j, s_j; \cdots \vec{p}_i, s_i; \cdots \vec{p}_m, s_m\rangle$$

$$|\vec{q}_1, t_1; \cdots \vec{q}_i, t_i; \cdots \vec{q}_j, t_j; \cdots \vec{q}_n, t_n\rangle = -|\vec{q}_1, t_1; \cdots \vec{q}_j, t_j; \cdots \vec{q}_i, t_i; \cdots \vec{q}_n, t_n\rangle \quad (3.46)$$

Hence it follows the Fermi-Dirac-statistics of the spin half particles. One can also construct the product space $\mathrm{H} = \mathrm{H}^{(-)} \otimes \mathrm{H}^{(+)}$ with the state vector

$$|\vec{p}_1, s_1; \vec{p}_2, s_2; \cdots; \vec{p}_m, s_m; \vec{q}_1, t_1; \vec{q}_2, t_2; \cdots; \vec{q}_n, t_n\rangle = |\vec{p}_1, s_1; \vec{p}_2, s_2; \cdots; \vec{p}_m, s_m\rangle$$

$$|\vec{q}_1, t_1; \vec{q}_2, t_2; \cdots; \vec{q}_n, t_n\rangle \in \mathrm{H} \quad (3.47)$$

This state can be created out of the vacuum

$$|\vec{p}_1, s_1; \vec{p}_2, s_2; \cdots; \vec{p}_m, s_m; \vec{q}_1, t_1; \vec{q}_2, t_2; \cdots; \vec{q}_n, t_n\rangle$$

$$= \frac{1}{\sqrt{n!\, m!}}\, \hat{b}^\dagger(\vec{p}_1, s_1)\, \hat{b}^\dagger(\vec{p}_2, s_2) \cdots \hat{b}^\dagger(\vec{p}_m, s_m) \hat{d}^\dagger(\vec{q}_1, t_1)\, \hat{d}^\dagger(\vec{q}_2, t_2) \cdots \hat{d}^\dagger(\vec{q}_n, t_n)\, |0\rangle$$

$$\Rightarrow \quad \langle\vec{p}_1, s_1; \vec{p}_2, s_2; \cdots; \vec{p}_m, s_m; \vec{q}_1, t_1; \vec{q}_2, t_2; \cdots; \vec{q}_n, t_n|$$

$$= \frac{1}{\sqrt{n!\, m!}}\, \langle 0|\hat{d}(\vec{q}_n, t_n) \cdots \hat{d}(\vec{q}_2, t_2)\, \hat{d}(\vec{q}_1, t_1)\, \hat{b}\vec{p}_m, s_m) \cdots \hat{b}(\vec{p}_2, s_2)\, \hat{b}(\vec{p}_1, s_1) \quad (3.48)$$

The normalisation is

$$\langle\vec{p}\,'_1, s'_1; \cdots; \vec{p}\,'_{m'}, s'_{m'}; \vec{q}\,'_1, t'_1; \cdots; \vec{q}\,'_{n'}, t'_{n'}|\vec{p}_1, s_1; \cdots; \vec{p}_m, s_m; \vec{q}_1, t_1; \cdots; \vec{q}_n, t_n\rangle$$

$$= \frac{\delta_{mm'}\delta_{nn'}}{m!\, n!} \det|\delta_{s_i s'_j}\, \delta^{(3)}(\vec{p}_i - \vec{p}\,'_j)| \, \det|\delta_{t_k t'_l}\, \delta^{(3)}(\vec{q}_k - \vec{q}\,'_l)| \quad (3.49)$$

The creation operators act on a multiple particle state as follows

$$\hat{b}^\dagger(\vec{p}, s)\, |\vec{p}_1, s_1; \cdots; \vec{p}_m, s_m; \vec{q}_1, t_1; \cdots; \vec{q}_n, t_n\rangle = (-1)^{i+1}\sqrt{m+1}$$

$$|\vec{p}_1, s_1; \cdots \vec{p}_i, s_i; \cdots; \vec{p}_m, s_m; \vec{q}_1, t_1; \cdots; \vec{q}_n, t_n\rangle \quad \vec{p}_i = \vec{p} \quad s_i = s$$

$$\hat{d}^\dagger(\vec{q}, t)\, |\vec{p}_1, s_1; \cdots; \vec{p}_m, s_m; \vec{q}_1, t_1; \cdots; \vec{q}_n, t_n\rangle = (-1)^{m+j+1}\sqrt{n+1}$$

$$|\vec{p}_1, s_1; \cdots; \vec{p}_m, s_m; \vec{q}_1, t_1; \cdots \vec{q}_j, t_j; \cdots \vec{q}_n, t_n\rangle \quad \vec{q}_j = \vec{q} \quad t_j = t \quad (3.50)$$

and for the annihilation operators

$$\hat{b}(\vec{p}, s)\, |\vec{p}_1, s_1; \cdots; \vec{p}_m, s_m; \vec{q}_1, t_1; \cdots; \vec{q}_n, t_n\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} (-1)^{i+1}$$

$$\delta_{ss_i}\,\delta^{(3)}(\vec{p}-\vec{p}_i)\,|\vec{p}_1,s_1;\cdots\vec{p}_{i-1},s_{i-1};\vec{p}_{i+1},s_{i+1};\cdots;\vec{p}_m,s_m;\vec{q}_1,t_1;\cdots;\vec{q}_n,t_n\rangle$$

$$\hat{d}(\vec{q},t)\,|\vec{p}_1,s_1;\cdots;\vec{p}_m,s_m;\vec{q}_1,t_1;\cdots;\vec{q}_n,t_n\rangle = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}(-1)^{m+j+1}$$

$$\delta_{tt_j}\,\delta^{(3)}(\vec{q}-\vec{q}_i)\,|\vec{p}_1,s_1;\cdots;\vec{p}_m,s_m;\vec{q}_1,t_1;\cdots\vec{q}_{j-1},t_{j-1};\vec{q}_{j+1},t_{j+1};\cdots\vec{q}_n,t_n\rangle \tag{3.51}$$

One can again define a number density operator for particles and anti-particles

$$\hat{N}_-(\vec{p},s) = \hat{b}^\dagger(\vec{p},s)\,\hat{b}(\vec{p},s) \qquad \hat{N}_+(\vec{p},s) = \hat{d}^\dagger(\vec{p},s)\,\hat{d}(\vec{p},s) \tag{3.52}$$

from which follows a number operator

$$\hat{N}_\pm = \sum_s \int d^3\vec{p}\,\hat{N}_\pm(\vec{p},s) \tag{3.53}$$

The commutation relations read

$$\begin{aligned}
[\hat{N}_-(\vec{p},s),\hat{b}(\vec{p}',s')] &= -\delta_{ss'}\,\delta^{(3)}(\vec{p}-\vec{p}')\,\hat{b}(\vec{p},s)\\
[\hat{N}_-(\vec{p},s),\hat{b}^\dagger(\vec{p}',s')] &= \delta_{ss'}\,\delta^{(3)}(\vec{p}-\vec{p}')\,\hat{b}^\dagger(\vec{p},s)\\
[\hat{N}_+(\vec{p},s),\hat{d}(\vec{p}',s')] &= -\delta_{ss'}\,\delta^{(3)}(\vec{p}-\vec{p}')\,\hat{d}(\vec{p},s)\\
[\hat{N}_+(\vec{p},s),\hat{d}^\dagger(\vec{p}',s')] &= \delta_{ss'}\,\delta^{(3)}(\vec{p}-\vec{p}')\,\hat{d}^\dagger(\vec{p},s)
\end{aligned} \tag{3.54}$$

with as property for the number density operator

$$\begin{aligned}
&\hat{N}_-(\vec{p},s) \quad |\vec{p}_1,s_1;\vec{p}_2,s_2;\cdots;\vec{p}_m,s_m;\vec{q}_1,t_1;\vec{q}_2,t_2;\cdots;\vec{q}_n,t_n\rangle =\\
&\Big\{\sum_{i=1}^{m}\delta_{ss_i}\,\delta^{(3)}(\vec{p}-\vec{p}_i)\Big\}\,|\vec{p}_1,s_1;\vec{p}_2,s_2;\cdots;\vec{p}_m,s_m;\vec{q}_1,t_1;\vec{q}_2,t_2;\cdots;\vec{q}_n,t_n\rangle\\
&\hat{N}_+(\vec{q},t) \quad |\vec{p}_1,s_1;\vec{p}_2,s_2;\cdots;\vec{p}_m,s_m;\vec{q}_1,t_1;\vec{q}_2,t_2;\cdots;\vec{q}_n,t_n\rangle =\\
&\Big\{\sum_{j=1}^{n}\delta_{tt_j}\,\delta^{(3)}(\vec{q}-\vec{q}_j)\Big\}\,|\vec{p}_1,s_1;\vec{p}_2,s_2;\cdots;\vec{p}_m,s_m;\vec{q}_1,t_1;\vec{q}_2,t_2;\cdots;\vec{q}_n,t_n\rangle
\end{aligned} \tag{3.55}$$

In the states all momenta and spins occur only once. This follows from the Fermi-Dirac statistics. This is in contrast to the Bose-Einstein statistics where an infinite number of particles with the same quantum numbers occur. For this we could not find a continuous representation. The four momenta and the charge can be written as follows.

$$\hat{P}^\mu = \sum_s \int d^3\vec{p}\, p^\mu \left(\hat{b}^\dagger(\vec{p},s)\,\hat{b}(\vec{p},s) - \hat{d}(\vec{p},s)\,\hat{d}^\dagger(\vec{p},s)\right)$$

$$\hat{Q} = e \sum_s \int d^3\vec{p} \left(\hat{b}^\dagger(\vec{p},s)\,\hat{b}(\vec{p},s) + \hat{d}(\vec{p},s)\,\hat{d}^\dagger(\vec{p},s)\right) \qquad e < 0 \tag{3.56}$$

This can be rewritten in

$$\hat{P}^\mu = \sum_s \int d^3\vec{p}\, p^\mu \left(\hat{N}_-(\vec{p},s) + \hat{N}_+(\vec{p},s) - \delta^{(3)}(0)\right)$$

$$\hat{Q} = e \sum_s \int d^3\vec{p} \left(\hat{N}_-(\vec{p},s) - \hat{N}_+(\vec{p},s) + \delta^{(3)}(0)\right) \tag{3.57}$$

Like in the case of the bosonic operators we have to normal order for the fermionic case. Define

$$: \hat{b}(\vec{p},s)\,\hat{b}^\dagger(\vec{p},s) := -\hat{b}^\dagger(\vec{p},s)\,\hat{b}(\vec{p},s)$$

$$: \hat{d}(\vec{p},s)\,\hat{d}^\dagger(\vec{p},s) := -\hat{d}^\dagger(\vec{p},s)\,\hat{d}(\vec{p},s) \tag{3.58}$$

The minus sign is coming from the anti commutator. Normal ordering eliminates the term $\delta^{(3)}(0)$. For example

$$\hat{b}(\vec{p},s)\,\hat{b}^\dagger(\vec{p},s) = \{\hat{b}(\vec{p},s), \hat{b}^\dagger(\vec{p},s)\} - \hat{b}^\dagger(\vec{p},s)\,\hat{b}(\vec{p},s)$$

$$= \delta^{(3)}(0) - \hat{b}^\dagger(\vec{p},s)\,\hat{b}(\vec{p},s) \tag{3.59}$$

The same holds for $\hat{d}(\vec{p},s)\,\hat{d}^\dagger(\vec{p},s)$. In general it means that for an arbitrary product of creation and annihilation operators the creation operators has to be left of the annihilation operator. Moreover one has to take into account that a minus sign appears after an odd number of interchanges.
Example:

$$: \hat{b}_1\,\hat{d}_2\,\hat{d}_3^\dagger\,\hat{b}_4^\dagger := -\hat{b}_4^\dagger\,\hat{d}_3^\dagger\,\hat{b}_1\,\hat{d}_2 = \hat{b}_4^\dagger\,\hat{d}_3^\dagger\,\hat{d}_2\,\hat{b}_1 \tag{3.60}$$

After normal ordering we get equations (3.57) in the following form

$$: \hat{P}^\mu := \sum_s \int d^3\vec{p}\, p^\mu \left(\hat{N}_-(\vec{p},s) + \hat{N}_+(\vec{p},s)\right)$$

$$: \hat{Q} := e \sum_s \int d^3\vec{p} \left(\hat{N}_-(\vec{p},s) - \hat{N}_+(\vec{p},s)\right) = e\left(\hat{N}_- - \hat{N}_+\right) \tag{3.61}$$

Further we have the property

$$[: \hat{Q} :, : \hat{P}^\mu :] = 0 \tag{3.62}$$

These operators act on the states as follows

$$: \hat{P}^\mu : \, |\vec{p}_1, s_1; \vec{p}_2, s_2; \cdots; \vec{p}_m, s_m; \vec{q}_1, t_1; \vec{q}_2, t_2; \cdots; \vec{q}_n, t_n\rangle$$

$$= \Big\{ \sum_{i=1}^{m} p_i^\mu + \sum_{j=1}^{n} q_j^\mu \Big\} \, |\vec{p}_1, s_1; \vec{p}_2, s_2; \cdots; \vec{p}_m, s_m; \vec{q}_1, t_1; \vec{q}_2, t_2; \cdots; \vec{q}_n, t_n\rangle$$

$$: \hat{Q} : \, |\vec{p}_1, s_1; \vec{p}_2, s_2; \cdots; \vec{p}_m, s_m; \vec{q}_1, t_1; \vec{q}_2, t_2; \cdots; \vec{q}_n, t_n\rangle$$

$$= e\,(m-n)\,|\vec{p}_1, s_1; \vec{p}_2, s_2; \cdots; \vec{p}_m, s_m; \vec{q}_1, t_1; \vec{q}_2, t_2; \cdots; \vec{q}_n, t_n\rangle \tag{3.63}$$

Finally we want to show the connexion between spin and statistics of fermions. Suppose that $\hat{\psi}$ and $\hat{\Pi}$ satisfy commutation relations instead of anti commutation relations. $b, b^\dagger, d, d^\dagger$ satisfy commutation relations. In this case the energy operator $: \hat{H} :$ looks after normal ordering

$$: \hat{H} := c : P^0 := \sum_s \int d^3\vec{p}\, E \left(b^\dagger(\vec{p}, s)\, b(\vec{p}, s) - d^\dagger(\vec{p}, s)\, d(\vec{p}, s) \right) \tag{3.64}$$

The eigenvalue for the anti-fermions then becomes

$$: \hat{H} : \, |\vec{q}_1, t_1; \vec{q}_2, t_2; \cdots; \vec{q}_n, t_n\rangle = -\sum_{i=1}^{n} E_i \, |\vec{q}_1, t_1; \vec{q}_2, t_2; \cdots; \vec{q}_n, t_n\rangle \tag{3.65}$$

Because $\sum_{i=1}^{n} E_i \to \infty$ for $n \to \infty$ are the eigenvalues for the energy operator $: \hat{H} :$ not bounded from below. This is unacceptable for a quantum theory where we have the requirement that the ground state (here the vacuum) has the lowest eigenvalue which has to be finite. We conclude that the spin half character of the Dirac field is relevant for the fact that we have to take anti commutation relations to quantise the field. The same holds for integer spin fields like the scalar field or the electromagnetic field. Here one has to take commutation relations to quantise the field for the same reason namely to avoid that the energy spectrum is unbounded from below. This observation is one of important successes of quantum field theory namely the connexion between spin and statistics.

3.4 quantisation of the electromagnetic field

Except for the polarisation vectors $\epsilon^\mu(\vec{k})$ is there large similarity between the electromagnetic field and the real scalar field $\phi(x)$. We shall only consider the quantisation of the electromagnetic field in the radiation gauge. In this case only physical polarisations occur. The quantised field becomes

$$\vec{A}(\vec{x}, t) = \sum_\alpha \int d^3\vec{k}\, c \sqrt{\frac{\hbar}{2\omega}} \left[\hat{a}(\vec{k}, \alpha)\, \vec{\epsilon}^{\,\alpha}(\vec{k})\, e^{-i\,kx} + \hat{a}^\dagger(\vec{k}, \alpha)\, \vec{\epsilon}^{\,*\alpha}(\vec{k})\, e^{i\,kx} \right]$$

$$\alpha = 1, 2 \quad \text{or} \quad \alpha = R, L \tag{3.66}$$

where $\hat{a}^\dagger(\vec{k}, \alpha)$ and $\hat{a}(\vec{k}, \alpha)$ are creation and annihilation operators which satisfy the relations

$$\begin{aligned}
&[\hat{a}(\vec{k}, \alpha), \hat{a}^\dagger(\vec{k}', \alpha')] = \delta_{\alpha\alpha'}\, \delta^{(3)}(\vec{k} - \vec{k}') \\
&[\hat{a}^\dagger(\vec{k}, \alpha), \hat{a}^\dagger(\vec{k}', \alpha')] = 0 \\
&[\hat{a}(\vec{k}, \alpha), \hat{a}(\vec{k}', \alpha')] = 0
\end{aligned} \tag{3.67}$$

The connexion between the creation and annihilation operators of the linear ($\alpha = 1, 2$) and circular polarisation ($\alpha = R, L$) is given by

$$\hat{a}(\vec{k}, R) = -\frac{1}{\sqrt{2}} \left(\hat{a}(\vec{k}, 1) - i\, \hat{a}(\vec{k}, 2) \right) \qquad \hat{a}(\vec{k}, L) = \frac{1}{\sqrt{2}} \left(\hat{a}(\vec{k}, 1) + i\, \hat{a}(\vec{k}, 2) \right) \tag{3.68}$$

The canonical commutation relations are equal to

$$\begin{aligned}
&[\hat{A}_i(\vec{x}, t), \hat{\Pi}_j(\vec{x}', t)] = \frac{1}{c}\, [\hat{A}_i(\vec{x}, t), \hat{E}_j(\vec{x}', t)] = \\
&-i\,\hbar \int \frac{d^3\vec{k}}{(2\pi)^3}\, e^{-i\,\vec{k}\cdot(\vec{x}-\vec{x}')} \left(\delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2} \right) \\
&[\hat{\Pi}_i(\vec{x}, t), \hat{\Pi}_j(\vec{x}', t)] = 0 \\
&[\hat{A}_i(\vec{x}, t), \hat{A}_j(\vec{x}', t)] = 0
\end{aligned} \tag{3.69}$$

Notice that the commutator is consistent with the radiation gauge condition

$$\partial_k \hat{A}_k = \vec{\nabla} \cdot \vec{\hat{A}} = 0 \tag{3.70}$$

Because the radiation gauge leads to physical polarisations of the electromagnetic field only physical states (here called photons) arise in the multi particle space so that it becomes a Hilbert space. This state is denoted by

$$|\vec{k}_1, \alpha_1; \vec{k}_2, \alpha_2; \cdots; \vec{k}_n, \alpha_n\rangle \in \mathrm{H} \quad \text{met} \quad \alpha_i = R, L \tag{3.71}$$

This can be created out of the vacuum

$$\begin{aligned}
&|\vec{k}_1, \alpha_1; \vec{k}_2, \alpha_2; \cdots; \vec{k}_n, \alpha_n\rangle = \frac{1}{\sqrt{n!}}\, a^\dagger(\vec{k}_1, \alpha_1)\, a^\dagger(\vec{k}_2, \alpha_2) \cdots a^\dagger(\vec{k}_n, \alpha_n)\, |0\rangle \\
\Rightarrow \quad &\langle \vec{k}_1, \alpha_1; \vec{k}_2, \alpha_2; \cdots; \vec{k}_n, \alpha_n| = \frac{1}{\sqrt{n!}}\, \langle 0|\, a(\vec{k}_n, \alpha_n) \cdots a(\vec{k}_2, \alpha_2)\, a(\vec{k}_1, \alpha_1)
\end{aligned} \tag{3.72}$$

Because the creation and annihilation operators commutate among each other the states are again symmetric under interchange of momenta and polarisations. The creation and annihilation act on the photon states as follows

$$
\begin{aligned}
&a^\dagger(\vec{k},\alpha)\,|\vec{k}_1,\alpha_1;\cdots;\vec{k}_n,\alpha_n\rangle = \sqrt{n+1}\,|\vec{k},\alpha;\vec{k}_1,\alpha_1;\cdots;\vec{k}_n,\alpha_n\rangle \\
&a(\vec{k},\alpha)\,|\vec{k}_1,\alpha_1;\cdots;\vec{k}_n,\alpha_n\rangle = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{\alpha}\delta_{\alpha\alpha_i}\,\delta^{(3)}(\vec{k}-\vec{k}_i) \\
&\times|\vec{k}_1,\alpha_1;\cdots;\vec{k}_{i-1},\alpha_{i-1};\vec{k}_{i+1},\alpha_{i+1};\cdots;\vec{k}_n,\alpha_n\rangle
\end{aligned}
\tag{3.73}
$$

Next we define the number density operator

$$
\hat{N}(\vec{k},\alpha) = a^\dagger(\vec{k},\alpha)\,a(\vec{k},\alpha) \tag{3.74}
$$

with the number operator

$$
\hat{N} = \sum_{\alpha}\int d^3\vec{k}\,\hat{N}(\vec{k},\alpha) \tag{3.75}
$$

the former acts on the multi photon state as follows

$$
\hat{N}(\vec{k},\alpha)\,|\vec{k}_1,\alpha_1;\cdots;\vec{k}_n,\alpha_n\rangle = \sum_{i=1}^{n}\sum_{\alpha}\delta_{\alpha\alpha_i}\,\delta^{(3)}(\vec{k}-\vec{k}_i)\,|\vec{k}_1,\alpha_1;\cdots;\vec{k}_n,\alpha_n\rangle \tag{3.76}
$$

The normalisation becomes

$$
\begin{aligned}
&\langle\vec{k}_1,\alpha_1;\cdots;\vec{k}_m,\alpha_m|\vec{k}_1,\alpha_1;\cdots;\vec{k}_n,\alpha_n\rangle = \frac{\delta_{mn}}{n!} \\
&\sum_{P(k_1',\alpha_1'\cdots k_n',\alpha_n')}\delta_{\alpha_1\alpha_1'}\,\delta^{(3)}(\vec{k}_1-\vec{k}_1')\cdots\delta_{\alpha_n\alpha_n'}\,\delta^{(3)}(\vec{k}_n-\vec{k}_n')
\end{aligned}
\tag{3.77}
$$

After normal ordering we can express $:\hat{P}^\mu:$ again in creation and annihilation operators

$$
:\hat{P}^\mu := \hbar\sum_{\alpha=R,L}\int d^3\vec{k}\,k^\mu\,\hat{N}(\vec{k},\alpha) \tag{3.78}
$$

It acts on the multi photon state as follows

$$
:\hat{P}^\mu:\,|\vec{k}_1,\alpha_1;\cdots;\vec{k}_n,\alpha_n\rangle = \hbar\left\{\sum_{i=1}^{n}k_i^\mu\right\}|\vec{k}_1,\alpha_1;\cdots;\vec{k}_n,\alpha_n\rangle \tag{3.79}
$$

4 Interacting fields

4.1 Green functions for the scalar field

From now on we shall put $\hbar = c = 1$. Green functions play an important role in Quantum Field Theory. They occur in the perturbative expansion of the S-matrix from which one can derive cross sections and decay widths of particles. They also appear in Statistical Mechanics and Condensed Matter where they represent the correlation functions from which one can derive quantities like susceptibility, magnetisation or specific heat. Since we are mainly interested in scalar fields we will derive the Green function together with other distributions for that case only. From Eq. (3.3) one can derive various distributions like the causal commutator

$$i\,\Delta(x-y) \equiv \Big[\hat{\phi}(x), \hat{\phi}(y)\Big] \tag{4.1}$$

or the Green function

$$i\,\Delta_F(x-y) \equiv \langle 0|T\Big(\hat{\phi}(x)\,\hat{\phi}(y)\Big)|0\rangle \tag{4.2}$$

where the time ordered product for Boson fields is defined by

$$T\Big(\hat{\phi}(x)\,\hat{\phi}(y)\Big) \equiv \theta(x_0-y_0)\,\hat{\phi}(x)\,\hat{\phi}(y) + \theta(y_0-x_0)\,\hat{\phi}(y)\,\hat{\phi}(x) \tag{4.3}$$

For convenience one can also define other distributions like

$$\begin{aligned}
\Delta_+(x-y) &\equiv \langle 0|\hat{\phi}(x)\,\hat{\phi}(y)|0\rangle \\
\Delta_-(x-y) &\equiv \langle 0|\hat{\phi}(y)\,\hat{\phi}(x)|0\rangle \\
\Delta_1(x-y) &\equiv \langle 0|\Big\{\hat{\phi}(x), \hat{\phi}(y)\Big\}|0\rangle
\end{aligned} \tag{4.4}$$

Let us first derive an expression for the causal commutator in Eq. (4.1) Insertion of the free fields into the commutator yields

$$\begin{aligned}
&i\,\Delta(x-y) = \\
&\frac{1}{(2\pi)^3}\int d^3\vec{p}\int d^3\vec{p}'\,\frac{1}{\sqrt{4E_pE_{p'}}}\,\Big(e^{i\,(p'\cdot y - p\cdot x)}\Big[a(\vec{p}), a^\dagger(\vec{p}')\Big] + e^{i\,(p\cdot x - p'\cdot y)}\Big[a^\dagger(\vec{p}), a(\vec{p}')\Big]\Big) \\
&= \frac{1}{(2\pi)^3}\int d^3\vec{p}\int d^3\vec{p}'\,\frac{1}{\sqrt{4E_pE_{p'}}}\,\delta^{(3)}(\vec{p}-\vec{p}')\,\Big(e^{i\,(p'\cdot y - p\cdot x)} - e^{i\,(p\cdot x - p'\cdot y)}\Big) \\
&= \frac{1}{(2\pi)^3}\int d^3\vec{p}\,\frac{1}{2E_p}\,\Big(e^{i\,p\cdot(y-x)} - e^{i\,p\cdot(x-y)}\Big)
\end{aligned} \tag{4.5}$$

Note that $\vec{p}' = \vec{p} \to E_{p'} = E_p$. Replace in the last exponent $\vec{p} \to -\vec{p}$

$$i\,\Delta(x-y) = -i\,\frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{E_p}\, e^{i\,p\cdot(x-y)}\,\sin E_p\,(x_0-y_0) \tag{4.6}$$

The latter we can write in a Lorentz invariant way

$$\begin{aligned} i\,\Delta(x-y) &= \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{2E_p}\, e^{i\,\vec{p}\cdot(\vec{x}-\vec{y})}\,\Big(e^{-i\,E_p\,(x_0-y_0)} - e^{-i\,(-E_p)\,(x_0-y_0)}\Big) \\ &= \frac{1}{(2\pi)^3}\int d^4p\,\delta(p^2-m^2)\,\epsilon(p^0)\,e^{-i\,p\cdot(x-y)} \end{aligned} \tag{4.7}$$

with

$$\delta(p^2-m^2) = \frac{1}{2E_p}\Big(\delta(p^0-E_p)+\delta(p^0+E_p)\Big) \qquad \epsilon(p^0) = \theta(p^0)-\theta(-p^0) \tag{4.8}$$

Therefore we can split up $\delta(p^2-m^2)$ into a positive and negative energy part. From Eq. (4.9) we can derive that for $x_0 = y_0$ the causal commutator $\Delta(x-y)$ vanishes. For those values $x-y$ is spacelike. Because of the Lorentz invariance of $\Delta(x-y)$ (see Eq. (4.7)) the latter vanishes for all spacelike distances. Further we can derive

$$\frac{\partial}{\partial y_0}\,i\,\Delta(x-y)|_{y_0=x_0} = i\,\delta^{(3)}(\vec{x}-\vec{y}) \tag{4.9}$$

Hence we obtain the canonical commutation relation

$$\begin{aligned} \Big[\hat{\phi}(\vec{x},x_0),\hat{\Pi}(\vec{y},x_0)\Big] &= \frac{\partial}{\partial y_0}\,\Big[\hat{\phi}(\vec{x},x_0),\hat{\phi}(\vec{y},y_0)\Big]\Big|_{y_0=x_0} = \frac{\partial}{\partial y_0}\,i\,\Delta(x-y)|_{y_0=x_0} \\ &= i\,\delta^{(3)}(\vec{x}-\vec{y}) \end{aligned} \tag{4.10}$$

where $\Pi(\vec{y},y_0)$ is the canonical momentum operator given by $\frac{\partial}{\partial y_0}\,\hat{\phi}(\vec{y},y_0)$. From Eq. (4.7) it also follows that the causal commutator satisfies the Klein Gordon equation

$$\Big(\Box_x + m^2\Big)\,i\,\Delta(x-y) = 0 \tag{4.11}$$

Similar expressions as the one presented in Eq. (4.7) can be given for the distributions in Eq. (4.4)

$$\begin{aligned} \Delta_+(x-y) &= \frac{1}{(2\pi)^3}\int d^4p\,\delta(p^2-m^2)\,\theta(p^0)\,e^{-i\,p\cdot(x-y)} \\ \Delta_-(x-y) &= \frac{1}{(2\pi)^3}\int d^4p\,\delta(p^2-m^2)\,\theta(-p^0)\,e^{-i\,p\cdot(x-y)} \\ \Delta_1(x-y) &= \frac{1}{(2\pi)^3}\int d^4p\,\delta(p^2-m^2)\,e^{-i\,p\cdot(x-y)} \end{aligned} \tag{4.12}$$

All expressions above satisfy the Klein Gordon equation as given in Eq. (4.11). Finally we compute this equation when it acts on the time-ordered product in Eq. (4.3). First we look at the effect on this product (see Eq. (4.3)) when it is differentiated twice w.r.t. the time coordinate x_0. Differentiating once w.r.t. x_0 yields

$$\frac{\partial}{\partial x_0}\Big(\theta(x_0-y_0)\,\hat\phi(x)\,\hat\phi(y)+\theta(y_0-x_0)\,\hat\phi(y)\,\hat\phi(x)\Big)=$$

$$\delta(x_0-y_0)\left[\hat\phi(x),\hat\phi(y)\right]+\left(\theta(x_0-y_0)\,\frac{\partial\hat\phi(x)}{\partial x_0}\,\hat\phi(y)+\theta(y_0-x_0)\,\hat\phi(y)\,\frac{\partial\hat\phi(x)}{\partial x_0}\right)= \tag{4.13}$$

The first term represented by the commutator equals zero because of

$$\left[\hat\phi(\vec x,x_0),\hat\phi(\vec y,x_0)\right]=0 \tag{4.14}$$

(see the beginning of section 3). Differentiating the second term above again w.r.t. x_0 gives

$$\frac{\partial}{\partial x_0}\left(\theta(x_0-y_0)\,\frac{\partial\hat\phi(x)}{\partial x_0}\,\hat\phi(y)+\theta(y_0-x_0)\,\hat\phi(y)\,\frac{\partial\hat\phi(x)}{\partial x_0}\right)=$$

$$\delta(x_0-y_0)\left[\hat\Pi(x),\hat\phi(y)\right]+\left(\theta(x_0-y_0)\,\frac{\partial^2\hat\phi(x)}{\partial^2 x_0}\,\hat\phi(y)+\theta(y_0-x_0)\,\hat\phi(y)\,\frac{\partial^2\hat\phi(x)}{\partial^2 x_0}\right) \tag{4.15}$$

Using the equal time commutator in Eq. (4.10) we obtain

$$\Box_x\Big(\theta(x_0-y_0)\,\hat\phi(x)\,\hat\phi(y)+\theta(y_0-x_0)\,\hat\phi(y)\,\hat\phi(x)\Big)=-i\,\delta^{(4)}(x-y)$$

$$+\Big(\theta(x_0-y_0)\,\Box_x\hat\phi(x)\,\hat\phi(y)+\theta(y_0-x_0)\,\hat\phi(y)\,\Box_x\hat\phi(x)\Big) \tag{4.16}$$

Using the equation of motion in Eq. (4.11) and sandwiching the expression between the vacuum state we finally obtain

$$\Big(\Box_x+m^2\Big)\,i\,\Delta_F(x-y)=-i\,\delta^{(4)}(x-y) \tag{4.17}$$

Because of the δ-function on the right-hand side $\Delta_F(x-y)$ becomes a Green's function. The equation above admits the following solution

$$i\,\Delta_F(x-y)=i\lim_{\epsilon\,\to\,0}\int\frac{d^4p}{(2\pi)^4}\,e^{-i\,p\cdot(x-y)}\,\frac{1}{p^2-m^2+i\,\epsilon}\qquad \epsilon\geq 0 \tag{4.18}$$

The ϵ occurs in order to regularize the pole at $p^2=m^2$. Very often one omits the limit symbol which is implicitly understood. The sign of ϵ is determined in such a way that it reproduces the answer which is obtained by substituting the free field in Eq. (4.2).

4.2 Perturbation theory

As an example we choose ϕ^4-theory. The Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\,\phi\,\partial^\mu\,\phi - \frac{1}{2}\,\mu^2\,\phi^2 - \frac{\lambda}{4!}\,\phi^4 \tag{4.19}$$

The Hamiltonian density is equal to

$$\mathcal{H} = \Pi\,\partial^0\,\phi - \mathcal{L} = \frac{1}{2}\,\partial_0\,\phi\,\partial^0\,\phi + \frac{1}{2}\,\vec{\nabla}\phi\,\vec{\nabla}\phi + \frac{1}{2}\,\mu^2\,\phi^2 + \frac{\lambda}{4!}\,\phi^4 \tag{4.20}$$

From which follows

$$\begin{aligned} \mathcal{H}_0 &= \frac{1}{2}\,\partial_0\,\phi\,\partial^0\,\phi + \frac{1}{2}\,\vec{\nabla}\phi\,\vec{\nabla}\phi + \frac{1}{2}\,\mu^2\,\phi^2 \\ \mathcal{V} &= \frac{\lambda}{4!}\,\phi^4 \end{aligned} \tag{4.21}$$

Suppose λ is a small parameter. In that case we can apply perturbation theory. In quantum field theory this is an assumption which in practice is not always correct. From the above we obtain the Lagrangian and Hamiltonian

$$L = \int d^3x\,\mathcal{L}(x) \qquad H = \int d^3x\,\mathcal{H}(x) \tag{4.22}$$

The Hamiltonian is split up as follows

$$H = H_0 + V \tag{4.23}$$

The solutions of H_0 are known but the solutions of H are not. In the Interaction picture the state vector is given by

$$|\Psi_I(t)\rangle = e^{i\,H_0\,t}\,|\Psi_S(t)\rangle = e^{i\,H_0\,t}\,e^{-i\,H\,t}\,|\Psi_S(0)\rangle \tag{4.24}$$

$|\Psi_S(t)\rangle$ is a time dependent state vector in the Schródinger picture. The time dependence in the interaction picture is given by the dynamics

$$A_I(t) = e^{i\,H_0\,t}\,A_S(t)\,e^{-i\,H_0\,t} \tag{4.25}$$

$A_S(t)$ is the operator in the Schrödinger picture and is time independent unless it explicitly depends on time. In that case it implies that the time dependence does not originate by the dynamics c.q. the Hamiltonian. The equations of motion in the interaction picture are

$$\begin{aligned} i\,\frac{d}{d\,t}\,|\Psi_I(t)\rangle &= H_I(t)\,|\Psi_I(t)\rangle \\ \frac{d\,A_I(t)}{d\,t} &= \frac{\partial\,A_I(t)}{\partial\,t} + i\,\Big[H_0, A_I(t)\Big] \end{aligned} \tag{4.26}$$

$$H_I = e^{i\,H_0\,t}\,V\,e^{-i\,H_0\,t} \tag{4.27}$$

The time dependence of $|\Psi_I(t)\rangle$ is given by H_I whereas the time dependence of $A_I(t)$ is given by H_0. Define the time evolution operator $U(t, t_0)$ by

$$|\Psi_I(t)\rangle = U(t,t_0)\,|\Psi_I(t_0)\rangle \tag{4.28}$$

From (4.26) we derive

$$i\,\frac{d\,U(t,t_0)}{d\,t} = H_I(t)\,U(t,t_0) \quad \text{met} \quad U(t_0,t_0) = 1 \tag{4.29}$$

The solution of the above equation is

$$U(t,t_0) = 1 - i\int_{t_0}^{t} dt'\,H_I(t')\,U(t',t_0) \tag{4.30}$$

Dyson series

$$U(t,t_0) = \sum_{n=0}^{\infty} U^{(n)}(t,t_0) \qquad U^{(0)}(t,t_0) = 1 \tag{4.31}$$

$$U^{(n)}(t,t_0) = \frac{(-i)^n}{n!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n\,T\Big(H_I(t_1)\,H_I(t_2)\cdots H_I(t_n)\Big) \tag{4.32}$$

or in shorthand notation

$$U(t,t_0) = T\left\{e^{-i\int_{t_0}^{t} dt'\,H_I(t')}\right\} \tag{4.33}$$

The definition of the time ordered product is

$$T\Big(H_I(t_1)\,H_I(t_2)\cdots H_I(t_n)\Big) = H_I(t_i)\,H_I(t_j)\cdots H_I(t_k) \quad \text{if} \quad t_i > t_j > \cdots > t_k \tag{4.34}$$

This can be rewritten in using θ-functions

$$T\Big(H_I(t_1)\,H_I(t_2)\Big) = \theta(t_1 - t_2)\,H_I(t_1)\,H_I(t_2) + \theta(t_2 - t_1)\,H_I(t_2)\,H_I(t_1)$$

$$2! \ \text{terms} \tag{4.35}$$

$$\begin{aligned} T\Big(H_I(t_1)\,H_I(t_2)\,H_I(t_3)\Big) &= \theta(t_1 - t_2)\,\theta(t_2 - t_3)\,H_I(t_1)\,H_I(t_2)\,H_I(t_3) \\ &\quad +\theta(t_2 - t_3)\,\theta(t_3 - t_1)\,H_I(t_2)\,H_I(t_3)\,H_I(t_1) \end{aligned}$$

$$
\begin{aligned}
&+\theta(t_3 - t_1)\,\theta(t_1 - t_2)\,H_I(t_3)\,H_I(t_1)\,H_I(t_2)\\
&+\theta(t_2 - t_1)\,\theta(t_1 - t_3)\,H_I(t_2)\,H_I(t_1)\,H_I(t_3)\\
&+\theta(t_3 - t_2)\,\theta(t_2 - t_1)\,H_I(t_3)\,H_I(t_2)\,H_I(t_1)\\
&+\theta(t_1 - t_3)\,\theta(t_3 - t_2)\,H_I(t_1)\,H_I(t_3)\,H_I(t_2)\\
&3!\ \text{terms}
\end{aligned}
\tag{4.36}
$$

$$
T\bigg(H_I(t_1)\,H_I(t_2)\cdots H_I(t_n)\bigg) \quad\Longrightarrow\quad \text{n! terms} \tag{4.37}
$$

Scattering matrix S

Physical picture: At $t = -\infty$ we have free particles which do not interact. They are described by free fields. For $-\infty < t < \infty$ the particles interact mutually. This interaction is described by H_I. For $t = \infty$ we have again free particles which do not interact. The latter are again described by free fields. The S-operator maps the incoming state ($t = -\infty$) on the outgoing state ($t = \infty$)

$$
\begin{aligned}
&|\Psi_I(\infty)\rangle = S\,|\Psi_I(-\infty)\rangle\\
&|i\rangle = |\Psi_I(-\infty)\rangle \quad \text{incoming state} \qquad |f\rangle = |\Psi_I(\infty)\rangle \quad \text{outgoing state}
\end{aligned}
\tag{4.38}
$$

The S-operator is then given by

$$
S = \lim_{T\,\to\,\infty} U\left(\frac{T}{2}, -\frac{T}{2}\right) = T\bigg\{\exp\bigg(-\,i\int_{-\infty}^{\infty} dt\,H_I(t)\bigg)\bigg\} \tag{4.39}
$$

This leads to the definition of the transition operator T

$$
S = 1 - 2\,\pi\,i\,\delta(E_i - E_f)\,T \tag{4.40}
$$

S is a unitary operator

$$
S\,S^\dagger = S^\dagger\,S = 1 \tag{4.41}
$$

This follows from

$$
H_I^\dagger = H_I \quad\Longrightarrow\quad U\,U^\dagger = U^\dagger\,U = 1 \tag{4.42}
$$

From this follows the unitary relation for the transition operator

$$
i\left(T_{fi} - T_{fi}^\dagger\right) = 2\,\pi \sum_n \delta(E_f - E_n)\,T_{fn}^\dagger\,T_{ni} \quad \text{with} \quad T_{fi} = \langle f|T|i\rangle \tag{4.43}
$$

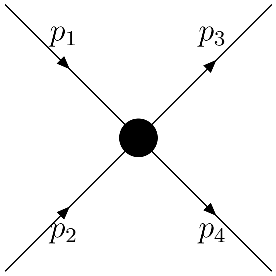


Figure 1: Scattering process for a scalar field theory $1 + 2 \to 3 + 4$.

Let us take for example ϕ^4-theory and calculate the S-matrix. Consider the quantum field operator

$$H_I = \int d^3x \left(\frac{\lambda}{4!}\, \phi^4(x) \right)_I \tag{4.44}$$

wherein the interaction picture $\phi(x)$ becomes a free field operator. The latter describes a scalar particle. We describe a process where two scalar particles come in and two scalar particles of the same kind go out.

$$|i\rangle = |\vec{p}_1, \vec{p}_2\rangle \qquad |f\rangle = |\vec{p}_3, \vec{p}_4\rangle \tag{4.45}$$

The S-matrix becomes

$$\begin{aligned} S_{fi} &= \langle f|S|i\rangle = \langle f|U(-\infty,\infty)|i\rangle = \sum_{n=0}^{\infty} \langle f|U^{(n)}(-\infty,\infty)|i\rangle = \langle f|i\rangle \\ &\quad + \langle f|U^{(1)}(-\infty,\infty)|i\rangle + \langle f|U^{(2)}(-\infty,\infty)|i\rangle + \cdots \end{aligned} \tag{4.46}$$

The first term equals

$$\langle f|i\rangle = \langle \vec{p}_3, \vec{p}_4|\vec{p}_1, \vec{p}_2\rangle = \delta^{(3)}(\vec{p}_1 - \vec{p}_3)\, \delta^{(3)}(\vec{p}_2 - \vec{p}_4) + \delta^{(3)}(\vec{p}_1 - \vec{p}_4)\, \delta^{(3)}(\vec{p}_2 - \vec{p}_3) \tag{4.47}$$

In lowest order of perturbation theory in λ we have

$$\begin{aligned} \langle f|U^{(1)}(-\infty,\infty)|i\rangle &= -i \int_{-\infty}^{\infty} dt\, \langle f|H_I(t)|i\rangle \\ &= -\frac{i\,\lambda}{4!} \int_{-\infty}^{\infty} dt \int d^3x\, \langle \vec{p}_3, \vec{p}_4|\phi^4(x)|\vec{p}_1, \vec{p}_2\rangle \end{aligned} \tag{4.48}$$

Substitution of the free field

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left[a(\vec{k})\, e^{-i\,k\cdot x} + a^\dagger(\vec{k})\, e^{i\,k\cdot x} \right] \tag{4.49}$$

leads to

$$
\begin{aligned}
\langle f|U^{(1)}(-\infty,\infty)|i\rangle \;&=\; -\frac{i\,\lambda}{4!}\int d^4x \int \frac{d^3k_1}{(2\pi)^{3/2}}\,\frac{1}{\sqrt{2\omega_{k_1}}}\int \frac{d^3k_2}{(2\pi)^{3/2}}\,\frac{1}{\sqrt{2\omega_{k_2}}} \\
&\int \frac{d^3k_3}{(2\pi)^{3/2}}\,\frac{1}{\sqrt{2\omega_{k_3}}}\int \frac{d^3k_4}{(2\pi)^{3/2}}\,\frac{1}{\sqrt{2\omega_{k_4}}}\,\langle \vec{p}_3,\vec{p}_4|\Big\{a(\vec{k}_1)\,e^{-i\,k_1\cdot x} \\
&+a^\dagger(\vec{k}_1)\,e^{i\,k_1\cdot x}\Big\}\Big\{a(\vec{k}_2)\,e^{-i\,k_2\cdot x}+a^\dagger(\vec{k}_2)\,e^{i\,k_2\cdot x}\Big\} \\
&\Big\{a(\vec{k}_3)\,e^{-i\,k_3\cdot x}+a^\dagger(\vec{k}_3)\,e^{i\,k_3\cdot x}\Big\}\Big\{a(\vec{k}_4)\,e^{-i\,k_4\cdot x} \\
&+a^\dagger(\vec{k}_4)\,e^{i\,k_4\cdot x}\Big\}\,|\vec{p}_1,\vec{p}_2\rangle
\end{aligned}
\tag{4.50}
$$

Remember $\omega_{k_i} = \sqrt{|\vec{k}_i|^2+\mu^2} = k_{i,0}$. In the products of a and $a^\dagger$ only contribute the following matrix elements

1

$$\langle \vec{p}_3,\vec{p}_4|a(\vec{k}_1)\,a(\vec{k}_2)\,a^\dagger(\vec{k}_3)\,a^\dagger(\vec{k}_4)|\vec{p}_1,\vec{p}_2\rangle \tag{4.51}$$

2

$$\langle \vec{p}_3,\vec{p}_4|a(\vec{k}_1)\,a^\dagger(\vec{k}_2)\,a(\vec{k}_3)\,a^\dagger(\vec{k}_4)|\vec{p}_1,\vec{p}_2\rangle \tag{4.52}$$

3

$$\langle \vec{p}_3,\vec{p}_4|a(\vec{k}_1)\,a^\dagger(\vec{k}_2)\,a^\dagger(\vec{k}_3)\,a(\vec{k}_4)|\vec{p}_1,\vec{p}_2\rangle \tag{4.53}$$

4

$$\langle \vec{p}_3,\vec{p}_4|a^\dagger(\vec{k}_1)\,a(\vec{k}_2)\,a(\vec{k}_3)\,a^\dagger(\vec{k}_4)|\vec{p}_1,\vec{p}_2\rangle \tag{4.54}$$

5

$$\langle \vec{p}_3,\vec{p}_4|a^\dagger(\vec{k}_1)\,a(\vec{k}_2)\,a^\dagger(\vec{k}_3)\,a(\vec{k}_4)|\vec{p}_1,\vec{p}_2\rangle \tag{4.55}$$

6

$$\langle \vec{p}_3,\vec{p}_4|a^\dagger(\vec{k}_1)\,a^\dagger(\vec{k}_2)\,a(\vec{k}_3)\,a(\vec{k}_4)|\vec{p}_1,\vec{p}_2\rangle \tag{4.56}$$

We have to rewrite equations 1-6 otherwise we cannot calculate the expectation values

1

$$\langle \vec{p}_3, \vec{p}_4 || a^\dagger(\vec{k}_3)\, a^\dagger(\vec{k}_4)\, a(\vec{k}_1)\, a(\vec{k}_2) + \delta^{(3)}(\vec{k}_2 - \vec{k}_3)\, a^\dagger(\vec{k}_4)\, a(\vec{k}_1)$$
$$+\delta^{(3)}(\vec{k}_2 - \vec{k}_4)\, a^\dagger(\vec{k}_3)\, a(\vec{k}_1) + \delta^{(3)}(\vec{k}_1 - \vec{k}_4)\, a^\dagger(\vec{k}_3)\, a(\vec{k}_2)$$
$$+\delta^{(3)}(\vec{k}_1 - \vec{k}_3)\, a^\dagger(\vec{k}_4)\, a(\vec{k}_2) + \delta^{(3)}(\vec{k}_2 - \vec{k}_3)\, \delta^{(3)}(\vec{k}_1 - \vec{k}_4)$$
$$+\delta^{(3)}(\vec{k}_1 - \vec{k}_3)\, \delta^{(3)}(\vec{k}_2 - \vec{k}_4)\, |\vec{p}_1, \vec{p}_2\rangle \tag{4.57}$$

2

$$\langle \vec{p}_3, \vec{p}_4 || a^\dagger(\vec{k}_2)\, a^\dagger(\vec{k}_4)\, a(\vec{k}_1)\, a(\vec{k}_3) + \delta^{(3)}(\vec{k}_3 - \vec{k}_4)\, a^\dagger(\vec{k}_2)\, a(\vec{k}_1)$$
$$+\delta^{(3)}(\vec{k}_1 - \vec{k}_4)\, a^\dagger(\vec{k}_2)\, a(\vec{k}_3) + \delta^{(3)}(\vec{k}_1 - \vec{k}_2)\, a^\dagger(\vec{k}_4)\, a(\vec{k}_3)$$
$$+\delta^{(3)}(\vec{k}_1 - \vec{k}_2)\, \delta^{(3)}(\vec{k}_3 - \vec{k}_4)\, |\vec{p}_1, \vec{p}_2\rangle \tag{4.58}$$

3

$$\langle \vec{p}_3, \vec{p}_4 || a^\dagger(\vec{k}_2)\, a^\dagger(\vec{k}_3)\, a(\vec{k}_1)\, a(\vec{k}_4) + \delta^{(3)}(\vec{k}_1 - \vec{k}_2)\, a^\dagger(\vec{k}_3)\, a(\vec{k}_4)$$
$$+\delta^{(3)}(\vec{k}_1 - \vec{k}_3)\, a^\dagger(\vec{k}_2)\, a(\vec{k}_4)\, |\vec{p}_1, \vec{p}_2\rangle \tag{4.59}$$

4

$$\langle \vec{p}_3, \vec{p}_4 || a^\dagger(\vec{k}_1)\, a^\dagger(\vec{k}_4)\, a(\vec{k}_2)\, a(\vec{k}_3) + \delta^{(3)}(\vec{k}_3 - \vec{k}_4)\, a^\dagger(\vec{k}_1)\, a(\vec{k}_2)$$
$$+\delta^{(3)}(\vec{k}_2 - \vec{k}_4)\, a^\dagger(\vec{k}_1)\, a(\vec{k}_3)\, |\vec{p}_1, \vec{p}_2\rangle \tag{4.60}$$

5

$$\langle \vec{p}_3, \vec{p}_4 || a^\dagger(\vec{k}_1)\, a^\dagger(\vec{k}_3)\, a(\vec{k}_2)\, a(\vec{k}_4) + \delta^{(3)}(\vec{k}_2 - \vec{k}_3)\, a^\dagger(\vec{k}_1)\, a(\vec{k}_4)\, |\vec{p}_1, \vec{p}_2\rangle \tag{4.61}$$

We can now, by interchanging of the integration variables, collect a great number of terms

$$\langle f | U^{(1)}(-\infty, \infty) | i \rangle = \int d^4x \int d^4x \int \frac{d^3k_1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{k_1}}} \int \frac{d^3k_2}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{k_2}}}$$
$$\int \frac{d^3k_3}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{k_3}}} \int \frac{d^3k_4}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{k_4}}}\, e^{i\,x\,(k_3+k_4-k_1-k_2)}$$

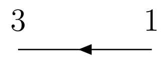


Figure 2: Feynman diagram

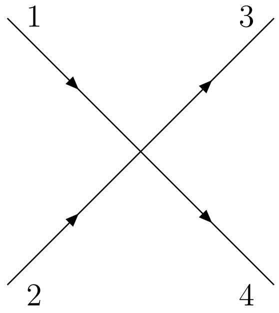


Figure 3: Vertex diagram in ϕ^4-theory.

$$
\begin{aligned}
&\Bigg[\left(-i\,\frac{\lambda}{4}\right)\,\langle\vec{p}_3,\vec{p}_4||a^\dagger(\vec{k}_3)\,a^\dagger(\vec{k}_4)\,a(\vec{k}_1)\,a(\vec{k}_2)\,|\vec{p}_1,\vec{p}_2\rangle \\
&+\left(-i\,\frac{\lambda}{2}\right)\,\delta^{(3)}(\vec{k}_2-\vec{k}_3)\,\langle\vec{p}_3,\vec{p}_4|a^\dagger(\vec{k}_4)\,a(\vec{k}_1)\,|\vec{p}_1,\vec{p}_2\rangle \\
&+\left(-i\,\frac{\lambda}{8}\right)\,\delta^{(3)}(\vec{k}_2-\vec{k}_3)\,\delta^{(3)}(\vec{k}_1-\vec{k}_4)\,\langle\vec{p}_3,\vec{p}_4||\vec{p}_1,\vec{p}_2\rangle\Bigg]
\end{aligned}
\tag{4.62}
$$

Use the following identities

$$
\begin{aligned}
a(\vec{k}_1)\,a(\vec{k}_2)\,|\vec{p}_1,\vec{p}_2\rangle &= a(\vec{k}_1)\,a(\vec{k}_2)\,a^\dagger(\vec{p}_1)\,a^\dagger(\vec{p}_2)\,|0\rangle = \Big\{\delta^{(3)}(k_1-p_1)\,\delta^{(3)}(k_2-p_2) \\
&\quad +\delta^{(3)}(k_1-p_2)\,\delta^{(3)}(k_2-p_1)\Big\}\,|0\rangle
\end{aligned}
\tag{4.63}
$$

likewise

$$
\langle\vec{p}_3,\vec{p}_4|\,a^\dagger(\vec{k}_3)\,a^\dagger(\vec{k}_4) = \langle 0|\Big\{\delta^{(3)}(k_3-p_3)\,\delta^{(3)}(k_4-p_4)+\delta^{(3)}(k_3-p_4)\,\delta^{(3)}(k_4-p_3)\Big\}
\tag{4.64}
$$

$$
a(\vec{k}_1)\,|\vec{p}_1,\vec{p}_2\rangle = \delta^{(3)}(k_1-p_1)\,|\vec{p}_2\rangle+\delta^{(3)}(k_1-p_2)\,|\vec{p}_1\rangle
\tag{4.65}
$$

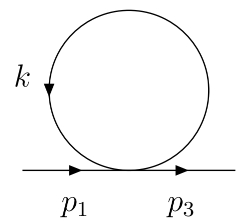


Figure 4: Tadpole diagram

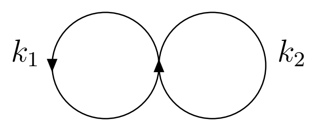


Figure 5: Vacuum bubble diagram

$$\langle \vec{p}_3, \vec{p}_4 |\, a^\dagger(\vec{k}_4) = \langle \vec{p}_3 | \delta^{(3)}(k_4 - p_4) + \langle \vec{p}_4 | \delta^{(3)}(k_4 - p_3) \tag{4.66}$$

Finally we get

$$\begin{aligned} \langle f | U^{(1)}(-\infty, \infty) | i \rangle \quad = \quad & \frac{(2\pi)^4}{(2\pi)^6} \frac{1}{\sqrt{16\, E_1\, E_2\, E_3\, E_4}} \, \delta^{(4)}(p_1 + p_2 - p_3 - p_4)\, (-i\,\lambda) \\ & + \frac{(2\pi)^4}{(2\pi)^6} \left\{ \frac{1}{2\, E_1} \, \delta^{(4)}(p_1 - p_4) \, \langle \vec{p}_3 | \vec{p}_2 \rangle + \frac{1}{2\, E_1} \, \delta^{(4)}(p_1 - p_3) \, \langle \vec{p}_4 | \vec{p}_2 \rangle \right. \\ & \left. + \frac{1}{2\, E_2} \, \delta^{(4)}(p_2 - p_4) \, \langle \vec{p}_3 | \vec{p}_1 \rangle + \frac{1}{2\, E_2} \, \delta^{(4)}(p_2 - p_3) \, \langle \vec{p}_4 | \vec{p}_1 \rangle \right\} \\ & \left(-i \, \frac{\lambda}{2} \right) \int \frac{d^3 k_2}{2\, \omega_2} + \frac{(2\pi)^4}{(2\pi)^6} \, \delta^{(0)}(0) \, \langle \vec{p}_3, \vec{p}_4 | \vec{p}_1, \vec{p}_2 \rangle \, \left(-i \, \frac{\lambda}{8} \right) \\ & \times \int \frac{d^3 k_1}{2\, \omega_1} \int \frac{d^3 k_2}{2\, \omega_2} \end{aligned} \tag{4.67}$$

In the language of Feynman diagrams the expresion

$$\langle \vec{p}_3 | \vec{p}_1 \rangle = \delta^{(3)}(\vec{p}_1 - \vec{p}_3) \tag{4.68}$$

is represented by the graph in Fig. 2. The first term in Eq. (4.67) i.e.

$$\delta^{(4)}(p_1 + p_2 - p_3 - p_4)\,(-i\,\lambda) \tag{4.69}$$

can be found in Fig. 3. The next term represented by

$$\frac{1}{2}\,\delta^{(4)}(p_1 - p_3)\left(-i\,\frac{\lambda}{(2\pi)^3}\right)\int\frac{d^3k}{2\,\omega} \tag{4.70}$$

is depicted in Fig. 4. Finally the term

$$\frac{1}{(2\pi)^6}\,\delta^{(4)}(0)\left(-i\,\frac{\lambda}{8}\right)\int\frac{d^3k_1}{2\,\omega_1}\int\frac{d^3k_2}{2\,\omega_2} \tag{4.71}$$

can be found in Fig. 5. Vacuum bubble diagrams arise even when $|i\rangle = |0\rangle$ and $|f\rangle = |0\rangle$ in other words $\langle 0|\phi^4(x)|0\rangle$. The S-matrix gets the form depicted in Fig. 6. Diagrams multiplied by vacuum bubbles are disconnected and therefore unacceptable for a physical S-matrix. Moreover the vacuum bubbles give a infinite contribution (the diagrams diverge). The latter also hold for the tadpole diagrams. However in this case the divergences can be removed via the renormalisation procedure. To obtain a physical S-matrix we have to redefine the naive S-matrix above. This goes via normal ordering. Take for example an arbitrary product of creation and annihilation operators. Normal ordering implies that one puts all creation operators left of the annihilation operators. In the case of fermions one has to watch out. One has to add an extra minus sign if the normal ordered product is obtained via an odd number of interchanges from the ordinary product.

$$:b\,d\,d^\dagger\,b^\dagger := -b^\dagger\,d^\dagger\,b\,d \qquad :b\,d\,d^\dagger\,b^\dagger := d^\dagger\,b^\dagger\,b\,d \tag{4.72}$$

The consequence is that the expectation value of normal ordered operators sandwiched between the vacuum state is zero.

$$\langle 0| : \phi^2(x) : |0\rangle = 0 \qquad \langle 0| : \phi^4(x) : |0\rangle = 0 \tag{4.73}$$

Watch that it is not necessary to apply normal ordering to products like

$$\phi(x)\,\phi(y) \quad \text{or} \quad :\phi^4(x)::\phi^4(y): \tag{4.74}$$

as long $x \neq y$ We have to give a new definition to the quantum Lagrangian density and Hamiltonian density. For ϕ^4-theory this means that

$$\begin{aligned} :\mathcal{L}(x): &= \frac{1}{2}\,:\partial_\mu\,\phi(x)\,\partial^\mu\,\phi(x): -\frac{1}{2}\,\mu^2\,:\phi^2(x): -\frac{\lambda}{4!}\,:\phi^4(x): \\ :\mathcal{H}(x): &= \frac{1}{2}\,:\partial_0\,\phi(x)\,\partial_0\,\phi(x): \frac{1}{2}\,:\vec{\nabla}\,\phi(x)\cdot\vec{\nabla}\,\phi(x): +\frac{1}{2}\,\mu^2\,:\phi^2(x): \\ &\quad +\frac{\lambda}{4!}\,:\phi^4(x): \end{aligned} \tag{4.75}$$

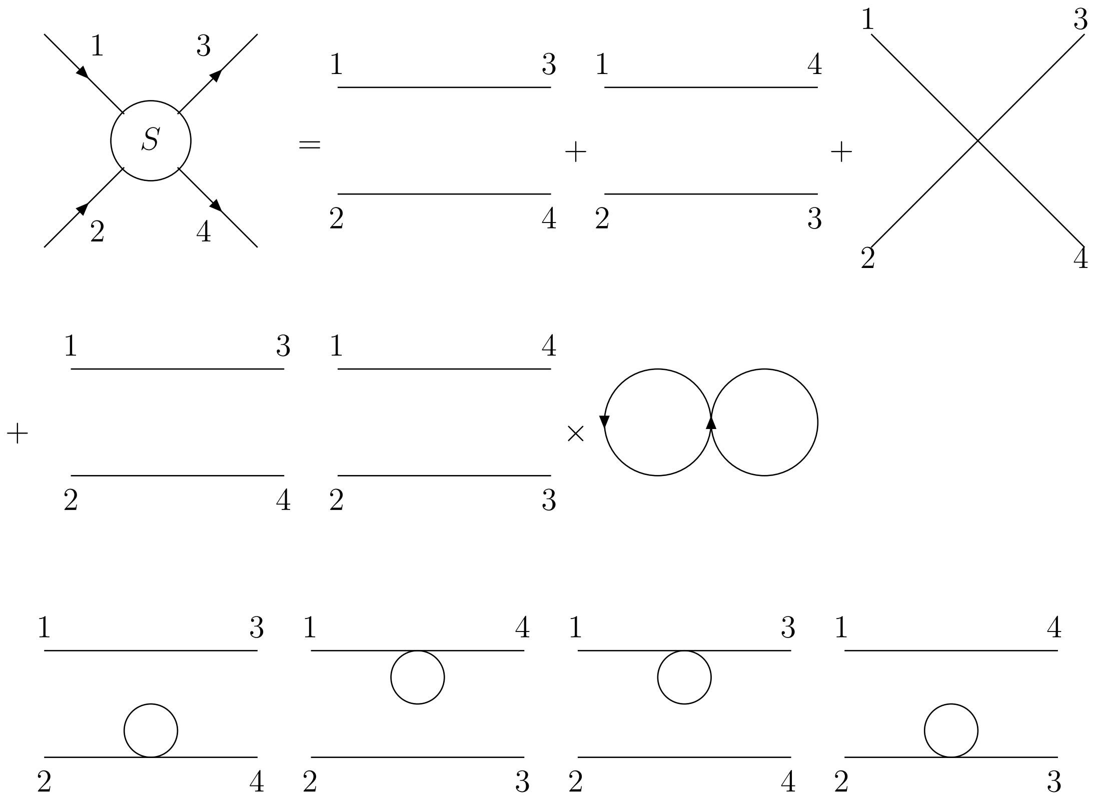


Figure 6: The lowest order diagrams of the S-matrix in ϕ^4-theory.

In lowest order perturbation theory the S-matrix becomes equal to

$$\langle \vec{p}_3, \vec{p}_4| : U^{(1)}(\infty, -\infty) : |\vec{p}_1, \vec{p}_2\rangle = -i\,\frac{\lambda}{4!} \int d^4x\, \langle \vec{p}_3, \vec{p}_4| : \phi^4(x) : |\vec{p}_1, \vec{p}_2\rangle \qquad (4.76)$$

We bring all creation operators to the left and the annihilation operators to the right. This implies that all terms proportional with $[a(\vec{k}_i), a^\dagger(\vec{k}_j)] = \delta^{(3)}(\vec{k}_i - \vec{k}_j)$ disappear (see page 63 1-5). The infinite contributions, represented by the bubble graphs, do not occur. However the tadpole graphs also disappear. The result in our example is

$$S_{fi} = \langle \vec{p}_3, \vec{p}_4 | \vec{p}_1, \vec{p}_2\rangle - i\,\lambda\, \frac{(2\pi)^4}{(2\pi)^6}\, \delta^{(4)}(p_1 + p_2 - p_3 - p_4)\, \frac{1}{\sqrt{16\, E_1\, E_2\, E_3\, E_4}} \qquad (4.77)$$

The transition operator becomes

$$S = 1 - 2\,\pi\, i\, \delta(E_1 + E_2 - E_3 - E_4)$$

$$T_{fi} = \frac{\lambda}{(2\pi)^3}\, \delta^{(3)}(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)\, \frac{1}{\sqrt{16\, E_1\, E_2\, E_3\, E_4}} \qquad (4.78)$$

For our convenience it is easier to define a Lorentz covariant transition operator M

$$S = 1 + (2\pi)^4\, \delta^{(4)}(p_1 + p_2 - p_3 - p_4)\, \frac{1}{(2\pi)^6}\, \frac{1}{\sqrt{16\, E_1\, E_2\, E_3\, E_4}}\, M \qquad (4.79)$$

Up to lowest order in λ

$$M = -i\,\lambda \qquad (4.80)$$

We now calculate the differential cross section of the process

$$p_1 + p_2 \quad \to \quad p_3 + p_4 \qquad (4.81)$$

The transition amplitude per unit volume per unit time is defined by

$$\begin{aligned} W_{fi} &= \frac{|\langle f | S - 1 | i\rangle|^2}{V\,T} \\ &= \frac{1}{V\,T}\, \frac{1}{(2\pi)^4}\, \frac{1}{16\, E_1\, E_2\, E_3\, E_4}\, \delta^{(4)}(0)\, \delta^{(4)}(p_1 + p_2 - p_3 - p_4)\, |M_{fi}|^2 \end{aligned} \qquad (4.82)$$

We use the identities

$$\delta^{(4)}(p) = \frac{1}{(2\pi)^4} \int d^4x\, e^{i\,p\,x} \qquad \delta^{(4)}(0) = \frac{1}{(2\pi)^4}\, V\,T$$

$$\int d^4x = \int_{-T/2}^{T/2} dt \underset{|\vec{x}| \le V}{\int d^3x} = T\,V \qquad |\delta^{(4)}(p)|^2 = \delta^{(4)}(0)\, \delta^{(4)}(p) \qquad (4.83)$$

$$W_{fi} = \frac{1}{(2\pi)^8}\,\frac{1}{16\,E_1\,E_2\,E_3\,E_4}\,\delta^{(4)}(p_1+p_2-p_3-p_4)\,|M_{fi}|^2 \tag{4.84}$$

De differential effective cross section is defined by

$$d\,\sigma = \frac{Vol}{|\vec{J}_i|}\,d\,N_f\,W_{fi} \tag{4.85}$$

where $|\vec{J}_i|$ is the flux of the incoming particles (e.g. $\vec{p}_i$). Vol^{-1} is the number of particles per volume unit in the target (e.g. $\vec{p}_2$). The number of particles which per time unit collide through a plane on each other is given by

$$|\vec{J}_i| = \frac{|\vec{v}_1 - \vec{v}_2|}{Vol} \tag{4.86}$$

for collinear process ($\vec{v}_1 \parallel \vec{v}_2$). The Vol is determined by the normalisation of the plane waves which appear in $\phi(x)$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}}\,\frac{1}{\sqrt{2\,\omega_k}}\left[a(\vec{k})\,e^{-i\,k\,x} + a^\dagger(\vec{k})\,e^{i\,k\,x}\right] \tag{4.87}$$

$$\text{plane wave} \quad \rightarrow \frac{1}{(2\pi)^{3/2}}\,e^{\pm i\,k\,x} \quad \rightarrow \quad Vol = (2\pi)^3 \tag{4.88}$$

$d\,N_f$ is the number of final states with momentum $\vec{p}_3$ in the volume element $d^3\vec{p}_3$ and momentum $\vec{p}_4$ in the volume element $d^3\vec{p}_4$ therefore $d\,N_f = d^3\vec{p}_3\,d^3\vec{p}_4$. Hence it follows

$$\begin{aligned} d\,\sigma \;=\; & \frac{1}{4\,E_1\,E_2\,|\vec{v}_1-\vec{v}_2|}\,\frac{d^3\vec{p}_3}{2E_3\,(2\pi)^3}\,\frac{d^3\vec{p}_4}{2E_4\,(2\pi)^3}\,(2\pi)^4\,\delta^{(4)}(p_1+p_2-p_3-p_4) \\ & |M_{fi}|^2 \end{aligned} \tag{4.89}$$

If $\vec{v}_1 \parallel \vec{v}_2$ (collinear collision) then is $E_1\,E_2\,|\vec{v}_1-\vec{v}_2|$ a Lorentz invariant. We shall prove this. We can put the direction of the collision along the z-axis

$$\begin{aligned} p_1 &= (E_1, 0, 0, p_{1,z}) \qquad p_2 = (E_2, 0, 0, p_{2,z}) \\ \vec{v}_1 &= \left(0, 0, \frac{p_{1,z}}{E_1}\right) \qquad \vec{v}_2 = \left(0, 0, \frac{p_{2,z}}{E_2}\right) \end{aligned} \tag{4.90}$$

$$E_1\,E_2\,|\vec{v}_1-\vec{v}_2| = E_2\,p_{1,z} - E_1\,p_{2,z}$$

$$(p_1\cdot p_2)^2 - m_1^2\,m2^2 = E_1^2\,E_2^2 - 2\,p_{1,z}\,p_{2,z}\,E_1\,E_2 + p_{1,z}^2\,p_{2,z}^2 - m_1^2\,m_2^2$$

$$p_{1,z}^2 = E_1^2 - m_1^2 \qquad p_{2,z}^2 = E_2^2 - m_2^2$$

$$(p_1 \cdot p_2)^2 - m_1^2\, m_2^2 = E_1^2\,(p_{2,z}^2 + m_2^2) - 2\,p_{1,z}\,p_{2,z}\,E_1\,E_2 + p_{1,z}^2\,(E_2^2 - m_2^2)$$
$$-m_1^2\,m_2^2$$
$$= E_1^2\,p_{2,z}^2 - 2\,p_{1,z}\,p_{2,z}\,E_1\,E_2 + p_{1,z}^2\,E_2^2 + m_2^2\,(E_1^2 - p_{1,z}^2 - m_1^2)$$
$$= (E_1\,p_{2,z} - E_2\,p_{1,z})^2$$
$$E_1\,E_2\,|\vec{v}_1 - \vec{v}_2| = \sqrt{(p_1 \cdot p_2)^2 - m_1^2\,m_2^2} \tag{4.91}$$

The last statement holds for any arbitrary frame. Further we have

$$\int \frac{d^3\vec{p}}{2\,E} = \int d^4p\,\theta(p^0)\,\delta(p^2 - m^2) \tag{4.92}$$

so that the cross section becomes

$$\int d\,\sigma \;=\; \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2\,m_2^2}} \int \frac{d^4p_3}{(2\pi)^3}\,\theta(p_{3,0})\,\delta(p_3^2 - m_3^2) \int \frac{d^4p_4}{(2\pi)^3}\,\theta(p_{4,0})$$
$$\delta(p_4^2 - m_4^2)\,(2\pi)^4\,\delta^{(4)}(p_1 + p_2 - p_3 - p_4)\,|M_{fi}|^2 \tag{4.93}$$

For example let us calculate the effective cross section for $M_{fi} = -i\,\lambda$. The centre of mass energy is equal to $\sqrt{s}$ where the Mandelstam variable is defined as

$$s = (p_1 + p_2)^2 \tag{4.94}$$

Proof: The centre of mass is given by

$$\vec{p}_1 + \vec{p}_2 = 0 \quad \rightarrow \quad p_1 + p_2 = (E_{CM}, 0, 0, 0)$$
$$E_{CM} = E_1 + E_2 = \sqrt{|\vec{p}_1|^2 + m_1^2|} + \sqrt{|\vec{p}_2|^2 + m_2^2|}$$
$$s = E_{CM}^2 \quad \rightarrow \quad \sqrt{s} = E_{CM} = E_1 + E_2 \tag{4.95}$$

Further we can rewrite

$$p_1 \cdot p_2 = \frac{1}{2}\,(s - m_1^2 - m_2^2) \quad \rightarrow \quad \sqrt{(p_1 \cdot p_2)^2 - m_1^2, m_2^2} = \frac{1}{2}\,\lambda^{1/2}(s, m_1^2, m_2^2) \tag{4.96}$$

The Källen function is equal to

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2\,x\,y - 2\,x\,z - 2\,y\,z \tag{4.97}$$

Integrate $\int d\,\sigma$ over p_4. Using the delta-function

$$\int d\,\sigma \;=\; \frac{1}{2\,\lambda^{1/2}(s, m_1^2, m_2^2)}\,\frac{\lambda^2}{(2\pi)^2} \int d^4p_3\,\theta(p_{3,0})\,\theta(p_{1,0} + p_{2,0} - p_{3,0})\,\delta(p_3^2 - m_3^2)$$
$$\delta\Big((p_1 + p_2 - p_3)^2 - m_4^2\Big) \tag{4.98}$$

We calculate the effective cross section in the centre of mass system (the z-axis is the direction of the collision)

$$p_1 = (E_1, 0, 0, p_{1,z}) \qquad p_2 = (E_2, 0, 0, -p_{1,z})$$
$$p_3 = (E_3, |\vec{p}_3|\, \sin\theta\, \cos\phi, |\vec{p}_3|\, \sin\theta\, \sin\phi, |\vec{p}_3|\, \cos\theta)$$
$$p_4 = (E_4, -|\vec{p}_3|\, \sin\theta\, \cos\phi, -|\vec{p}_3|\, \sin\theta\, \sin\phi, -|\vec{p}_3|\, \cos\theta) \tag{4.99}$$

$$p_{1,z}^2 = E_1^2 - m_1^2 = E_2^2 - m_2^2 = (E_1 - \sqrt{s})^2 - m_2^2$$
$$E_1 = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}} \qquad E_2 = \frac{s + m_2^2 - m_1^2}{2\sqrt{s}}$$
$$\rightarrow \quad |\vec{p}_1| = |\vec{p}_2| = \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{2\sqrt{s}} \tag{4.100}$$

The measure can be written as

$$d^4 p_3 = |\vec{p}_3|^2\, d\,|\vec{p}_3|\, d\, \cos\theta\, d\,\phi\, d\,p_{3,0} \tag{4.101}$$

so that the cross section becomes

$$\begin{aligned}\int d\,\sigma \quad &= \quad \frac{\lambda^2}{8\,\pi^2\,\lambda^{1/2}(s, m_1^2, m_2^2)} \int d\,|\vec{p}_3|\, d\, \cos\theta\, d\,\phi\, d\,p_{3,0}\, |\vec{p}_3|^2\, \theta(p_{3,0})\, \theta(\sqrt{s} - p_{3,0}) \\ &\quad \delta(p_{3,0}^2 - |\vec{p}_3|^2 - m_3^2)\, \delta(s - 2\,\sqrt{s}\, p_{3,0} + m_3^2 - m_4^2) \\ &= \frac{\lambda^2}{16\,\pi^2\,\lambda^{1/2}(s, m_1^2, m_2^2)} \int d\,\phi\, d\, \cos\theta\, d\,|\vec{p}_3|\, |\vec{p}_3|^2\, \theta\left(\frac{s + m_3^2 - m_4^2}{2\,\sqrt{s}}\right) \\ &\quad \theta\left(\frac{s + m_4^2 - m_3^2}{2\,\sqrt{s}}\right)\, \delta\left(|\vec{p}_3|^2 - \frac{\lambda(s, m_3^2, m_4^2)}{4\,s}\right)\end{aligned} \tag{4.102}$$

Use a property of the delta-functions

$$\delta(f(x)) = \sum_{i=1}^{n} \frac{\delta(x - a_i)}{|f'(a_i)|} \quad \text{with} \quad f(a_i) = 0 \tag{4.103}$$

$$\begin{aligned}\delta\left(|\vec{p}_3|^2 - \frac{\lambda(s, m_3^2, m_4^2)}{4\,s}\right) \quad &= \quad \frac{\sqrt{s}}{\lambda^{1/2}(s, m_3^2, m_4^2)} \left\{\delta\left(|\vec{p}_3| - \frac{\lambda^{1/2}(s, m_3^2, m_4^2)}{2\,\sqrt{s}}\right)\right. \\ &\quad \left. +\delta\left(|\vec{p}_3| + \frac{\lambda^{1/2}(s, m_3^2, m_4^2)}{2\,\sqrt{s}}\right)\right\}\end{aligned} \tag{4.104}$$

The last term vanishes. Integration over $|\vec{p}_3|$ gives

$$\int d\,\sigma = \frac{\lambda^2}{64\,\pi^2} \frac{\lambda^{1/2}(s, m_3^2, m_4^2)}{\lambda^{1/2}(s, m_1^2, m_2^2)} \frac{1}{s}\,\theta(s + m_3^2 - m_4^2)\,\theta(s + m_4^2 - m_3^2) \int d\,\cos\theta\, d\,\phi \tag{4.105}$$

From the theta-functions we can conclude $s \geq 0$. From the lambda-functions follows

$$\lambda(s, m_1^2, m_2^2) = \left\{s - (m_1 + m_2)^2\right\}\left\{s - (m_1 - m_2)^2\right\}$$
$$\lambda(s, m_3^2, m_4^2) = \left\{s - (m_3 + m_4)^2\right\}\left\{s - (m_3 - m_4)^2\right\} \tag{4.106}$$

$$s \geq (m_1 + m_2)^2 \quad \text{or} \quad s \leq (m_1 - m_2)^2$$
$$s \geq (m_3 + m_4)^2 \quad \text{or} \quad s \leq (m_3 - m_4)^2 \tag{4.107}$$

Because of

$$\sqrt{s} = \sqrt{|\vec{p}_1|^2 + m_1^2} + \sqrt{|\vec{p}_1|^2 + m_2^2} > 0 \qquad \vec{p}_1 = -\vec{p}_2 \quad \Longrightarrow \quad \sqrt{s} \geq m_1 + m_2 \tag{4.108}$$

In the same way because $\vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4 = \vec{0}$ it also holds

$$\sqrt{s} = \sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{|\vec{p}_4|^2 + m_4^2} > 0 \quad \Longrightarrow \quad \sqrt{s} \geq m_3 + m_4 \tag{4.109}$$

We have now the condition $\sqrt{s} \geq max(m_1 + m_2, m_3 + m_4)$ otherwise $d\,\sigma = 0$

$$\frac{d\,\sigma}{d\,\Omega} = \frac{\lambda^2}{64\,\pi^2\,s} \frac{\lambda^{1/2}(s, m_3^2, m_4^2)}{\lambda^{1/2}(s, m_1^2, m_2^2)} \quad \text{with} \quad d\,\Omega = d\,\cos\theta\,d\,\phi \tag{4.110}$$

In the case of ϕ^4-theory we have to deal with identical scalar bosons which have the same mass μ. The differential cross section becomes

$$\frac{d\,\sigma}{d\,\Omega} = \frac{\lambda^2}{64\,\pi^2\,s} \tag{4.111}$$

and the total effective cross section is equal to

$$\sigma_{tot} = \int d\,\Omega\,\frac{d\,\sigma}{d\,\Omega} = \int_{-1}^{1} d\,\cos\theta \int_0^{2\pi} d\,\phi\,\frac{\lambda^2}{64\,\pi^2\,s} = \frac{\lambda^2}{16\,\pi\,s} \tag{4.112}$$

Identical particle effect

If the outgoing particles 3 and 4 are identical one cannot distinguish which particle appears in the detector provided one integrates over all angles. The measured total cross section is then two times smaller than the calculated one.

$$\sigma_{tot} = \frac{\lambda^2}{32\,\pi\,s} \tag{4.113}$$

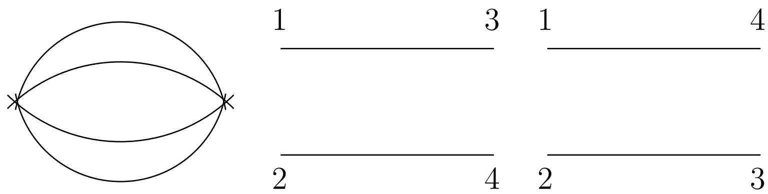


Figure 7: The bubble graph contribution to the S-matrix in ϕ^4-theory.

In general: Suppose we have a $2 \to n$ process with n_i identical particles of species i and $\sum_{i=1}^{N} n_i = n$ the the total cross section becomes

$$\sigma_{tot} = \prod_{i=1}^{N} \frac{1}{n_i!} \frac{1}{\sqrt{(p_1 \cdot p_2)^2 - m_1^2\, m_2^2}} \prod_{j=1}^{n} \int \frac{d^3\vec{k}_j}{2E_j\,(2\pi)^3} (2\pi)^4\, \delta^{(4)}(p_1 + p_2 - \sum_{i=1}^{n} k_i) \\ |M_{fi}|^2 \tag{4.114}$$

Besides the effective cross section there exists another interesting quantity i.e. the decay width Γ. Suppose we have a decay process $1 \to 3 + 4$. This process can only happen if $m_1 \geq m_3 + m_4$. The average life time τ_1 of particle 1 is given by $\tau_1 = 1/\Gamma_1$. The S-matrix for the decay $1 \to 3 + 4$ is given by

$$S = 1 + (2\pi)^4\, \delta^{(4)}(p_1 - p_3 - p_4)\, \frac{1}{(2\pi)^{9/2}}\, \frac{1}{8\, E_1\, E_3\, E_4}\, M_{fi} \tag{4.115}$$

The transition probability is the equal to

$$\begin{aligned} W_{fi} &= \frac{|\langle f|S-1|i\rangle|^2}{V\,T} = \frac{1}{V\,T}\, \frac{1}{2\pi}\, \frac{1}{8\, E_1\, E_3\, E_4}\, \delta^{(4)}(0)\, \delta^{(4)}(p_1 - p_3 - p_4)\, |M_{fi}|^2 \\ &= \frac{1}{(2\pi)^5}\, \frac{1}{8\, E_1\, E_3\, E_4}\, \delta^{(4)}(p_1 - p_3 - p_4)\, |M_{fi}|^2 \end{aligned} \tag{4.116}$$

The decay width is defined by ($Vol = (2\pi)^3$)

$$d\,\Gamma = Vol\, d\, N_f\, W_{fi} = \frac{1}{2E_1} \int \frac{d^3\vec{p}_3}{2E_3\,(2\pi)^3} \int \frac{d^3\vec{p}_4}{2E_4\,(2\pi)^3}\, (2\pi)^4\, \delta^{(4)}(p_1 - p_3 - p_4) \\ |M_{fi}|^2 \tag{4.117}$$

In the case that there are n_i identical particles in the final state we get again the extra factor $\prod_{i=1}^{N} 1/n_i!$. Further the decay width will be always calculated in the rest frame

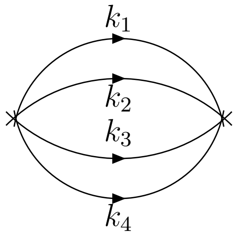


Figure 8: The bubble graph in ϕ^4-theory.

of the decaying particle. From the latter follows $E_1 = m_1$. The generalisation of $d\,\Gamma$ for $1 \to n$ is trivial.

In the example above ($\phi^4(x)$-theory) we have calculated the S-matrix up to order λ. The normal ordering procedure eliminates all contributions which arise by interchanging creation and annihilation operators in a product of fields which were defined in the same space time point. For example $\phi^4(x)$, $\phi^2(x)$. However this not the end of the problems encountered in a relativistic quantum field theory. This arises if we go to second order in λ. The second order term in the S-matrix is

$$\begin{aligned}\langle f|U^{(2)}(\infty,-\infty)|i\rangle &= -\frac{1}{2!}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\, \langle f|T\Big(: H_I(t_1) :: H_I(t_2) : \Big)|i\rangle \\ &= -\frac{1}{2!}\left(\frac{\lambda}{4!}\right)^2 \int d^4x \int d^4y\, \langle f|T\Big(: \phi^4(x) :: \phi^4(y) : \Big)|i\rangle\end{aligned} \tag{4.118}$$

$$\begin{aligned}\langle f|T\Big(: \phi^4(x) :: \phi^4(y) : \Big)|i\rangle &= \theta(x_0 - y_0)\,\langle f| : \phi^4(x) :: \phi^4(y) : |i\rangle \\ &+ \theta(y_0 - x_o)\,\langle f| : \phi^4(y) :: \phi^4(x) : |i\rangle\end{aligned} \tag{4.119}$$

A long calculation shows that the following terms appear in the S-matrix The first term in $\langle f|U^{(2)}(\infty,-\infty)|i\rangle$ is represent by the bubble graph in Fig. 7

$$-\frac{\lambda^2}{24}\,(2\pi)^4\,\delta^{(4)}(0)\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\int\frac{d^4k_3}{(2\pi)^4}\,\frac{i}{k_1-\mu^2+i\,\varepsilon}\,\frac{i}{k_2-\mu^2+i\,\varepsilon}$$

$$\frac{i}{k_3-\mu^2+i\,\varepsilon}\,\frac{i}{(k_1+k_2+k_3)^2-\mu^2+i\,\varepsilon}\,\langle p_3,p_4|p_1,p_2\rangle \tag{4.120}$$

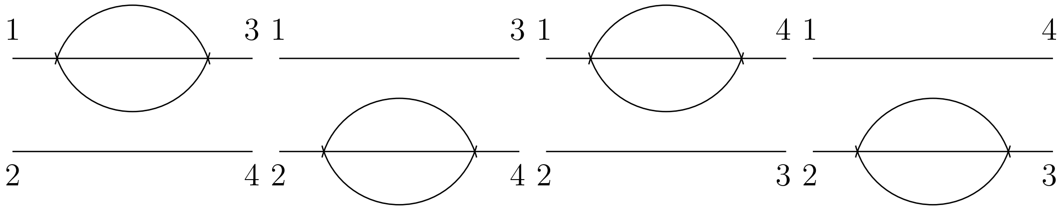


Figure 9: The self energy contribution to the S-matrix in ϕ^4-theory.

This contribution is represented by the graph in Fig. 8

$$\frac{(-i\,\lambda)^2}{4!}\prod_{i=1}^{4}\int\frac{d^4k_i}{(2\pi)^4}\,(2\pi)^4\,\delta^{(4)}(\sum_{j=1}^{4}k_j)\,\left(\frac{i}{k_l^2-\mu^2+i\,\varepsilon}\right) \tag{4.121}$$

We see that even after normal ordering vacuum bubbles appear which are ultra-violet divergent. This can be attributed to contractions of creation and annihilation operators between $:\phi^4(x):$ and $:\phi^4(y):$. They also occur in

$$\langle 0|T\Big(:\phi^4(x)::\phi^4(y):\Big)|0\rangle = -\frac{\lambda^2}{24}\,(2\pi)^4\,\delta^{(4)}(0)\,\prod_{i=1}^{4}\int\frac{d^4k_i}{(2\pi)^4}\,\delta^{(4)}(\sum_{j=1}^{4}k_j) \left(\frac{1}{k_i^2-\mu^2+i\,\varepsilon}\right) \tag{4.122}$$

Moreover this vacuum expectation value is infinite. The second term in $\langle f|U^{(2)}(\infty,-\infty)|i\rangle$ is a correction to the self energy. It occurs 4 times (see Fig. 9)

$$
\begin{aligned}
& i\,\frac{\lambda^2}{6}\,\frac{2\pi}{2E_1}\,\delta^{(4)}(p_1-p_3)\,\langle p_4|p_2\rangle\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\,\frac{1}{k_1-\mu^2+i\,\varepsilon}\,\frac{1}{k_2-\mu^2+i\,\varepsilon} \\
& \frac{1}{(k_1+k_2+p_3)-\mu^2+i\,\varepsilon} \\
& +i\,\frac{\lambda^2}{6}\,\frac{2\pi}{2E_1}\,\delta^{(4)}(p_1-p_3)\,\langle p_3|p_2\rangle\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\,\frac{1}{k_1-\mu^2+i\,\varepsilon}\,\frac{1}{k_2-\mu^2+i\,\varepsilon} \\
& \frac{1}{(k_1+k_2+p_4)-\mu^2+i\,\varepsilon} \\
& +i\,\frac{\lambda^2}{6}\,\frac{2\pi}{2E_2}\,\delta^{(4)}(p_1-p_4)\,\langle p_4|p_1\rangle\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\,\frac{1}{k_1-\mu^2+i\,\varepsilon}\,\frac{1}{k_2-\mu^2+i\,\varepsilon}
\end{aligned}
$$

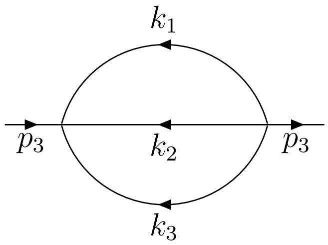


Figure 10: The self energy in ϕ^4-theory.

$$\frac{1}{(k_1+k_2+p_3)-\mu^2+i\,\varepsilon}$$

$$+i\,\frac{\lambda^2}{6}\,\frac{2\pi}{2E_2}\,\delta^{(4)}(p_2-p_3)\,\langle p_3|p_1\rangle\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\,\frac{1}{k_1-\mu^2+i\,\varepsilon}\,\frac{1}{k_2-\mu^2+i\,\varepsilon}$$

$$\frac{1}{(k_1+k_2+p_4)-\mu^2+i\,\varepsilon} \tag{4.123}$$

The expression for the self energy equals (see Fig. 10)

$$\frac{(-i\,\lambda)^2}{3!}\prod_{l=1}^{3}\int\frac{d^4k_l}{(2\pi)^4}\,(2\pi)^4\,\delta^{(4)}\left(\sum_{j=1}^{3}k_j+p_3\right)\left(\frac{i}{k_l^2-\mu^2+i\,\varepsilon}\right) \tag{4.124}$$

these self energy graphs are divergent in the ultra-violet region. However they do not appear in $\langle 0|T(|:\phi^4(x)::\phi^4(y):|0\rangle$ Watch the similarity between the contribution above and the tadpole graph in Fig. 4. The tadpole graph disappeared after normal ordering. This procedure does not work for self energy graphs. The last contribution to $\langle f|U^{(2)}(\infty,-\infty)|i\rangle$ can be found in Fig. 11

$$\frac{1}{2}\,\frac{\lambda^2}{(2\pi)^2}\,\frac{1}{\sqrt{16\,E_1\,E_2\,E_3\,E_4}}\,\delta^{(4)}(p_1+p_2-p_3-p_4)$$

$$\left[\int\frac{d^4k_1}{(2\pi)^4}\,\frac{i}{k_1-\mu^2+i\,\varepsilon}\,\frac{i}{(k_1+p_3+p_4)^2-\mu^2+i\,\varepsilon}\right.$$

$$+\int\frac{d^4k_1}{(2\pi)^4}\,\frac{i}{k_1-\mu^2+i\,\varepsilon}\,\frac{i}{(k_1+p_3-p_1)^2-\mu^2+i\,\varepsilon}$$

$$\left.+\int\frac{d^4k_1}{(2\pi)^4}\,\frac{i}{k_1-\mu^2+i\,\varepsilon}\,\frac{i}{(k_1+p_4-p_1)^2-\mu^2+i\,\varepsilon}\right] \tag{4.125}$$

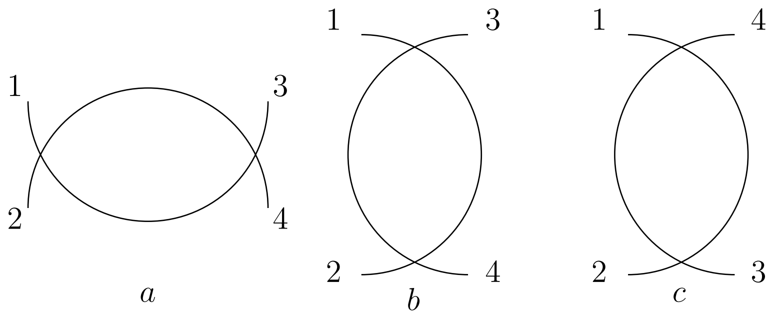


Figure 11: The four point function contribution to the S-matrix in ϕ^4-theory.

The graph in Fig. 11a is represented in Fig. 12.

$$-\frac{(-i\,\lambda)^2}{2!}\prod_{l=1}^{2}\int\frac{d^4k_l}{(2\pi)^4}\,(2\pi)^4\,\delta^{(4)}\left(\sum_{j=1}^{2}k_j+p_3+p_4\right)\left(\frac{i}{k_l^2-\mu^2+i\,\varepsilon}\right) \tag{4.126}$$

These graphs are also ultraviolet divergent. The latter are the only ones which contribute to M_{fi}. All integrals above are represented in momentum space. They are called Feynman integrals. Before determining M_{fi} we will first deal with the proper definition of the S-matrix wherein all vacuum bubbles vanish. Definition of the S-matrix: Definition I.

$$\langle f|S|i\rangle_I=\frac{\langle f|T\{\exp(-i\int_{-\infty}^{\infty}dt\ :H_I(t):)\}|i\rangle}{\langle 0|T\{\exp(-i\int_{-\infty}^{\infty}dt\ :H_I(t):)\}|0\rangle} \tag{4.127}$$

$$\begin{aligned}
&\langle f|S|i\rangle_I\\
&=\frac{\langle f|i\rangle-i\,\langle f|\int dt_1\ :H_I(t_1):i\rangle-1/2\,\langle f|\int dt_1\int dt_2\,T((:H_I(t_1)::H_I(t_1):)|i\rangle}{\langle 0|0\rangle-i\,\langle 0|\int dt_1\ :H_I(t_1):|0\rangle-1/2\,\langle 0|\int dt_1\int dt_2\,T(:H_I(t_1)::H_I(t_1):)|0\rangle}\\
&=\frac{\delta_{fi}+\lambda\,c_1+\lambda^2\,(a_2\,\delta_{fi}+b_2+c_2)}{1+\lambda^2\,a_2}=\delta_{fi}+\lambda\,c_1+\lambda^2\,(b_2+c_2)
\end{aligned} \tag{4.128}$$

Notice that $\langle 0|\int dt_1\ :H_I(t_1):|0\rangle=0$.

This follows from the assumption that the coupling constant λ is small. The various contributions c_1, a_2, b_2, c_2 are shown in Fig. 13.

$$\begin{aligned}
&\left\{\delta_{fi}+\lambda\,c_1+\lambda^2\left(a_2\,\delta_{fi}+b_2+c_2\right)\right\}\left\{1-a_2\,\lambda^2\right\}=\delta_{fi}+\lambda\,c_1+\lambda^2\left(b_2+c_2\right)\\
&+\cdots O(\lambda^3)
\end{aligned} \tag{4.129}$$

One can prove that in all orders the vacuum bubbles in the new S-matrix vanish.

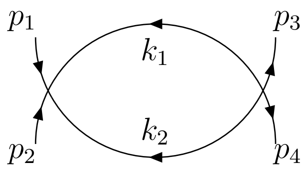


Figure 12: The four point function contribution in ϕ^4-theory.

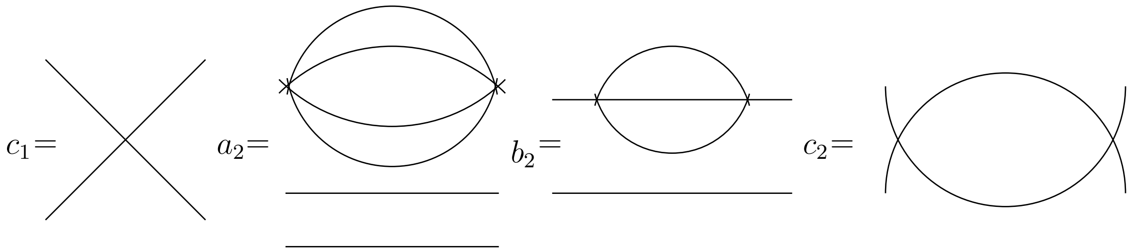


Figure 13: The various contributions to the S-matrix in ϕ^4-theory.

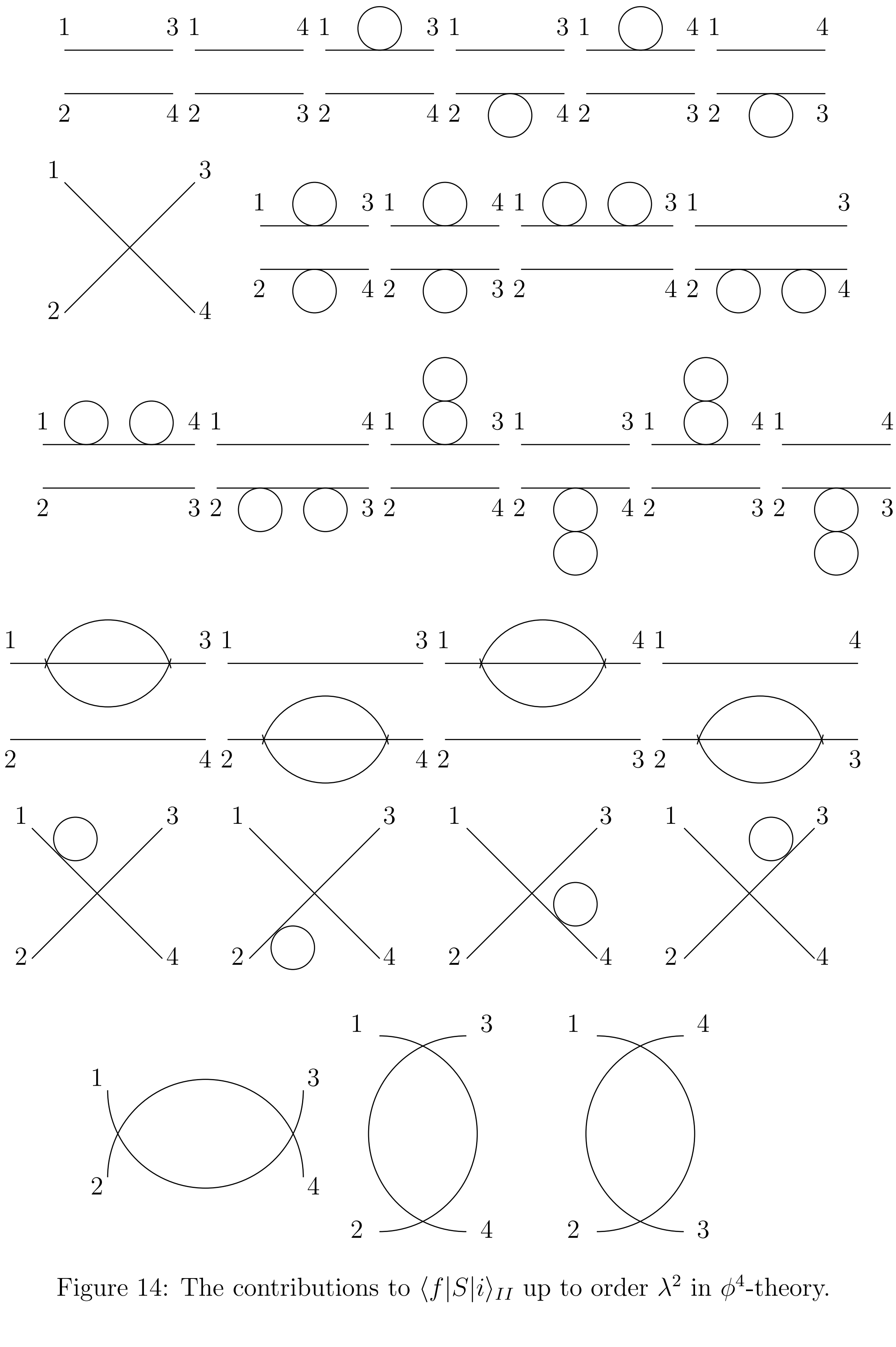


Figure 14: The contributions to $\langle f|S|i\rangle_{II}$ up to order λ^2 in ϕ^4-theory.

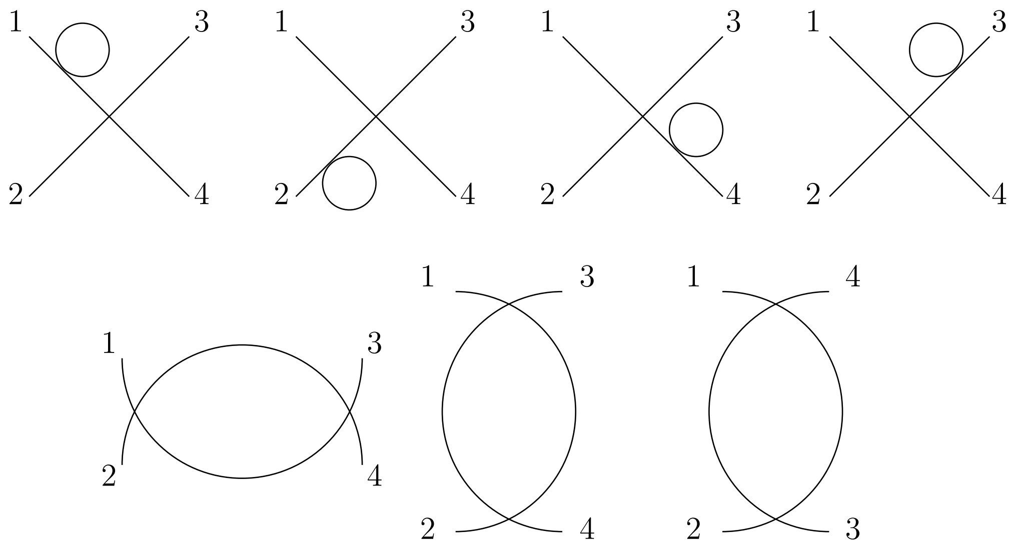


Figure 15: Connected graphs in order λ^2 contributing to M_{fi} in ϕ^4-theory.

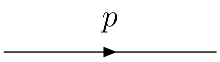
There are people who prefer no normal ordering. This is related to the renormalisation procedure. We shall talk about this later on. Suppose we do not perform the normal ordering procedure that there is a second definition for the S-matrix:
Definition II

$$\langle f|S|i\rangle_{II} = \frac{\langle f|T\{\exp(-i\,\int_{-\infty}^{\infty} dt\, H_I(t))\}|i\rangle}{\langle 0|T\{\exp(-i\,\int_{-\infty}^{\infty} dt\, H_I(t))\}|0\rangle} \tag{4.130}$$

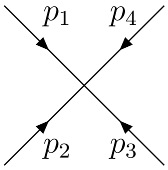
In the latter case all vacuum bubbles disappear but the tadpole graphs appear again.

$$\begin{aligned} &\langle f|S|i\rangle_{II} \\ &= \frac{\langle f|i\rangle - i\,\langle f|\int dt_1\, H_I(t_1)i\rangle - 1/2\,\langle f|\int dt_1\, \int dt_2\, T((H_I(t_1)\, H_I(t_1))|i\rangle}{\langle 0|0\rangle - i\,\langle 0|\int dt_1\, H_I(t_1)|0\rangle - 1/2\,\langle 0|\int dt_1\, \int dt_2\, T(H_I(t_1)\,:H_I(t_1))|0\rangle} \end{aligned} \tag{4.131}$$

but now $-i\,\langle 0|\int dt_1\, H_I(t_1)|0\rangle$ is equal to Fig. 5. Definition II has the following consequences for $\langle f|S|i\rangle$. How $\langle f|S|i\rangle_{II}$ looks like can be inferred up to order λ^2 from Fig. 14. Definition I of the S-matrix (with normal ordering) follows from definition II by omitting all tadpole graphs in Fig. 14. Diagrams with a closed line are called loop diagrams. They represent the quantum fluctuations of the theory. The diagrams without loops are the tree graphs. In the non relativistic limit the latter become equal to the classical result.
The covariant transition matrix is given by all graphs which have the factor



$i\tilde{\Delta}_F(p) = \frac{i}{p^2-\mu^2+i\,\varepsilon}$



$-i\,\lambda\,(2\pi)^4\,\delta^{(4)}(p_1+p_2+p_3+p_4)$

Figure 16: The Feynman rules for ϕ^4-theory.

$$\frac{(2\pi)^4}{(2\pi)^6}\,\frac{1}{\sqrt{16\,E_1\,E_2\,E_3\,E_4}}\,\delta^{(4)}(p_1+p_2-p_3-p_4) \tag{4.132}$$

In lowest order in λ this given by the graph in Fig. 3. In next order in λ this is given by the Feynman diagrams in Fig. 15. Notice that with normal ordering de diagrams with tadpoles disappear. The Feynman rules in ϕ^4 theory for M_{fi} read (see Fig. 16)

1 Draw at a given order in λ all topological inequivalent connected diagrams.

2 Each internal line is represented by the factor $i\,\tilde{\Delta}_F(p)$

3 Each vertex where four lines with the momenta p_1, p_2, p_3, p_4 come together is represented by a factor $-i\,\lambda$. Here is imposed the condition that $\sum_{i=1}^{4} p_i = 0$

4 Integrate over every internal momentum k. The measure is equal to $d^4k/(2\pi)^4$

5 A combinatorial factor $1/n!$ if n internal lines start on one vertex and end on another vertex.

In this way one obtains the expression for $(2\pi)^4\,\delta^{(4)}(\sum_{i=1}^{m} p_i)\,M_{fi}$.

4.3 Ultraviolet singularities and renormalization

While considering the transition amplitude M_{fi} we still encounter beyond lowest order ultraviolet divergences (see Fig. 15). Let us analyse where these divergences come from and to remove them in a consistent way. Notice that the latter is only possible in so called renormalizable field theories where ϕ^4-theory is an example. Let us therefore study the four point function in Fig. 12. In momentum space this function reads

$$\tilde{\Gamma}^{(4)}(p_1,\cdots,p_4)\,(2\pi)^4\,\delta^{(4)}\left(\sum_{j=1}^{4} p_j\right) = \frac{1}{2}\,\lambda^2\,(2\pi)^4\,\delta^{(4)}\left(\sum_{j=1}^{4} p_j\right)$$

$$\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2 - m^2 + i\,\epsilon}\,\frac{1}{(k+p_1+p_2)^2 - m^2 + i\,\epsilon} \tag{4.133}$$

We perform again a Wick rotation $k_0 = i\,k_4$, $p_{j,0} = i\,p_{j,4}$ $(j = 1\cdots 4)$ so that the four point-function becomes

$$\tilde{\Gamma}^{(4)}(p_1,\cdots,p_4) = i\,\frac{\lambda^2}{2}\int \frac{d^4k_E}{(2\pi)^4}\frac{1}{k_E^2 + m^2}\,\frac{1}{(k_E + p_{1,E} + p_{2,E})^2 + m^2} \tag{4.134}$$

Approximate the integrand for $k_E \to \infty$ then the integral looks like

$$-i\frac{\lambda^2}{16\pi^2}\int_0^\infty \frac{dk_E}{k_E} \tag{4.135}$$

which is logarithmic divergent at $k_E = \infty$. The divergence at $k_E = 0$ is artificial because we neglected masses. If there would be no masses then we also have an infrared divergences at $k_E = 0$. From the observations above we infer the following. The ultraviolet divergence corresponds with $k_E = \infty$ (large momentum) whereas the infrared divergence corresponds with $k_E = 0$ (small momentum). Since we want to compute the loop graphs we have to find a method to regularise these divergences. The only problem is that this procedure is arbitrary. This arbitrariness is reflected by the introduction of a mass scale in the integral as we will show below. Among the methods to regularise the integrals we adopt n-dimensional regularization. It boils down to the reformulation of the field theory in n dimensions. The consequence is that the coupling constant acquires a dimension which is the scale mentioned above. Let us first determine the dimension of all quantities.

$$\dim\, m = \mu \quad\rightarrow\quad \dim\, x = \mu^{-1} \qquad \dim\, d^n x = \mu^{-n}$$

$$\dim\, S[\phi] = \int d^n x\,\mathcal{L} = 1 \quad\rightarrow\quad \dim\,\mathcal{L} = \mu^n$$

$$\dim\,\partial_\mu = \mu$$

$$\dim\,\mu^2\,\phi^2(x) = \mu^n \quad\rightarrow\quad \dim\,\phi(x) = \mu^{(n-2)/2}$$

$$\dim\,\lambda_n\,\phi^4(x) = \mu^n \quad\rightarrow\quad \dim\,\lambda_n = \mu^{4-n} \qquad \lambda_n = \lambda\,\mu^{4-n} \tag{4.136}$$

Because the Feynman propagator for massive theories is much more simple in momentum space than in normal space we will carry out all integrals in momentum space. The Feynman rules in below Eq. (4.132) have to be modified as follows

3. For every vertex in Fig. 16 we have to assign the factor

$$-i\,\lambda\,(2\pi)^4\,\delta^{(4)}(p_1+p_2+p_3+p_4) \quad \rightarrow \quad -i\,\lambda_n\,(2\pi)^n\,\delta^{(n)}(p_1+p_2+p_3+p_4) \tag{4.137}$$

4. Integrate over each internal momentum k which has the measure

$$\frac{d^4k}{(2\pi)^4} \quad \rightarrow \quad \frac{d^nk}{(2\pi)^n} \tag{4.138}$$

which together with the other rules mentioned in 1, 2, 5 at the end of the previous section determines the scattering amplitude

$$M_{fi}\,(2\pi)^n\,\delta^{(n)}\left(\sum_{j=1}^{n} p_j\right) \tag{4.139}$$

As an example we compute the scattering amplitude M_{fi} for the reaction

$$p_1+p_2 \quad \rightarrow \quad p_3+p_4 \tag{4.140}$$

where the incoming scalar particles indicated by the momenta p_1 and p_2 belong to the initial state $|i\rangle$ and the outgoing particles, indicated by the momenta p_3 and p_4 belong to the initial state $\langle f|$. Ignoring the tadpole graphs in the first row of Fig. 15 we have up to order λ^2 the scattering amplitude M_{fi} represented by the graphs in Figs. 3 and 11 (definition I). It is given by (see Eq. (4.133))

$$M_{fi} = -i\,\lambda_n + J(s) + J(t) + J(u) \tag{4.141}$$

with

$$J(s) = \frac{\lambda_n^2}{2}\int \frac{d^nk}{(2\pi)^n}\,\frac{1}{k^2-m^2+i\,\epsilon}\,\frac{1}{(k+p_1+p_2)^2-m^2+i\,\epsilon} \tag{4.142}$$

If $n<4$ this integral is defined. We will now calculate this expression by introducing Feynman parameters. This trick reads

$$\frac{1}{a^ib^j} = \frac{\Gamma(i+j)}{\Gamma(i)\,\Gamma(i)}\int_0^1 dx\,\frac{x^{i-1}\,(1-x)^{j-1}}{[a\,x+b\,(1-x)]^{i+j}} \tag{4.143}$$

The Gamma function is defined as

$$\Gamma(x) = \int_0^\infty dt\,t^{x-1}\,e^{-t} \qquad \Gamma(x+1) = x\,\Gamma(x)$$

$$\Gamma(n+1) = n! \quad \text{with} \quad n \in \mathcal{N} \tag{4.144}$$

Further $\Gamma(x)$ has poles for $x=-n$ with $n \geq 0$. Using the Feynman parameter trick the expression in Eq. (4.142) equals

$$J(s) = \frac{\lambda_n^2}{2}\int \frac{d^nk}{(2\pi)^n}\int_0^1 dx\,\frac{1}{[k^2+2\,x\,k\cdot(p_1+p_2)+x.(p_1+p_2)^2-m^2+i\,\epsilon]^2} \tag{4.145}$$

Define

$$k' = k + x(p_1 + p_2) \qquad s = (p_1 + p_2)^2 \tag{4.146}$$

$$J(s) = \frac{\lambda_n^2}{2} \int \frac{d^n k'}{(2\pi)^n} \int_0^1 dx \, \frac{1}{[k'^2 + x\,(1-x)\,s - m^2 + i\,\epsilon]^2} \tag{4.147}$$

If $a = m^2 - x\,(1-x)\,s > 0$ we can drop $i\,\epsilon$. Let us calculate integrals of the type

$$I_1 = \int \frac{d^n k}{(2\pi)^n} \, \frac{1}{k^2 - a} \quad \text{with} \quad a > 0 \tag{4.148}$$

Perform a Wick rotation: $k_0 = i\,k_{E,n}$ so that $d^n k = i\,d^n k_E$ and $k^2 = -k_E^2$

$$I_1 = -i \int \frac{d^n k}{(2\pi)^n} \, \frac{1}{k_E^2 + a} \quad \text{with} \quad d^n k_E = \frac{2\,\pi^{n/2}}{\Gamma(n/2)} \, k_E^{n-1} \, dk_E \tag{4.149}$$

$$I_1 = -i \, \frac{2\,\pi^{n/2}}{(2\pi)^n \, \Gamma(n/2)} \int_0^\infty dk_E \, \frac{k_E^{n-1}}{k_E^2 + a} \tag{4.150}$$

Substitute $k_E = \sqrt{ax}$ $(a > 0)$

$$I_1 = -i \, \frac{\pi^{n/2}}{(2\pi)^n \, \Gamma(n/2)} \, a^{n/2-1} \int_0^\infty dx \, x^{n/2-1} \, (1+x)^{-1} \tag{4.151}$$

Substitute $x = y/(1-y)$

$$I_1 = -i \, \frac{\pi^{n/2}}{(2\pi)^n \, \Gamma(n/2)} \, a^{n/2-1} \int_0^1 dy \, y^{n/2-1} \, (1-y)^{-n/2} = -i \, \frac{\pi^{n/2}}{(2\pi)^n} \, \Gamma(1-n/2) \, a^{n/2-1} \tag{4.152}$$

Differentiate the expression above $l-1$ times w.r.t. a we get

$$I_l = \int \frac{d^n k}{(2\pi)^n} \, \frac{1}{(k^2 - a)^l} = (-1)^l \, i \, \frac{\pi^{n/2}}{(2\pi)^n} \, \frac{\Gamma(l - n/2)}{\Gamma(l)} \, a^{n/2-l} \tag{4.153}$$

For a theory with spin we also need the following integrals.

$$\int \frac{d^n k}{(2\pi)^n} \, \frac{k_\mu}{(k^2 - a)^l} = 0$$

$$\int \frac{d^n k}{(2\pi)^n} \, \frac{k_\mu}{(k^2 + 2\,k \cdot p - a)^l} = (-1)^{l+1} \, i \, \frac{\pi^{n/2}}{(2\pi)^n} \, \frac{\Gamma(l - n/2)}{\Gamma(l)} \, (a + p^2)^{n/2-l} \, p_\mu$$

$$\int \frac{d^n k}{(2\pi)^n} \, \frac{k_\mu \, k_\nu}{(k^2 + 2\,k \cdot p - a)^l} = (-1)^l \, i \, \frac{\pi^{n/2}}{(2\pi)^n} \, \frac{\Gamma(l - n/2 - 1)}{\Gamma(l)}$$

$$\times \Big\{ p_\mu \, p_\nu \, (l - n/2 - 1) - \frac{1}{2} \, g_{\mu\nu} \, (a + p^2) \Big\} \, (a + p^2)^{n/2-l} \tag{4.154}$$

Notice that $g^{\mu\nu}\, g_{\mu\nu} = n$. For the case $a = m^2 - x\,(1-x)\,s < 0$ we have to include the effect of $i\,\epsilon$. The integral I_1 in Eq. (4.150) becomes

$$I_1 = -i\,\frac{2\,\pi^{n/2}}{(2\pi)^n\,\Gamma(n/2)} \int_0^\infty dk_E\,\frac{k_E^{n-1}}{k_E^2 + a - i\,\epsilon} =$$

$$-i\,\frac{2\,\pi^{n/2}}{(2\pi)^n\,\Gamma(n/2)} \left\{ P \int_0^\infty dk_E\,\frac{k_E^{n-1}}{k_E^2 + a} + i\,\pi \int_0^\infty dk_E\,k_E^{n-1}\,\delta(k_E^2 + a) \right\} \quad (4.155)$$

The first integral we already evaluated. The second integral becomes equal to

$$\int_0^\infty dk_E\,k_E^{n-1}\,\delta(k_E^2 + a) = \frac{1}{2}\int_0^\infty dk_E^2\,(k_E^2)^{n/2-1}\,\delta(k_E^2 + a) = \frac{1}{2}\,(-a)^{n/2-1} \quad (4.156)$$

Hence for general a we can write

$$I_1 = -i\,\frac{\pi^{n/2}}{(2\pi)^n} \left\{ \Gamma(1 - n/2)\,|a|^{n/2-1} + i\,\frac{\pi}{\Gamma(n/2)}\;\; (-a)^{n/2-1}\Big|_{a<0} \right\} \quad (4.157)$$

Differentiation w.r.t. a yields

$$I_l = i\,\frac{\pi^{n/2}}{(2\pi)^n}\,(-1)^l \left\{ \Gamma(l - n/2)\,|a|^{n/2-l} + i\,\frac{\pi}{\Gamma(n/2 - l)}\;\; (-a)^{n/2-l}\Big|_{a<0} \right\} \quad (4.158)$$

Let us first calculate $J(s)$ in Eq. (4.147) for $s < 0$ $(a > 0)$

$$J_I(s) = i\,\frac{\lambda_n^2}{2}\,\frac{\pi^{n/2}}{(2\pi)^n}\,\Gamma(2 - n/2) \int_0^1 dx\,(m^2 - x\,(1-x)\,s)^{n/2-2} \quad (4.159)$$

Since we are interested in the limit $n \to 4$ we expand de Gamma function around $n - 4$

$$\begin{aligned}
\Gamma(1+x) &= \Gamma(1) + x\,\Gamma'(1) + \frac{x^2}{2}\,\Gamma''(1) + \cdots = 1 - \gamma_E\,x + \frac{1}{2}\left(\gamma_E^2 + \frac{\pi^2}{6}\right)x^2 + \cdots \\
&= e^{-\gamma_E\,x}\left(1 + \frac{\pi^2}{6}\,x^2 + \cdots\right) \qquad (4.160)
\end{aligned}$$

where $\gamma_E = 0.5772156649\cdots$ is the Euler Mascheroni constant. Define $\omega = n - 4$ then $\lambda_n = \lambda^{-\omega}$ and the integral equals

$$J_I(s) = i\,\frac{\lambda^2}{32\pi^2}\,\mu^{-\omega}\left[-\frac{2}{\omega} - \gamma_E + \ln 4\pi - \int_0^1 dx\,\ln\left(\frac{m^2 - x\,(1-x)\,s}{\mu^2}\right)\right] \quad (4.161)$$

The calculation of the last integral is straightforward and we obtain

$$\begin{aligned}
J_I(s) &= i\,\frac{\lambda^2}{32\pi^2}\,\mu^{-\omega}\left[-\frac{2}{\omega} - \gamma_E + \ln 4\pi - \ln\frac{m^2}{\mu^2} + 2 \right. \\
&\left. + \sqrt{1 - 4m^2/s}\,\ln\left(\frac{\sqrt{1 - 4m^2/s} - 1}{\sqrt{1 - 4m^2/s} + 1}\right)\right] \qquad (4.162)
\end{aligned}$$

For $0 < s < 4m^2$ we have to analytically continuate $J(s)$

$$\sqrt{1-4m^2/s} = i\,\sqrt{4m^2/s-1} \tag{4.163}$$

$$\begin{aligned} J_{II}(s) &= i\,\frac{\lambda^2}{32\pi^2}\,\mu^{-\omega}\left[-\frac{2}{\omega} - \gamma_E + \ln 4\pi - \ln\frac{m^2}{\mu^2} + 2\right. \\ & \left. -2\,\sqrt{4m^2/s-1}\,\arctan\left(\frac{1}{\sqrt{4m^2/s-1}}\right)\right] \quad J_I(o) = J_{II}(0) \end{aligned} \tag{4.164}$$

For $s > 4\,m^2$ (III) we have two options.
a) We continuate $J_{II}(s)$ analytically to $s > 4\,m^2$. This means that we take into account the $i\,\epsilon$ in $J(s)$

$$\begin{aligned} & -2\sqrt{\frac{4\,(m^2-i\,\epsilon)}{s} - 1}\,\arctan\left(\frac{1}{\sqrt{\frac{4\,(m^2-i\,\epsilon)}{s}-1}}\right) = \\ & \sqrt{1-4\,m^2/s}\,\ln\left(\frac{1-\sqrt{1-4\,m^2/s}}{1+\sqrt{1-4\,m^2/s}}\right) - \pi\,i\,\theta(s-4\,m^2)\,\sqrt{1-4\,m^2/s} \end{aligned} \tag{4.165}$$

b) We take into account the second term in formula (4.158)

$$\begin{aligned} & \frac{i\,\mu^{-2\,\omega}\,\lambda^2}{32\pi^2}\,[-i\,\pi\int_0^1 dx\,\theta(x\,(1-x)\,s-m^2)] = \frac{i\,\mu^{-2\,\omega}\,\lambda^2}{32\pi^2}\Big[-i\,\pi\,\theta(s-4\,m^2)\int_{x_1}^{x_2} dx\Big] \\ & = \frac{i\,\mu^{-2\,\omega}\,\lambda^2}{32\pi^2}\Big[-i\,\pi\,\theta(s-4\,m^2)\,\sqrt{1-4\,m^2/s}\Big] \end{aligned} \tag{4.166}$$

In this way the whole expression for $J_{III}(s)$ becomes

$$\begin{aligned} J_{III}(s) = \frac{i\,\mu^{-\omega}\,\lambda^2}{32\pi^2}&\left[-\frac{2}{\omega} - \gamma_E + \ln 4\,pi - \ln\frac{m^2}{\mu^2} + 2 + \sqrt{1-4\,m^2/s}\right. \\ & \left.\ln\left(\frac{1-\sqrt{1-4\,m^2/s}}{1+\sqrt{1-4\,m^2/s}}\right) - i\,\pi\,\theta(s-4\,m^2)\,\sqrt{1-4\,m^2/s}\right] \end{aligned} \tag{4.167}$$

Resuming

1. We have found a way to define a Feynman integral. This procedure is called regularization.

2. It Appears that the Feynman integral is an analytical function which has cuts.

3. In the Feynman integral is a term that diverges for $n \to 4$ ($\omega \to 0$). This term cannot contribute to physical quantities like effective cross sections.

Let us take for example de effective cross section of the process $p_1 + p_2 \to p_3 + p_4$ calculated in higher order in λ. We take definition 1 for the S-matrix because here no tadpole occur. In lowest order in λ we have for M_{fi} the diagram in Fig. 3. This diagram is enhanced in order λ^2 by the the diagrams in Fig. 11. Definition: Mandelstam variables

$$s = (p_1 + p_2)^2 \qquad t = (p_1 - p_3)^3 \qquad u = (p_1 - p_4)^2 \qquad s + t + u = 4\,m^2 \quad (4.168)$$

$$|\vec{p}_1| = |\vec{p}_2| = |\vec{p}_3| = |\vec{p}_4| = \frac{1}{2}\,sqrts - 4\,m^2 \quad (4.169)$$

In the physical region

$$\begin{aligned} & s = 4\,(|\vec{p}_1|^2 + m^2) \geq 4\,m^2 \qquad t = -2\,|\vec{p}_1|^2\,(1 - \cos\theta) \leq 0 \\ & u = -2\,|\vec{p}_1|^2\,(1 + \cos\theta) \leq 0 \end{aligned} \quad (4.170)$$

$$\begin{aligned} M_{fi} &= -i\,\lambda\,\mu^{-\varepsilon}\left[1 + \frac{3\,\lambda}{32\pi^2}\left\{\frac{2}{\omega} + \gamma_E - \ln 4\pi + \ln\frac{m^2}{\mu^2} - \frac{1}{3}\,F_{III}(s) - \frac{1}{3}\,F_I(t)\right.\right. \\ & \left.\left. -\frac{1}{3}\,F_I(u)\right\}\right] \end{aligned} \quad (4.171)$$

Watch that $F_{III}(s)$ has a real as well as an imaginary part because $s \geq 4\,m^2$. However $F_I(t)$ and $F_I(u)$ have no imaginary parts since $t < 0$ and $u < 0$.

$$\begin{aligned} |M_{fi}|^2 &= \lambda^2\,\mu^{-2\,\varepsilon}\left[1 + \frac{3\,\lambda}{16\,\pi^2}\left\{\frac{2}{\omega} + \gamma_E - \ln 4\,\pi + \ln\frac{m^2}{\mu^2} - \frac{1}{3}\,\Re F_{III}(s) - \frac{1}{3}\,F_I(t)\right.\right. \\ & \left.\left. -\frac{1}{3}\,F_I(u)\right\}\right] \end{aligned} \quad (4.172)$$

Therefore the cross section becomes

$$\begin{aligned} d\,\sigma &= \frac{1}{4}\,\frac{1}{\sqrt{(p_1.p_2)^2 - m_1^2\,m_2^2}} \int \frac{d^4p_3}{(2\pi)^3}\,\theta(p_{3,0})\,\delta(p_3^2 - m_3^2) \int \frac{d^3p_4}{(2\pi)^3}\,\theta(p_{4,0})\,\delta(p_4^2 - m_4^2) \\ & (2\pi)^4\,\delta^{(4)}(p_1 + p_2 - p_3 - p_4)\,|M_{fi}|^2 \end{aligned} \quad (4.173)$$

$$\begin{aligned} \frac{d\,\sigma}{d\,\Omega} &= \frac{\lambda^2\,\mu^{-2\,\varepsilon}}{64\pi^2 s}\left[1 + \frac{3\lambda}{16\pi^2}\left\{\frac{2}{\omega} + \gamma_E - \ln 4\pi + \ln\frac{m^2}{\mu^2} - \frac{1}{3}\,\Re F_{III}(s) - \frac{1}{3}\,F_I(t)\right.\right. \\ & \left.\left. -\frac{1}{3}\,F_I(u)\right\}\right] \end{aligned} \quad (4.174)$$

If we assume that this effective cross section describes the experimental data can λ be no physical parameter because $2/\omega \to \infty$ when $n \to 4$. In order to observe a physical coupling constant we have to define a renormalized coupling constant. This is called renormalisation of the coupling constant. This is done as follows Define the renormalized coupling constant $\lambda_R(i)$ by

$$\lambda = \lambda_R(i) \left[1 - \frac{3\lambda_r(i)}{32\pi^2} \left\{ \frac{2}{\omega} + C_i \right\} + O(\lambda_R^2(i)) \right] \tag{4.175}$$

The renormalized cross section becomes now

$$\begin{aligned} \frac{d\,\sigma^R}{d\,\Omega} &= \frac{\lambda_R^2(i)}{64\pi^2 s} \left[1 + \frac{3\lambda_R(i)}{16\pi^2} \left\{ \gamma_E - \ln 4\pi - C_i + \ln \frac{m^2}{\mu^2} - \frac{1}{3} \Big(\Re F_{III}(s) + F_I(t) \right. \right. \\ &\qquad \left. \left. + F_I(u) \Big) \right\} + O(\lambda_R^2(i)) \right] \Bigg\} \end{aligned} \tag{4.176}$$

The arbitrary constant C_i indicates that the perturbation series for $d\,\sigma^R$ is renormalization dependent. This dependence is indicated by the parameter i. However nature is not sensitive to the renormalisation choice. Remind that the coupling constant is not given by the theory. We have to determine it experimentally. The cross section is symmetric in t and u or equivalently $\cos\theta \Leftrightarrow -\cos\theta$. If we fit the cross section to the data then we get a whole list of values for $\lambda_R(i)$ that depends on C_i. We have to watch the following.

a. The C_i cannot be chosen so that $\lambda_R(i)$ becomes too large (condition $\lambda_R(i) \ll 1$) otherwise the perturbation series diverges.

b. It can happen that the form of the curve disagrees with the measured values. In this case one has to take higher powers in $\lambda_R(i)$ into account.

c. It is also possible that the form never agrees with the experimental data. Higher order terms cannot bring agreement. One has to take $\lambda_R(i)$ so large that one can get a conflict with a.. The conclusion is then that the theory is wrong.

d. However even if the cross section is in agreement with the data we are not there. We have to show that for any arbitrary other process the data can be accurately predicted with the same renormalisation scheme i.

If we use the same renormalization scheme as in the previous process then we can use the value for the coupling constant $\lambda_R(i)$ in the new process. In this we can give an absolute prediction for the latter process.

Remark
We can absorb the constant C_i in μ^2 by defining $\mu^2 = \mu_i^2\ \exp(-C_i)$. In the literature one

sees instead of $\lambda_R(i)$ the coupling constant $\lambda_R(\mu_i^2)$ or simply $\lambda_R(\mu^2)$ (μ^2 is a continuous parameter). We can now renormalize in various ways

$$\begin{aligned} C_i &= \ln \frac{\mu_i^2}{\mu^2} \\ \lambda &= \lambda_R(\mu_i^2)\left[1 - \frac{3\lambda_R(\mu_i^2)}{32\pi^2}\left\{\frac{2}{\omega} + \ln \frac{\mu_i^2}{\mu^2}\right\}\right] \\ &= \lambda_R(\mu_j^2)\left[1 - \frac{3\lambda_R(\mu_j^2)}{32\pi^2}\left\{\frac{2}{\omega} + \ln \frac{\mu_j^2}{\mu^2}\right\}\right] \end{aligned} \tag{4.177}$$

so that we can express $\lambda_R(\mu_j^2)$ into $\lambda_R(\mu_i^2)$ by

$$\lambda_R(\mu_j^2) = \frac{\lambda_R(\mu_i^2)}{1 - (3\lambda_R(\mu_i^2)/32\pi^2)\,(\ln \mu_j^2/\mu_i^2)} \tag{4.178}$$

Take μ_i fixed and vary μ_j. Let $\mu_j^2 \to 0$ then $\lambda_R(\mu_j^2) \to 0$. The zero is IR stable point. Besides coupling constant renormalization there exists also mass renormalization. In lowest order this given by the diagrams in the first row of Fig. 15. Finally we want to make another remark. Perturbation theory will not finally determine coupling constant renormalization. Therefore the coupling constant is unknown. More information is obtained by non-perturbative arguments for instance lattice calculations.

$$\lambda = \lambda_R(\mu)\, Z_\lambda \left(\frac{\Lambda}{\mu}\right) \tag{4.179}$$

Λ is a cut off or an inverse lattice distance in lattice theories. Suppose λ is finite then either

$$\lim_{\Lambda \to \infty} Z_\lambda = \text{finite} \quad \text{for} \quad \lambda_R \neq 0 \tag{4.180}$$

or

$$\lim_{\Lambda \to \infty} Z_\lambda = \text{infinite} \quad \text{for} \quad \lambda_R = 0 \tag{4.181}$$

Non-perturbative arguments in ϕ^4-theory lead to the result that $\lambda_R = 0$.

5 Spontaneous symmetry breaking

5.1 Nambu-Goldstone mechanism

<u>**Goldstone theorem**</u>

If the Lagrange density $\mathcal{L}$ is invariant under the global symmetry group G (n-parameter group) whereas the Hilbert space $\mathcal{H}$ is invariant under the subgroup $H \subset G$ (H is a m-parameter group) then there are n-m massless (Goldstone) spin zero bosons in the spectrum of $\mathcal{H}$.

Suppose we have the following Lagrange density which is globally invariant under the group $SU(2)$.

$$\begin{aligned}\mathcal{L} &= i\,\bar{\psi}_i(x)\,\gamma_\mu\,\partial^\mu\,\psi_i(x) - M\,\bar{\psi}_i(x)\,\psi_i(x) + \frac{1}{2}\,\partial\phi_a(x)\,\partial^\mu\phi_a(x) - \frac{1}{2}\,\mu^2\,|\phi_a(x)|^2 \\ &\quad + g\,\bar{\psi}_i(x)\,(\tau_a)_{ij}\,\psi_j(x)\,\phi_a(x) - \frac{\lambda}{4}\,|\phi_a(x)|^2\,|\phi_b(x)|^2 \end{aligned} \tag{5.1}$$

The Dirac fields ψ_i are in the fundamental representation $i = 1, 2$ and the scalar fields ϕ_a are in the adjoint representation $a = \pm, 0$.

$$\vec{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\phi_+ + \phi_-) \\ \frac{i}{\sqrt{2}}(\phi_- - \phi_+) \\ \phi_0 \end{pmatrix} \tag{5.2}$$

$$\vec{\phi}_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi \\ i\,\phi \\ 0 \end{pmatrix} \qquad \vec{\phi}_- = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi \\ -i\,\phi \\ 0 \end{pmatrix} \qquad \vec{\phi}_0 = \begin{pmatrix} 0 \\ 0 \\ \phi \end{pmatrix} \tag{5.3}$$

$\vec{\psi}$ is a doublet and $\vec{\phi}$ is a triplet.

$$\phi_0 = \phi_3 \qquad \phi_\pm = \frac{1}{\sqrt{2}}\,(\phi_1 \pm i\,\phi_2)$$

$$|\phi_a|^2 = \phi_1^2 + \phi_2^2 + \phi_3^2 = 2\,\phi_+\,\phi_- + \phi_0^2 \tag{5.4}$$

Under the group the fields transform as

$$\psi'_k(x) = \Big[\exp(-i\,\alpha_a\,\tau_a)\Big]_{kl}\,\psi_l(x) \qquad \phi'_k(x) = \Big[\exp(-i\,\alpha_a\,T_a)\Big]_{kl}\,\phi_l(x) \tag{5.5}$$

with the matrices

$$\tau_+ = \frac{1}{2}\,(\tau_1 + i\,\tau_2) \qquad \tau_- = \frac{1}{2}\,(\tau_1 - i\,\tau_2) \qquad \tau_0 = \tau_3$$

$$T_+ = \frac{1}{\sqrt{2}}\,(T_1 + i\,T_2) \qquad T_- = \frac{1}{\sqrt{2}}\,(T_1 - i\,T_2) \qquad T_0 = T_3 \tag{5.6}$$

$$\tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \tau_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.7}$$

$$T_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -i \\ 1 & i & 0 \end{pmatrix} \quad T_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{pmatrix} \quad T_0 = i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.8}$$

Further we have

$$\begin{aligned} &T_+\,\vec{\phi}_+ = 0 \qquad T_+\,\vec{\phi}_0 = -\vec{\phi}_+ \qquad T_+\,\vec{\phi}_- = \vec{\phi}_0 \\ &T_0\,\vec{\phi}_+ = \vec{\phi}_+ \qquad T_0\,\vec{\phi}_0 = 0 \qquad T_0\,\vec{\phi}_- = -\vec{\phi}_- \\ &T_-\,\vec{\phi}_+ = -\vec{\phi}_0 \qquad T_-\,\vec{\phi}_0 = \vec{\phi}_- \qquad T_-\,\vec{\phi}_- = 0 \end{aligned} \tag{5.9}$$

and

$$\tau_a\,\phi_a = \tau_1\,\phi_1 + \tau_2\,\phi_2 + \tau_3\,\phi_3 = \tau_0\,\phi_0 + \tau_+\,\phi_- + \tau_-\,\phi_+ \tag{5.10}$$

The Noether current equals ($i,j = 1,2,\ a,b,c = 1,2,3$)

$$j_a^\mu(x) = \bar{\psi}(x)_i\,\gamma^\mu \left(\tau_a\right)_{ij} \psi_j(x) + i\,\phi_b(x)^\dagger\,\overleftrightarrow{\partial^\mu} \left(T_a\right)_{bc} \phi_c(x) \tag{5.11}$$

The Lagrangian above has the global symmetry group $SU(2)$. Now we have to construct a Hilbert space $\mathcal{H}$ such that the fields are invariant under $U(1)$. Let us consider the potential

$$V(|\vec{\phi}|) = \frac{1}{2}\,\mu^2\,|\vec{\phi}|^2 + \frac{\lambda}{4}\,|\vec{\phi}|^4 \tag{5.12}$$

with $\lambda > 0$ ($V(|\phi|)$ has be bounded from below). There are now two possibilities

A $\mu^2 > 0$: All scalar fields are massive. Moreover $\mathcal{H}$ is symmetric under $G = SU(2)$. Further $\langle 0|\vec{\Phi}|0\rangle = 0$

B $\mu^2 < 0$: One field is massive and two fields are massless. Moreover $\mathcal{H}$ is symmetric under $G = U(1)$. Further $\langle 0|\vec{\Phi}|0\rangle = \sqrt{-\mu^2/\lambda}$.

The minimum of the potential in case B is given by $\partial V(|\vec{\phi}|)/\partial|\vec{\phi}| = 0$. From which follows $|\vec{\phi}| = \sqrt{-\mu^2/\lambda}$. Notice that all solutions $\vec{\phi}(x)$ for which $|\vec{\phi}| = \sqrt{-\mu^2/\lambda} = v$ lead to a minimum of $V(|\vec{\phi}|)$. From case B we can derive the following properties of the vacuum $|0\rangle$

a $|\langle 0|\vec{\Phi}|0\rangle| = \sqrt{-\mu^2/\lambda}$

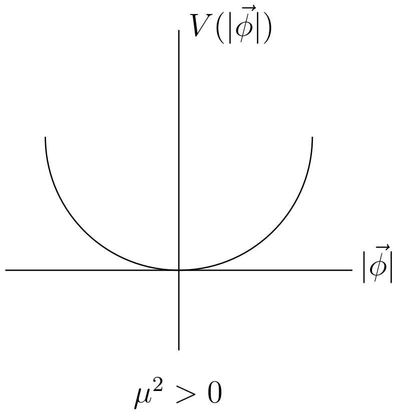


Figure 17: Case $\mu^2 > 0$.

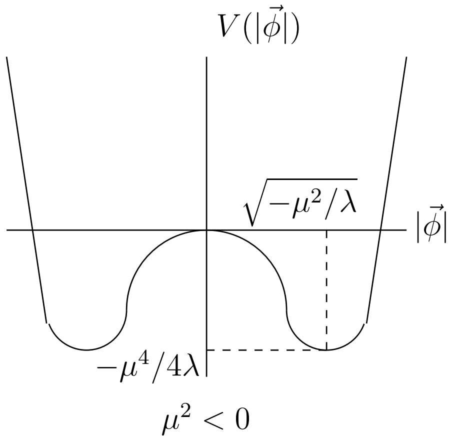


Figure 18: Case $\mu^2 < 0$.

b $|0\rangle$ is not anymore invariant under G but under the subgroup H. The latter is called the stabiliser of $|0\rangle$.

$$\hat{U}(g)|0\rangle \neq |0\rangle \qquad g \in G \quad g \notin H \quad \text{but} \quad \hat{U}(h)|0\rangle = |0\rangle \quad h \in H \tag{5.13}$$

c From b it follows that there an infinite number of vacuums $|0\rangle^i$ with $\hat{U}(g)|0\rangle^i = |0\rangle^j$ $(i \neq j)$ but with $\hat{U}(h)|0\rangle^i = |0\rangle^i$

d ${}^i\langle 0|0\rangle^i = \infty$

Proof:

a:
This is evident. The vacuum is always given by the expectation value of Φ for which $V(|\langle 0|\vec{\Phi}|0\rangle|)$ is a minimum.
b:
Define $\langle 0|\vec{\Phi}(x)|0\rangle = \vec{v}$ with $|\vec{v}| = \sqrt{-\mu^2/\lambda}$. The vector $\vec{v}$ is invariant under H but not invariant under $g \in G$ with $g \notin H$. Hence it follows that the ground state $|0\rangle$ is not invariant under $\hat{U}(g)$ with $g \notin H$ but invariant under $\hat{U}(h)$ with $h \in H$. This means $\hat{U}(g)|0\rangle \neq |0\rangle$ with $g \notin H$ but it is invariant under $\hat{U}(h)|0\rangle = |0\rangle$ with $h \in H$. This can be shown in the following way. The Wigner theorem states

$$\hat{U}(g)^\dagger\, \vec{\Phi}(x)\, \hat{U}(g) = e^{-i\,\alpha_a\, T_a}\, \vec{\Phi}(x) \tag{5.14}$$

with $\hat{U}(g)$ is unitary. Further is $\vec{\Phi}$ no singlet under the transformation $e^{-i\,\alpha_a\, T_a}$ in other words

$$\hat{U}\left(e^{-i\,\alpha_a\, T_a}\right)^\dagger\, \vec{\Phi}(x)\, \hat{U}\left(e^{-i\,\alpha_a\, T_a}\right) = e^{-i\,\alpha_a\, T_a}\, \vec{\Phi}(x) \neq \vec{\Phi}(x) \tag{5.15}$$

From Eq. (5.14) it follows that

$$\langle \hat{U}(g)\, 0|\vec{\Phi}(x)|\hat{U}(g)\, 0\rangle = e^{-i\,\alpha_a\, T_a}\, \langle 0|\vec{\Phi}(x)|0\rangle \tag{5.16}$$

Suppose $\hat{U}(g)|0\rangle = |0\rangle$. In this case we have

$$\langle 0|\vec{\Phi}(x)|0\rangle = e^{-i\,\alpha_a\, T_a}\, \langle 0|\vec{\Phi}(x)|0\rangle \qquad e^{-i\,\alpha_a\, T_a} = g \in G \quad \text{and} \quad g \notin H$$

$$\Longrightarrow \quad \langle 0|\vec{\Phi}(x)|0\rangle = 0 \tag{5.17}$$

because $\langle 0|\vec{\Phi}(x)|0\rangle$ is no singlet under the group. But that contradicts $\langle 0|\vec{\Phi}(x)|0\rangle = \vec{v} \neq 0$. Therefore $\hat{U}(g)|0\rangle = |0\rangle$ cannot be true. Conclusion

$$\hat{U}(g)|0\rangle = |0\rangle' \neq |0\rangle \tag{5.18}$$

However

$$\langle 0|\vec{\Phi}(x)|0\rangle = e^{-i\,\alpha_a\, T_a}\, \langle 0|\vec{\Phi}(x)|0\rangle \quad \text{if} \quad e^{-i\,\alpha_a\, T_a} = h \in H \quad \Longrightarrow \quad \hat{U}(h)|0\rangle = |0\rangle \tag{5.19}$$

c:
Because there exist an infinite number of transformations $g = \exp(-i\,\alpha_a\,T_a)$ there must also exist an infinite number of vacuums $|0\rangle^i$ which are transformed in each other. In other words i in

$$^i\langle 0|\vec{\Phi}|0\rangle^i = \vec{v} \tag{5.20}$$

is infinite
d:

$$\hat{U}(g) = \hat{U}\left(e^{-i\,\alpha_a\,T_a}\right) = e^{-i\,\alpha_a\,\hat{D}(T_a)} \qquad \hat{D}(T_a)^\dagger = \hat{D}(T_a) \quad \text{because} \quad \hat{U}^{-1} = \hat{U}^\dagger \tag{5.21}$$

$\hat{D}(T_a)$ is hermitian and a representation of the generator T_a in the Hilbert space. Further is

$$\hat{D}(T_a) = \int d^3\vec{x}\,\hat{J}^0_a(x) \tag{5.22}$$

$\hat{J}^\mu_a$ is a quantum Noether current in Hilbert space. We show that $^i\langle 0|0\rangle^i = \infty$. First $\hat{D}(T_a)|0\rangle^i = |0\rangle^j$ for arbitrary a (notice if $T_a \in \mathcal{L}(H)$ then $\hat{D}(T_a)|0\rangle^i = 0$).

$$^j\langle 0|0\rangle^j =^i \langle 0|\hat{D}(T_a)\,\hat{D}(T_a)|0\rangle^i = \int d^3\vec{x}\,^i\langle 0|\hat{J}^0_a(x)\,\hat{D}(T_a)|0\rangle^i \tag{5.23}$$

translational invariance

$$\hat{J}^0_a(x) = e^{-i\,\hat{P}\,x}\,\hat{J}^0_a(0)\,e^{i\,\hat{P}\,x} \tag{5.24}$$

Further

$$\left[\hat{P}, \hat{D}(T_a)\right] = 0 \quad \text{because} \quad \left[\hat{P}, \hat{U}(g)\right] = 0 \tag{5.25}$$

$$^j\langle 0|0\rangle^j = \int d^3\vec{x}\,^i\langle 0|\hat{J}^0_a(0)\,\hat{D}(T_a)|0\rangle^i = \infty \tag{5.26}$$

Conclusion: the space characterised by $^i\langle 0|\vec{\Phi}|0\rangle^i \neq 0$ is not a good Hilbert space (For an unbroken symmetry is $\hat{D}(T_a)|0\rangle = 0$ therefore the above argument does not hold). We construct a good Hilbert space which is invariant under H. As vacuum we choose one of the old vacuums $|0\rangle^i$. Which i we take does not matter.
Further we define a new scalar quantum field

$$\tilde{\vec{\Phi}}(x) = \vec{\Phi}(x) - \vec{v}\,1 \tag{5.27}$$

such that

$$^i\langle 0|\tilde{\vec{\Phi}}|0\rangle^i = 0 \qquad \hat{U}(h)|0\rangle^i = |0\rangle^i \tag{5.28}$$

The classical field $\vec{\tilde{\phi}}(x)$ which corresponds to $\vec{\tilde{\Phi}}(x)$ is equal to $\vec{\tilde{\phi}}(x) = \vec{\phi}(x) - \vec{v}$. Expressed in the new fields $\vec{\tilde{\phi}}(x)$ the Lagrange density becomes

$$\begin{aligned}
\mathcal{L} &= i\,\bar{\psi}_k\,\gamma^\mu\,\partial_\mu\,\psi_k - M\,\bar{\psi}_k\,\psi_k + \frac{1}{2}\,\partial_\mu\vec{\tilde{\phi}}\cdot\partial^\mu\vec{\tilde{\phi}} + g\,\bar{\psi}_i\,(\tau_a)_{ij}\,\psi_j\,v_a + g\,\bar{\psi}_i(\tau_a)_{ij}\psi_j\,\tilde{\phi}_a \\
&\quad -\frac{1}{2}\,\mu^2\,|\vec{\tilde{\phi}}|^2 - \mu^2\,\vec{\tilde{\phi}}\cdot\vec{v} - \frac{1}{2}\,\mu^2\,|\vec{v}|^2 - \frac{\lambda}{4}\left\{|\vec{\tilde{\phi}}|^2 + 2\,\vec{v}\cdot\vec{\tilde{\phi}} + |\vec{v}|^2\right\}^2 |_{\mu^2=-\lambda\,|\vec{v}^2|} \\
&= i\,\bar{\psi}_k\,\gamma^\mu\,\partial_\mu\,\psi_k - \bar{\psi}_i\left(M\,\delta_{ij} - g\,(\vec{\tau}\cdot\vec{v})_{ij}\right)\psi_j + \frac{1}{2}\,\partial_\mu\vec{\tilde{\phi}}\cdot\partial^\mu\vec{\tilde{\phi}} - \lambda\left(\vec{v}\cdot\vec{\tilde{\phi}}\right)^2 \\
&\quad -\frac{\lambda}{4}\,|\vec{\tilde{\phi}}|^4 - \lambda\,\vec{v}\cdot\vec{\tilde{\phi}}\,|\vec{\tilde{\phi}}|^2 + \frac{\lambda}{4}\,|\vec{v}|^4
\end{aligned} \tag{5.29}$$

The minimum of the potential $V(|\vec{\tilde{\phi}}|)$ is now $-\lambda/4\,|\vec{v}|^4$ for $|\vec{\tilde{\phi}}| = 0$. Hence it follows that $\langle 0|\vec{\tilde{\Phi}}|0\rangle = 0$ (there is one vacuum $|0\rangle^i = |0\rangle$ for a specific i). In our example was the field $\vec{\tilde{\phi}}_0$ invariant under $H = U(1)$ with $T_0 \in \mathcal{L}(U(1))$.

$$\vec{v} = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \qquad \vec{\tilde{\phi}}_0 = \begin{pmatrix} 0 \\ 0 \\ \tilde{\phi} \end{pmatrix} \qquad \vec{\tilde{\phi}} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\tilde{\phi}_+ + \tilde{\phi}_-) \\ \frac{i}{\sqrt{2}}(\tilde{\phi}_- - \tilde{\phi}_+) \\ \tilde{\phi}_0 \end{pmatrix} \tag{5.30}$$

From which follows $T_0\,\vec{v} = 0$ and $\exp(-i\,\alpha_0\,T_0)\,\vec{v} = \vec{v}$. Further $\exp(-i\,\alpha_0\,T_0)\,\vec{\tilde{\phi}}_0 = \vec{\tilde{\phi}}_0$. The consequence is that $\vec{v}\cdot\vec{\tilde{\phi}} = v\,\tilde{\phi}_0$ and $\tilde{\phi}_0$ has a mass $2\,\lambda\,v^2 = -2\,\mu^2 > 0$. $\tilde{\phi}_+$ and $\tilde{\phi}_-$ are the massless fields and they represent the Goldstone bosons. For the fermions we have the mass matrix

$$M = \begin{pmatrix} M - g\,v & 0 \\ 0 & M + g\,v \end{pmatrix} \qquad \vec{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{5.31}$$

ψ_1 and ψ_2 have mass $M - g\,v$ and $M + g\,v$ respectively. Therefore the degeneracy is lifted.

Conclusions:

a For case II the Hilbert space $\mathcal{H}$ is not invariant under the group G (here $SU(2)$) but under its subgroup H (here $U(1)$). If dim G= n and dim H= m then the number of Goldstone bosons is equal to dim G/H= $n - m$.

b The Lagrange density $\mathcal{L}$ expressed in $\vec{\tilde{\phi}}$ is still invariant under G!! The replacement of $\vec{\phi}$ by $\vec{\tilde{\phi}} + \vec{v}$ does not break the symmetry. We have still conserved Noether currents j_a^μ for all a.

c The fields which are left invariant by the subgroup H are those fields which describe massive particles. The fields which belong to the co-set G/H are massless (Goldstone bosons). Notice that G/H is not a group unless H is a normal subgroup and the Goldstone bosons are therefore not representations of G/H.

The Goldstone theorem is only correct for global symmetries. In the case of local symmetries the Goldstone bosons vanish and they appear as the longitudinal component of the massive gauge boson. The massive scalar field describes then the Higgs boson.

5.2 Higgs mechanism

If the symmetry group G is local then the Noether currents disappear and the arguments which leads to the Goldstone mechanism does hold anymore. That this happens is shown by P. Higgs. We illustrate this via a simple example. Choose the local gauge group $U(1)$. we take the Lagrangian density which is invariant under $U(1)$

$$\mathcal{L} = -\frac{1}{4}\, F_{\mu\nu}(x)\, F^{\mu\nu}(x) + \Big(D_\mu\, \phi(x)\Big)^*\, D^\mu\, \phi(x) - \mu^2\, \phi^*(x)\, \phi(x) - \lambda\, \Big(\phi^*(x)\, \phi(x)\Big)^2 \tag{5.32}$$

with

$$D_\mu\, \phi(x) = \Big(\partial_\mu + i\, e\, A_\mu(x)\Big)\, \phi(x) \qquad (D_\mu\, \phi(x))^* = \Big(\partial_\mu - i\, e\, A_\mu(x)\Big)\, \phi^*(x) \tag{5.33}$$

Further the Lagrangian density is invariant under the transformations

$$\phi'(x) = e^{i\, e\, \alpha(x)}\, \phi(x) \qquad \phi'^*(x) = e^{-i\, e\, \alpha(x)}\, \phi(x)^* \tag{5.34}$$

Let us express the complex fields into real fields

$$\phi(x) = \frac{1}{\sqrt{2}}\, (\phi_1(x) + i\, \phi_2(x)) \qquad \phi(x)^* = \frac{1}{\sqrt{2}}\, (\phi_1(x) - i\, \phi_2(x)) \tag{5.35}$$

Under gauge transformations the real fields transform like

$$\begin{aligned} \phi_1'(x) &= \cos(e\, \alpha(x))\, \phi_1(x) - \sin(e\, \alpha(x))\, \phi_2(x) \\ \phi_2'(x) &= \sin(e\, \alpha(x))\, \phi_1(x) + \cos(e\, \alpha(x))\, \phi_2(x) \end{aligned} \tag{5.36}$$

For $\lambda > 0$ and $\mu^2 < 0$ we get again spontaneous symmetry breaking. The minimum of the potential is at

$$|\phi(x)| = \sqrt{\frac{-\mu^2}{2\lambda}} = \frac{v}{\sqrt{2}} \qquad |\phi(x)| = \frac{1}{\sqrt{2}}\, \sqrt{\phi_1(x)^2 + \phi_2(x)^2} \tag{5.37}$$

There are again an infinite number of degenerated vacuum which are characterised by

$$\langle 0|\Phi(x)|0\rangle = \frac{v}{\sqrt{2}}\, e^{i\,\delta} \tag{5.38}$$

for arbitrary δ. Let us choose a specific vacuum en build around that vacuum a new quantum theory. Since the symmetry group is $U(1)$ is totally broken it does not matter which vacuum is chosen. There remains one Goldstone boson. Choose $\delta = 0$ then

$$\langle 0|\Phi_1(x)|0\rangle = v \quad \text{and} \quad \langle 0|\Phi_2(x)|0\rangle = 0$$

$$\tilde{\phi}_1(x) = \phi_1(x) - v \qquad \tilde{\phi}_2(x) = \phi_2(x) \tag{5.39}$$

ϕ_2 becomes the Goldstone boson field. the Lagrange density expressed in the new fields reads

$$\begin{aligned}
\mathcal{L} \;=\; & -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} + \frac{1}{2}\,\partial_\mu\,\phi_1\,\partial^\mu\,\phi_1 + \frac{1}{2}\,\partial_\mu\,\phi_2\,\partial^\mu\,\phi_2 + \frac{1}{2}\,e^2\,A_\mu\,A^\mu\,(\phi_1^2+\phi_2^2) \\
& + e\,A^\mu\,(\phi_1\,\partial_\mu\,\phi_2 - \phi_2\,\partial_\mu\,\phi_1) - \frac{\mu^2}{2}\,(\phi_1^2+\phi_2^2) - \frac{\lambda}{4}\,(\phi_1^2+\phi_2^2)^2 \\
& = -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} + \frac{1}{2}\,\partial_\mu\,\tilde{\phi}_1\,\partial^\mu\,\tilde{\phi}_1 + \frac{1}{2}\,\partial_\mu\,\tilde{\phi}_2\,\partial^\mu\,\tilde{\phi}_2 + \frac{1}{2}\,e^2\,v^2\,A_\mu\,A^\mu \\
& + \frac{1}{2}\,e^2\,A_\mu\,A^\mu\,(\tilde{\phi}_1^2+\tilde{\phi}_2^2) + e^2\,v\,A_\mu\,A^\mu\,\tilde{\phi}_1 + e\,v\,A^\mu\,\partial_\mu\,\tilde{\phi}_2 + e\,A^\mu\,(\tilde{\phi}_1\,\partial_\mu\,\tilde{\phi}_2 \\
& - \tilde{\phi}_2\,\partial_\mu\,\tilde{\phi}_1) - \frac{\lambda}{4}\,(\tilde{\phi}_1^2+\tilde{\phi}_2^2)^2 - \lambda\,v\,\tilde{\phi}_1\,(\tilde{\phi}_1^2+\tilde{\phi}_2^2) - \frac{\mu^2}{2}\,(\tilde{\phi}_1^2+\tilde{\phi}_2^2) - \mu^2\,v\,\tilde{\phi}_1 \\
& - \frac{1}{2}\,\mu^2\,v^2 - \lambda\,v^2\,\tilde{\phi}_1^2 - \frac{\lambda}{4}\,v^4 - \frac{\lambda}{2}\,v^2\,(\tilde{\phi}_1^2+\tilde{\phi}_2^2) - \lambda\,v^3\,\tilde{\phi}_1
\end{aligned} \tag{5.40}$$

After substitution of $\mu^2 = -\lambda\,v^2$ all terms linear in $\tilde{\phi}_1$ disappear and we have

$$\begin{aligned}
\mathcal{L} \;=\; & -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} + \frac{1}{2}\,\partial_\mu\,\tilde{\phi}_1\,\partial^\mu\,\tilde{\phi}_1 + \frac{1}{2}\,e^2\,v^2\,(A_\mu + \frac{1}{ev}\,\partial_\mu\,\tilde{\phi}_2)\,(A^\mu + \frac{1}{ev}\,\partial^\mu\,\tilde{\phi}_2) \\
& + \frac{1}{2}\,e^2\,A_\mu\,A^\mu\,(\tilde{\phi}_1^2+\tilde{\phi}_2^2) + e^2\,v\,A_\mu\,A^\mu\,\tilde{\phi}_1 + e\,A^\mu\,(\tilde{\phi}_1\,\partial_\mu\,\tilde{\phi}_2 - \tilde{\phi}_2\,\partial_\mu\,\tilde{\phi}_1) + \mu^2\,\tilde{\phi}_1^2 \\
& + \frac{\lambda}{4}\,v^4 - \frac{\lambda}{4}\,(\tilde{\phi}_1^2+\tilde{\phi}_2^2)^2 - \lambda\,v\,\tilde{\phi}_1\,(\tilde{\phi}_1^2+\tilde{\phi}_2^2)
\end{aligned} \tag{5.41}$$

For $A_\mu = 0$ we have a field $\tilde{\phi}_1$ with mass $-2\,\mu^2$ ($\mu^2 < 0$) and a massless Goldstone field $\tilde{\phi}_2$.
For $A_\mu \neq 0$ we have two possibilities

1 A massless gauge field A_μ, a massless Goldstone field $\tilde{\phi}_2$ and massive field $\tilde{\phi}_1$. This is called the Goldstone mode.

2 A massive gauge field $B_\mu = A_\mu + 1/ev\,\partial_\mu\,\tilde{\phi}_2$ with mass $e\,v$ and a massive scalar field $\tilde{\phi}_1$ with mass $-2\,\mu^2$. This is called the Higgs mode. The massive scalar field

is called the Higgs field. The vanished massless scalar field $\tilde{\phi}_2$ is called the would be Goldstone field.

The most people adopt the second point of view. In the Higgs mode is there a cancellation between the scalar polarisation of the gauge field and the would be Goldstone field. In this case only the three polarisations (two transversal and one longitudinal) of the massive gauge field contribute. We shall now rewrite the Lagrangian density. Define the massive vector field

$$B_\mu = A_\mu + \frac{1}{e\,v}\,\partial_\mu\,\tilde{\phi}_2 \qquad \partial_\mu\,B_\nu - \partial_\nu\,B_\mu = \partial_\mu\,A_\nu - \partial_\nu\,A_\mu = F_{\mu\nu} \tag{5.42}$$

The Lagrangian density becomes

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} + \frac{1}{2}\,\partial_\mu\,\tilde{\phi}_1\,\partial^\mu\,\tilde{\phi}_1 + \frac{1}{2}\,e^2\,v^2\,B_\mu\,B^\mu + \frac{1}{2}\,e^2\,B_\mu\,B^\mu\,(\tilde{\phi}_1^2 + \tilde{\phi}_2^2) \\ & -\frac{e}{v}\,B_\mu\,\partial^\mu\,\tilde{\phi}_2\,(\tilde{\phi}_1^2 + \tilde{\phi}_2^2) + e^2\,v\,B_\mu\,B^\mu\,\tilde{\phi}_1 - 2\,e\,B_\mu\,\partial^\mu\,\tilde{\phi}_2\,\tilde{\phi}_1 + e\,B^\mu\,(\tilde{\phi}_1\,\partial_\mu\,\tilde{\phi}_2 \\ & -\tilde{\phi}_2\,\partial_\mu\,\tilde{\phi}_1) + \frac{1}{2\,v^2}\,\partial_\mu\,\tilde{\phi}_2\,\partial^\mu\,\tilde{\phi}_2\,(\tilde{\phi}_1^2 + \tilde{\phi}_2^2) + \mu^2\,\tilde{\phi}_1^2 + \frac{\lambda}{4}\,v^4 - \frac{\lambda}{4}\,(\tilde{\phi}_1^2 + \tilde{\phi}_2^2)^2 \\ & -\lambda\,v\,\tilde{\phi}_1\,(\tilde{\phi}_1^2 + \tilde{\phi}_2^2) + \frac{1}{v}\,\tilde{\phi}_2\,\partial_\mu\,\tilde{\phi}_2\,\partial^\mu\,\tilde{\phi}_1 \end{aligned} \tag{5.43}$$

with $m_B = e\,v$ th mass of the vector boson and $m_H^2 = -2\,\mu^2$ the mass of the Higgs boson. In this case the field $\tilde{\phi}_2$ has become unphysical. Notice that the symmetry is still present in $\mathcal{L}$ it is only hidden. Only the Hilbert space is not symmetric anymore. It is now convenient to rewrite the Lagrange density such that only physical fields occur. There is one to one correspondence between the classical and the quantum fields. Because we can choose a gauge such that that $\tilde{\phi}_2$ in the Lagrange density disappears. This is achieved in the following way. Define polar fields

$$\rho(x) = \sqrt{\phi_1^2(x) + \phi_2^2(x)} \qquad \theta(x) = \arctan\frac{\phi_2(x)}{\phi_1(x)} \tag{5.44}$$

$$\phi(x) = \frac{1}{\sqrt{2}}\,\rho(x)\,e^{i\,\theta(x)} \qquad \phi^*(x) = \frac{1}{\sqrt{2}}\,\rho(x)\,e^{-i\,\theta(x)} \tag{5.45}$$

or

$$\phi_1(x) = \rho(x)\,\cos\theta(x) \qquad \phi_2(x) = \rho(x)\,\sin\theta(x) \tag{5.46}$$

The choice

$$\langle 0|\Phi_1(x)|0\rangle = v \quad \text{and} \quad \langle 0|\Phi_2(x)|0\rangle = 0 \tag{5.47}$$

corresponds with

$$\langle 0|\mathrm{P}(x)|0\rangle = v \quad \text{and}\langle 0|\Theta(x)|0\rangle = 0 \tag{5.48}$$

Therefore $\mathrm{P}(x)$ describes the Higgs field and $\Theta(x)$ the would be Goldstone field. Write now

$$\phi(x) = \frac{1}{\sqrt{2}}\left(\tilde{\rho}(x)+v\right)e^{i\,\theta(x)} \qquad \phi^*(x) = \frac{1}{\sqrt{2}}\left(\tilde{\rho}(x)+v\right)e^{-i\,\theta(x)} \tag{5.49}$$

and the Lagrange density expressed into the new field becomes

$$\begin{aligned}\mathcal{L} \;&=\; -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} + \frac{1}{2}\,\partial_\mu\,\tilde{\rho}\,\partial_\mu\,\tilde{\rho} + \frac{e^2}{2}\left(\tilde{\rho}+v\right)^2\left(A_\mu + \frac{1}{e}\,\partial_\mu\,\theta\right)\left(A^\mu + \frac{1}{e}\,\partial^\mu\,\theta\right) + \mu^2\,\tilde{\rho}^2 \\ &\quad -\frac{\lambda}{4}\,\tilde{\rho}^4 - \lambda\,v\,\tilde{\rho}^3 + \frac{1}{4}\,\lambda\,v^4\end{aligned} \tag{5.50}$$

Define again

$$B_\mu(x) = A_\mu(x) + \frac{1}{e}\,\partial_\mu\,\theta(x) \tag{5.51}$$

In this way we remove $\theta(x)$ from the Lagrangian density. This can be also achieved by the following gauge transformation

$$\begin{aligned}&\phi'(x) = e^{-i\,\theta(x)}\,\phi(x) \quad \rightarrow \quad \rho'(x) = e^{-i\,\theta(x)}\left(\rho(x)\,e^{i\,\theta(x)}\right) = e^{-i\,\theta(x)}\left(\tilde{\rho}(x)+v\right)e^{i\,\theta(x)} \\ &= \tilde{\rho}(x) + v \qquad \rho(x) = \tilde{\rho}(x) + v\end{aligned} \tag{5.52}$$

$$A'_\mu(x) = \tilde{A}_\mu(x) = A_\mu(x) + \frac{1}{e}\,\partial_\mu\,\theta(x) \tag{5.53}$$

Notice that this gauge transformation is not arbitrary because $\alpha(x) = \theta(x)$. Hence it is a gauge choice. It makes $\tilde{A}_\mu(x)$ massive. Finally the Lagrange density becomes in this gauge

$$\begin{aligned}\mathcal{L} \;&=\; -\frac{1}{4}\,\tilde{F}_{\mu\nu}\,\tilde{F}^{\mu\nu} + \frac{1}{2}\,m_A^2\,\tilde{A}_\mu\,\tilde{A}^\mu + \frac{1}{2}\,\partial_\mu\,\tilde{\rho}\,\partial_\mu\,\tilde{\rho} + \frac{e^2}{2}\,\tilde{\rho}^2\,\tilde{A}_\mu\,\tilde{A}^\mu + v\,e^2\,\tilde{\rho}\,\tilde{A}_\mu\,\tilde{A}^\mu - \frac{1}{2}\,m_H^2\,\tilde{\rho}^2 \\ &\quad -\frac{\lambda}{4}\,\tilde{\rho}^4 - \lambda\,v\,\tilde{\rho}^3 + \frac{1}{4}\,\lambda\,v^4\end{aligned} \tag{5.54}$$

The masses of the vector boson and the Higgs boson are

$$m_A = e\,v \qquad m_H^2 = -2\,\mu^2 = 2\,\lambda\,v^2 \tag{5.55}$$

respectively.

5.3 Landau-Ginzburg theory

We will derive the Landau-Gimzburg Hamiltonian using the partition function for an ferromagnet. In order to define the model we consider a lattice, in any number of dimensions n, and on each site we place a two-valued variable $s_i = \pm 1$. The lattice spacing is a. To a given configuration of the spins, of which there are 2^N (where N is the number of sites) we ascribe the energy

$$E\{s_i\} = -\sum_{i,j} \frac{1}{2} V_{ij}\, s_i\, s_j - \sum_i h_i\, s_i \tag{5.56}$$

For a ferromagnet $V_{ij} > 0$ over the range a. Therefore we get lower energies to configurations with parallel spins (equal sign). The variable h_i is the value of an external magnetic field at the site i. Hence the partition function is given by

$$Z[H_i] = \sum_{s_i} \exp\Big\{\frac{\beta}{2} \sum_{i,j} V_{ij}\, s_i\, s_j + \sum_i H_i\, s_i\Big\} \tag{5.57}$$

where $\beta = 1/kT$ and $H_i = \beta\, h_i$. In this discrete model $Z[H_i]$ generates all the correlation functions. The average magnetisation at site i is

$$M_i = \langle s_i \rangle_{H_i=0} = Z^{-1} \frac{\partial Z}{\partial H_i} |_{H_i=0} \tag{5.58}$$

Usually $M_i = 0$ at $H_i = 0$ because of the symmetry in the energy $E\{s_i\}$ under $s_i \to -s_i$. Unless one encounters spontaneous symmetry breaking. The susceptibility is given by

$$\chi = \frac{\partial M_i}{\partial H_i} |_{H_i=0} = \sum_{i,j} \Big(\langle s_i\, s_j \rangle - \langle s_i \rangle\, \langle s_j \rangle\Big) = \sum_{i,j} \langle \Big(s_i - \langle s_i \rangle\Big) \Big(s_j - \langle s_j \rangle\Big)\rangle \tag{5.59}$$

The mean energy is equal to

$$\langle E_i \rangle = \langle -\frac{1}{2} V_{ij}\, s_i\, s_j \rangle = -\frac{\partial \ln Z[0]}{\partial \beta} \tag{5.60}$$

and the specific heat

$$C_v = \frac{\partial \langle E_i \rangle}{\partial T} = -k\,\beta^2 \frac{\partial \langle E_i \rangle}{\partial \beta} = k\,\beta^2 \frac{\partial^2 \ln Z[0]}{\partial \beta^2} \tag{5.61}$$

Insertion of the partition function (see Eq. (5.56)) provides us with the result

$$\begin{aligned} C_v &= k\,\beta^2 \left[\langle \Big(\frac{1}{2} V_{ij}\, s_i\, s_j\Big)\Big(\frac{1}{2} V_{kl}\, s_k\, s_l\Big)\rangle - \langle \frac{1}{2} V_{ij}\, s_i\, s_j \rangle^2 \right] \\ &= k\,\beta^2 \left[\sum_{ik} \langle \Big(E_i - \langle E_i \rangle\Big) \Big(E_k - \langle E_k \rangle\Big)\rangle \right] \end{aligned} \tag{5.62}$$

Put $H_i = 0$ in Eq. (5.56) and use the following identity

$$\prod_{i=1}^{N} \int_{-\infty}^{\infty} d\phi_i \exp\{-\frac{1}{2} \sum_{i,j} \phi_i \, (\beta \, V_{ij})^{-1} \, \phi_j + \sum_i \phi_i \, s_i\}$$

$$= (2\pi)^{N/2} \, (\det \beta V)^{1/2} \, \exp\{\frac{\beta}{2} \sum_{i,j} V_{ij} \, s_i \, s_j\} \tag{5.63}$$

Hence

$$Z[0] = \sum_{s_i} \exp\{\frac{\beta}{2} \sum_{i,j} V_{ij} \, s_i \, s_j\} = (2\pi)^{-N/2} \, (\det \beta V)^{-1/2} \sum_{s_i} \prod_{i=1}^{N} \int_{-\infty}^{\infty} d\phi_i$$

$$\exp\{-\frac{1}{2} \sum_{i,j} \phi_i \, (\beta \, V_{ij})^{-1} \, \phi_j + \sum_i \phi_i \, s_i\} \tag{5.64}$$

Further we have

$$\sum_{s_i} \exp\{\sum_i \phi_i \, s_i\} = \prod_i \Big(\exp\{\phi_i\} + \exp\{-\phi_i\} \Big) = \prod_i 2 \, \cosh \phi_i$$

$$= 2^N \, \exp\{\sum_i \ln(\cosh \phi_i)\} \tag{5.65}$$

Hence

$$Z[0] = 2^N \, (2\pi)^{-N/2} \, (\det \beta V)^{-1/2}$$

$$\times \prod_{i=1}^{N} \int_{-\infty}^{\infty} d\phi_i \, \exp\{-\frac{1}{2} \sum_{i,j} \phi_i \, (\beta \, V_{ij})^{-1} \, \phi_j + \sum_i \ln(\cosh \phi_i)\} \tag{5.66}$$

Expand

$$\ln \cosh \phi_i = \frac{1}{2} \, \phi_i^2 - \frac{1}{12} \, \phi_i^4 + \cdots = \frac{1}{2} \, \phi_i^2 - \sum_{k=2}^{\infty} \frac{M_{2k}}{(2k)!} \, \phi_i^{2k} \tag{5.67}$$

The partition function becomes

$$Z[0] = 2^N \, (2\pi)^{-N/2} \, (\det \beta V)^{-1/2}$$

$$\times \prod_{i=1}^{N} \int_{-\infty}^{\infty} d\phi_i \, \exp\Big\{ -\frac{1}{2} \sum_{i,j} \phi_i \, (\beta \, V_{ij})^{-1} \, \phi_j + \frac{1}{2} \sum_i \phi_i^2 - \sum_{k=2}^{\infty} \frac{M_{2k}}{(2k)!} \, \phi_i^{2k} \Big\} \tag{5.68}$$

We now go in the limit of long wavelength approximation. In this limit it is legitimate to replace

$$\sum_i = \int \frac{d^n x}{a^n} \qquad \phi(x) = a^{-(n-2)/2} \, \phi_i \qquad \lambda_{2k} = a^{n(k-1)-2k} \, M_{2k} \tag{5.69}$$

so that $Z[0]$ becomes

$$
\begin{aligned}
Z[0] \;&=\; C \int \mathcal{D}\phi(x)\,\exp\Big\{\int d^n x\Big[-\frac{a^{-2}}{2}\,\phi(x)\,(\beta\,V(x))^{-1}\,\phi(x) + \frac{a^{-2}}{2}\,\phi^2(x) \\
&\quad -\sum_{k=2}^{\infty}\frac{\lambda_{2k}}{(2k)!}\,\phi^{2k}(x)\Big]\Big\} \qquad (5.70)
\end{aligned}
$$

where C is an arbitrary constant. To see this we go from position space to momentum space

$$
\begin{aligned}
Z[0] \;&=\; C \int \mathcal{D}\phi(q)\,\exp\Big\{\int d^n q\Big[-\frac{a^{-2}}{2}\,\phi(q)\,(\beta\,V(q))^{-1}\,\phi(q) + \frac{a^{-2}}{2}\,\phi^2(q) \\
&\quad -\sum_{k=2}^{\infty}\frac{\lambda_{2k}}{(2k)!}\,\phi^{2k}(q)\Big]\Big\} \qquad (5.71)
\end{aligned}
$$

Since $V(q) > 0$ over a range a we have $\max V(q) = V(0)$ and for $qa \ll 1$ we may expand according to $V(q) = V(0)(1-(qa)^2+\cdots)$. Insertion in (5.70) provides us with the form

$$
\begin{aligned}
Z[0] \;&=\; C \int \mathcal{D}\phi(q)\,\exp\Big\{\int d^n q\Big[-\frac{a^{-2}}{2}\,\phi(q)\,\frac{1-\beta V(0)+\beta V(0)\,(qa)^2}{\beta V(0)}\,\phi(q) \\
&\quad -\sum_{k=2}^{\infty}\frac{\lambda_{2k}}{(2k)!}\,\phi^{2k}(q)\Big]\Big\} \qquad (5.72)
\end{aligned}
$$

where we neglected higher powers in qa in numerator and denominator. Define

$$
\mu^2 = \frac{1-\beta V(0)}{\beta V(0)}\,a^{-2} \qquad (5.73)
$$

We get

$$
Z[0] = C \int \mathcal{D}\phi(q)\,\exp\Big\{\int d^n q\Big[-\frac{1}{2}\,\phi(q)\,q^2\,\phi(q) - \frac{\mu^2}{2}\,\phi^2(q) - \sum_{k=2}^{\infty}\frac{\lambda_{2k}}{(2k)!}\,\phi^{2k}(q)\Big]\Big\} \qquad (5.74)
$$

or in position space

$$
\begin{aligned}
Z[0] \;&=\; C \int \mathcal{D}\phi(x)\,\exp\Big\{\int d^n x\Big[-\frac{1}{2}\,\nabla\phi(x)\cdot\nabla\phi(x) - \frac{\mu^2}{2}\,\phi^2(x) - \sum_{k=2}^{\infty}\frac{\lambda_{2k}}{(2k)!}\,\phi^{2k}(x)\Big]\Big\} \\
&= \int \mathcal{D}\phi(x)\,\exp\Big\{-\int d^n x\, L(\phi,\nabla\phi)\Big\} \qquad (5.75)
\end{aligned}
$$

So the above expression can be used to calculate fluctuations of wave number q such that $qa \ll 1$. It is assumed that the singular part of $G \sim -\ln Z[0]$ (which is the free energy) comes from the low q part and this is the region where the second order phase transition occurs. The argument of the exponent is the Lagrangian but since we are in Euclidian space it is also called Hamiltonian (see Eq. (4.21)). It was original proposed by Landau without the $\frac{1}{2}\nabla\phi(x)\cdot\nabla\phi(x)$ term. However if we want to calculate correlation functions we also need the latter term to account for fluctuations. This was proposed by Ginzburg. Hence the Landau-Ginzburg Hamiltonian. Before we proceed some comments have to be made. The first comment concerns the number of dimensions. In one dimension a phase transition cannot occur but the above Hamiltonian leads to phase transition. Therefore we conclude that the procedure outlined above is wrong in one dimension. In two dimensions phase transitions do occur with Eq. (5.56) the specific heat is logarithmic i.e. $\sim \ln(T-T_c)$. This quantities with the Hamiltonian above behave power like as $(T-T_c)^{-b}$. Therefore in this case the Landau-Ginzburg theory leads to a wrong prediction too. In three and larger dimension the latter theory is presumably right and from now on we shall apply it for $n \geq 3$.

Add an external magnetic field H_0 to the Hamiltonian H as follows

$$Z[H_0] = C \int \mathcal{D}\phi(x)\, \exp \int d^n x \,\Big(H(\phi, \nabla\phi) + H_0(x)\,\phi(x) \Big) \tag{5.76}$$

Further we make the Landau approximation and put $\phi(x)$ independent of x. The free energy reads

$$G = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\,\phi^4 - H_0\,\phi \tag{5.77}$$

We assume that the fluctuations in ϕ are small so that we can neglect higher powers in ϕ. Because of (5.58) ϕ can be interpreted as the magnetisation. We determine ϕ by minimising G in the case that $H_0 = 0$.

$$\mu^2 > 0 \qquad \frac{\partial G}{\partial \phi} = 0 \quad \rightarrow \quad \phi = 0 \quad \text{for} \quad T > T_c$$

$$\mu^2 < 0 \qquad \frac{\partial G}{\partial \phi} = 0 \quad \rightarrow \quad \phi = \pm\sqrt{-6\frac{\mu^2}{\lambda}} \quad \text{for} \quad T < T_c \tag{5.78}$$

This implies that in the paramagnetic case $\phi = 0$ whereas in the ferromagnetic case $\phi \neq 0$ even if $H_0 = 0$. We make now the following assumptions

$$\mu^2 \;=\; a\,(T-T_c) \quad \rightarrow \quad \phi = \sqrt{6\frac{a}{\lambda}}\,(T_c - T)^{1/2} \sim (T_c - T)^{\beta}$$

$$\beta = \frac{1}{2} \quad \text{Landau value} \tag{5.79}$$

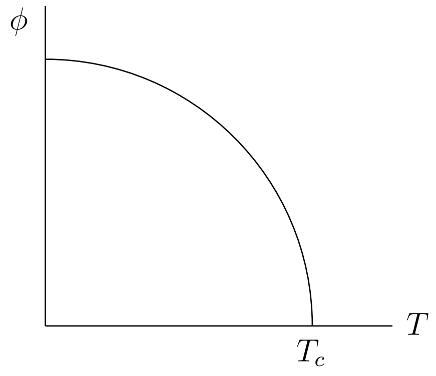


Figure 19: The magnetisation ϕ as a function of T.

The quantity β is called a critical exponent. The magnetic field becomes at $H_0 = 0$ and $T = T_c$

$$H = \frac{\partial G}{\partial \phi} = \frac{\lambda}{6}\,\phi^3 \quad \rightarrow \quad \phi \sim H^{1/3} = H^{1/\delta} \qquad \delta = 3 \quad \text{Landau value} \tag{5.80}$$

and δ is a critical exponent. We shall now calculate the susceptibility. Differentiate $\partial G/\partial\phi$ with respect to H but $T \neq T_c$

$$H = \frac{\partial\, G}{\partial\, \phi} = -H_0 + \mu^2\,\phi + \frac{\lambda}{6}\,\phi^3$$

$$1 = \mu^2\,\frac{\partial\phi}{\partial H} + \frac{1}{2}\,\phi^2\,\frac{\partial\phi}{\partial H} \tag{5.81}$$

We get now

$$\begin{aligned} T > T_c \qquad & \chi = \frac{\partial\phi}{\partial H} = \frac{1}{\mu^2} = \frac{1}{a\,(T - T_c)} \sim (T - T_c)^{-\gamma} \quad \gamma = 1 \quad \text{Landau value} \\ T < T_c \qquad & \chi = \frac{\partial\phi}{\partial H} = \frac{1}{\mu^2} = \frac{1}{a\,(T - T_c)} \sim (T - T_c)^{-\gamma} \quad \gamma = 1 \quad \text{Landau value} \end{aligned} \tag{5.82}$$

where we have neglected higher powers in ϕ in the limit $T \rightarrow T_c$. The free energy becomes for $H_0 = 0$

$$\begin{aligned} T > T_c \qquad & \phi = 0 \quad \rightarrow \quad G = 0 \\ T < T_c \qquad & \phi = \pm\sqrt{-6\frac{\mu^2}{\lambda}} \quad \rightarrow \quad G = -\frac{3\,a^2}{2\,\lambda}\,(T - T_c)^2 \end{aligned} \tag{5.83}$$

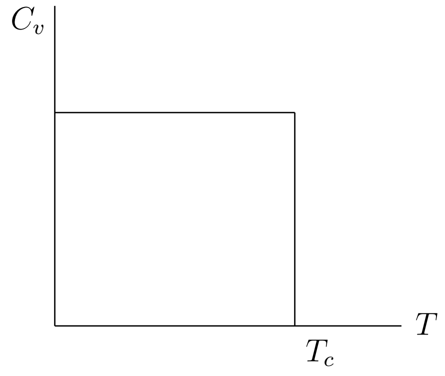


Figure 20: The specific heat C_v as a function of T.

The specific heat C_v is

$$T > T_c \quad \rightarrow \quad C_v = -2\,k\,T\,\frac{\partial\, G}{\partial\, T}|_{T=T_c} - k\,T^2\,\frac{\partial^2\, G}{\partial\, T^2}|_{T=T_c} = 0$$

$$T < T_c \quad \rightarrow \quad C_v = -2\,k\,T\,\frac{\partial\, G}{\partial\, T}|_{T=T_c} - k\,T^2\,\frac{\partial^2\, G}{\partial\, T^2}|_{T=T_c} = 3\,k\,T_c^2\,\frac{a^2}{\lambda}$$

$$\sim (T - T_c)^{-\alpha} \qquad \alpha = 0 \quad \text{Landau value} \tag{5.84}$$

The Landau theory neglects the fluctuations of ϕ. Hence it fails when these fluctuations are large compared to ϕ. To account for fluctuations let us add the kinetic term $\frac{1}{2}\,\nabla\phi(x)\cdot\nabla\phi(x)$ to the Hamiltonian (see (5.75)).

$$\begin{aligned} Z[H_0] &= C\int \mathcal{D}\phi(x)\,\exp\Big\{\int d^n x\Big[-\frac{1}{2}\,\nabla\phi(x)\cdot\nabla\phi(x) - \frac{\mu^2}{2}\,\phi^2(x) \\ &\quad -\frac{\lambda}{4!}\,\phi^4(x) + H_0(x)\,\phi(x)\Big]\Big\} \end{aligned} \tag{5.85}$$

Let us first put $\lambda = 0$ (free field theory). The path integral becomes

$$Z[H_0] = C\int \mathcal{D}\phi\,\exp\Big[\int d^n x\Big\{\frac{1}{2}\,\phi(x)\,\triangle_x\,\phi(x) - \frac{\mu^2}{2}\,\phi^2(x) + H_0(x)\,\phi(x)\Big\}\Big] \tag{5.86}$$

Define

$$G_0^{-1}(x,y) = G_0^{-1}(x-y) = -\Big(\triangle_x - \mu^2\Big)\,\delta^{(n)}(x-y) \tag{5.87}$$

The path integral can be rewritten as

$$Z[H_0] = C\int \mathcal{D}\phi\,\exp\Big[-\frac{1}{2}\int d^n x\int d^n y\,\Big\{\frac{1}{2}\,\phi(x)\,G_0(x,y)\,\phi(y)\Big\}$$

$$+\int d^n x\, H_0(x)\, \phi(x)\Big] \tag{5.88}$$

Using Eq. (5.63)

$$Z[H_0] = \frac{C}{\det^{1/2} G_0^{-1}}\, \exp\left[\frac{1}{2}\int d^n x \int d^n y\, H_0(x)\, G_0(x,y)\, H_0(y)\right] \tag{5.89}$$

Choose C in such a way that $Z[0]=1$.

$$G_0^{-1}(x,y) = \int \frac{d^n p}{(2\pi)^n}\, e^{-i\,p\cdot(x-y)}\left(p^2+\mu^2\right) \tag{5.90}$$

or

$$G_0(x,y) = \int \frac{d^n p}{(2\pi)^n}\, e^{-i\,p\cdot(x-y)}\, \frac{1}{p^2+\mu^2} \tag{5.91}$$

The correlation function becomes

$$\langle \phi(x)\, \phi(y)\rangle = Z[H_0]^{-1}\, \frac{\delta^2\, Z[H_0]}{\delta\, H_0(x)\, \delta\, H_0(y)}\bigg|_{H_0(x_i)=0} = G_0(x,y) \tag{5.92}$$

In momentum space (inverse Fourier transform) this equals

$$\begin{aligned} \tilde{G}_0(k) &= \int d^n x\, e^{i\,k\,(x-y)}\, G_0(x,y) = \frac{1}{k^2+\mu^2} \underset{k\,\gg\,\mu}{=} k^{-2} = k^{-2+\eta} \\ &\eta = 0 \quad \text{Landau value} \end{aligned} \tag{5.93}$$

The quantity η is a critical exponent which in zeroth order in λ is zero. The susceptibility χ is

$$\tilde{G}_0(0) = \mu^{-2} = (T-T_c)^{-\gamma} \quad \gamma = 1 \quad \text{Landau value} \tag{5.94}$$

Notice that the correlation length $\xi = \mu^{-1}$ it is defined by

$$\xi^2 = \frac{\int dx\, x^2\, G_0(x)}{\int dx\, G_0(x)} = \frac{-\partial\, \tilde{G}_0(k)/\partial\, k^2}{\tilde{G}_0(k)}|_{k=0} = \mu^{-2}$$

$$G(x,y) = \int \frac{d^n k}{(2\pi)^n}\, e^{-i\,k\cdot(x-y)}\, \frac{1}{k^2+\mu^2} \sim \frac{e^{-\mu\,|x-y|}}{|x-y|^{n-2}} \quad \text{with} \quad \mu = \xi^{-1} \tag{5.95}$$

From the behaviour of μ with respect to T we infer

$$\xi = (T-T_c)^{-\nu} = (T-T_c)^{-1/2} \quad \nu = \frac{1}{2} \quad \text{Landau value} \tag{5.96}$$

Inserting the correlation length $k = \xi^{-1}$ into Eq. (5.93) we obtain

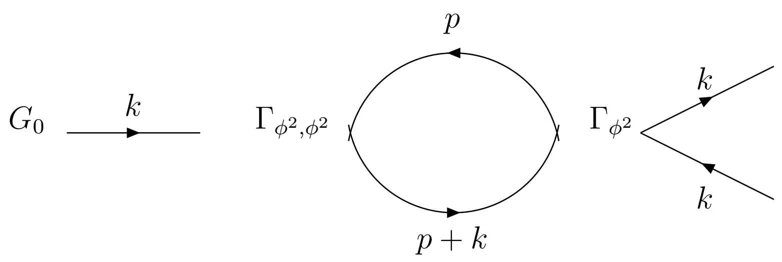


Figure 21: Lowest order contribution to the Green's function G_0, Γ_{ϕ^2,ϕ^2} and Γ_{ϕ^2}.

$$(T - T_c)^{\nu\,(\eta-2)} \tag{5.97}$$

Equating this with Eq. (5.93) we get the fluctuation dissipation theorem

$$\gamma = \nu\,(2-\eta) \tag{5.98}$$

The specific heat is derived from $C_v = \Gamma_{\phi^2,\phi^2}$ with

$$\Gamma_{\phi^2,\phi^2} = -\frac{\partial^2 \ln Z[H_0]}{\partial(\mu^2)^2} = -\left[\langle\frac{1}{2}\int d^n x\,\phi^2(x)\,\frac{1}{2}\int d^n y\,\phi^2(y)\rangle - \langle\frac{1}{2}\int d^n x\,\phi(x)\rangle^2\right] \tag{5.99}$$

where $\mu^2 = a\,(T - T_c)$. We can also write this as

$$\Gamma_{\phi^2,\phi^2}(k,\mu) = \int \frac{d^n x}{(2\pi)^n}\, e^{i\,k\cdot x}\,\langle\phi^2(x)\,\phi^2(0)\rangle \tag{5.100}$$

This expression is equal to

$$\begin{aligned}\Gamma_{\phi^2,\phi^2}(k,\mu) &= \int \frac{d^n p}{(2\pi)^n}\,\frac{1}{\left(p^2+\mu^2\right)\left((p+k)^2+\mu^2\right)} \\ &= i\,S_n\,\Gamma(2-n/2)\int_0^1 dx\,[k^2\,x\,(1-x)+\mu^2]^{n/2-2}\end{aligned} \tag{5.101}$$

substitute $x = 1/2 - y/2$

$$\Gamma_{\phi^2,\phi^2}(k,\mu) = i\,S_n\,\Gamma(2-n/2)\int_0^1 dy\left[\frac{k^2}{4}\,(1-y^2)+\mu^2\right]^{n/2-2} \tag{5.102}$$

Substitute $y = \sqrt{t}$ then we can cast the integral into a hypergeometric function $F_{1,2}(a,b,c;z)$

$$\Gamma_{\phi^2,\phi^2}(k,\mu)$$

$$= \frac{i}{2}\, S_n\, \Gamma(2-n/2)\, \left(\frac{k^2}{4}+\mu^2\right)^{n/2-2} \int_0^1 dt\, t^{-1/2}\left[1-\frac{k^2}{k^2+4\mu^2}\,t\right]^{n/2-2}$$

$$= i\, S_n\, \Gamma(2-n/2)\, \left(\frac{k^2}{4}+\mu^2\right)^{n/2-2} F_{1,2}\left(2-n/2, 1, 3/2; \frac{k^2}{k^2+4\mu^2}\right)$$

$$= i\, S_n\, \Gamma(2-n/2)\, (\mu^2)^{n/2-2}\, F_{1,2}\left(2-n/2, 1, 3/2; -\frac{k^2}{4\mu^2}\right) \sim \mu^{n-4} \qquad (5.103)$$

for $k = 0$. In the limit $\mu \to 0$ we have to make a Kummer transformation on the hypergeometric function

$$\Gamma_{\phi^2,\phi^2}(k,\mu)$$

$$= i\, S_n\, \Gamma(2-n/2)\, (\mu^2)^{n/2-2}\, F_{1,2}\left(2-n/2, 1, 3/2; -\frac{k^2}{4\mu^2}\right) =$$

$$= i\, S_n\, \frac{\Gamma(3/2)\,\Gamma(2-n/2)\,\Gamma(n/2-1)}{\Gamma(n/2-1/2)} \left(\frac{k^2}{4}\right)^{n/2-2}$$

$$\times F_{1,2}\left(2-n/2, 3/2-n/2, 2-n/2; -\frac{4\mu^2}{k^2}\right)$$

$$+ i\, S_n\, \frac{\Gamma(3/2)\,\Gamma(2-n/2)\,\Gamma(n/2-1)}{\Gamma(1/2)\,\Gamma(n/2-1/2)} \left(\frac{4\mu^2}{k^2}\right) (\mu^2)^{n/2-2}$$

$$\times F_{1,2}\left(1, 1/2, n/2; -\frac{4\mu^2}{k^2}\right) \sim k^{n-4} \quad \text{if} \quad \mu = 0 \qquad (5.104)$$

We can now derive the exponents

$$\Gamma_{\phi^2,\phi^2}(k,0) = k^{n-4} = k^{\lambda} \qquad \lambda = n-4 \quad \text{Landau value} \qquad (5.105)$$

and

$$\Gamma_{\phi^2,\phi^2}(0,\mu) = \mu^{n-4} = (T-T_c)^{n/2-2} = (T-T_c)^{-\alpha} \qquad \alpha = 2-n/2 \qquad (5.106)$$

Finally we have the Greens function

$$\Gamma_{\phi^2}(k,\mu) = \int d^n x\, e^{i\,k\cdot x} \int d^n y\, e^{i\,k\cdot y}\, \langle \phi(x)\,\phi^2(y)\,\phi(0)\rangle = k^{\mu} \qquad (5.107)$$

In zeroth order perturbation theory we have

$$\Gamma_{\phi^2}(k,\mu) = 1 \quad \to \quad \mu = 0 \quad \text{Landau value} \qquad (5.108)$$

There exist a relation between γ, η and μ which is called hyper scaling.

$$\frac{1}{\gamma} = 1 - \frac{\mu}{2-\eta} \tag{5.109}$$

This remains satisfied if we turn the coupling constant λ unequal to zero. Also an other relation remains preserved

$$\alpha = 2 - \nu\, n \tag{5.110}$$

5.4 The ε expansion and the renormalisation group.

Until now we have only calculated the critical exponents for $N = 1$. It would not be difficult to generalize them to an order parameter having dimension N and the symmetry group is in this case $O(N)$. Further we take $\lambda \neq 0$ and the Ginzburg-Landau partition function reads

$$\begin{aligned} Z[H_0] &= C \int \mathcal{D}\phi(x) \, \exp\Big\{ \int d^n x \Big[-\frac{1}{2} \sum_{i=1}^{N} \nabla\phi_i(x) \cdot \nabla\phi_i(x) - \frac{\mu^2}{2} \sum_{i=1}^{N} \phi_i^2(x) \\ & \quad - \frac{\lambda}{4!} (\sum_{i=1}^{N} \phi_i^2(x))^2 + \sum_{i=1}^{N} H_i(x)\, \phi_i(x) \Big] \Big\} \end{aligned} \tag{5.111}$$

We see that for $\mu^2 = a\,(T - T_c)$ and $T < T_c$ we get again a second order phase transition. The symmetry group $O(N)$ breaks down to a smaller group. In other words we get more order. Hence the name order parameter for ϕ_i. If $T \to T_c$ the mass μ approaches zero and the correlation length $\xi = \mu^{-1}$ becomes infinite. Therefore at the critical point we are at a maximally ordered state. At this critical temperature the thermodynamic functions show scaling in the form

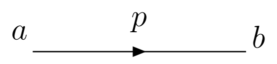
$$C = (T - T_c)^{-b} \tag{5.112}$$

and b is called the critical exponents. Examples are for $n = 3$:

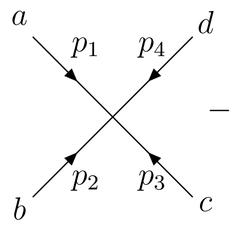
$N = 0$	Self avoiding walk	
$N = 1$	Ising	Ferro magnetism
$N = 2$	λ point of Helium	Super fluidity
$N = 3$	Heisenberg model	
$N = \infty$	Spherical model	

The Feynman rules of the partition function in Eq. (5.111) (generating functional for the connected graphs) are:

1 Draw at a given order in λ all topological inequivalent connected diagrams.



$G_{0,ab}(p) = \frac{\delta_{ab}}{p^2+\mu^2}$



$-\lambda_n \, I_{abcd} \, (2\pi)^4 \, \delta^{(4)}(p_1 + p_2 + p_3 + p_4)$

Figure 22: The Feynman rules for Euclidian ϕ^4-theory.

2 Each internal line with momentum p is represented by the factor $G_{0,ab}(p)$

3 Each vertex where four lines with the momenta p_1, p_2, p_3, p_4 come together is represented by a factor $-\lambda_n \, I_{abcd}$ with $I_{abcd} = \frac{1}{3}\Big(\delta_{ab}\,\delta_{cd} + \delta_{ac}\,\delta_{bd} + \delta_{ad}\,\delta_{bc}\Big)$. Here is imposed the condition that $\sum_{i=1}^{4} p_i = 0$

4 Integrate over every internal momentum k. The measure is equal to $d^n k/(2\pi)^n$

5 A combinatorial factor $1/n!$ if n internal lines start on one vertex and end on another vertex.

We will now calculate the first order corrections to the critical exponents. We start with the vertex correction and set the mass $\mu = 0$ (the ultra violet divergence will not depend on μ^2). In lowest order the vertex is given by

$$\Lambda^{(0)}_{abcd} = -\lambda_n \, I_{abcd} \tag{5.113}$$

The first order correction to $\Lambda^{(0)}$ is exhibited in Fig. 22. The result for the first graph in Fig. 23 is equal to

$$\Lambda^{(1)}_{abcd}(p) = \frac{1}{2}\,\lambda_n^2 \, I_{ablm} \, I_{lmcd} \int \frac{d^n k}{(2\pi)^n} \, \frac{1}{k^2 \, (k+p)^2} \tag{5.114}$$

Let us first calculate $I_{ablm} \, I_{lmcd}$. The result is

$$I_{ablm} \, I_{lmcd} = \frac{1}{9}\Big[(N+4)\,\delta_{ab}\,\delta_{cd} + 2\,\delta_{ac}\,\delta_{bd} + 2\,\delta_{ad}\,\delta_{bc}\Big] \tag{5.115}$$

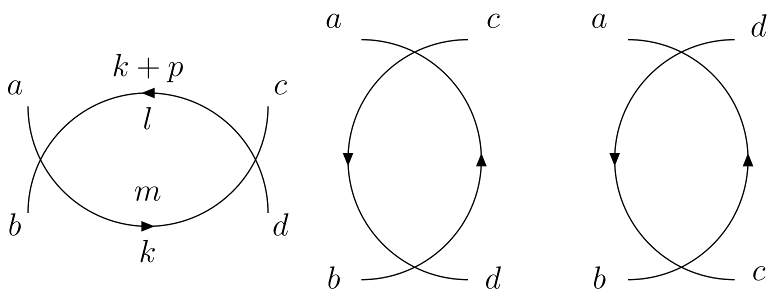


Figure 23: The first order contribution to $\Gamma^{(4)}_{abcd}$.

Then we compute the momentum dependence of $\Lambda^{(1)}$. Using Feynman parameters

$$\begin{aligned}
\Lambda^{(1)}_{abcd}(p) &= \frac{1}{2}\, I_{ablm}\, I_{lmcd}\, \frac{\lambda_n^2}{(2\pi)^n} \int d^n k \int_0^1 dx\, \frac{1}{\left[k^2 + 2\,x\,k\cdot p + x\,p^2\right]^2} \\
&= \frac{1}{2}\, I_{ablm}\, I_{lmcd}\, \lambda_n^2\, \frac{\pi^{n/2}}{(2\pi)^n}\, \Gamma(2-n/2) \int_0^1 dx \left(x\,(1-x)\right)^{n/2-2} (p^2)^{n/2-2} \\
&= \frac{1}{2}\, I_{ablm}\, I_{lmcd}\, \lambda_n^2\, \frac{\pi^{n/2}}{(2\pi)^n}\, \frac{\Gamma(2-n/2)\,\Gamma^2(n/2-1)}{\Gamma(n-2)}\, (p^2)^{n/2-2}
\end{aligned} \tag{5.116}$$

Define

$$C_n = \frac{\pi^{n/2}}{(2\pi)^n} \frac{\Gamma(3-n/2)\,\Gamma^2(n/2-1)}{\Gamma(n-2)} \tag{5.117}$$

and

$$\varepsilon = 4 - n \qquad \lambda_n = \lambda\, M^{\varepsilon} \tag{5.118}$$

We get

$$\Lambda^{(1)}_{abcd}(p) = \frac{1}{2}\, I_{ablm}\, I_{lmcd}\, \lambda_n\, (C_n\, \lambda) \left[\frac{2}{\varepsilon} - \ln \frac{p^2}{M^2}\right] \tag{5.119}$$

The other two graphs in Fig. 23 give the same except for the group theoretical factor. The second graph yields

$$I_{ackl}\, I_{klbd} = \frac{1}{9}\left[(N+4)\,\delta_{ac}\,\delta_{bd} + 2\,\delta_{ab}\,\delta_{cd} + 2\,\delta_{ad}\,\delta_{bc}\right] \tag{5.120}$$

and the third graph gives

$$I_{adkl}\, I_{klbc} = \frac{1}{9}\left[(N+4)\,\delta_{ad}\,\delta_{bc} + 2\,\delta_{ac}\,\delta_{bd} + 2\,\delta_{ab}\,\delta_{cd}\right] \tag{5.121}$$

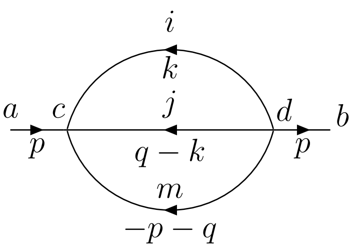


Figure 24: The first order contribution to the self energy $\Gamma^{(2)}_{ab}$.

Adding all three contributions from Fig. 23 we get for $\Lambda^{(1)}_{abcd}$

$$\Lambda^{(1)}_{abcd} = \frac{N+8}{6} \, I_{abcd} \, \lambda_n \, (C_n \, \lambda) \left[\frac{2}{\varepsilon} - \ln \frac{p^2}{M^2} \right] \tag{5.122}$$

The first order correction to $\Gamma^{(2)}_{ab}(p)$ is given by Fig. 24. Notice that the tadpole graph in Fig. 4 yields zero for $\mu = 0$ The expression for this graph is

$$\begin{aligned} G_{1,ab}(p) &= G_{0,ac}(p) \, \Pi_{cd}(p) \, G_{0,db}(p) \\ \Pi_{cd}(p) &= \frac{1}{3!} \, I_{cijm} \, I_{dijm} \, \lambda_n^2 \int \frac{d^n k}{(2\pi)^n} \int \frac{d^n q}{(2\pi)^n} \, \frac{1}{(q+p)^2 \, (k-q)^2 \, k^2} \end{aligned} \tag{5.123}$$

The group factor is

$$I_{cijm} \, I_{dijm} = \frac{1}{9} \, (3\, N + 6) \, \delta_{cd} \tag{5.124}$$

The contribution to the two point function becomes

$$\Pi_{cd}(p) \quad = \quad \frac{N+2}{18} \, \delta_{cd} \int \frac{d^n q}{(2\pi)^n} \, \frac{1}{(q+p)^2} \, \tilde{\Lambda}(q) \tag{5.125}$$

with

$$\tilde{\Lambda}(q) = \lambda_n \, (C_n \, \lambda_n) \, \frac{2}{\varepsilon} \, (q^2)^{n/2-2} \tag{5.126}$$

Therefore

$$\Pi_{cd}(p) \quad = \quad \frac{N+2}{18} \, \delta_{cd} \, \lambda_n \, (C_n \, \lambda_n) \, \frac{2}{\varepsilon} \int \frac{d^n q}{(2\pi)^n} \, \frac{1}{(q+p)^2 \, (q^2)^{2-n/2}} \tag{5.127}$$

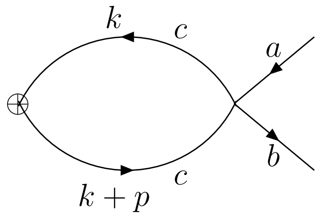


Figure 25: The first order contribution to the Γ_{ab,ϕ^2}.

Let us work out the integral.

$$\int \frac{d^n q}{(2\pi)^n} \frac{1}{(q+p)^2 \, (q^2)^{2-n/2}} = \frac{\Gamma(3-n/2)}{\Gamma(2-n/2)} \int \frac{d^n q}{(2\pi)^n} \int_0^1 dy \, \frac{(1-y)^{1-n/2}}{(q^2 + 2\, y\, q\cdot p + y\, p^2)^{3-n/2}}$$

$$= \frac{\pi^{n/2}}{(2\pi)^n} \frac{\Gamma(3-n)}{\Gamma(2-n/2)} (p^2)^{n-3} \int_0^1 dy\, y^{n-3} \, (1-y)^{n/2-2}$$

$$= \frac{\pi^{n/2}}{(2\pi)^n} \frac{\Gamma(3-n)\,\Gamma(n-2)\,\Gamma(n/2-1)}{\Gamma(2-n/2)\,\Gamma(3n/2-3)} (p^2)^{n-3}$$

$$= -\frac{1}{2}\, C_n \, \frac{\Gamma(5-n)\,\Gamma^2(n-3)}{\Gamma(3n/2-5)\,\Gamma(n/2-1)\,\Gamma^2(3-n/2)} \, \frac{n-3}{(3n/2-4)\,(3n/2-5)} \, (p^2)^{n-3}$$

$$= -\frac{1}{4}\, C_n \, (1 + 5/4\, \varepsilon + \cdots) \, (p^2)^{n-3} \tag{5.128}$$

Therefore Π_{cd} becomes

$$\Pi_{cd}(p) = \frac{N+2}{18}\, \delta_{cd} \left(C_n\, \lambda\right)^2 p^2 \left[-\frac{1}{2\,\varepsilon} + \frac{1}{2} \ln \frac{p^2}{M^2} - \frac{5}{8} \right] \tag{5.129}$$

Finally we compute Γ_{ϕ^2} (Eq. (5.106)) up to first order

$$\Gamma^{(1)}_{ab,\phi^2} = \frac{1}{2}\, I_{ccab}\, \lambda_n \int \frac{d^n k}{(2\pi)^n} \, \frac{1}{k^2\,(k+p)^2} \tag{5.130}$$

The group theoretical factor becomes

$$I_{ccab} = \frac{N+2}{3}\, \delta_{ab} \tag{5.131}$$

and the two point function $\Gamma^{(1)}_{ab,\phi^2}$ is equal to

$$\Gamma^{(1)}_{ab,\phi^2} = \frac{N+2}{6}\, C_n\, \lambda \left[\frac{2}{\varepsilon} - \ln \frac{p^2}{M^2} \right] \tag{5.132}$$

Collecting all terms we can write the correlation functions corrected up to one loop order

$$\begin{aligned}
\Gamma^{(4)}_{abcd}(p) &= \Lambda^{(0)}_{abcd}(p) + \Lambda^{(1)}_{abcd}(p) + \cdots = -\lambda_n\, I_{abcd}\left[1 - \frac{N+8}{6}\, C_n\, \lambda \left\{\frac{2}{\varepsilon} - \ln\frac{p^2}{M^2}\right\}\right. \\
&\quad \left. + \cdots\right] \\
\Gamma^{(2)}_{ab}(p) &= G_{0,ab}(p) + G_{0,ac}(p)\,\Pi_{cd}(p)\,G_{0,db}(p) + \cdots = \frac{1}{p^2}\,\delta_{ab}\left[1 - \frac{N+2}{18}\, C_n^2\, \lambda^2\right. \\
&\quad \left.\times\left\{\frac{1}{2\,\varepsilon} - \frac{1}{2}\ln\frac{p^2}{M^2} + \frac{5}{8}\right\} + \cdots\right] \\
\Gamma_{ab,\phi^2}(p) &= \Gamma^{(0)}_{ab,\phi^2}(p) + \Gamma^{(1)}_{ab,\phi^2}(p) + \cdots = \delta_{ab}\left[1 - \frac{N+2}{6}\, C_n\, \lambda \left\{\frac{2}{\varepsilon} - \ln\frac{p^2}{M^2}\right\}\right. \\
&\quad \left. + \cdots\right]
\end{aligned} \tag{5.133}$$

We have now to renormalize to get rid of the infinity exhibited by the pole term $1/\varepsilon$

$$\begin{aligned}
\Gamma^{(4),R}_{abcd}(p) &= Z_1\, \Gamma^{(4)}_{abcd}(p) \\
\Gamma^{(2),R}_{ab}(p) &= Z_3^{-1}\, \Gamma^{(2)}_{ab}(p) \\
\Gamma^{R}_{ab,\phi^2}(p) &= Z_{\phi^2}\, \Gamma_{ab,\phi^2}(p)
\end{aligned} \tag{5.134}$$

Up to first order

$$\begin{aligned}
Z_1 &= 1 + C_n\, \lambda_R\, \frac{N+8}{3}\, \frac{1}{\varepsilon} \\
Z_3 &= 1 - C_n^2\, \lambda_R^2\, \frac{N+2}{36}\, \frac{1}{\varepsilon} \\
Z_{\phi^2} &= 1 + C_n\, \lambda_R\, \frac{N+2}{3}\, \frac{1}{\varepsilon}
\end{aligned} \tag{5.135}$$

en the finite Green's function equal

$$\begin{aligned}
\Gamma^{(4),R}_{abcd}(p) &= -M^{n-4}\, \lambda_R\, I_{abcd}\left[1 + \frac{N+8}{6}\, C_n\, \lambda_R\, \ln\frac{p^2}{M^2} + \cdots\right] \\
\Gamma^{(2),R}_{ab}(p) &= \frac{1}{p^2}\,\delta_{ab}\left[1 + \frac{N+2}{36}\, C_n^2\, \lambda_R^2 \left\{\frac{1}{2}\ln\frac{p^2}{M^2} - \frac{5}{8}\right\} + \cdots\right]
\end{aligned}$$

$$\Gamma^R_{ab,\phi^2}(p) \;=\; \delta_{ab}\left[1 + \frac{N+2}{6}\, C_n\, \lambda_R\, \ln\frac{p^2}{M^2} + \cdots\right] \tag{5.136}$$

The Greens functions satisfy a renormalisation group equation (Callan-Symanzik equation) which we will now derive. We have defined the renormalization group constants Z_1, Z_3 and Z_λ as follows

$$\begin{aligned} L \;=\; & \int d^n x \Big[\frac{1}{2}\sum_{i=1}^N \nabla\phi_i(x)\cdot\nabla\phi_i(x) + \frac{\mu^2}{2}\sum_{i=1}^N \phi_i^2(x) + \frac{\lambda}{4!}(\sum_{i=1}^N \phi_i^2(x))^2 \\ & + J_{\phi^2}\sum_{i=1}^N \phi_i^2(x) + \sum_{i=1}^N H_i(x)\,\phi_i(x)\Big] \end{aligned} \tag{5.137}$$

We make the substitution

$$\phi_i(x) = Z_3(M)^{1/2}\,\phi_{i,R}(x) \tag{5.138}$$

Note that Z_3 is independent of i. The Lagrangian becomes

$$\begin{aligned} L \;=\; & \int d^n x \Big[\frac{1}{2}\, Z_3 \sum_{i=1}^N \nabla\phi_{i,R}(x)\cdot\nabla\phi_{i,R}(x) + \frac{\mu^2}{2}\, Z_3 \sum_{i=1}^N \phi_{i,R}^2(x) \\ & + \frac{\lambda}{4!}\, Z_3^2\,(\sum_{i=1}^N \phi_{i,R}^2(x))^2 + J_{\phi^2}\, Z_{\phi^2}\sum_{i=1}^N \phi_{i,R}^2(x) \\ & + Z_3^{1/2}\sum_{i=1}^N H_i(x)\,\phi_{i,R}(x)\Big] \end{aligned} \tag{5.139}$$

On the other hand

$$\begin{aligned} L \;=\; & \int d^n x \Big[\frac{1}{2}\, Z_3 \sum_{i=1}^N \nabla\phi_{i,R}(x)\cdot\nabla\phi_{i,R}(x) + \frac{\mu_R^2}{2}\, Z_\mu \sum_{i=1}^N \phi_{i,R}^2(x) \\ & + \frac{\lambda_R}{4!}\, M^{4-n}\, Z_1\,(\sum_{i=1}^N \phi_{i,R}^2(x))^2 + J_{\phi^2}\, Z_{\phi^2}\sum_{i=1}^N \phi_{i,R}^2(x) \\ & + Z_3^{1/2}\sum_{i=1}^N H_i(x)\,\phi_{i,R}(x)\Big] \end{aligned} \tag{5.140}$$

We obtain the relations

$$\lambda = M^{4-n}\,\frac{Z_1(\lambda_R)}{Z_3(\lambda_R)^2}\,\lambda_R = M^\varepsilon\, Z_\lambda(M)\,\lambda_R \tag{5.141}$$

$$\mu^2 = \frac{Z_\mu}{Z_3}\,\mu_R^2 \tag{5.142}$$

$$\phi_i^2(x) = Z_{\phi_i^2}\, \phi_{i,R}^2(x) \tag{5.143}$$

The bare coupling constant λ is independent of the renormalization scale M.

$$
\begin{aligned}
M\,\frac{d\,\lambda}{d\,M} &= 0 \quad \Longrightarrow \\
0 &= \varepsilon\, M^{\varepsilon}\, \lambda_R(M)\, Z_\lambda(M) + M^{1+\varepsilon}\, \frac{d\,\lambda_R(M)\, Z_\lambda(M)}{d\,M} \\
0 &= \varepsilon\, \lambda_R(M)\, Z_\lambda(M) + \frac{\partial\,\lambda_R(M)\, Z_\lambda(M)}{\partial\,\lambda_R}\, M\, \frac{\partial\,\lambda_R}{\partial\,M} \qquad (5.144)
\end{aligned}
$$

Define the beta-function

$$\beta\,(\lambda_R) = M\,\frac{\partial\,\lambda_R(M)}{\partial\,M} = -\frac{\varepsilon\, Z_\lambda(M)\,\lambda_R}{\partial\,\lambda_R\, Z_\lambda(M)/\partial\,\lambda_R} \tag{5.145}$$

One can show that this function is finite in all orders in perturbation theory for $\varepsilon \to 0$. Following the same arguments for the unrenormalized Green function one can derive

$$
\begin{aligned}
\Gamma^{(n)}_{i_1,i_2,\cdots i_n}(x_1, x_2, \cdots x_n) &= \langle \phi_{i_1}(x_1)\, \phi_{i_2}(x_2) \cdots \phi_{i_n}(x_n) \rangle \\
\Gamma^{(n),R}_{i_1,i_2,\cdots i_n}(x_1, x_2, \cdots x_n) &= \langle \phi_{i_1,R}(x_1)\, \phi_{i_2,R}(x_2) \cdots \phi_{i_n,R}(x_n) \rangle \\
\Gamma^{(n)}_{i_1,i_2,\cdots i_n}(x_1, x_2, \cdots x_n) &= Z_3^{n/2}\, \Gamma^{(n),R}_{i_1,i_2,\cdots i_n}(x_1, x_2, \cdots x_n) \\
\Gamma^{(n)}_{i_1,i_2,\cdots i_n}(p_1, p_2, \cdots p_n) &= Z_3^{n/2}\, \Gamma^{(n),R}_{i_1,i_2,\cdots i_n}(p_1, p_2, \cdots p_n) \qquad (5.146)
\end{aligned}
$$

For massless theories we have

$$
\begin{aligned}
&M\,\frac{d\,\Gamma^{(n)}_{i_1,i_2,\cdots i_n}(p_1, p_2, \cdots p_n)}{d\,M} = 0 \quad \longrightarrow \\
&\left[M\,\frac{\partial}{\partial\,M} + \beta\,(\lambda_R)\, \frac{\partial}{\partial\,\lambda_R} + \frac{n}{2}\, M\, \frac{\partial\, \ln Z_3(M)}{\partial\,M} \right] \Gamma^{(n),R}_{i_1,i_2,\cdots i_n}(p_1, p_2, \cdots p_n) = 0
\end{aligned}
\tag{5.147}
$$

Define the anomalous dimension of the field $\phi_{i,R}(x)$ by

$$\gamma_\phi\,(\lambda_R) = \frac{1}{2}\, M\, \frac{\partial\, \ln Z_3(M)}{\partial\,M} = \frac{1}{2}\, \beta(\lambda_R)\, \frac{\partial\, \ln Z_3(M)}{\partial\,\lambda_R} \tag{5.148}$$

which is finite in all orders in the limit $\varepsilon \to 0$. The result is

$$\left[M\,\frac{\partial}{\partial\,M} + \beta\,(\lambda_R)\, \frac{\partial}{\partial\,\lambda_R} + \frac{n}{2}\, \gamma_\phi\,(\lambda_R) \right] \Gamma^{(n),R}_{i_1,i_2,\cdots i_n}(p_1, p_2, \cdots p_n) = 0 \tag{5.149}$$

For simplicity we shall only consider two-point functions which are only dependent on one momentum ($n = 2$). Equation (5.149) becomes

$$\left(-\frac{\partial}{\partial\, t} + \beta(\lambda_R)\,\frac{\partial}{\partial\, \lambda_R} + \gamma_\phi(\lambda_R)\right)\,\Gamma^R_{ij}(\lambda_R, t) = 0 \qquad t = \frac{1}{2}\,\ln\frac{p^2}{M^2} \tag{5.150}$$

The equation above admits the following solution

$$\Gamma^R_{ij}\left(\lambda_R(M), \frac{1}{2}\ln\frac{p^2}{M^2}\right) = \Gamma^R_{ij}(\bar\lambda(t), 0)\;\exp\left\{\int_{\bar\lambda(0)}^{\bar\lambda(t)} dx\,\frac{\gamma_\phi(x)}{\beta(x)}\right\} \tag{5.151}$$

Here $\bar\lambda(t)$ is called the running coupling constant which satisfies the equation

$$\left(-\frac{\partial}{\partial\, t} + \beta(\lambda_R)\,\frac{\partial}{\partial\, \lambda_R}\right)\,\bar\lambda(t) = 0 \quad \text{with} \quad \bar\lambda(0) = \lambda_R(M) \equiv \lambda_R \tag{5.152}$$

Eq. (5.152) can be also written as an integral equation

$$\int_{\lambda_R}^{\bar\lambda(t)} dx\,\frac{1}{\beta(x)} = t \tag{5.153}$$

Taking the partial derivative of the equation above

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{\lambda_R(M)}^{\bar\lambda(t)} dx\,\frac{1}{\beta(x)} = 1 \\
\Longrightarrow \quad &\frac{\partial\,\bar\lambda(t)}{\partial t}\,\frac{1}{\beta(\bar\lambda(t)} = 1
\end{aligned} \tag{5.154}$$

Hence it follows that

$$\frac{\partial\,\bar\lambda(t)}{\partial t} = \beta(\bar\lambda(t)) \tag{5.155}$$

To show that one can obtain Eq. (5.153) from Eq. (5.152) we differentiate the former w.r.t. λ_R

$$\begin{aligned}
&\frac{\partial\,\bar\lambda(t)}{\partial\,\lambda_R}\,\frac{1}{\beta(\bar\lambda(t))} - \frac{1}{\beta(\lambda_R)} = 0 \\
\Longrightarrow \quad &\beta(\lambda_R)\,\frac{\partial\,\bar\lambda(t)}{\partial\,\lambda_R} - \beta(\bar\lambda(t)) = 0 \\
\Longrightarrow \quad &\beta(\lambda_R)\,\frac{\partial\,\bar\lambda(t)}{\partial\,\lambda_R} - \frac{\partial\bar\lambda(t)}{\partial t} = 0
\end{aligned} \tag{5.156}$$

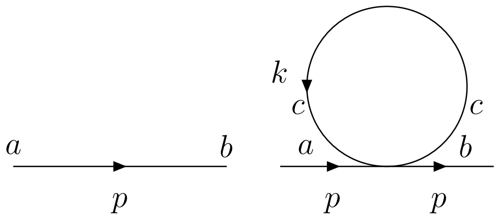


Figure 26: Corrections to the propagator $G(p)$.

Suppose the beta-function has a zero for small momenta ($p^2 \to 0$ then $t \to -\infty$). The zero shows up in the following way

$$\frac{\partial\,\bar{\lambda}(t)}{\partial\, t} = \beta(\bar{\lambda}(t)) = \beta_0\,(\lambda(t) - \lambda_0) \tag{5.157}$$

The solution is

$$\lambda(t) = \lambda_0 + e^{\beta_0\, t} \quad\to\quad \beta_0 > 0 \tag{5.158}$$

So β_0 must be positive which is the case for ϕ^4-theory. The solution of the Callan-Symanzik equation is

$$\Gamma^R_{ij}\left(\lambda_R(M), \frac{1}{2}\ln\frac{p^2}{M^2}\right) \sim \Gamma^R_{ij}(\lambda_0, 0)\,\left(\frac{p}{M}\right)^{\gamma_0\,\lambda_0^2} \tag{5.159}$$

where we have chosen

$$\gamma_\phi(\lambda_R) = \gamma_0\,\lambda_R^2 + \cdots \tag{5.160}$$

This is called scaling. So in second order phase transitions if we are at the critical temperature the correlation length $\xi = (T - T_c)^{-\nu} \to \infty$. In other words the momenta p go to zero. At that point the correlation functions show scaling. The coefficient of this scaling functions i.e. $\gamma_0\,\lambda_0^2$ are the critical exponents. If we compare Eq. (5.93) with (5.148) we conclude

$$\eta = 2\,\gamma_\phi(\lambda_0) \tag{5.161}$$

is two times the anomalous dimension of the ϕ field. According to Eq. (5.145) the beta-function up to one-loop order is

$$\begin{aligned}\beta(\lambda_R) &= -\varepsilon\,\lambda_R + \lambda_R^2\,C_n\,\frac{N+8}{3} = \frac{C_n\,(N+8)\,\lambda_R}{3}\,(\lambda_R - \lambda_0) \\ \beta_0 &= \frac{C_n\,(N+8)\,\lambda_R}{3} > 0\end{aligned} \tag{5.162}$$

The zero is

$$\lambda_0 = \frac{3\,\varepsilon}{C_n\,(N+8)} \tag{5.163}$$

The critical exponents are

$$\begin{aligned}
\eta \;&=\; \beta(\lambda_R)\,\frac{\partial\,\ln Z_3(M)}{\partial\,\lambda_R} = \Big(-\varepsilon\,\lambda_R + \lambda_R^2\,C_n\,\frac{N+8}{3}\Big)\Big(-\lambda_R\,C_n^2\,\frac{N+2}{36}\,\frac{1}{\varepsilon}\Big) \\
&= \frac{(N+2)}{2\,(N+8)^2}\,\varepsilon^2
\end{aligned} \tag{5.164}$$

The critical exponent μ (Eq. (5.106)) is equal to the anomalous dimension of the operator $\phi_{i,R}^2$

$$\begin{aligned}
\mu \;&=\; -\beta(\lambda_R)\,\frac{\partial\,\ln(Z_{\phi^2}\,Z_3^{-1})}{\partial\,\lambda_R} = -\Big(-\varepsilon\,\lambda_R + \lambda_R^2\,C_n\,\frac{N+8}{3}\Big)\Big(C_n\,\frac{N+2}{3}\,\frac{1}{\varepsilon}\Big) \\
&= \frac{N+2}{N+8}\,\varepsilon
\end{aligned} \tag{5.165}$$

From the hyper scaling equation Eq. (5.109) we get (up to order ε we can neglect η)

$$\gamma = 1 + \frac{\mu}{2} + \cdots = 1 + \frac{N+2}{2(N+8)}\,\varepsilon \tag{5.166}$$

The γ can be also calculated from corrections to the two-point function $\Gamma^{(2)}(p)$ (see Fig. 26)

$$\Gamma^{(2)}(0) = \mu^{-2}\left[1 - \lambda_R\,C_n\,\frac{N+2}{6}\Big(\frac{2}{\varepsilon} - \ln\frac{\mu^2}{M^2} - 1\Big)\right] \tag{5.167}$$

From which we can derive

$$\Gamma^{(2),R}(0) = \mu^{-2}\left[1 + \lambda_R\,C_n\,\frac{N+2}{6}\Big(\ln\frac{\mu^2}{M^2} + 1\Big)\right] \tag{5.168}$$

$$Z_\mu = 1 + \lambda_R\,C_n\,\frac{N+2}{3}\,\frac{1}{\varepsilon} \tag{5.169}$$

$$\gamma_\mu(\lambda_R) = \beta(\lambda_R)\,\frac{\partial\,\ln(Z_m\,Z_3^{-1})}{\partial\,\lambda_R} = -C_n\,\frac{N+2}{3}\,\lambda_R \tag{5.170}$$

From which we derive (see (5.94))

$$\gamma = 1 - \frac{1}{2}\,\gamma_m = 1 + \frac{N+2}{2\,(N+8)}\,\varepsilon \tag{5.171}$$

Which confirms hyper scaling. The critical exponent α is derived from the hyperscaling relation $\gamma = \nu\,(2-\eta)$. In next to leading order this implies ($\eta = \mathcal{O}(\varepsilon^2)$)

$$\nu = \frac{1}{2} + \frac{N+2}{4\,(N+8)}\,\varepsilon \tag{5.172}$$

The critical exponent of the specific heat is $\alpha = 2 - n\,\nu$

$$\alpha = 2 - \frac{n}{2} - \frac{n\,(N+2)}{4\,(N+8)}\,\varepsilon = \frac{\varepsilon}{2} - \frac{N+2}{N+8}\,\varepsilon = \frac{4-N}{2\,(N+8)}\,\varepsilon \tag{5.173}$$

If we put the dimension of the physical system $n = 3$, therefore $\varepsilon = 4 - n = 1$, we obtain

order parameter	ε-expansion	high temperature series
$N = 1$	$\eta = 0.019$	$\eta = 0.041 \pm 0.010$
	$\gamma = 1.167$	$\gamma = 1.250 \pm 0.003$
	$\alpha = 0.167$	$\alpha = 0.110 \pm 0.005$
$N = 2$	$\eta = 0.020$	
	$\gamma = 1,200$	
$N = 3$	$\eta = 0.021$	$\eta = 0.043 \pm 0.014$
	$\gamma = 1.227$	$\gamma = 1.375 \pm 0.040$

Comparison of the ε expansion with the high temperature series.

5.5 The $1/N$ expansion.

Besides the ε expansion we can make another one i.e. the $1/N$ expansion. In this case we take $N \to \infty$. In that case the vertex correction behaves as follows see Fig. 27. The coupling constant in the large N limit is defined as

$$\Lambda^{(0)}_{abcd} = -\lambda_n\,\delta_{ab}\,\delta_{cd} \tag{5.174}$$

We get now a geometric series. All other graphs in a given order are suppressed by powers in N. Only the bubble graphs dominate the perturbation series (see Fig. 28). The bubble sum in Fig. 27 provides us with the effective vertex

$$\Lambda_{abcd}(p,\mu^2)_= - \frac{\lambda_n}{1+\lambda_n\,N\,\Pi(p,\mu^2)}\,\delta_{ab}\,\delta_{cd} \qquad \text{with}$$

$$\Pi(p,\mu^2) = \frac{1}{2}\int \frac{d^n q}{(2\pi)^n}\,\frac{1}{(q^2+\mu^2)\Big((q+p)^2+\mu^2\Big)} \tag{5.175}$$

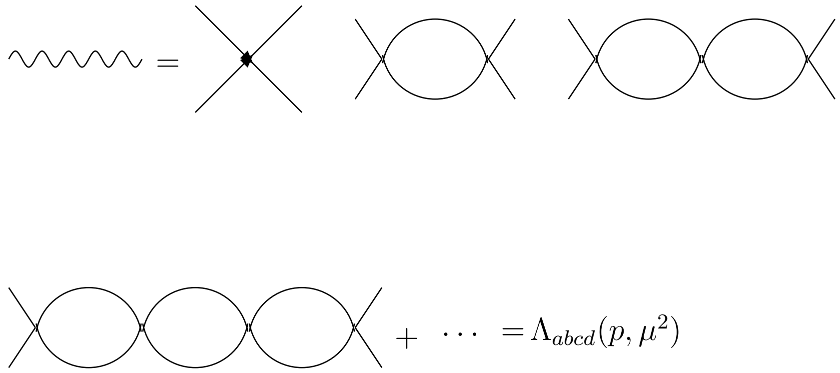


Figure 27: Correction to the vertex in the limit $N \to \infty$.

The effective vertex will now be the variable in which we perturb. As an example we will calculate the critical exponent η. The latter is inferred from the graph in Fig. 29. First we put $\mu = 0$ in Eq. (5.101). The effective vertex $\Lambda(p, 0)$ becomes

$$\Lambda_{abcd}(p, 0) = -\frac{\lambda_n}{1 + \lambda_n \, N \, \Pi(p, 0)} \, \delta_{ab} \, \delta_{cd} \qquad \text{with}$$

$$\Pi(p, 0) = \frac{1}{2} \int \frac{d^n q}{(2\pi)^n} \, \frac{1}{q^2 \, (q + p)^2} = S_n^{-1} \, p^{n-4} \quad \text{and}$$

$$S_n = 2^{n+1} \, \pi^{n/2} \, \frac{\Gamma(n-2)}{\Gamma(2 - n/2) \, \Gamma^2(n/2 - 1)} \tag{5.176}$$

The graph in Fig. 29 is equal to

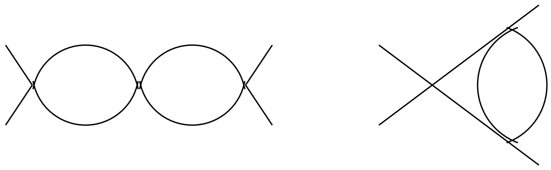
$$\Sigma_{ab}(k) = -\delta_{ab} \int \frac{d^n p}{(2\pi)^n} \, \frac{\lambda_n}{1 + \lambda_n \, N \, \Pi(p, 0)} \, \frac{1}{(p + k)^2} \tag{5.177}$$

We are interested in the infrared behaviour $k \to 0$. Therefore we approximate the integral by providing a cut off

$$\Sigma_{ab}(k) = -\delta_{ab} \int^{\lambda_n} \frac{d^n p}{(2\pi)^n} \, \frac{1}{N \, \Pi(p, 0)} \, \frac{1}{(p + k)^2} \tag{5.178}$$

We can also remove the cut off and apply m-dimensional regularization

$$\Sigma_{ab}(k) = -\delta_{ab} \, (\lambda_n)^{n-m} \int \frac{d^m p}{(2\pi)^m} \, \frac{1}{N \, \Pi(p, 0)} \, \frac{1}{(p + k)^2} \tag{5.179}$$



a *b*

Figure 28: Graph a is of the order N^2 whereas graph b is of the order of N.

The integration is straightforward

$$\Sigma_{ab}(k) = -\frac{\delta_{ab}}{N}\, S_n\, (\lambda_n)^{n-m} \int \frac{d^m p}{(2\pi)^m}\, \frac{1}{(p^2)^{n/2-2}\,(p+k)^2} \tag{5.180}$$

Using Feynman parameters

$$\Sigma_{ab}(k) = -\frac{\delta_{ab}}{N}\, S_n\, (\lambda_n)^{n-m}\, \frac{\Gamma(n/2-1)}{\Gamma(n/2-2)} \int \frac{d^m p}{(2\pi)^m} \int_0^1 dx\, \frac{(1-x)^{n/2-3}}{(p^2 + 2\,x\,p\,k + x\,k^2)^{n/2-1}} \tag{5.181}$$

We are allowed to translate the momentum

$$\Sigma_{ab}(k) = -\frac{\delta_{ab}}{N}\, S_n\, (\lambda_n)^{n-m}\, \frac{\Gamma(n/2-1)}{\Gamma(n/2-2)} \int \frac{d^m p}{(2\pi)^m} \int_0^1 dx\, \frac{(1-x)^{n/2-3}}{(p^2 + x\,(1-x)\,k^2)^{n/2-1}} \tag{5.182}$$

$$\begin{aligned}\Sigma_{ab}(k) &= -\frac{\delta_{ab}}{N}\, S_n\, \frac{\pi^{m/2}}{(2\pi)^m}\, \frac{\Gamma(n/2-m/2-1)}{\Gamma(n/2-2)}\, (\lambda_n)^{n-m} \int_0^1 dx\, x^{m/2-n/2+1}\,(1-x)^{m/2-2} \\ &\quad \times (k^2)^{m/2-n/2+1}\end{aligned} \tag{5.183}$$

$$\begin{aligned}\Sigma_{ab}(k) &= -\frac{\delta_{ab}}{N}\, S_n\, \frac{\pi^{m/2}}{(2\pi)^m}\, (\lambda_n)^{n-m}\, \frac{\Gamma(n/2-m/2-1)}{\Gamma(n/2-2)}\, \frac{\Gamma(m/2-1)\,\Gamma(m/2+2-n/2)}{\Gamma(m+1-n/2)} \\ &\quad \times (k^2)^{m/2-n/2+1}\end{aligned} \tag{5.184}$$

$$\begin{aligned}\Sigma_{ab}(k) &= -\frac{\delta_{ab}}{N}\, S_n\, \frac{\pi^{m/2}}{(2\pi)^m}\, (\lambda_n)^{n-m}\, \frac{\Gamma(n/2-m/2-1)}{\Gamma(n/2-2)}\, \frac{\Gamma(m/2-1)\,\Gamma(m/2+2-n/2)}{\Gamma(m+1-n/2)} \\ &\quad \times (k^2)^{m/2-n/2+1}\end{aligned} \tag{5.185}$$

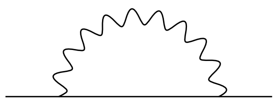


Figure 29: Self energy diagram to order $1/N$.

Put $m = n - \varepsilon$

$$\Sigma_{ab}(k) = \frac{\delta_{ab}}{N} S_n \frac{\pi^{n/2-\varepsilon/2}}{(2\pi)^{n-\varepsilon}} \lambda_n^{\varepsilon} \frac{2\,\Gamma(1+\varepsilon/2)}{(1-\varepsilon/2)\,\varepsilon} \frac{\Gamma(n/2-1-\varepsilon/2)\,\Gamma(2-\varepsilon/2)}{\Gamma(n/2+1-\varepsilon)\,\Gamma(n/2-2)} \times (k^2)^{1-\varepsilon/2} \tag{5.186}$$

Since we are only interested in the logarithmic term we put $\varepsilon = 0$.

$$\Sigma_{ab}(k) = \frac{\delta_{ab}}{N} \frac{4}{\varepsilon} \frac{\Gamma(n-2)}{\Gamma(2-n/2)\,\Gamma(n/2-1)\,\Gamma(n/2+1)\,\Gamma(n/2-2)} k^2 \Big[1 + \frac{\varepsilon}{2} \ln \frac{\lambda_n^2}{k^2}\Big] \tag{5.187}$$

We can now remove the pole term because it is fictitious (it is not a real ultra-violet singularity). Including the lowest order term we get (see Eq. (5.93))

$$\begin{aligned} G_{ab}(k) &= \delta_{ab} \frac{1}{k^2} \Big[1 + \frac{2\,\Gamma(n-2)}{N\,\Gamma(2-n/2)\,\Gamma(n/2-1)\,\Gamma(n/2+1)\,\Gamma(n/2-2)} \ln \frac{\lambda_n^2}{k^2}\Big] \\ &= \delta_{ab} \frac{1}{k^2} \Big[1 - \frac{\eta}{2} \ln \frac{\lambda_n^2}{k^2}\Big] \end{aligned} \tag{5.188}$$

Hence

$$\eta = -\frac{4}{N} \frac{\Gamma(n-2)}{\Gamma(2-n/2)\,\Gamma(n/2-1)\,\Gamma(n/2+1)\,\Gamma(n/2-2)} \tag{5.189}$$

Substitute $n = 4 - \varepsilon$ then

$$\eta = \frac{1}{2} \frac{\varepsilon^2}{N} \tag{5.190}$$

If we take the ε expansion for η in Eq. (5.164) and the limit $N \to \infty$ we get exactly the same formula.

6 Quantum electrodynamics

6.1 Green's function for the Dirac field

The Dirac field is given by

$$\hat{\psi}_\alpha(x) = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\,E_p}} \Big[b(\vec{p},s)\, u_\alpha(\vec{p},s)\, e^{-i\,p\cdot x} + d^\dagger(\vec{p},s)\, v_\alpha(\vec{p},s)\, e^{i\,p\cdot x} \Big] \tag{6.1}$$

with the anti-commutation relations

$$\Big\{ b(\vec{p},s), b^\dagger(\vec{p}',s') \Big\} = \delta_{ss'}\, \delta^{(3)}(\vec{p}-\vec{p}') \qquad \Big\{ d(\vec{p},s), d^\dagger(\vec{p}',s') \Big\} = \delta_{ss'}\, \delta^{(3)}(\vec{p}-\vec{p}') \tag{6.2}$$

The other anti-commutation relations are zero. We compute the following distributions

$$\begin{aligned} -i\,S(x-x') &= \Big\{ \hat{\psi}(x), \hat{\bar{\psi}}(x') \Big\} = \frac{1}{(2\pi)^3} \sum_{s,s'} \int d^3p \int d^3p' \frac{1}{\sqrt{4\,E_p E_{p'}}} \Big[e^{i\,(p'\cdot x' - p\cdot x)} \\ &\quad \times u(\vec{p},s)\, \bar{u}(\vec{p}',s') \Big\{ b(\vec{p},s), b^\dagger(\vec{p}',s') \Big\} + e^{-i\,(p'\cdot x' - p\cdot x)}\, v(\vec{p},s)\, \bar{v}(\vec{p}',s') \\ &\quad \times \Big\{ d^\dagger(\vec{p},s), d(\vec{p}',s') \Big\} \Big] \end{aligned} \tag{6.3}$$

We normalise the spinors in a different way and choose Eq. (1.66). Therefore the projection operators become

$$\sum_s u(\vec{p},s)\, \bar{u}(\vec{p},s) = \not{p} + m \qquad \sum_s v(\vec{p},s)\, \bar{v}(\vec{p},s) = \not{p} - m \tag{6.4}$$

The distribution becomes

$$S(x-x') = \frac{i}{(2\pi)^3} \int \frac{d^3\vec{p}}{2\,E_p} \Big\{ (\not{p}+m)\, e^{i\,p\cdot(x'-x)} + (\not{p}-m)\, e^{-i\,p\cdot(x'-x)} \Big\} \tag{6.5}$$

Notice that S is a four by four matrix. Substitute in the last term $\vec{p}$ by $-\vec{p}$ then

$$\begin{aligned} S(x-x') &= \frac{i}{(2\pi)^3} \int \frac{d^3\vec{p}}{2\,E_p} \Big\{ (\not{p}+m)\, e^{i\,p^0\cdot(x_0'-x_0)} - \Big(\gamma_0(-p^0) - \vec{\gamma}\cdot\vec{p} + m\Big) \\ &\quad \times e^{i\,(-p^0)\cdot(x_0'-x_0)} \Big\}\, e^{-i\,\vec{p}\cdot(\vec{x}'-\vec{x})} \\ &= \frac{i}{(2\pi)^3} \int d^4p\, \epsilon(p^0)\, \delta(p^2-m^2)\, (\not{p}+m)\, e^{i\,p\cdot(x'-x)} \end{aligned} \tag{6.6}$$

with $p^0 = E_p$. The distribution satisfies the Dirac equation

$$\Big(i\,\gamma^\mu\,\partial_\mu - m)\,S(x - x') = 0 \tag{6.7}$$

Next we define the positive and negative energy contributions to the Feynman propagator

$$\begin{aligned}
S_+(x - x') &= \langle 0|\hat{\psi}(x)\,\hat{\bar{\psi}}(x')|0\rangle = \frac{1}{(2\pi)^3}\int d^4p\,\theta(p^0)\,\delta(p^2 - m^2)\,(\not{p} + m)\,e^{-i\,p\cdot(x-x')} \\
S_-(x - x') &= \langle 0|\hat{\bar{\psi}}(x')\,\hat{\psi}(x)|0\rangle = -\frac{1}{(2\pi)^3}\int d^4p\,\theta(-p^0)\,\delta(p^2 - m^2)\,(\not{p} + m)\,e^{-i\,p\cdot(x-x')} \\
&= \frac{1}{(2\pi)^3}\int d^4p\,\theta(p^0)\,\delta(p^2 - m^2)\,(\not{p} - m)\,e^{-i\,p\cdot(x'-x)}
\end{aligned} \tag{6.8}$$

Both satisfy the Dirac equation

$$\Big(i\,\gamma^\mu\,\partial_\mu - m)\,S_\pm(x - x') = 0 \tag{6.9}$$

S can be expressed into $S_\pm$ as follows

$$-i\,S(x - x') = S_+(x - x') + S_-(x - x') \tag{6.10}$$

The time ordered product for the fermion field

$$i\,S_F(x - x') = \langle 0|T\Big(\hat{\psi}(x)\,\hat{\bar{\psi}}(x)\Big)|0\rangle \tag{6.11}$$

or Feynman propagator is a Green's function because it satisfies

$$\Big(i\,\gamma^\mu\,\partial_\mu - m\Big)\,i\,S_F(x - x') = i\,\delta^{(4)}(x - x') \tag{6.12}$$

Proof:

$$T\Big(\hat{\psi}(x)\,\hat{\bar{\psi}}(x')\Big) = \theta(x_0 - x_0')\,\hat{\psi}(x)\,\hat{\bar{\psi}}(x') - \theta(x_0' - x_0)\,\hat{\bar{\psi}}(x')\,\hat{\psi}(x) \tag{6.13}$$

or

$$i\,S_F(x - x') = \theta(x_0 - x_0')\,S_+(x - x') - \theta(x_0' - x_0)\,S_-(x - x') \tag{6.14}$$

We now use

$$\frac{\partial}{\partial\,x_0}\,\theta(x_0 - x_0') = \delta(x_0 - x_0') \qquad \frac{\partial}{\partial\,x_0}\,\theta(x_0' - x_0) = -\delta(x_0 - x_0') \tag{6.15}$$

so that

$$\begin{aligned}
\frac{\partial}{\partial\,x_0}\,i\,S_F(x - x') &= -\delta(x_0 - x_0')\,i\,S(x - x') + \theta(x_0 - x_0')\,\frac{\partial}{\partial\,x_0}\,S_+(x - x') \\
&\quad -\theta(x_0' - x_0)\,\frac{\partial}{\partial\,x_0}\,S_-(x - x')
\end{aligned} \tag{6.16}$$

Further is

$$\partial^i \theta(x_0 - x_0') = 0 \tag{6.17}$$

We obtain

$$\begin{aligned}\Big(i\,\gamma_\mu\,\partial^\mu - m\Big)\,i\,S_F(x-x') &= \delta(x_0 - x_0')\,i\,\gamma_0\Big(-\,i\,S(x-x')\Big) + \theta(x_0 - x_0')\\ &\times\Big(i\,\gamma_\mu\,\partial^\mu - m\Big)\,S_+(x-x') - \theta(x_0' - x_0)\,\Big(i\,\gamma_\mu\,\partial^\mu - m\Big)\\ &\times S_-(x-x') = \gamma_0\,\delta(x_0 - x_0')\,S(x-x') \end{aligned}\tag{6.18}$$

Using the expression for $S(x-x')$

$$\begin{aligned}\Big(i\,\gamma_\mu\,\partial^\mu - m\Big)\,i\,S_F(x-x') &= \frac{1}{(2\pi)^3}\int\frac{d^3\vec{p}}{2\,E_p}\bigg[\gamma_0\,\Big(\gamma_0\,p^0 + \gamma_i\,p^i + m\Big)\,e^{i\,p^0\,(x_0'-x_0)}\\ &+\gamma_0\,\Big(\gamma_0\,p^0 - \gamma_i\,p^i - m\Big)\,e^{-i\,p^0\,(x_0'-x_0)}\bigg]\,e^{i\,\vec{p}\cdot(\vec{x}-\vec{x}')}\\ &\times\delta(x_0 - x_0') = \frac{i}{(2\pi)^3}\int d^3\vec{p}\,e^{i\,\vec{p}\cdot(\vec{x}-\vec{x}')}\,\delta(x_0 - x_0')\\ &= i\,\delta^{(4)}(x-x') \end{aligned}\tag{6.19}$$

The canonical anti-commutation relation is derived from

$$-i\,S(x-x')\,\gamma_0|_{x_0=x_0'} = \Big\{\hat{\psi}(x),\hat{\psi}^\dagger(x')\Big\} = i\,\delta^{(3)}(\vec{x}-\vec{x}') \tag{6.20}$$

Furthermore we can write for $S_F(x-x')$

$$i\,S_F(x-x') = i\int\frac{d^4p}{(2\pi)^4}\,e^{-i\,p\cdot(x-x')}\,\frac{\not{p}+m}{p^2 - m^2 + i\,\varepsilon} \tag{6.21}$$

This satisfies the aforementioned differential equation Eq. (6.19). Finally we have the distribution

$$\begin{aligned}-S_1(x-x') &= \langle 0|\Big[\hat{\psi}(x),\hat{\bar{\psi}}(x')\Big]|0\rangle = S_+(x-x') - S_-(x-x')\\ &= \frac{1}{(2\pi)^3}\int d^4p\,\delta(p^2 - m^2)\,\bigg[\theta(p^0)\,(\not{p}+m)\,e^{-i\,p\cdot(x-x')}\\ &+\theta(-p^0)\,(\not{p}+m)\,e^{-i\,p\cdot(x-x')}\bigg]\\ &= \frac{1}{(2\pi)^3}\int d^4p\,e^{-i\,p\cdot(x-x')}\,\delta(p^2 - m^2)\,(\not{p}+m) \end{aligned}\tag{6.22}$$

The relations between the the scalar and fermion Green's functions are

$$\begin{aligned}
S_\pm(x-x') &= \pm\,(i\,\gamma^\mu\,\partial_\mu + m)\,\Delta_\pm(x-x') \\
-i\,S(x-x') &= (i\,\gamma^\mu\,\partial_\mu + m)\,i\,\Delta(x-x') \\
-i\,S_1(x-x') &= (i\,\gamma^\mu\,\partial_\mu + m)\,i\,\Delta_1(x-x') \\
i\,S_F(x-x') &= (i\,\gamma^\mu\,\partial_\mu + m)\,i\,\Delta_F(x-x')
\end{aligned} \tag{6.23}$$

Hence it follows that

$$\begin{aligned}
&-i\,S(x-x') = \Big\{\hat\psi(x), \hat{\bar\psi}(x')\Big\} = 0 \quad \text{when} \quad (x-x')^2 < 0 \\
&\text{because} \quad S(x-x') = 0 \quad \text{when} \quad (x-x')^2 < 0
\end{aligned} \tag{6.24}$$

micro causality

6.2 Green's function for the electromagnetic field

In defining the Feynman propagator for the electromagnetic field one encounters a problem that is related to gauge invariance Define

$$i\,D_F^{\mu\nu} = \langle 0|T\Big(A^\mu(x)\,A^\nu(x')\Big)|0\rangle \tag{6.25}$$

Because of Bose Einstein statistics

$$T\Big(A^\mu(x)\,A^\nu(x')\Big) = \theta(x_0 - x_0')\,A^\mu(x)\,A^\nu(x') + \theta(x_0' - x_0)\,A^\nu(x')\,A^\mu(x') \tag{6.26}$$

Further we assume that in analogy with the scalar field and the fermion field $i\,D_F^{\mu\nu}$ is a Green's function that satisfies the following differential equation

$$L_{\mu\nu}(x)\,i\,D_F^{\mu\nu}(x-x') = i\,\delta_\mu^\lambda\,\delta^{(4)}(x-x') \tag{6.27}$$

with the equation of motion

$$L_{\mu\nu}(x)\,A^\nu(x) = 0 \tag{6.28}$$

$L_{\mu\nu}$ follows from the Euler Lagrange equation

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} \quad \text{with} \quad F^{\mu\nu} = \partial^\mu\,A^\nu(x) - \partial^\nu\,A^\mu(x) \\
&\partial^\nu\,\frac{\partial\,\mathcal{L}}{\partial\,\partial^\nu\,A^\mu} = \frac{\partial\,\mathcal{L}}{\partial\,A^\mu} \quad \to \quad \partial^\nu\,F_{\mu\nu} = 0
\end{aligned} \tag{6.29}$$

or

$$\left(g_{\mu\nu}\,\Box - \partial_\mu\,\partial_\nu\right) A^\nu(x) = 0 \quad\Longrightarrow\quad L_{\mu\nu}(x) = g_{\mu\nu}\,\Box - \partial_\mu\,\partial_\nu \tag{6.30}$$

We have to invert $L_{\mu\nu}$ in other words

$$D_F^{\nu\lambda} = i\,\left(L^{-1}\right)^{\nu\mu}\,\delta_\mu^\lambda\,\delta^{(4)}(x-x') \tag{6.31}$$

Unfortunately L^{-1} does not exist this is because $\mathcal{L}$ is locally gauge invariant. $F^{\mu\nu}$ is invariant under the local gauge transformation

$$A_\mu(x) \quad\rightarrow\quad A_\mu(x) + \partial_\mu\,\Lambda(x) \quad\Longrightarrow\quad F_{\mu\nu}(x) \quad\rightarrow\quad F_{\mu\nu}(x) \tag{6.32}$$

This implies that if

$$L_{\mu\nu}(x)\,A^\nu(x) = 0 \tag{6.33}$$

holds it is also true that

$$L_{\mu\nu}(x)\left(A^\nu(x) + \partial^\nu\,\Lambda(x)\right) = 0 \tag{6.34}$$

in other words

$$L_{\mu\nu}(x)\,\partial^\nu\,\Lambda(x) = 0 \tag{6.35}$$

This implies that $L_{\mu\nu}$ has an eigenvalue which is zero. Therefore $L_{\mu\nu}$ is not invertible. To find a solution for $D_F^{\mu\nu}$ we have to break the gauge invariance. This implies that we have to make a gauge choice for A^μ. Let us make a covariant gauge choice

$$\mathcal{L}' = \mathcal{L} - \frac{1}{2\alpha}\left(\partial_\mu\,A^\mu(x)\right)^2 \tag{6.36}$$

Application of the Euler-Lagrange equations yields

$$L_{\mu\nu}(x) = g_{\mu\nu}\,\Box - \left(1 - \frac{1}{\alpha}\right)\,\partial_\mu\,\partial_\nu \tag{6.37}$$

Because $\mathcal{L}'$ is not invariant under local gauge transformations $\partial^\nu\,\Lambda$ is not an eigenvector anymore with eigenvalue zero. We can now invert $L_{\mu\nu}$

$$\left(g_{\mu\nu}\,\Box - \left(1 - \frac{1}{\alpha}\right)\,\partial_\mu\,\partial_\nu\right) i\,D_F^{\nu\lambda}(x) = i\,\delta_\mu^\lambda\,\delta^{(4)}(x-x') \tag{6.38}$$

Take the following general solution

$$D_F^{\nu\lambda} = \frac{1}{(2\pi)^4}\int d^4k\,e^{-i\,k\cdot(x-x')}\,\frac{1}{k^2 + i\,\varepsilon}\left(a\,g^{\nu\lambda} + b\,\frac{k^\nu\,k^\lambda}{k^2}\right) \tag{6.39}$$

Substitution in Eq. (6.38) yields

$$\frac{1}{(2\pi)^4} \int d^4k \, e^{-i\,k\cdot(x-x')} \, \frac{1}{k^2 + i\,\varepsilon} \left(- g_{\mu\nu}\, k^2 + \left(1 - \frac{1}{\alpha}\right) k_\mu \, k_\nu \right) \left(a\, g^{\nu\lambda} + b\, \frac{k^\nu\, k^\lambda}{k^2} \right)$$

$$= \frac{1}{(2\pi)^4} \int d^4k \, e^{-i\,k\cdot(x-x')} \, \delta_\mu^\lambda \tag{6.40}$$

Hence we have the solution

$$a = -1 \qquad a\left(1 - \frac{1}{\alpha}\right) - \frac{b}{\alpha} = 0 \quad \rightarrow \quad b = 1 - \alpha \tag{6.41}$$

The Feynman propagator becomes

$$D_F^{\nu\lambda}(x - x') = -\frac{1}{(2\pi)^4} \int d^4k \, e^{-i\,k\cdot(x-x')} \, \frac{1}{k^2 + i\,\varepsilon} \left(g^{\nu\lambda} - (1-\alpha)\, \frac{k^\nu\, k^\lambda}{k^2} \right) \tag{6.42}$$

Notice that the photon is massless. Two gauge choices are very popular. The Feynman gauge is given by $\alpha = 1$ and the Landau gauge is given by $\alpha = 0$. We could also choose the radiation gauge. However this is not covariant. In this gauge the electromagnetic field is equal to

$$\vec{A}(\vec{x}, t) = \sum_{\alpha=1,2} \int \frac{d^3\,\vec{k}}{(2\pi)^{3/2}} \, \frac{1}{\sqrt{2\,\omega}} \left[a(\vec{k}, \alpha)\, \vec{\epsilon}^{(\alpha)}(\vec{k})\, e^{-i\,k\cdot x} + a^\dagger(\vec{k}, \alpha)\, \vec{\epsilon}^{(\alpha)}(\vec{k})\, e^{i\,k\cdot x} \right] \tag{6.43}$$

Define

$$\epsilon^{(\alpha),\mu}(\vec{k}) = \left(0, \vec{\epsilon}^{(\alpha)}(\vec{k})\right) \tag{6.44}$$

so that we can write A^μ in a pseudo covariant way

$$A^\mu(x) = \sum_{\alpha=1,2} \int \frac{d^3\,\vec{k}}{(2\pi)^{3/2}} \, \frac{1}{\sqrt{2\,\omega}} \left[a(\vec{k}, \alpha)\, \epsilon^{(\alpha),\mu}(\vec{k})\, e^{-i\,k\cdot x} + a^\dagger(\vec{k}, \alpha)\, \epsilon^{(\alpha),\mu}(\vec{k})\, e^{i\,k\cdot x} \right] \tag{6.45}$$

The sum over the transverse polarisations is

$$\sum_\alpha \epsilon^{(\alpha),i}(\vec{k})\, \epsilon^{(\alpha),j}(\vec{k}) = \delta_{ij} - \frac{k_i\, k_j}{|\vec{k}|^2} \tag{6.46}$$

Define

$$n = (1, 0, 0, 0) \qquad |\vec{k}|^2 = (k \cdot n)^2 - k^2 \tag{6.47}$$

then

$$P^{\mu\nu}(k) = \sum_\alpha \epsilon^{(\alpha),\mu}(k)\, \epsilon^{(\alpha),\nu}(k) = -g^{\mu\nu} - \frac{n^2\, k^\mu\, k^\nu - k\cdot n\, (k^\mu\, n^\nu + n^\mu\, k^\nu) + k^2\, n^\mu\, n^\nu}{(k\cdot n)^2 - k^2} \tag{6.48}$$

Therefore the Feynman propagator in this gauge looks like

$$i\, D_F^{\mu\nu}(x - x') = \frac{1}{(2\pi)^4} \int d^4k \, e^{-i\,k\cdot(x-x')} \, \frac{i}{k^2 + i\,\varepsilon} \, P^{\mu\nu}(k) \tag{6.49}$$

In the subsequent sections we will choose the covariant gauge. Notice that S-matrix elements are independent of any gauge choice but Green's functions are not.

6.3 Feynman rules for QED

The Lagrangian density in the case of a spin half field (electron, muon) is given by

$$\begin{aligned}\mathcal{L}_{\text{QED}}(x) &= \bar{\psi}(x)\,\gamma_\mu\,\partial^\mu\,\psi(x) - m\,\bar{\psi}(x)\,\psi(x) - e\,\bar{\psi}(x)\,\gamma_\mu\,\psi(x)\,A^\mu(x) \\ &\quad -\frac{1}{4}\,F_{\mu\nu}(x)\,F^{\mu\nu}(x) \end{aligned} \tag{6.50}$$

with

$$F_{\mu\nu}(x) = \partial_\mu\,A_\nu(x) - \partial_\nu\,A_\mu(x) \tag{6.51}$$

$\mathcal{L}(x)$ is invariant under local gauge transformations

$$\begin{aligned} A_\mu(x) &\;\to\; A'_\mu(x) = A_\mu(x) - \frac{1}{e}\,\partial_\mu\,\alpha(x) \\ \psi(x) &\;\to\; \psi'(x) = e^{i\,\alpha(x)}\,\psi(x) \end{aligned} \tag{6.52}$$

The Feynman rules are given by

1 Draw at a given order in e all topological inequivalent connected graphs.

2 Each internal photon line is represented by the propagator $i\,\tilde{D}_F^{\mu\nu}(k)$

$$i\,\tilde{D}_F^{\mu\nu}(k) = -\frac{i}{k^2 + i\,\varepsilon}\left(g^{\mu\nu} - (1-\alpha)\,\frac{k^\mu\,k^\nu}{k^2}\right) \tag{6.53}$$

3 Each internal fermion line is represented by the propagator $i\,\tilde{S}_F(p)$

$$i\,\tilde{S}_F(p) = \frac{i\,(\not{p} + m)}{p^2 - m^2 + i\,\varepsilon} = \frac{i}{\not{p} - m} \tag{6.54}$$

4 Each fermion-fermion-photon vertex with incoming momenta p_1, p_2, p_3 is given by

$$-i\,e\,\gamma_\mu\,(2\pi)^4\,\delta^{(4)}(p_1 + p_2 + p_3) \tag{6.55}$$

5 Each incoming fermion line gets the spinor $u(\vec{p}, s)$. Each outgoing fermion line gets the spinor $\bar{u}(\vec{p}, s)$. Each incoming anti-fermion line gets the anti-spinor $\bar{v}(\vec{p}, s)$. Each outgoing anti-fermion line gets the anti-spinor $v(\vec{p}, s)$.

6 Each incoming or outgoing photon gets the polarisation vector $\epsilon_\mu^{(\alpha)}(k)$.

7 Integrate over each internal loop momentum k with the measure

$$\int \frac{d^4k}{(2\pi)^4} \tag{6.56}$$

8 A factor -1 is needed for each internal fermion loop.

These rules provide us with the amplitude $M_{fi}\,(2\pi)^4\,\delta^{(4)}(\sum_i p_i)$. There are some additional comments. We get a relative minus sign if one Feynman diagram is obtained from another one by exchanging two identical fermions. Finally we get an extra minus sign if an incoming (outgoing) fermion is exchanged with an outgoing (incoming) anti-fermion. The latter is called crossing. Finally the anti-electron is called the positron.

6.4 Lowest order processes in QED

Our first application will be electron-electron scattering (Möller scattering)

$$e^-(p_1, s_1) + e^-(p_2, s_2) \quad \rightarrow \quad e^-(p_3, s_3) + e^-(p_4, s_4) \tag{6.57}$$

The amplitude M_{fi} is equal to

$$\begin{aligned} M_{fi} \;=\; & i\,(-i\,e)^2 \Big[\bar{u}(p_3, s_3)\,\gamma_\mu\,u(p_1, s_1)\,\tilde{D}_F^{\mu\nu}(p_1 - p_3)\,\bar{u}(p_4, s_4)\,\gamma_\nu\,u(p_2, s_2) \\ & -\bar{u}(p_3, s_3)\,\gamma_\mu\,u(p_2, s_2)\,\tilde{D}_F^{\mu\nu}(p_2 - p_3)\,\bar{u}(p_4, s_4)\,\gamma_\nu\,u(p_1, s_1)\Big] \end{aligned} \tag{6.58}$$

Notice that apparently M_{fi} depends on the gauge choice parameter α. However it turns out that this is not the case. This we will show below. Substitute $i\,\tilde{D}_F^{\mu\nu}$

$$\begin{aligned} M_{fi} \;=\; & i\,e^2 \Bigg[\frac{\Big\{\bar{u}(p_3, s_3)\,\gamma^\mu\,u(p_1, s_1)\Big\}\,\Big\{\bar{u}(p_4, s_4)\,\gamma_\mu\,u(p_2, s_2)\Big\}}{(p_1 - p_3)^2 + i\,\varepsilon} \\ & - \frac{\Big\{\bar{u}(p_3, s_3)\,\gamma^\mu\,u(p_2, s_2)\Big\}\,\Big\{\bar{u}(p_4, s_4)\,\gamma_\mu\,u(p_1, s_1)\Big\}}{(p_2 - p_3)^2 + i\,\varepsilon} \\ & -(1-\alpha)\,\frac{\Big\{\bar{u}(p_3, s_3)\,(\not{p}_3 - \not{p}_1)\,u(p_1, s_1)\Big\}\,\Big\{\bar{u}(p_4, s_4)\,(\not{p}_3 - \not{p}_1)\,u(p_2, s_2)\Big\}}{(p_1 - p_3)^2\,\Big[(p_1 - p_3)^2 + i\,\varepsilon\Big]} \\ & +(1-\alpha)\,\frac{\Big\{\bar{u}(p_3, s_3)\,(\not{p}_3 - \not{p}_2)\,u(p_2, s_2)\Big\}\,\Big\{\bar{u}(p_4, s_4)\,(\not{p}_3 - \not{p}_2)\,u(p_1, s_1)\Big\}}{(p_2 - p_3)^2\,\Big[(p_2 - p_3)^2 + i\,\varepsilon\Big]}\Bigg] \end{aligned} \tag{6.59}$$

Substitute at the end in the last two terms $p_3 - p_1 = p_2 - p_4$ and $p_3 - p_2 = p_1 - p_4$. Because of current conservation the last two terms in M_{fi} are equal to zero. Proof:

$$\begin{aligned} & (p_3 - p_1)_\mu\,\bar{u}(p_3, s_3)\,\gamma^\mu\,u(p_1, s_1) = \bar{u}(p_3, s_3)\,(\not{p}_3 - \not{p}_1)\,u(p_1, s_1) \\ & = \bar{u}(p_3, s_3)\,\Big[(\not{p}_3 - m) - (\not{p}_1 - m)\Big]\,u(p_1, s_1) = 0 \end{aligned} \tag{6.60}$$

idem

$$\bar{u}(p_4, s_4)\,(\not{p}_2 - \not{p}_4)\,u(p_2, s_2) = 0 \tag{6.61}$$

etc. In other words M_{fi} is independent of the gauge choice. Therefore only the term $-i\,g_{\mu\nu}/k^2$ in $\tilde{D}_F^{\mu\nu}(k)$ contributes. In general Feynman graphs depend on α. However S-matrix elements or transition amplitudes M_{fi} never depend on α. On the other hand Green's function depend on α. The calculation of the effective cross section for Möller scattering proceeds as follows. Call the Mandelstam variables

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \qquad t = (p_1 - p_3)^2 = (p_2 - p_4)^2 \\ u &= (p_2 - p_3)^2 = (p_1 - p_4)^2 \end{aligned} \tag{6.62}$$

The amplitude squared reads

$$\begin{aligned} |M_{fi}|^2 &= e^4 \Big[\Big\{ u(p_3, s_3)\,\bar{u}(p_3, s_3)\,\gamma^\mu\,u(p_1, s_1)\,\bar{u}(p_1, s_1)\,\gamma^\nu \Big\} \Big\{ u(p_4, s_4)\,\bar{u}(p_4, s_4)\,\gamma_\mu \\ &\quad u(p_2, s_2)\,\bar{u}(p_2, s_2)\,\gamma_\nu \Big\} \frac{1}{t^2} + \Big\{ u(p_3, s_3)\,\bar{u}(p_3, s_3)\,\gamma^\mu\,u(p_2, s_2)\,\bar{u}(p_2, s_2)\,\gamma^\nu \Big\} \\ &\quad \Big\{ u(p_4, s_4)\,\bar{u}(p_4, s_4)\,\gamma_\mu\,u(p_1, s_1)\,\bar{u}(p_1, s_1)\,\gamma_\nu \Big\} \frac{1}{u^2} - \Big\{ u(p_3, s_3)\,\bar{u}(p_3, s_3)\,\gamma^\mu \\ &\quad u(p_1, s_1)\,\bar{u}(p_1, s_1)\,\gamma_\nu\,u(p_4, s_4)\,\bar{u}(p_4, s_4)\,\gamma_\mu\,u(p_2, s_2)\,\bar{u}(p_2, s_2)\,\gamma^\nu \Big\} \frac{1}{t\,u} \\ &\quad - \Big\{ u(p_3, s_3)\,\bar{u}(p_3, s_3)\,\gamma^\nu\,u(p_2, s_2)\,\bar{u}(p_2, s_2)\,\gamma_\mu\,u(p_4, s_4)\,\bar{u}(p_4, s_4)\,\gamma_\nu \\ &\quad u(p_1, s_1)\,\bar{u}(p_1, s_1)\,\gamma^\mu \Big\} \frac{1}{t\,u} \Big] \end{aligned} \tag{6.63}$$

Since $t < 0$ and $u < 0$ we can omit $i\,\varepsilon$. We consider the cross section where we sum over all spins in the final state 3 and 4. Therefore we look at the production of electrons where each spin is allowed. Furthermore the incoming electrons unpolarised. This implies that we also sum over the initial spins 1 and 2 provided we average over the initial spins. The average factor for each incoming particle with spin s_i is $1/(2\,s_i + 1)$. Notice that

$$\sum_s u(p, s)\,\bar{u}(p, s) = \not{p} + m \quad \text{and} \quad \sum_s v(p, s)\,\bar{v}(p, s) = \not{p} - m \tag{6.64}$$

Using the above we have

$$\overline{|M_{fi}|^2} = \frac{1}{(2\,s_1 + 1)\,(2\,s_2 + 1)} \sum_{s_1, s_2, s_3, s_4} |M_{fi}|^2 = \frac{e^4}{4} \Big[Tr \Big\{ (\not{p}_3 + m)\,\gamma^\mu\,(\not{p}_1 + m)\,\gamma^\nu \Big\}$$

$$\times Tr\left\{(\not p_4 + m)\,\gamma_\mu\,(\not p_2 + m)\,\gamma_\nu\right\}\frac{1}{t^2} + Tr\left\{(\not p_3 + m)\,\gamma^\mu\,(\not p_2 + m)\,\gamma^\nu\right\}$$

$$\times Tr\left\{(\not p_4 + m)\,\gamma_\mu\,(\not p_1 + m)\,\gamma_\nu\right\}\frac{1}{u^2} - Tr\left\{(\not p_3 + m)\,\gamma^\mu\,(\not p_1 + m)\,\gamma_\nu\,(\not p_4 + m)\right.$$

$$\left.\times\gamma_\mu\,(\not p_2 + m)\,\gamma^\nu\right\}\frac{1}{t\,u} - Tr\left\{(\not p_3 + m)\,\gamma^\nu\,(\not p_2 + m)\,\gamma_\mu\,(\not p_4 + m)\,\gamma_\nu\right.$$

$$\left.\left.\times(\not p_2 + m)\,\gamma^\mu\right\}\frac{1}{t\,u}\right] \tag{6.65}$$

We can calculate these traces. Use the following relations in n dimensions.

$$\left\{\gamma_\mu, \gamma_\nu\right\} = 2\,g_{\mu\nu} \qquad \gamma_\mu\,\gamma^\mu = \delta^\mu_\mu = n \qquad \gamma_\mu\,\gamma_\alpha\,\gamma^\mu = (2-n)\,\gamma_\alpha$$

$$\gamma_\mu\,\gamma_\alpha\,\gamma_\beta\,\gamma^\mu = 4\,g_{\alpha\beta} + (n-4)\,\gamma_\alpha\,\gamma_\beta$$

$$\gamma_\mu\,\gamma_\alpha\,\gamma_\beta\,\gamma_\lambda\,\gamma^\mu = -2\,\gamma_\lambda\,\gamma_\beta\,\gamma_\alpha - (n-4)\,\gamma_\alpha\,\gamma_\beta\,\gamma_\lambda \qquad \left\{\gamma_5, \gamma_\mu\right\} = 0 \tag{6.66}$$

The traces are

$$Tr(\text{odd number of }\gamma\text{-matrices}) = 0 \qquad Tr\,1 = n \qquad Tr\,\gamma_\alpha\,\gamma_\beta = 4\,g_{\alpha\beta}$$

$$Tr\,\gamma_\alpha\,\gamma_\beta\,\gamma_\lambda\,\gamma_\rho = 4\left\{g_{\alpha\beta}\,g_{\lambda\rho} - g_{\alpha\lambda}\,g_{\beta\rho} + g_{\alpha\rho}\,g_{\beta\lambda}\right\}$$

$$Tr\,\gamma_5 = 0 \qquad Tr\,\gamma_5\,\gamma_\alpha\,\gamma_\beta = 0 \tag{6.67}$$

The generalisation to n dimensions is only necessary if we calculate loop integrals which contain ultraviolet divergences (n dimensional regularisation) otherwise we put $n = 4$. Let us calculate the traces which occur above

$$Tr\,(\not p_3 + m)\,\gamma^\mu\,(\not p_1 + m)\,\gamma^\nu = Tr\,\not p_3\,\gamma^\mu\,\not p_1\,\gamma^\nu + m^2\,Tr\,\gamma^\mu\,\gamma^\nu$$

$$= 4\left[p_3^\mu\,p_1^\nu + p_3^\nu\,p_1^\mu + g^{\mu\nu}\,(m^2 - p_1\cdot p_3)\right]$$

$$Tr\,(\not p_4 + m)\,\gamma_\mu\,(\not p_2 + m)\,\gamma_\nu = 4\left[p_{4,\mu}\,p_{2,\nu} + p_{4,\nu}\,p_{2,\mu} + g_{\mu\nu}\,(m^2 - p_2\cdot p_4)\right]$$

$$Tr\,(\not p_3 + m)\,\gamma^\mu\,(\not p_2 + m)\,\gamma^\nu = 4\left[p_3^\mu\,p_2^\nu + p_3^\nu\,p_2^\mu + g^{\mu\nu}\,(m^2 - p_2\cdot p_3)\right]$$

$$Tr\,(\not p_4 + m)\,\gamma_\mu\,(\not p_1 + m)\,\gamma_\nu = 4\left[p_{4,\mu}\,p_{1,\nu} + p_{4,\nu}\,p_{1,\mu} + g_{\mu\nu}\,(m^2 - p_1\cdot p_4)\right] \tag{6.68}$$

$$Tr\,(\not p_3 + m)\,\gamma^\mu\,(\not p_1 + m)\,\gamma_\nu\,(\not p_4 + m)\,\gamma_\mu\,(\not p_2 + m)\,\gamma^\nu = Tr\left\{\not p_3\,\gamma^\mu\,\not p_1\,\gamma_\nu\ + m\,\gamma^\mu\,\not p_1\,\gamma_\nu\right.$$

$$+m\,\not{p}_3\,\gamma^\mu\,\gamma_\nu + m^2\,\gamma^\mu\,\gamma_\nu\Big\}\Big\{\not{p}_4\,\gamma_\mu\,\not{p}_2\,\gamma^\nu + m\,\gamma_\mu\,\not{p}_2\,\gamma^\nu + m\,\not{p}_4\,\gamma_\mu\,\gamma^\nu + m^2\,\gamma_\mu\,\gamma^\nu\Big\} =$$

$$Tr\,\not{p}_3\,\gamma^\mu\,\not{p}_1\,\gamma_\nu\,\not{p}_4\,\gamma_\mu\,\not{p}_2\,\gamma^\nu + m^2\,Tr\,\not{p}_3\,\gamma^\mu\,\not{p}_1\,\gamma_\nu\,\gamma_\mu\,\gamma^\nu + m^2\,Tr\,\gamma^\mu\,\not{p}_1\,\gamma_\nu\,\gamma_\mu\,\not{p}_2\,\gamma^\nu$$

$$+m^2\,Tr\,\gamma^\mu\,\not{p}_1\,\gamma_\nu\,\not{p}_4\,\gamma_\mu\,\gamma^\nu + m^2\,Tr\,\not{p}_3\,\gamma^\mu\,\gamma_\nu\,\gamma_\mu\,\not{p}_2\,\gamma^\nu + m^2\,Tr\,\not{p}_3\,\gamma^\mu\,\gamma_\nu\,\not{p}_4\,\gamma_\mu\,\gamma^\nu$$

$$+m^2\,Tr\,\gamma^\mu\,\gamma_\nu\,\not{p}_4\,\gamma_\mu\,\not{p}_2\,\gamma^\nu + m^4\,Tr\,\gamma^\mu\,\gamma_\nu\,\gamma_\mu\,\gamma^\nu = -2\,Tr\,\not{p}_3\,\gamma^\mu\,\not{p}_1\,\not{p}_2\,\gamma_\mu\,\not{p}_4$$

$$+4\,m^2\,p_{1,\nu}\,Tr\,\not{p}_3\,\gamma^\nu + 4\,m^2\,p_{2,\mu}\,Tr\,\gamma^\mu\,\not{p}_1 + 4\,m^2\,p_{4,\mu}\,Tr\,\gamma^\mu\,\not{p}_1 + 4\,m^2\,p_{2,\mu}\,Tr\,\not{p}_3\,\gamma^\mu$$

$$+4\,m^2\,p_{4,\mu}\,Tr\,\not{p}_3\,\gamma^\mu + 4\,m^2\,p_{4,\mu}\,Tr\,\not{p}_2\,\gamma^\nu - 2\,m^4\,Tr\,\gamma^\mu\,\gamma_\mu$$

$$= -32\,(p_1\cdot p_2)\,(p_3\cdot p_4) + 16\,m^2\,\Big(p_1\cdot p_3 + p_1\cdot p_2 + p_1\cdot p_4 + p_2\cdot p_3 + p_3\cdot p_4$$

$$+p_2\cdot p_4\Big) - 32\,m^4 = -8\,(s-2\,m^2)^2 + 32\,m^2\,(s-m^2) - 32\,m^4$$

$$= -8\,(s-2\,m^2)\,(s-6\,m^2) \tag{6.69}$$

$$Tr\,(\not{p}_3+m)\,\gamma^\nu\,(\not{p}_2+m)\,\gamma_\mu\,(\not{p}_4+m)\,\gamma_\nu\,(\not{p}_1+m)\,\gamma^\mu = -8\,(s-2\,m^2)^2$$

$$+32\,m^2\,(s-m^2) - 32\,m^4 = -8\,(s-2\,m^2)\,(s-6\,m^2) \tag{6.70}$$

$$Tr\,\Big\{(\not{p}_3+m)\,\gamma^\mu\,(\not{p}_1+m)\,\gamma^\nu\Big\}\,Tr\,\Big\{(\not{p}_4+m)\,\gamma_\mu\,(\not{p}_2+m)\,\gamma_\nu\Big\} = 16\,\Big[2\,(p_1\cdot p_2)\,(p_3\cdot p_4)$$

$$+2\,(p_1\cdot p_4)\,(p_2\cdot p_3) + m^2\,\Big(4\,m^2 - 2\,(p_1\cdot p_3) - 2\,(p_2\cdot p_4)\Big)\Big] = 8\,(s-4\,m^2)^2$$

$$+8\,(u-4\,m^2)^2 - 64\,m^4 \tag{6.71}$$

$$Tr\,\Big\{(\not{p}_3+m)\,\gamma^\mu\,(\not{p}_2+m)\,\gamma^\nu\Big\}\,Tr\,\Big\{(\not{p}_4+m)\,\gamma_\mu\,(\not{p}_1+m)\,\gamma_\nu\Big\} = 8\,(s-4\,m^2)^2$$

$$+8\,(t-4\,m^2)^2 - 64\,m^4 \tag{6.72}$$

The spin average of the amplitude squared is then equal to

$$\begin{aligned}\overline{|M_{fi}|^2} \;&=\; 2\,e^4\,\Bigg[\frac{(s-4\,m^2)^2 + (u-4\,m^2)^2 - 8\,m^4}{t^2} + \frac{(s-4\,m^2)^2 + (t-4\,m^2)^2 - 8\,m^4}{u^2} \\ &\quad + \frac{2\,(s-2\,m^2)\,(s-6\,m^2)}{t\,u}\Bigg] = 16\,e^4\,\Bigg[\frac{(s/2-m^2)\,(s/4-m^2)}{t^2\,u^2} \\ &\quad - \frac{s^2/2 - s\,m^2 - m^4}{t\,u} + \frac{1}{4}\Bigg]\end{aligned} \tag{6.73}$$

Here we have used $s + t + u = 4\,m^2$. Because the kinematics is the same as for scalar particle scalar particle scattering (see Eq. (4.111)) we can write the effective cross section (replace on page 78 in Eq. (4.111) λ^2 by $\overline{|M_{fi}|^2}$).

$$\frac{d\,\sigma}{d\,\Omega} = \frac{1}{64\,\pi^2\,s}\,\overline{|M_{fi}|^2} \tag{6.74}$$

Define the fine structure constant $\alpha = e^2/4\,\pi$ the cross section becomes equal to

$$\begin{aligned} \frac{d\,\sigma}{d\,\Omega} &= \frac{\alpha^2}{2\,s}\left[\frac{(s-4\,m^2)^2 + (u-4\,m^2)^2 - 8\,m^4}{t^2} + \frac{(s-4\,m^2)^2 + (t-4\,m^2)^2 - 8\,m^4}{u^2}\right. \\ &\left.+\frac{2\,(s-2\,m^2)\,(s-6\,m^2)}{t\,u}\right] \end{aligned} \tag{6.75}$$

On page 77 we have calculated the energies and momenta of the incoming and outgoing particles. Here we have for all i the value $m_i = m$. This implies

$$E_i = \frac{1}{2}\,\sqrt{s} \qquad |\vec{p}_i| = \frac{1}{2}\,\sqrt{s - 4\,m^2} \equiv |\vec{p}| \tag{6.76}$$

where $|\vec{p}|$ is the centre of mass momentum and E_i the centre of mass energy for each particle i. The Mandelstam variables become

$$\begin{aligned} s &= 4\left(|\vec{p}|^2 + m^2\right) \qquad t = -2\,|\vec{p}|^2\,(1-\cos\theta) = -4\,|\vec{p}|^2\,\sin^2\frac{\theta}{2} \\ u &= -2\,|\vec{p}|^2\,(1+\cos\theta) = -4\,|\vec{p}|^2\,\cos^2\frac{\theta}{2} \end{aligned} \tag{6.77}$$

In the ultra relativistic limit $|\vec{p}| \gg m$ when $s \sim 4\,|\vec{p}|^2$ the cross section becomes

$$\left(\frac{d\,\sigma}{d\,\Omega}\right)^{\mathrm{UR}} = \frac{\alpha^2}{8\,|\vec{p}|^2}\left[\frac{1+\cos^4\theta/2}{\sin^4\theta/2} + \frac{1+\sin^4\theta/2}{\cos^4\theta/2} + \frac{2}{\sin^2\theta/2\,\cos^2\theta/2}\right] \tag{6.78}$$

Notice that in the forward direction $\theta = 0$ and in the backward direction $\theta = \pi$ the cross section becomes infinite. This is because in the t-channel and in the u-channel a massless particle (here the photon) is exchanged. In the non-relativistic limit where $|\vec{p}| \ll m$ and $s \sim 4\,m^2$, $-t \ll m^2$, $-u \ll m^2$ the cross section becomes

$$\left(\frac{d\,\sigma}{d\,\Omega}\right)^{\mathrm{NR}} = \frac{\alpha^2\,m^2}{16\,|\vec{p}|^4}\left[\frac{1}{\sin^4\theta/2} + \frac{1}{\cos^4\theta/2} - \frac{1}{\sin^2\theta/2\,\cos^2\theta/2}\right] \tag{6.79}$$

Because of the divergences which occur at $\theta = 0, \pi$ we cannot compute the total cross section. The cross section for the process

$$e^+(p_1', s_1') + e^+(p_2', s_2') \quad \rightarrow \quad e^+(p_3', s_3') + e^+(p_4', s_4') \tag{6.80}$$

is identical to that for Möller scattering.

The electron-positron annihilation process (Bhabha scattering)

$$e^-(p_1, s_1) + e^+(p_2', s_2') \quad \rightarrow \quad e^+(p_3', s_3') + e^-(p_4, s_4) \tag{6.81}$$

can be obtained from the scattering amplitude for electron-electron scattering via crossing $p_2 \rightarrow -p_3'$ and $p_3 \rightarrow -p_2$. The Mandelstam variables s', t', u' are

$$\begin{aligned} s &= (p_1 + p_2)^2 \quad \rightarrow \quad t' = (p_1 - p_3')^2 \\ t &= (p_1 - p_3)^2 \quad \rightarrow \quad s' = (p_1 + p_2')^2 \\ u &= (p_1 - p_4)^2 \quad \rightarrow \quad u' = (p_1 - p_4)^2 \end{aligned} \tag{6.82}$$

Under crossing the amplitude becomes

$$\begin{aligned} \overline{|M_{fi}|^2} &= 2\, e^4 \left[\frac{(t' - 4\, m^2)^2 + (u' - 4\, m^2)^2 - 8\, m^4}{s'^2} \right. \\ & \left. + \frac{(t' - 4\, m^2)^2 + (s' - 4\, m^2)^2 - 8\, m^4}{u'^2} + \frac{2\, (s' - 2\, m^2)\, (s' - 6\, m^2)}{s'\, u'} \right] \end{aligned} \tag{6.83}$$

The cross section becomes

$$\frac{d\, \sigma}{d\, \Omega} = \frac{1}{64\, \pi^2\, s'}\, \overline{|M_{fi}|^2} \tag{6.84}$$

In the centre of mass system we have

$$\begin{aligned} s' &= 4 \left(|\vec{p}|^2 + m^2 \right) \qquad t' = -2\, |\vec{p}|^2\, (1 + \cos\theta) = -4\, |\vec{p}|^2\, \cos^2 \frac{\theta}{2} \\ u' &= -2\, |\vec{p}|^2\, (1 - \cos\theta) = -4\, |\vec{p}|^2\, \sin^2 \frac{\theta}{2} \end{aligned} \tag{6.85}$$

In the ultra relativistic limit $|\vec{p}| \gg m$

$$\left(\frac{d\, \sigma}{d\, \Omega} \right)^{\mathrm{UR}} = \frac{\alpha^2}{8\, |\vec{p}|^2} \left[\frac{1 + \cos^4 \frac{\theta}{2}}{\sin^4 \frac{\theta}{2}} - \frac{2\, \cos^4 \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} + \frac{1 + \cos^2 \theta}{2} \right] \tag{6.86}$$

Notice that also in this case the cross section diverges if $\theta = 0$. Therefore the total cross section does not exist.

The photon-electron scattering (Compton scattering) process is given by

$$\gamma(p_1, i) + e^-(p_2, s_2) \rightarrow \gamma(p_3, j) + e^-(p_4, s_4) \tag{6.87}$$

The transition amplitude becomes

$$M_{fi} = i\,(-i\,e)^2\,\bar{u}(p_4,s_4)\,\epsilon_j^{\nu}(p_3)\left[\frac{\gamma_\nu\,(\not{p}_1+\not{p}_2+m)\,\gamma_\mu}{(p_1+p_2)^2-m^2+i\,\varepsilon}+\frac{\gamma_\mu\,(\not{p}_2-\not{p}_3+m)\,\gamma_\nu}{(p_2-p_3)^2-m^2+i\,\varepsilon}\right]$$
$$\times u(p_2,s_2)\,\epsilon_i^{\mu}(p_1) = \epsilon_j^{\nu}(p_3)\,T_{\mu\nu}\,\epsilon_i^{\mu}(p_1) \qquad (6.88)$$

with

$$T_{\mu\nu} = -i\,e^2\,\bar{u}(p_4,s_4)\left[\frac{\gamma_\nu\,(\not{p}_1+\not{p}_2+m)\,\gamma_\mu}{(p_1+p_2)^2-m^2+i\,\varepsilon}+\frac{\gamma_\mu\,(\not{p}_2-\not{p}_3+m)\,\gamma_\nu}{(p_2-p_3)^2-m^2+i\,\varepsilon}\right]u(p_2,s_2) \qquad (6.89)$$

The transverse polarisations are given $i,j=1,2$. Further we have the Ward identity

$$p_1^{\mu}\,T_{\mu\nu}=0 \qquad p_3^{\nu}\,T_{\mu\nu}=0 \qquad (6.90)$$

The proof of the Ward identity involves the equations of motion

$$(\not{p}_2-m)\,u(p_2,s_2)=0 \qquad u(p_4,s_4)\,(\not{p}_4-m) \qquad (6.91)$$

The ward identity implies that we can make a gauge transformation

$$A^{\mu} \quad\rightarrow\quad A'^{\mu}=A^{\mu}-\frac{1}{e}\,\partial^{\mu}\,\alpha(x) \qquad \epsilon_i^{\mu}(k) \quad\rightarrow\quad \epsilon_i'^{\mu}(k)=\epsilon_i^{\mu}(k)+\lambda\,k^{\mu} \qquad (6.92)$$

Under this transformation M_{fi} is invariant

$$\epsilon_j'^{\nu}(p_3)\,T_{\mu\nu}\,\epsilon_i'^{\mu}(p_1)=\left(\epsilon_j^{\nu}(p_3)+\lambda\,p_3^{\nu}\right)T_{\mu\nu}\left(\epsilon_i^{\mu}(p_1)+\lambda\,p_1^{\mu}\right)=\epsilon_j^{\nu}(p_3)\,T_{\mu\nu}\,\epsilon_i^{\mu}(p_1) \qquad (6.93)$$

Because $k_\mu\,\epsilon_i^{\mu}(k)=0$ $(i=1,2)$ we can choose an arbitrary vector n_μ for which $n_\mu\,\epsilon_i^{\mu}(k)=0$. This is equivalent with a gauge choice $n\cdot A=0$. We have in a given Lorentz frame four mutual perpendicular vectors i.e. $k_\mu, n_\mu, \epsilon_1^{\mu}(k), \epsilon_2^{\mu}(k)$. For on-shell photons we have the relation

$$P^{\mu\nu}(k)=\sum_i \epsilon_i^{\mu}(k)\,\epsilon_i^{\nu}(k) = -g^{\mu\nu}+\frac{k^{\mu}\,n^{\nu}+n^{\mu}\,k^{\nu}}{n\cdot k}-\frac{n^2\,k^{\mu}\,k^{\nu}}{(n\cdot k)^2} \qquad (6.94)$$

If we now sum again over all polarisation of the in- and outgoing photons and average over the incoming polarisations then

$$\overline{|M_{fi}|^2}=\frac{1}{2}\,P^{\nu\sigma}(p_3)\,T_{\mu\nu}\,T_{\lambda\sigma}\,P^{\mu\lambda}(p_1) \qquad (6.95)$$

From de Ward identity follows

$$\overline{|M_{fi}|^2}=\frac{1}{2}\,(-g^{\nu\sigma})\,T_{\mu\nu}\,T_{\lambda\sigma}\,(-g^{\mu\lambda})=\frac{1}{2}\,T_{\mu\nu}\,T^{\mu\nu} \qquad (6.96)$$

If $T_{\mu\nu}$ satisfies the Ward identity then we are allowed to replace $P^{\mu\nu}$ by $-g^{\mu\nu}$. Summing also over the spins of the incoming and outgoing electron and average over the initial spins we get

$$\begin{aligned}
\overline{|M_{fi}|^2} &= \frac{e^4}{4}\Bigg[\frac{Tr\,(\not p_4+m)\,\gamma_\nu\,(\not p_1+\not p_2+m)\,\gamma_\mu\,(\not p_2+m)\,\gamma^\mu\,(\not p_1+\not p_2+m)\,\gamma^\nu}{(s-m^2)^2} \\
&+\frac{Tr\,(\not p_4+m)\,\gamma_\mu\,(\not p_2-\not p_3+m)\,\gamma_\nu\,(\not p_2+m)\,\gamma^\nu\,(\not p_2-\not p_3+m)\,\gamma^\mu}{(u-m^2)^2} \\
&+\frac{Tr\,(\not p_4+m)\,\gamma_\nu\,(\not p_1+\not p_2+m)\,\gamma_\mu\,(\not p_2+m)\,\gamma^\nu\,(\not p_2-\not p_3+m)\,\gamma^\mu}{(s-m^2)\,(u-m^2)} \\
&+\frac{Tr\,(\not p_4+m)\,\gamma_\mu\,(\not p_2-\not p_3+m)\,\gamma_\nu\,(\not p_2+m)\,\gamma^\mu\,(\not p_1+\not p_2+m)\,\gamma^\nu}{(s-m^2)\,(u-m^2)}\Bigg]
\end{aligned} \tag{6.97}$$

The Mandelstam variables are

$$\begin{aligned}
s &= (p_1+p_2)^2 = m^2+2\,p_1\cdot p_2 = (p_3+p_4)^2 = m^2+2\,p_3\cdot p_4 \\
t &= (p_1-p_3)^2 = -2\,p_1\cdot p_3 = (p_2-p_4)^2 = 2\,m^2-2\,p_2\cdot p_4 \\
u &= (p_1-p_4)^2 = m^2-2\,p_1\cdot p_4 = (p_2-p_3)^2 = m^2-2\,p_2\cdot p_3 \\
& \quad s+t+u = 2\,m^2
\end{aligned} \tag{6.98}$$

The traces are

$$\begin{aligned}
& Tr\,(\not p_4+m)\,\gamma_\nu\,(\not p_1+\not p_2+m)\,\gamma_\mu\,(\not p_2+m)\,\gamma^\mu\,(\not p_1+\not p_2+m)\,\gamma^\nu \\
& = 8\Big(-s\,u+3\,s\,m^2+u\,m^2+m^4\Big) \\
& Tr\,(\not p_4+m)\,\gamma_\mu\,(\not p_2-\not p_3+m)\,\gamma_\nu\,(\not p_2+m)\,\gamma^\nu\,(\not p_2-\not p_3+m)\,\gamma^\mu \\
& = 8\Big(-s\,u+3\,u\,m^2+s\,m^2+m^4\Big) \\
& Tr\,(\not p_4+m)\,\gamma_\nu\,(\not p_1+\not p_2+m)\,\gamma_\mu\,(\not p_2+m)\,\gamma^\nu\,(\not p_2-\not p_3+m)\,\gamma^\mu \\
& = 8\,m^2\Big(s+u+2\,m^2\Big) \\
& Tr\,(\not p_4+m)\,\gamma_\mu\,(\not p_2-\not p_3+m)\,\gamma_\nu\,(\not p_2+m)\,\gamma^\mu\,(\not p_1+\not p_2+m)\,\gamma^\nu \\
& = 8\,m^2\Big(s+u+2\,m^2\Big)
\end{aligned} \tag{6.99}$$

The amplitude squared becomes

$$\overline{|M_{fi}|^2} = 2\,e^4 \left[\frac{-s\,u + 3\,s\,m^2 + u\,m^2 + m^4}{(s-m^2)^2} + \frac{-s\,u + 3\,u\,m^2 + s\,m^2 + m^4}{(u-m^2)^2} + \frac{2\,m^2\,(s+u+2\,m^2)}{(s-m^2)\,(u-m^2)} \right] \tag{6.100}$$

In the centre of mass frame the energies and momenta get the following form

$$E_1 = \frac{s-m^2}{2\,\sqrt{s}} \qquad E_2 = \frac{s+m^2}{2\,\sqrt{s}}$$

$$E_3 = \frac{s-m^2}{2\,\sqrt{s}} \qquad E_4 = \frac{s+m^2}{2\,\sqrt{s}}$$

$$|\vec{p}_i| = \frac{s-m^2}{2\,\sqrt{s}} \tag{6.101}$$

The two Mandelstam variables become

$$t = \frac{(s-m^2)^2}{2\,s}\,(1-\cos\theta) \qquad u = \frac{m^4}{s} - \frac{(s-m^2)^2}{2\,s}\,(1+\cos\theta) \tag{6.102}$$

The differential cross section in the centre of mass frame is equal to (replace λ^2 by $\overline{|M_{fi}|^2}$ in Eq. (4.111))

$$\begin{aligned} \frac{d\,\sigma}{d\,\Omega} &= \frac{1}{64\,\pi^2\,s}\,\overline{|M_{fi}|^2} \\ &= \frac{\alpha^2}{2\,s} \left[\frac{-s\,u + 3\,s\,m^2 + u\,m^2 + m^4}{(s-m^2)^2} + \frac{-s\,u + 3\,u\,m^2 + s\,m^2 + m^4}{(u-m^2)^2} + \frac{2\,m^2\,(s+u+2\,m^2)}{(s-m^2)\,(u-m^2)} \right] \end{aligned} \tag{6.103}$$

We can now calculate the total cross section because the particle in the u-channel is massive

$$\begin{aligned} \sigma_{tot} &= \int d\,\Omega\,\frac{d\,\sigma}{d\,\Omega} = \int_{-1}^{1} d\,\cos\theta \int_0^{2\pi} d\,\phi\,\frac{d\,\sigma}{d\,\Omega} \\ &= \frac{\pi\,\alpha^2}{s} \left[\left\{ -\frac{16\,m^6}{(s-m^2)^3} - \frac{24\,m^4}{(s-m^2)^2} - \frac{6\,m^2}{s-m^2} + 2 \right\} \ln\frac{s}{m^2} + \frac{16\,m^4}{(s-m^2)^2} + \frac{16\,m^2}{s-m^2} + \frac{m^2}{s} + 1 \right] \end{aligned} \tag{6.104}$$

In the non-relativistic limit ($s \sim m^2$)

$$\sigma_{tot}^{\mathrm{NR}} = \frac{8\,\pi\,\alpha^2}{3\,m^2} \quad \text{Thomson cross section} \tag{6.105}$$

$r_0 = \alpha/m$ is the classical radius of the electron ($r_0 = 2.82.10^{-15}\,m$). In the ultra relativistic limit ($s \gg m^2$) we have

$$\sigma_{tot}^{\mathrm{UR}} = \frac{2\,\pi\,\alpha^2}{s}\left[\ln\frac{s}{m^2} + \frac{1}{2}\right] \tag{6.106}$$

From Compton scattering we can derive the amplitude of the two other processes via crossing. They are given by fermion pair creation

$$\gamma(p_1, i) + \gamma(p_2', j) \to e^+(p_3', s_3') + e^-(p_4, s_4) \tag{6.107}$$

and fermion pair annihilation

$$e^+(p_1', s_1') + e^-(p_2, s_2) \to \gamma(p_3, i) + \gamma(p_4', j) \tag{6.108}$$

If we start with fermion pair creation we have to substitute for the process in Eq. (6.87)

$$p_3 \quad \to \quad -p_2' \qquad p_2 \quad \to \quad -p_3' \quad \Longrightarrow \quad \overline{|M_{fi}|^2} \quad \to \quad -\overline{|M_{fi}|^2} \tag{6.109}$$

The Mandelstam variables become

$$\begin{aligned}
s &= (p_1 + p_2)^2 \quad \to \quad t' = (p_1 - p_3')^2 \\
t &= (p_1 - p_3)^2 \quad \to \quad s' = (p_1 + p_2')^2 \\
u &= (p_1 - p_4)^2 \quad \to \quad u' = (p_1 - p_4)^2
\end{aligned} \tag{6.110}$$

In the centre of mass system the energies and momenta are

$$\begin{aligned}
&E_1 = E_{2'} = \frac{1}{2}\sqrt{s'} \qquad E_{3'} = E_4 = \frac{1}{2}\sqrt{s'} \\
&|\vec{p}_1| = |\vec{p}_{2'}| = \frac{1}{2}\sqrt{s'} \qquad |\vec{p}_{3'}| = |\vec{p}_4| = \frac{1}{2}\sqrt{s' - 4\,m^2}
\end{aligned} \tag{6.111}$$

and the Mandelstam variables become

$$\begin{aligned}
t' &= -\frac{s'}{2} + m^2 + \frac{1}{2}\sqrt{s'(s' - 4\,m^2)}\,\cos\theta \\
u' &= -\frac{s'}{2} + m^2 - \frac{1}{2}\sqrt{s'(s' - 4\,m^2)}\,\cos\theta
\end{aligned} \tag{6.112}$$

The differential cross section becomes (see Eq. (4.110))

$$\begin{aligned}\frac{d\,\sigma}{d\,\Omega} &= \frac{1}{64\,\pi^2\,s'}\,\frac{\sqrt{s'(s'-4\,m^2)}}{s'}\,\overline{|M_{fi}|^2}\\ &= \frac{\alpha^2}{2\,s'}\sqrt{1-\frac{4\,m^2}{s'}}\left[\frac{t'\,u'-3\,t'\,m^2-u'\,m^2-m^4}{(t'-m^2)^2}+\frac{t'\,u'-3\,u'\,m^2-t\,m^2-m^4}{(u'-m^2)^2}\right.\\ &\left.+\frac{2\,m^2\,(s'-m^2)}{(t'-m^2)\,(u'-m^2)}\right] \end{aligned} \tag{6.113}$$

The total cross section is equal to

$$\begin{aligned}\sigma_{tot} &= \frac{4\,\pi\,\alpha^2}{s'}\left[-\left(1+\frac{4\,m^2}{s'}\right)\sqrt{1-\frac{4\,m^2}{s'}}+\left(1+\frac{4\,m^2}{s'}-\frac{8\,m^4}{s'^2}\right)\right.\\ &\left.\times\ln\frac{1+\sqrt{1-4m^2/s'}}{1-\sqrt{1-4m^2/s'}}\right] \end{aligned} \tag{6.114}$$

Notice that fermion pair creation is a typical relativistic process which has no counter part in the non-relativistic limit. Near threshold it behaves like

$$\sigma_{tot} \underset{s'\,\sim\,4\,m^2}{=} \frac{\pi\,\alpha^2}{m^2}\sqrt{1-\frac{4\,m^2}{s'}} \tag{6.115}$$

and in the ultra relativistic limit the total cross section behaves like

$$\sigma_{tot}^{\mathrm{UR}} \underset{s'\,\gg\,m^2}{=} \frac{4\,\pi\,\alpha^2}{s'}\left[\ln\frac{s'}{m^2}-1\right] \tag{6.116}$$

The amplitude for fermion pair annihilation Eq. (6.108) is obtained from the process in Eq. (6.87) by substituting

$$p_4 \;\rightarrow\; -p_1' \qquad p_1 \;\rightarrow\; -p_4' \;\Longrightarrow\; \overline{|M_{fi}|^2} \;\rightarrow\; -\overline{|M_{fi}|^2} \tag{6.117}$$

The Mandelstam variables become

$$\begin{aligned} s &= (p_1+p_2)^2 \quad\rightarrow\quad t' = (p_2-p_4')^2\\ t &= (p_1-p_3)^2 \quad\rightarrow\quad s' = (p_3+p_4')^2\\ u &= (p_2-p_3)^2 \quad\rightarrow\quad u' = (p_2-p_3)^2 \end{aligned} \tag{6.118}$$

In the centre of mass system the energies and momenta are

$$E_{1'} = E_2 = \frac{1}{2}\sqrt{s'} \qquad E_3 = E_{4'} = \frac{1}{2}\sqrt{s'}$$

$$|\vec{p}_{1'}| = |\vec{p}_2| = \frac{1}{2}\sqrt{s' - 4\,m^2} \qquad |\vec{p}_3| = |\vec{p}_{4'}| = \frac{1}{2}\sqrt{s'} \tag{6.119}$$

and the Mandelstam variables become

$$t' = -\frac{s'}{2} + m^2 + \frac{1}{2}\sqrt{s'(s' - 4\,m^2)}\,\cos\theta$$

$$u' = -\frac{s'}{2} + m^2 - \frac{1}{2}\sqrt{s'(s' - 4\,m^2)}\,\cos\theta \tag{6.120}$$

The differential cross section reads

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{1}{64\,\pi^2}\frac{1}{\sqrt{s'(s'-4\,m^2)}}\overline{|M_{fi}|^2} \\ &= \frac{\alpha^2}{2\,s'}\frac{1}{\sqrt{1-4m^2/s'}}\left[\frac{t'\,u' - 3\,t'\,m^2 - u'\,m^2 - m^4}{(t'-m^2)^2} + \frac{t'\,u' - 3\,u'\,m^2 - t'\,m^2 - m^4}{(u'-m^2)^2}\right. \\ &\left. + \frac{2\,m^2\,(s'-m^2)}{(t'-m^2)\,(u'-m^2)}\right]\end{aligned} \tag{6.121}$$

The total cross section is equal to

$$\begin{aligned}\sigma_{tot} &= \frac{1}{2}\int d\Omega\,\frac{d\sigma}{d\Omega} \\ &= \frac{2\,\pi\,\alpha^2}{s' - 4\,m^2}\left[-\left(1 + \frac{4\,m^2}{s'}\right)\sqrt{1 - \frac{4\,m^2}{s'}} + \left(1 + \frac{4\,m^2}{s'} - \frac{8\,m^4}{s'^2}\right)\right. \\ &\left. \times \ln\frac{1+\sqrt{1-4m^2/s'}}{1-\sqrt{1-4m^2/s'}}\right]\end{aligned} \tag{6.122}$$

The factor 1/2 is due to the fact that we have two identical particles (here photons) in the final state. Like in the previous case this process is relativistic and it has no non-relativistic counter part. Near threshold it behaves like

$$\sigma_{tot} \underset{s' \sim 4\,m^2}{=} \frac{\pi\,\alpha^2}{2\,m^2}\frac{1}{\sqrt{1 - 4\,m^2/s'}} \tag{6.123}$$

and in the ultra relativistic limit the total cross section equals

$$\sigma_{tot}^{\mathrm{UR}} \underset{s' \gg m^2}{=} \frac{2\,\pi\,\alpha^2}{s'}\left[\ln\frac{s'}{m^2} - 1\right] \tag{6.124}$$