THE DUAL SUPERCONDUCTOR PICTURE FOR CONFINEMENT

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INTRODUCTION

At short distances QCD vacuum mimics the Fock space ground state of perturbation theory: deep inelastic scattering experiments, jet production in high energy reactions, QCD sum rules provide empirical evidence for that.

In fact colour is confined, and quarks and gluons never appear as free particles in asymptotic states: the ground state is very different from the perturbative vacuum. Its exact structure is not known. Many models have been attempted to give an approximate description of it: some of them are based on "mechanisms", i.e. they assume that the degrees of freedom relevant for large distance physics behave as other well understood physical systems.

The most attractive mechanism for colour confinement is *Dual superconductivity of type II of QCD vacuum*[1, 2]. Dual means interchange of electric with magnetic with respect to ordinary superconductors. The idea is that the chromoelectric field in the region of space between a $Q\bar{Q}$ pair is constrained by the dual Meissner effect into Abrikosov[3] flux tubes, with constant energy per unit length. The energy is then proportional to the distance

$$E = \sigma R \tag{1}$$

and this means confinement.

The lattice is the ideal tool to study (at least numerically) large distance phenomena from first principles. There is indeed evidence from lattice simulations that:

1) The string tension exists: large Wilson loops $W(R,T)$ describing a pair of static quarks at a distance R propagating for a time T, obey the area law[4]

$$W(R,T) \simeq \exp(-\sigma RT) \tag{2}$$

Since in general $W(R,T) \simeq \exp(-V(R)T)$, the observed behaviour (Eq.(2)) confirms confinement as defined by Eq.(1).

2) Chromoelectric flux tubes have been observed, joining a $Q\bar{Q}$ pair propagating in Wilson loops[5, 6]. Their transverse size is ~ 0.5 fm.

3) String like modes of these flux tubes have been detected[7].

4) Particles belonging to higher representations than quarks also experience a string tension at intermediate distances, which, for $SU(2)$, depends on the colour spin J as[8]

$$\sigma_J = kJ(J+1)$$

or

$$\frac{\sigma_J}{\sigma_{1/2}} = \frac{3}{4}J(J+1) \tag{3}$$

Observations 1)-3) support the idea of dual superconductivity. We will discuss in detail the implications of 4) in the following.

The problem we address in these lectures is: can dual superconductivity of QCD vacuum be directly tested?

The plan of the lectures is as follows. We will recall basic superconductivity and its order parameter, in order to clarify what we are looking for. We will then define dual superconductivity and its disorder parameter. We will construct the disorder parameter for $U(1)$ pure gauge theory.

We will then check the construction with the $X-Y$ 3d model (liquid He4). Subsequently we will revisit the abelian projection, which reduces the problem of dual superconductivity in QCD to a $U(1)$ problem. The Heisenberg ferromagnet will prove a useful laboratory to check this procedure. We will finally show how dual superconductivity can be directly detected in $SU(2)$ and $SU(3)$ gauge theories.

A discussion of the results and of their physical consequences is contained in the final section.

BASIC SUPERCONDUCTIVITY: THE ORDER PARAMETER[9]

A relativistic version of a superconductor is the abelian Higgs model

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + (D_\mu\Phi)^\dagger(D_\mu\Phi) - V(\Phi) \tag{4}$$

$F_{\mu\nu}$ is the electromagnetic field strength, D_μ is the covariant derivative

$$D_\mu\Phi = (\partial_\mu - iqA_\mu)\Phi \tag{5}$$

and $V(\Phi)$ the potential of the scalar field

$$V(\Phi) = \frac{1}{4}\left(\Phi^\dagger\Phi - \mu^2\right)^2 \tag{6}$$

If μ^2 is positive the field Φ has a nonzero vacuum expectation value. Since Φ is a charged field this is nothing but a spontaneous breaking of the $U(1)$ symmetry related to charge conservation. The ground state is a superposition of states with different electric charges, a phenomenon which is usually called "condensation" of charges.

A convenient parametrization of Φ is

$$\Phi = \rho e^{i\theta q} \qquad \rho = \rho^\dagger > 0 \tag{7}$$

Under gauge transformations

$$A_\mu \to A_\mu - \partial_\mu \alpha \qquad \theta \to \theta + \alpha \tag{8}$$

The covariant derivative (5) reads in this notation

$$D_\mu \Phi = \mathrm{e}^{\mathrm{i}\theta q} \left[\partial_\mu - \mathrm{i}q(A_\mu - \partial_\mu \theta)\right] \rho \tag{9}$$

The quantity $\tilde{A}_\mu = A_\mu - \partial_\mu \theta$ is gauge invariant. Moreover

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu \tag{10}$$

The equation of motion reads, neglecting loop corrections (or looking at $\mathcal{L}$ as an effective lagrangian)

$$\partial_\mu F^{\mu\nu} + \frac{\tilde{m}^2}{2} \tilde{A}_\nu = 0 \qquad \tilde{m} = \sqrt{2} q \langle \Phi \rangle \tag{11}$$

In the gauge $A_0 = 0$ a static configuration has $\partial_0 \vec{A} = 0$, $\partial_0 \Phi = 0$ so that $E_i = F_{0i} = 0$. Eq.(11) implies that

$$\vec{\nabla} \wedge \vec{H} + \frac{\tilde{m}^2}{2} \vec{\tilde{A}} = 0 \tag{12}$$

The term $\tilde{m}^2/2\, \vec{\tilde{A}}$ in Eq.(12) is a consequence of spontaneous symmetry breaking and is an stationary electric current (London current). A persistent current with $\vec{E} = 0$, means $\rho = 0$ since $\rho \vec{j} = \vec{E}$ and hence superconductivity.

The curl of Eq.(12), reads

$$\nabla^2 \vec{H} - \frac{\tilde{m}^2}{2} \vec{H} = 0 \tag{13}$$

The magnetic field has a finite penetration depth $1/\tilde{m}$, and this is nothing but the Meissner effect. The key parameter is $\langle \Phi \rangle$, which is the order parameter for superconductivity: it signals spontaneous breaking of charge conservation.

Besides $\lambda_A = 1/\tilde{m}$ there is another parameter with dimension of a length, $\lambda_\Phi = \mu^{-1}$. If $\lambda_A \geq \sqrt{2} \lambda_\Phi$ the superconductor is called of the second kind otherwise it is of the first kind. For a superconductor of the first kind there is a Meissner effect for an external magnetic field $H < H_c$ (critical field); for $H > H_c$ the field penetrates the bulk and superconductivity is destroyed. For a superconductor of the second kind instead a penetration by Abrikosov flux tubes of transverse size $1/\tilde{m}$ is energetically favoured. Flux tubes repel each other. By increasing the external field the number of flux tubes increases. When they touch each other the field penetrates the bulk and superconductivity is destroyed.

In a dual superconductor the role of the electric and magnetic field is interchanged. The $U(1)$ symmetry related to magnetic charge conservation is spontaneously broken, i.e. monopoles condense in the vacuum. An order parameter for dual superconductivity will then be the vacuum expectation value of a field carrying nonzero magnetic charge.

Monopoles

The equations of motion for the electromagnetic field in the presence of an electric current j_μ and of a magnetic current j^M_μ are

$$\begin{aligned} \partial^\mu F_{\mu\nu} &= j_\nu \\ \partial^\mu F^*_{\mu\nu} &= j^M_\nu \end{aligned} \tag{14}$$

If both j_μ and j^M_μ are zero (no charges, no monopoles) photons are free, and the equations of motion are invariant under the transformation

$$\begin{aligned} F_{\mu\nu} &\to \cos\theta\, F_{\mu\nu} + \sin\theta\, F^*_{\mu\nu} \\ F^*_{\mu\nu} &\to \cos\theta\, F^*_{\mu\nu} - \sin\theta\, F_{\mu\nu} \end{aligned} \tag{15}$$

for any θ. In particular if $\theta = \pi/2$, Eq.'s (15) give $\vec{E} \to \vec{H}$, $\vec{H} \to -\vec{E}$ which is known as a duality transformation. In nature $j^M_\mu = 0$. The general solution of Eq.(14) is then written in terms of vector potential A_μ

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{16}$$

and $\partial^\mu F^*_{\mu\nu} = 0$ is identically satisfied (Bianchi identities).

If a monopole exists Bianchi identities are violated. However they can be preserved, and with them the description in terms of A_μ, by considering the monopole as the end point of a thin solenoid (Dirac string) connecting it to infinity: the flux of the Coulomb like magnetic field, $\vec{H} = \frac{M}{4\pi}\frac{\vec{r}}{r^3}$, $\Phi(H) = M$ is conveyed to infinity by the string[10].

The string is invisible if the parallel transport of any electric charge around it is trivial:

$$\mathrm{e}^{\mathrm{i}q\oint \vec{A}\mathrm{d}\vec{x}} = \mathrm{e}^{\mathrm{i}q\Phi(H)} = \mathrm{e}^{\mathrm{i}2\pi n} = 1$$

or

$$qM = 2\pi n \tag{17}$$

Eq.(17) is known as the Dirac quantization condition, and constrains the $U(1)$ group to be compact. If one insists to describe the system in terms of A_μ, monopoles are non-local objects with nontrivial topology. One could introduce dual vector potential B_μ, such that $F^*_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. Dual Bianchi identities would read $\partial^\mu F_{\mu\nu} = 0$, monopoles would be pointlike but electric charges could only exist if dual strings were attached to them.

There is another manifestation of duality, which originates from statistical mechanics. The prototype example is the 2d Ising model. The model is defined on a square lattice, by associating to each site i a field $\sigma(i)$ wich can assume 2 values, say ± 1. The action can be written

$$\mathcal{L} = -\beta \sum_{i,j} \sigma(i)\sigma(j) \tag{18}$$

the sum running on nearest neighbours. The partition function $K[\beta]$ is known exactly in the thermodynamical limit. At high β (low temperatures) the system is magnetized $\langle\sigma(i)\rangle \neq 0$; at low β it is disordered. A dual description can be given of the same system, by associating to each link (dual lattice site) a variable σ^* with value -1 if the values of σ in the sites connected by the link are the same, $+1$ otherwise.

It can be rigorously proven that the partition function in terms of the new variables has the same form as the original one $K[\beta]$

$$K^*[\beta] = K[\beta^*] \tag{19}$$

with the only change

$$\beta \to \beta^* \qquad \beta^* = \frac{1}{2}\mathrm{arcsinh}\left(\frac{1}{\sinh 2\beta}\right) \simeq \frac{1}{\beta} \tag{20}$$

A relation like Eq.(19) is called a duality relation. It maps high temperature (strong coupling) regime of K^* to the low temperature (weak coupling) of K[11]. Similar relations

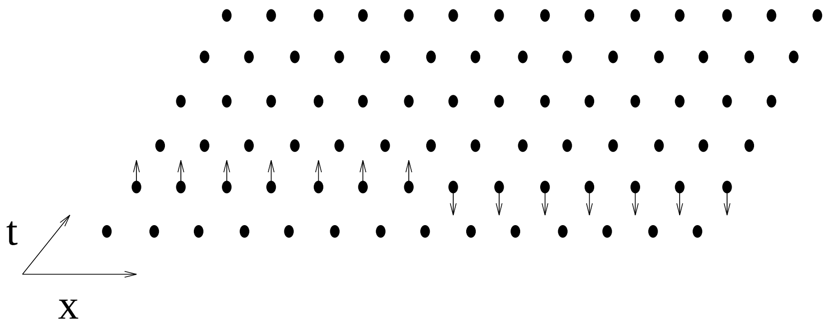

Figure 1. A kink in the Ising model.

have been recently discovered in SUSY QCD with $N = 2$ supersymmetry, and more generally in models of string theory[12].

If we look at the Ising model as the euclidean version of some 1+1 dimensional field theory, a configuration at fixed t, with $\sigma(i) = -1$, $i < i_0$ $\sigma(i) = +1$, $i \geq i_0$ will appear on the dual lattice as a single spin up. This configuration has topology, it is a kink, see fig.1. The excitations of the dual lattice are kinks. At low temperature the system is magnetized, $\langle\sigma\rangle \neq 0$ and very few kinks are present. At high temperature $\langle\sigma\rangle = 0$, but using the duality relation $\langle\sigma^*\rangle \neq 0$. $\langle\sigma^*\rangle$ is called a disorder parameter, as opposed to $\langle\sigma\rangle$, which is the order parameter. The relation

$$\langle\sigma\rangle \cdot \langle\sigma^*\rangle = 0 \tag{21}$$

can be proven in the thermodynamical limit. $\langle\sigma^*\rangle$ signals the condensation of kinks.

We shall next address the study of monopole condensation in $U(1)$ compact gauge theory. We shall define a disorder parameter for this system, which describes the condensation of monopoles in the vacuum at high temperature (low β). The parameter will be the v.e.v. of an operator with nonzero monopole charge, and will thus signal dual superconductivity. The same construction will then be used for non-abelian gauge theories after abelian projection.

MONOPOLE CONDENSTATION IN COMPACT $U(1)$: A DISORDER PARAMETER[13]

Like any other gauge theory, compact $U(1)$ is defined in terms of parallel transport along the links joining nearest neighbours on the lattice

$$U_\mu(n) = \exp(\mathrm{i}eaA_\mu(n)) \tag{22}$$

a being the lattice spacing.

In the following we shall denote $eaA_\mu(n)$ as $\theta_\mu(n)$. The action is written in terms of the parallel transport $\Pi_{\mu\nu}$ around the elementary square of the lattice in the plane μ,ν

$$\begin{aligned}\Pi_{\mu\nu} &= \exp\mathrm{i}\left[\theta_\mu(n) + \theta_\nu(n+\hat\mu) - \theta_\mu(n+\hat\mu) - \theta_\nu(n)\right] \equiv \exp(-\mathrm{i}\theta_{\mu\nu}(n)) \\ \theta_{\mu\nu}(n)) &= \Delta_\mu\theta_\nu(n) - \Delta_\nu\theta_\mu(n) \underset{a\to 0}{\simeq} a^2 eF_{\mu\nu}\end{aligned} \tag{23}$$

$$S = \beta \sum_{n,\mu<\nu} (1 - \cos\theta_{\mu\nu}) \tag{24}$$

As $a \to 0$

$$S \simeq \beta \sum_{n,\mu\neq\nu} \frac{\theta^2_{\mu\nu}(n)}{4} \tag{25}$$

and Eq.(24) describes photons if $\beta = 1/e^2$.

The generating functional of the theory (partition function) is

$$Z(\beta) = \int \prod_{n,\mu} \left[\frac{d\,\theta_\mu(n)}{2\pi}\right] \exp(-S) \tag{26}$$

The theory is compact, since S depends on the cos of the angular variables, and is invariant under change of variables

$$\theta_\mu(n) \to \theta_\mu(n) + f_\mu(n) \tag{27}$$

with arbitrary $f_\mu(n)$. A special case of Eq.(27) are gauge transformations.

A critical β, $\beta_c \simeq 1.011$ exists such that for $\beta > \beta_c$ the theory describes free photons. For $\beta < \beta_c$ electric charge is confined: Wilson loops obey the area law[14] Eq.(2) and flux tubes are observed[15].

A variant of the theory is provided by the Villain action

$$\exp(-S) = \sum_m \exp\left[-\frac{\beta}{2}\sum_{n,\mu<\nu} |\theta_{\mu\nu} - 2\pi m|^2\right] \tag{28}$$

For this variant, condensation of monopoles has rigorously been proven as a mechanism of confinement[16]. Recently the proof has been extended to more general forms of the action, including[17] the Wilson action Eq.(23). Monopoles are identified and counted by the following procedure[18].

Since by construction $\pi \leq \theta_\mu(n) \leq \pi$, it follows from Eq.(23) that

$$-4\pi \leq \theta_{\mu\nu}(n) \leq 4\pi$$

$\theta_{\mu\nu}$ can be redefined modulo an integer multiple of 2π, $n_{\mu\nu}$, as

$$\theta_{\mu\nu} = \bar{\theta}_{\mu\nu} + 2\pi n_{\mu\nu} \qquad -\pi < \bar{\theta}_{\mu\nu} \leq \pi \tag{29}$$

and the monopole current as

$$\rho^M_\mu = \frac{1}{6}\varepsilon_{\mu\nu\rho\sigma}\Delta_\nu n_{\rho\sigma} \equiv \partial_\nu F^*_{\mu\nu} \tag{30}$$

The total number of monopoles is

$$N^M = \sum_n \rho^M_0(n) \tag{31}$$

N^M is large in the confined phase, and drops to zero in the deconfined phase. It has sometimes been identified with the disorder parameter for monopole condensation. Of course N^M commutes with the monopole charge, and therefore cannot signal by any means spontaneous breaking of magnetic $U(1)$.

The disorder parameter

The basic idea of the construction[11, 19] of a disorder parameter is the simple formula for translations

$$\mathrm{e}^{ipa}|x\rangle = |x + a\rangle$$

If we identify in our field theory

$$\begin{aligned} x &\to \vec{A}(\vec{x}, t) \\ p &\to \vec{E}(\vec{x}, t) = -\mathrm{i}\frac{\delta}{\delta \vec{A}(\vec{x}, t)} \end{aligned} \tag{32}$$

then the operator

$$\mu(\vec{y},t) = \exp\left[\frac{\mathrm{i}}{e}\int d^3x\,\vec{E}(\vec{x},t)\vec{b}(\vec{x}-\vec{y})\right] \tag{33}$$

operating on field states in the Schrödinger representation will give

$$\mu|\vec{A}(\vec{x},t)\rangle = |\vec{A}(\vec{x},t) + \frac{1}{e}\vec{b}(\vec{x}-\vec{y})\rangle$$

i.e. it will add a monopole to any field configuration provided that $\vec{b}$ is the vector potential describing the field produced by the monopole

$$\frac{1}{e}\vec{b}(\vec{r}) = \frac{1}{e}\frac{m}{2\pi}\frac{\vec{r}\wedge\vec{n}}{r(r-\vec{r}\cdot\vec{n})} \tag{34}$$

The gauge has been chosen to have the Dirac string in the direction $\vec{n}$. μ does not dependent on the choice of the gauge for $\vec{b}$ if $\vec{E}$ obeys Gauss' law.

On a lattice[20], after Wick rotation, and with the identification

$$a^2 E^i = \frac{1}{e}\mathrm{Im}\Pi^{0i} + \mathcal{O}(a^4) \simeq \frac{1}{e}\sin\theta^{0i} \tag{35}$$

$$\mu(\vec{y},n_0) = \exp\left[-\beta\sum_{\vec{n}} b^i(\vec{n}-\vec{y})\sin\theta^{0i}(\vec{n},n_0)\right] \tag{36}$$

Here $b_i(\vec{n})$ is the discretized transcription of Eq.(34) and the β in front comes from the normalization $1/e$ in Eq.(22) times the $1/e$ appearing in the monopole charge.

A better definition (compactified) which shifts the angle θ_{0i} and not $\sin\theta_{0i}$ is[13]

$$\mu(\vec{y},n_0) = \exp\beta\sum_n\left[S(\theta^{0i}(\vec{n},n_0) + b^i(\vec{n}-\vec{y})) - S(\theta^{0i}(\vec{n},n_0))\right] \tag{37}$$

which reduces to Eq.(36) at first order in $\vec{b}$. We denote by S in Eq.(37) the density of the action. The action can be any of form, e.g. Eq.(24) or (28), provided Eq.(25) is satisfied.

If an arbitrary number of monopoles or antimonopoles is created at time n_0, then $b^i(\vec{n}-\vec{y})$ has to be replaced by the corresponding field configuration. To compute correlation functions of operators at different times, the rule is

$$\langle\mu(\vec{y}_1,n_1^0),\ldots\mu(\vec{y}_k,n_k^0)\rangle = \frac{1}{Z}\int\prod\left[\frac{d\theta_\mu(n)}{2\pi}\right]\exp\left(\beta(S+\Delta S)\right) \tag{38}$$

where $S+\Delta S$ is obtained by replacing in the action the plaquettes $\Pi^{0i}(\vec{n},n_\ell^0) = 1 - \cos(\theta^{0i}(\vec{n},n_\ell^0))$ by $1-\cos(\theta^{0i}(\vec{n},n_\ell^0) + b^i(\vec{n}-\vec{y}_\ell))$. $(1\le\ell\le k)$.

In particular we will study the correlator

$$\mathcal{D}(x_0) = \langle\mu(\vec{0},x_0)\mu(\vec{0},0)\rangle \tag{39}$$

At large enough x_0

$$\mathcal{D}(x_0) \underset{|x_0|\to\infty}{\simeq} A\exp(-M|x_0|) + \langle\mu\rangle^2 \tag{40}$$

The last equality follows from the cluster property, translation invariance and C invariance of the vacuum. Our aim will be to extract M and $\langle\mu\rangle$ from numerical determinations of $\mathcal{D}(x_0)$. $\langle\mu\rangle$ is the disorder parameter: a nonzero value of it in the thermodynamic limit signals dual superconductivity. M is the mass of the lightest excitation with the quantum numbers of a monopole and is a lower limit to the mass of the effective Higgs field which produces superconductivity.

As explained above

$$\mathcal{D}(x_0) = \frac{1}{Z[S]} Z[S + \Delta S] \tag{41}$$

where $S + \Delta S$ is obtained from S by the change

$$\theta^{i0}(\vec{n}, 0) \to \theta^{i0}(\vec{n}, 0) + b^i(\vec{n}) \tag{42}$$

$$\theta^{i0}(\vec{n}, x_0) \to \theta^{i0}(\vec{n}, x_0) - b^i(\vec{n}) \tag{43}$$

and $b^i(\vec{n})$ is defined by Eq.(34). Since

$$\theta^{i0}(\vec{n}, 0) = -\theta^i(\vec{n}, 1) + \theta^i(\vec{n}, 0) + \theta^0(\vec{n} + \hat{\imath}, 0) - \theta^0(\vec{n}, 0)$$

the replacement, Eq.(42), amounts to the change

$$\theta^i(\vec{n}, 1) \to \theta^i(\vec{n}, 1) - b_i(\vec{n}) \equiv \bar{\theta}^i(\vec{n}, 1) \tag{44}$$

The change, Eq.(44), can be reabsorbed in a redefinition of variables of the Feynman integral defining $Z[S + \Delta S]$ which leaves the measure invariant [Eq.(27)]. As a consequence

$$\theta^{ij}(\vec{n}, 1) \to \theta^{ij}(\vec{n}, 1) + \Delta^i b^j - \Delta^j b^i$$

meaning that a monopole is added at $x_0 = 1$. Moreover

$$\theta^{0i}(\vec{n}, 1) \to \theta^{0i}(\vec{n}, 1) + b^i(\vec{n})$$

Again this change can be reabsorbed by a change of variables

$$\theta^i(\vec{n}, 2) \to \theta^i(\vec{n}, 2) - b_i(\vec{n})$$

after which

$$\theta^{ij}(\vec{n}, 2) \to \theta^{ij}(\vec{n}, 2) + \Delta^i b^j - \Delta^j b^i$$

and

$$\theta^{i0}(\vec{n}, 2) \to \theta^{i0}(\vec{n}, 2) + b^i(\vec{n})$$

The construction can be repeated till $n_0 = x_0$, when the addition of $b_i(\vec{n})$ to $\theta^{i0}(\vec{n}, n_0)$ cancels with the term $-b_i(\vec{n})$ in Eq.(43). The change $Z[S + \Delta S]$ amounts to add a monopole on the site $\vec{n} = \vec{y}$, propagating from time 0 to time x_0.

Measuring $\mathcal{D}(x_0)$, to extract M and $\langle \mu \rangle$, is nontrivial due to large fluctuations. ΔS is the change of the action on a spatial volume V: it fluctuates roughly as $\sqrt{V}$, which means a fluctuation $\sim \exp(\sqrt{V})$ for $\langle \mu \rangle$. A way out of this difficulty is to measure, instead of $\mathcal{D}(x_0)$, the quantity

$$\rho(x_0) = \frac{d}{d\beta} \ln \mathcal{D}(n_0) = \langle S \rangle_S - \langle S + \Delta S \rangle_{S+\Delta S} \tag{45}$$

The last equality trivially follows from the definition of Z. $\langle S \rangle_S$ means average of S computed by weighting with the action S. Since $\mathcal{D}(x_0)_{\beta=0} = 1$

$$\mathcal{D}(x_0) = \exp\left(\int_0^\beta \rho(\beta) d\,\beta\right) \tag{46}$$

As $x_0 \to \infty$

$$\rho \underset{x_0 \to \infty}{\simeq} 2 \frac{d}{d\beta} \ln\langle \mu \rangle + C \exp(-M x_0) \tag{47}$$

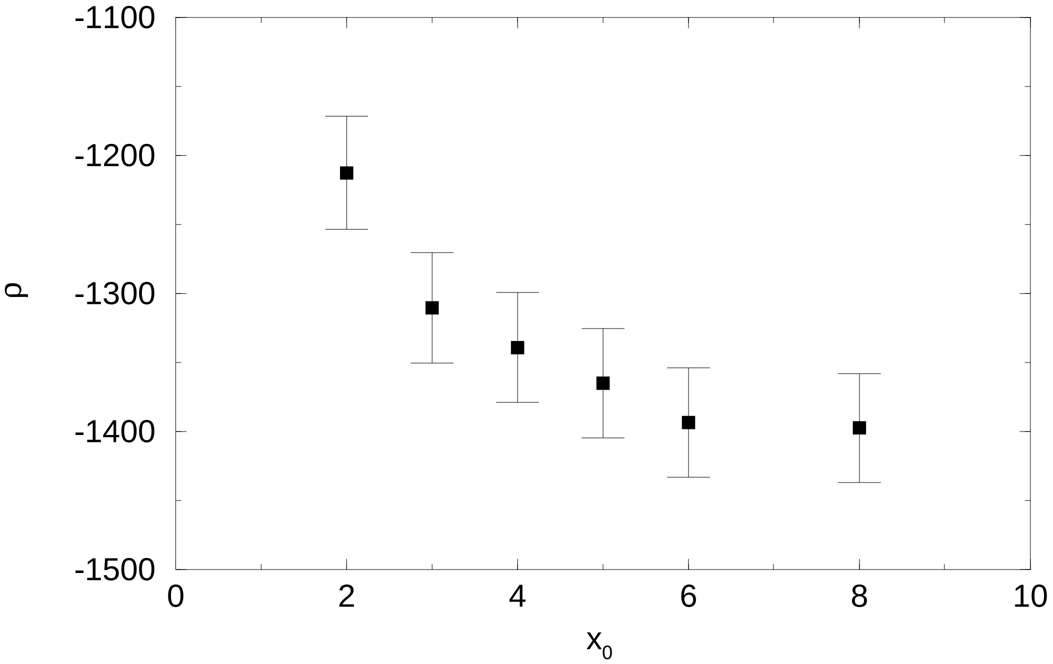

Figure 2. Monopole antimonopole correlation. (Lattice $8^3 \times 16$.)

Fig.2 shows the behaviour of ρ as a function of x_0 consistent with Eq.(47). In fig.3 $\rho(\infty) \equiv 2\frac{d}{d\beta}\ln\langle\mu\rangle$ is shown as a function of β. A huge negative peak appears at the phase transition, which, according to the definitions Eq.(45), Eq.(46) reflects a sharp decrease of $\langle\mu\rangle$. This can be appreciated from fig.4 where a direct measurement of $\langle\mu\rangle$ is shown, even with large errors.

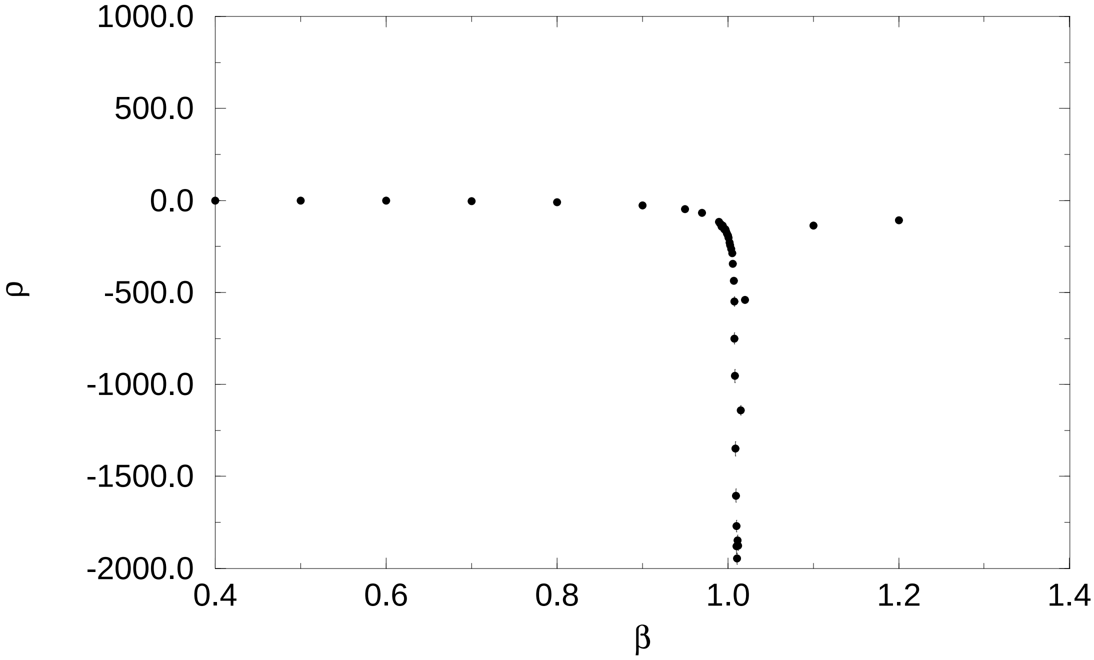

Figure 3. ρ_∞ as a function of β. The negative peak signals the phase transition. (Lattice $8^3 \times 16$.)

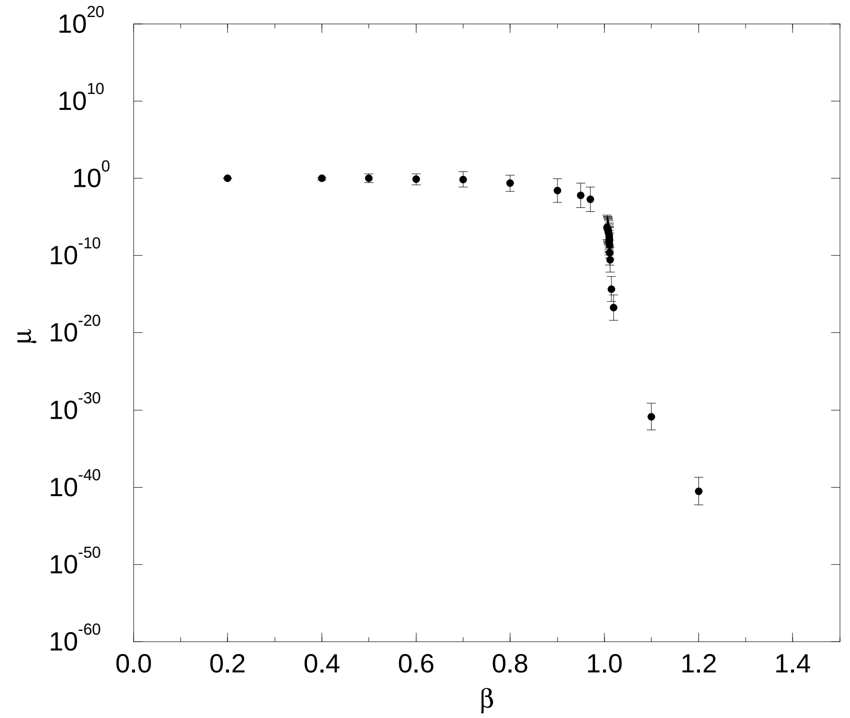

Figure 4. $\langle \mu \rangle$ v.s. β. (Lattice $10^3 \times 20$.)

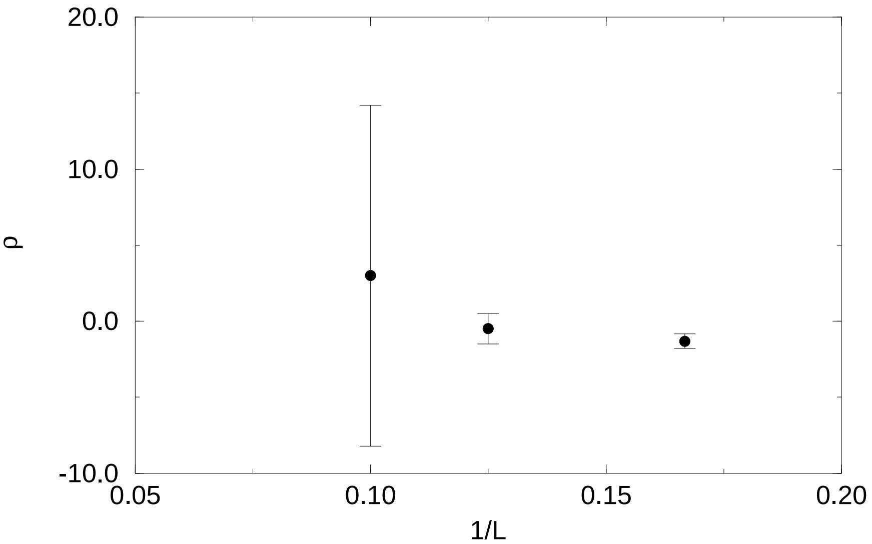

Figure 5. ρ_∞ versus $1/L$ for $\beta = 1.009$.

At $\beta > \beta_c$ the system describes free photons: μ and $\rho(\infty)$ can be computed in perturbation theory by a gaussian integration. The numerical result for a lattice $L^3 \times 2L$ is

$$\rho_\infty = -10.1 \cdot L + 9.542 \qquad (\beta > \beta_c) \tag{48}$$

ρ_∞ tends to $-\infty$ in the thermodynamical limit $L \to \infty$, or $\langle \mu \rangle \to 0$. In fact $\langle \mu \rangle$ is an analytic function of β at finite volume, and cannot be exactly zero for $\beta > \beta_c$, since it would be identically zero everywhere. Only as $V \to \infty$, Lee - Yang singularities develop and $\langle \mu \rangle = 0$ for $\beta > \beta_c$ (48), as a respectable order parameter. For $\beta < \beta_c$, ρ_∞ tends to a finite value as $V \to \infty$. This can be seen from fig.5 but it is also a theorem proven in ref.(16), which generalizes the result of ref.(15) for the Villain action.

For $\beta \sim \beta_c$ there is a phase transition, which is weak first order or second order. In any case the correlation length will grow as $\beta \to \beta_c$ in a certain interval of $(\beta_c - \beta)$ with some effective critical index ν:

$$\xi \simeq (\beta_c - \beta)^{-\nu} \tag{49}$$

A finite size scaling analysis can be done as follow. In general, by dimensional reasons

$$\mu = \mu\left(\frac{L}{\xi}, \frac{a}{\xi}\right) \tag{50}$$

As $\beta \to \beta_c$, $a/\xi \to 0$ and $\mu \simeq \mu(L/\xi, 0)$ or

$$\mu = f\left(L^{1/\nu}(\beta_c - \beta)\right) \tag{51}$$

which implies in turn that

$$\rho = \frac{d}{d\beta}\ln\langle\mu\rangle = -L^{1/\nu}\frac{f'(L^{1/\nu}(\beta_c - \beta))}{f(L^{1/\nu}(\beta_c - \beta))} \tag{52}$$

or $\rho L^{-1/\nu}$ is a universal function of $L^{1/\nu}(\beta_c - \beta)$.

Data from lattices of different size will lay on the same universal curve only for appropriate values of β_c and ν. The best values can be then determined. We obtain

$$\beta_c = 1.01160(5) \qquad \nu = 0.29(2) \tag{53}$$

$\mu \simeq (\beta_c - \beta)^{\delta}$ as $\beta \to \beta_c$, and

$$\frac{\rho}{L^{1/\nu}} \simeq -\frac{\delta}{L^{1/\nu}(\beta_c - \beta)}$$

For δ we obtain $\delta = 1.1 \pm 0.2$.

Our result is then that $\langle\mu\rangle \neq 0$ for $\beta < \beta_c$ in the thermodynamic limit, i.e. that the system is a dual superconductor for $\beta < \beta_c$.

We have also measured the penetration depth of the electric field, i.e. the mass of the photon, m, by the method of ref.(18). m properly scales as $\beta \to \beta_c$ with index ν, and the indication is that it is substantially smaller than M, fig.6, or that the superconductor is type II.

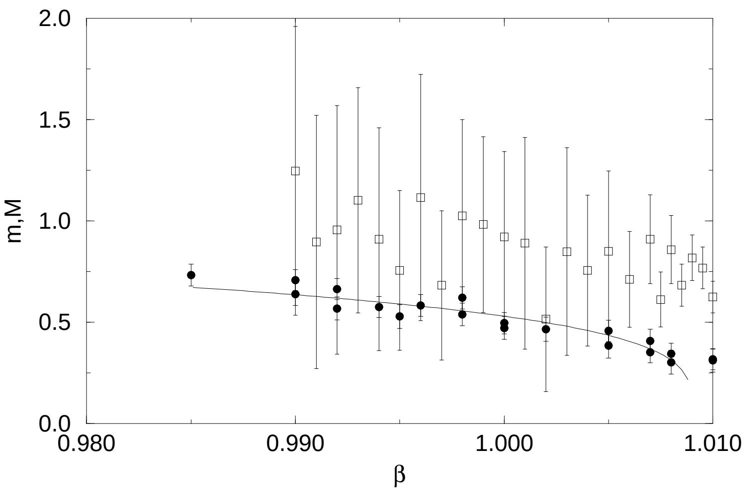

Figure 6. Mass of the monopole M (squares), and mass of the dual photon m (circles) vs. β.

An alternative method to demonstrate dual superconductivity is to detect the London current in the flux tube configurations[15] between $Q\bar{Q}$. For a detailed comparison of our approach with ref.16 we refer to ref.13.

As for a direct determination of $\langle\mu\rangle$ it can be shown that $\langle\mu\rangle$ has a gaussian distribution in the sense of the central limit theorem[13]. However, due to the exponential dependence on ΔS the expectation value of $\langle\mu\rangle$ is not centered at the minimum of ΔS. Let $\Pi(y)$ be the distribution of $y = \beta\Delta S - \beta\langle\Delta\bar{S}\rangle$, the distribution probability for μ is

$$P(\mu) = \Pi\left(\ln(\frac{\mu}{\bar{\mu}})\right) d\ln(\frac{\mu}{\bar{\mu}})$$

If $\Pi(y)$ is gaussian with width σ_y,

$$\langle\mu\rangle = \overline{\mu}\exp(\frac{\sigma_y^2}{2}) \qquad \sigma_\mu = \overline{\mu}\exp(\sigma_y^2)$$

The fluctuations σ_μ are larger than $\langle\mu\rangle$.

A careful analysis requires to account for higher cumulants. The displacement of the maximum of $P(\mu)$ with respect to $\langle\mu\rangle$ should be kept into account when computing the so called constrained effective potential.

As a final comment it can be shown that our construction gives the same result as that of ref.16 in the case of the Villain action. We have thus a disorder parameter which is a reliable tool to detect dual superconductivity.

3d $X-Y$ MODEL[21] (LIQUID He_4)

We have successfully repeated the construction for the $X-Y$ model in 3d, where vortices condense to produce the phase transition. The result can be checked against experiment (liquid He_4). The model is defined on a 3d cubic lattice. An angle $\theta(i)$ is defined on each site i. The action reads

$$S = \beta\sum_i\sum_{\mu=0}^{2}[1-\cos(\Delta_\mu\theta(i))] \tag{54}$$

$\Delta_\mu\theta(i) = \theta(i+\hat{\mu}) - \theta(i)$. The partition function is

$$Z = \int\prod_i\frac{d\theta(i)}{2\pi}\exp(-S) \tag{55}$$

Z is a periodic functional of $\theta(i)$ with period 2π (compactness). Z is invariant under the change of variables

$$\theta(i)\to\theta(i)+f(i) \tag{56}$$

with arbitrary $f(i)$ and so is any correlator of compact fields, in spite of the fact that the transformation (56) is not a symmetry of the action. In running the indices in Eq.(56) from 0 to 2 we anticipate that we shall consider the theory as the euclidean version of a 2+1 dimensional field theory.

As $\beta\to\infty$

$$S \simeq \frac{\beta}{2}\sum_{\mu,i}[\Delta_\mu(\theta_i)]^2 \tag{57}$$

and the theory describes a massless scalar field. At $\beta=\beta_c=0.454$ a 2nd order phase transition takes place. Below β_c vortices are expected to condense in the vacuum. Like for monopoles, condensation has always been demonstrated by the drop of the density of vortices when β rises through β_c. We will show instead that a spontaneous symmetry breaking of the $U(1)$ symmetry related to the conservation of vortex number takes place and that a legitimate disorder parameter can be defined.

We define

$$A_\mu = \partial_\mu \theta \tag{58}$$

Under the transformation, Eq.(56), A_μ undergoes a gauge transformation

$$A_\mu \to A_\mu + \partial_\mu f \tag{59}$$

The invariance under Eq.(56) means gauge invariance if the theory is phrased in terms of A_μ. From Eq.(58) it follows

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 0 \tag{60}$$

In fact Eq.(60) is valid apart from singularities. In terms of A_μ

$$\theta(x) = \exp(\mathrm{i}\int_C^x A_\mu dx^\mu) \tag{61}$$

and, if Eq.(60) holds, the choice of the path C used in Eq.(61) is irrelevant. A current j_μ can be defined as the dual of $F_{\mu\nu}$:

$$j_\mu = \varepsilon_{\mu\alpha\beta}\partial^\alpha A^\beta \tag{62}$$

and

$$\partial^\mu j_\mu = 0 \tag{63}$$

is the analog of the Bianchi identities. The conserved quantity associated to Eq.(63) is the vorticity

$$Q = \int d^2x\ j^0(\vec{x},t) = \int d^2x(\vec{\nabla}\wedge\vec{A})_0 = \oint \vec{A}\cdot d\vec{x} \tag{64}$$

Since $\vec{A} = \vec{\nabla}\theta$, single-valuedness of the action implies that $Q = 2\pi n$. If there are no singularities it follows from Eq.(58) that $\vec{\nabla}\wedge\vec{A} = 0$, or from Eq.(64) that $Q = 0$.

There exist however configurations with nontrivial vorticity. An example is

$$\bar{\theta}_{(q)}(\vec{x}-\vec{y}) = q\arctan\frac{(x_2-y_2)}{(x_1-y_1)} \tag{65}$$

For this configuration

$$A_0 = 0 \qquad \vec{A} = \frac{q}{r}\vec{\nu}_\theta \tag{66}$$

$\vec{\nu}_\theta$ being the unit vector in the polar coordinate direction θ. If $\vec{A}$ is the field of velocities, $\bar{\theta}_{(q)}$ describes a vortex. For this configuration $j^0 = 2\pi\delta^{(2)}(\vec{x}-\vec{y})$ and

$$\oint_C \vec{A}\cdot d\vec{x} = 2\pi q \tag{67}$$

if the path C encloses $\vec{y}$ once.

A disorder parameter describing condensation of vortices can be defined. In the continuum this is nothing but addition to the field $\theta(x)$ of any configuration, the field $\bar{\theta}$ describing a vortex. The conjugate momentum to θ being $\beta\sin\partial_0\theta = \delta\mathcal{L}/\delta(\partial_0\theta)$

$$\mu(\vec{y},t) = \exp\left[\mathrm{i}\beta\int d^2x\ \sin(\partial_0\theta(\vec{x},t))\bar{\theta}_{(q)}(\vec{x}-\vec{y})\right] \tag{68}$$

analogous to Eq.(38).

After Wick rotation the compactified version of Eq.(68) becomes (see Eq.(37))

$$\mu(\vec{y},t) = \exp\left\{-\beta\sum_{\vec{n}}\left[\cos(\Delta_0\theta(\vec{n},t) - \bar{\theta}_{(q)}(\vec{n}-\vec{y})) - \cos(\Delta_0\theta(\vec{n},t)\right]\right\} \tag{69}$$

When computing correlation functions of $\mu(\vec{y},t)$, Eq.(69) produces a change of the term in the action containing $\Delta_0\theta$ at time t from $\Delta_0\theta(\vec{n},t)$ to $\Delta_0\theta(\vec{n},t) - \bar{\theta}_{(q)}(\vec{n}-\vec{y})$. The correlator

$$\mathcal{D}(t) = \langle\mu(\vec{0},t)\mu(\vec{0},0)\rangle \tag{70}$$

will behave as

$$\mathcal{D}(t) \underset{t\to\infty}{\simeq} \langle\mu\rangle^2 + A\mathrm{e}^{-Mt} \tag{71}$$

and, from the definition of μ

$$\mathcal{D}(t) = \frac{Z[S+\Delta S]}{Z[S]} \tag{72}$$

The $S+\Delta S$ is obtained from S by the modification

$$\Delta_0\theta(\vec{n},0) \to \Delta_0\theta(\vec{n},0) - \bar{\theta}_{(q)}(\vec{n}-\vec{y}) \tag{73}$$

$$\Delta_0\theta(\vec{n},t) \to \Delta_0\theta(\vec{n},t) + \bar{\theta}_{(q)}(\vec{n}-\vec{y}) \tag{74}$$

A change of variables $\theta(\vec{n},1) \to \theta(\vec{n},1) - \bar{\theta}_{(q)}$ reabsorbs the modification in $\Delta_0\theta(\vec{n},0)$, but shifts $\Delta_i\theta(\vec{n},1) = A_i(\vec{n},1)$ by a vortex $\Delta_i\bar{\theta}_{(q)} = A_i^{(q)}$ and sends

$$\Delta_0\theta(\vec{n},1) \to \Delta_0\theta(\vec{n},1) - \bar{\theta}_{(q)}(\vec{n}-\vec{y})$$

Similarly to what was done with $U(1)$ monopoles the construction can be repeated till $n_0 = t$ is reached and $-\bar{\theta}_{(q)}$ cancels $\bar{\theta}_{(q)}$ of Eq.(74).

$\mathcal{D}(t)$ describes a vortex at $\vec{y}=0$ propagating in time from 0 to t. Instead of $\mathcal{D}(t)$, $\rho(t) = \frac{d}{d\beta}\ln\mathcal{D}(t)$ can be studied. At large t

$$\rho(t) \simeq \rho + A\mathrm{e}^{-Mt} \tag{75}$$

$$\rho = 2\frac{d}{d\beta}\ln\langle\mu\rangle \qquad \langle\mu\rangle = \exp\left[\frac{1}{2}\int_0^\beta \rho_\infty(\beta)\,d\beta\right] \tag{76}$$

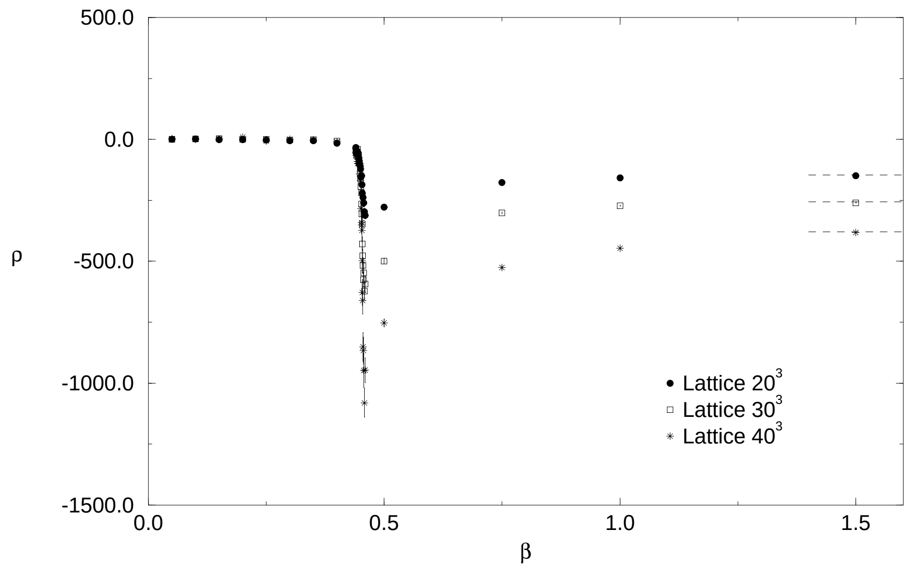

Figure 7. ρ vs β for $X-Y$ model. The peak is at the transition to a superfluid.

From Eq.(76) the disorder parameter $\langle\mu\rangle$ can be determined. The typical behaviour of ρ is shown in fig.6. In the thermodynamical limit, $\langle\mu\rangle \neq 0$ signals spontaneous symmetry breaking of the symmetry Eq.(63), and hence condensation of vortices. At large β the theory is free and ρ can be computed by a gaussian integration. The result for a cubic lattice of size L is

$$\rho = -11.33 \cdot L + 72.669 \tag{77}$$

As $L \to \infty$ this implies $\langle\mu\rangle = 0$. For $\beta < \beta_c$, ρ tends to a finite value, as $L \to \infty$.

Around β_c a finite size scaling analysis can be done as in the $U(1)$ model giving

$$\nu = 0.669 \pm 0.065 \qquad [.670(7)] \tag{78}$$

$$\beta_c = 0.4538 \pm 0.0003 \qquad [.45419(2)] \tag{79}$$

$$\delta = 0.740 \pm 0.029 \tag{80}$$

δ is the index of μ. The numbers in parenthesis on the right are the accepted determinations by other methods[22]. The agreement is good.

MONOPOLES IN QCD. REVISITING THE ABELIAN PROJECTION[23]

At the classical level monopoles in non-abelian gauge theories can be defined by the usual multipole expansion[24, 25]. At large distances the magnetic monopole field obeys abelian equations. By a suitable choice of gauge the direction of the Dirac string can be chosen, and magnetic charges are identified by a diagonal matrix in the fundamental representation with positive or negative integer eigenvalues. For $SU(N)$, $N-1$ magnetic charges exist, which correspond to a group $U(1)^{N-1}$.

This classification coincides with the so called abelian projection. Let $\vec{\Phi} \cdot \vec{\sigma}$ be any field belonging to the adjoint representation: in what follows we shall refer to $SU(2)$ for the sake of simplicity. For $SU(N)$ the procedure is analogous with some formal complication. A gauge transformation $U(x)$ which diagonalizes $\vec{\Phi}\cdot\vec{\sigma}$ is called an abelian projection. $U(x)$ will be singular at the locations where $\vec{\Phi} = 0$. These locations are world lines of $U(1)$ Dirac monopoles. $U(1)$ is the residual invariance under rotation around the axis of $\vec{\Phi} \cdot \vec{\sigma} = \Phi\sigma_3$ after diagonalization.

These monopoles are supposed to condense and produce dual superconductivity. There is a large arbitrariness in the choice of $\vec{\Phi}\cdot\vec{\sigma}$, i.e. in the identification of monopoles. We will discuss this point in detail below.

The meaning of the abelian projection can be better understood[26] in the Georgi Glashow model[27]. This is an $SO(3)$ gauge theory coupled to a triplet Higgs

$$\mathcal{L} = -\frac{1}{4}\vec{G}_{\mu\nu}\vec{G}_{\mu\nu} + (D_\mu\vec{\Phi})^\dagger(D_\mu\vec{\Phi}) - V(\vec{\Phi}) \tag{81}$$

with

$$D_\mu\vec{\Phi} = (\partial_\mu - g\vec{W}_\mu\wedge)\vec{\Phi}$$

and

$$V(\vec{\Phi}) = \frac{\lambda}{4}\left[(\vec{\Phi}^2)^2 - \mu^2\vec{\Phi}^2\right]$$

The model admits monopoles as soliton solutions in the spontaneous broken phase where $\langle\vec{\Phi}\rangle = \vec{\Phi}^0 \neq 0$.

Usually a fixed (space independent) frame of reference is used in colour space, e.g. $\vec{\xi}_i^0$ $(i = 1, 2, 3)$, the unit vectors of an orthogonal cartesian frame. $\vec{\xi}_i^0 \wedge \vec{\xi}_j^0 = \varepsilon_{ijk}\vec{\xi}_k^0$. One can, however, define a Body Fixed Frame (BFF) by 3 orthogonal unit vectors $\vec{\xi}_i(x)$

$$\vec{\xi}_i \wedge \vec{\xi}_j = \varepsilon_{ijk}\vec{\xi}_k \tag{82}$$

such that $\vec{\xi}_3(x) = \hat{\Phi}(x)$ is parallel to the direction $\hat{\Phi}(x)$ of the $\vec{\Phi}$ field. This system is defined up to a rotation around $\vec{\xi}_3$. A gauge group element $R(x)$ exists such that

$$\vec{\xi}_i(x) = R^{-1}(x)\vec{\xi}_i^0 \tag{83}$$

By construction $R(x)$ is the gauge transformation which implements the abelian projection, since $R(x)\vec{\xi}_3 = \vec{\xi}_3^0$. Since $\vec{\xi}_i^2 = 1$, $\partial_\mu \vec{\xi}_i = \vec{\omega}_\mu \wedge \vec{\xi}_i$ or

$$D_\mu(\omega)\vec{\xi}_i \equiv (\partial_\mu - \vec{\omega}_\mu \wedge)\vec{\xi}_i = 0 \tag{84}$$

Eq.(84) implies $[D_\mu, D_\nu] = G_{\mu\nu}(\vec{\omega}) = 0$. The field $\vec{\omega}_\mu$ is a pure gauge. The last equality reads explicitly

$$\partial_\mu \vec{\omega}_\nu - \partial_\nu \vec{\omega}_\mu + \vec{\omega}_\mu \wedge \vec{\omega}_\nu = 0 \tag{85}$$

Eq.(85) is true apart from singularities. As a consequence of Eq.(85)

$$\vec{\Phi}(x) = P \exp\left[\mathrm{i} \int_C^x \vec{\omega}_\mu \cdot \vec{T}\, dx^\mu\right] \vec{\Phi}^0 \tag{86}$$

Due to Eq.(85) the integral in Eq.(86) is independent of the path C. This is true apart from singularities which can give a nontrivial connection to space time.

A 't Hooft[28]-Polyakov[29] monopole configuration has a zero of $\vec{\Phi}(x)$ at the location of the monopole, and in that point the abelian projection $R(x)$ has a singularity.

The singularities of $\vec{\omega}_\mu$ can be studied by expressing $\vec{\xi}(x)$ in terms of polar coordinates $(\theta(x), \psi(x))$, with polar axis 3 in colour space. The singularities come from the fact that $\psi(x)$ is not defined at the sites where $\theta(x) = 0, \pi$. In terms of $\vec{G}_{\mu\nu}(\omega)$ the potentially singular term at $\theta(x) = 0, \pi$ is[26]

$$F^3_{\mu\nu}(x) = -\cos\theta(x)(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\psi(x) \tag{87}$$

The singularity exists where $\theta = 0, \pi$ or at the sites where $\hat{\Phi}$ is in the direction 3, and the field is parallel to $\hat{\Phi}$ in colour space and abelian. The singularity is a string with flux $\pm 2n\pi$ $(n = 0, 1 \ldots)$. The abelian field is related to the residual $U(1)$ symmetry along the 3d axis. The field strength is the abelian part of $\vec{F}_{\mu\nu}$, or

$$\mathcal{F}_{\mu\nu} = \hat{\Phi} \cdot \vec{G}_{\mu\nu} - \frac{1}{g}\hat{\Phi} \cdot (D_\mu\hat{\Phi} \wedge D_\nu\hat{\Phi}) \tag{88}$$

Indeed, in the abelian projected frame $D_\mu\hat{\Phi} = g\vec{A}_\mu \wedge \vec{\xi}_3^0$ and

$$\frac{1}{g}\hat{\Phi} \cdot (D_\mu\vec{\Phi} \wedge D_\nu\vec{\Phi}) = g(\vec{W}_\mu \wedge \vec{W}_\nu) \cdot \vec{\Phi}$$

which cancels the non-abelian term of $\hat{\Phi} \cdot \vec{G}_{\mu\nu}$ in Eq.(88). $\mathcal{F}_{\mu\nu}$ is nothing but a covariant expression for the $U(1)$ field identified by the abelian projection. It is a gauge invariant quantity. For a 't Hooft-Polyakov monopole configuration $\mathcal{F}_{\mu\nu}$ is the field of a pointlike Dirac monopole, located at the zero of $\vec{\Phi}(x)$.

In QCD there are no fundamental Higgs fields. The idea is that monopoles could be defined by any composite field $\vec{\Phi}(x)$ in the adjoint representation[23]. No unique criterion is known for the choice of the operator $\vec{\Phi}$ in QCD. Popular choices are[23]:

1) $\hat{\Phi}(x)$ is the Polyakov line

$$P(\vec{x}, x_0) = \oint_{C,x_0}^{x_0} P\exp(\mathrm{i}\vec{A}_\mu \cdot \vec{T}\, dx^\mu) \tag{89}$$

The path C being the line $\vec{x} =$ constant along the time axis closing at infinity by periodic boundary conditions.

2) Any component $\vec{F}_{\mu\nu}$ of the field strength tensor.

3) The operator implicitly defined by the maximization of

$$\sum_{\mu,n} \mathrm{Tr}\left\{\sigma_3 U_\mu(n)\sigma_3 U_\mu^\dagger(n)\right\} \tag{90}$$

This choice is known as maximal abelian projection.

4) For $SU(3)$ $(F_{\mu\nu})^2$ since, contrary to the case of $SU(2)$, it is not a singlet, due to the d algebra.

For each of these choices a $U(1)$ gauge field (actually 2 for $SU(3)$) can be identified, which couples to monopoles, and a creation operator of monopoles can be constructed on the same lines as for compact $U(1)$.

A strategy to answer the question whether QCD vacuum is a dual superconductor is to detect the condensation of the monopoles defined by different abelian projections in the confined phase and across the transition to the deconfined phase. As shown in the previous sections a reliable tool (disorder parameter) exists for that, which has been successfully tested in systems which are well understood. Before proceeding to that we will test the ideas of this section on a simple system: the Heisenberg ferromagnet.

The Heisenberg ferromagnet[30]

The action is

$$\mathcal{L} = \frac{1}{2}\sum_{\mu=0}^{2}\sum_x \Delta_\mu\vec{n}(x)\cdot\Delta_\mu\vec{n}(x) = \sum_{\mu=0}^{2}\sum_x(1 - \vec{n}(x+\hat{\mu})\cdot\vec{n}(x)) \tag{91}$$

x runs on a cubic 3d lattice and $\vec{n}$ is a vector of unit length, $\vec{n}^2 = 1$.

We shall look at the model as a 2+1 dimensional field theory. The Feynman functional

$$Z = \int\prod_x d\Omega(x)\ \exp(-\beta S) \tag{92}$$

has a much bigger symmetry than the lagrangian. Any local rotation $\vec{n}(x) \to R(x)\vec{n}(x)$ even if it does not leave $\mathcal{L}$ invariant, is reabsorbed in a change of variables in the Feynman integral, leaving the measure invariant. Assuming constant boundary condition at infinity, $\vec{n}_0$, we can write

$$\vec{n}(x) = R^{-1}(x)\vec{n}_0 \tag{93}$$

R^{-1} is determined up to a rotation along $\vec{n}$. As in the Georgi Glashow model a gauge field $\vec{\omega}_\mu$ can be defined by introducing a Body Fixed Frame $\vec{\xi}_i$, with $\vec{\xi}_3 = \vec{n}$. Then

$$\partial_\mu\vec{\xi}_i = \vec{\omega}_\mu \wedge \vec{\xi}_i$$

or

$$D_\mu\vec{\xi}_i = (\partial_\mu - \mathrm{i}\vec{\omega}\wedge)\vec{\xi}_i = 0$$

which implies $F_{\mu\nu}(\omega) = 0$, apart from singularities and

$$\vec{n}(x) = P \exp\left[\mathrm{i} \int_{\infty, C}^{x} \vec{\omega}_\mu \cdot \vec{T}\, dx^\mu\right] \vec{n}_0 \tag{94}$$

independent of the path C, $\vec{\omega}_\mu$ being a pure gauge. Again this is true apart from singularities. A conserved current exists,

$$\vec{j}_\mu = \frac{1}{8\pi} \varepsilon_{\mu\alpha\beta} \partial_\alpha \vec{n} \wedge \partial_\beta \vec{n} \tag{95}$$

$\vec{j}_\mu$ is parallel to $\vec{n}$, since both $\partial_\alpha \vec{n}$ and $\partial_\beta \vec{n}$ are orthogonal to it. The corresponding conserved quantity is

$$Q = \int d^2x j^0(x) = \frac{1}{4\pi} \int d^2x\ \vec{n} \cdot (\partial_1 \vec{n} \wedge \partial_2 \vec{n}) \tag{96}$$

which is nothing but the topological charge of the 2 dimensional version of the model. Instantons of the 2 dimensional model look as solitons of the 2+1 dimensional one.

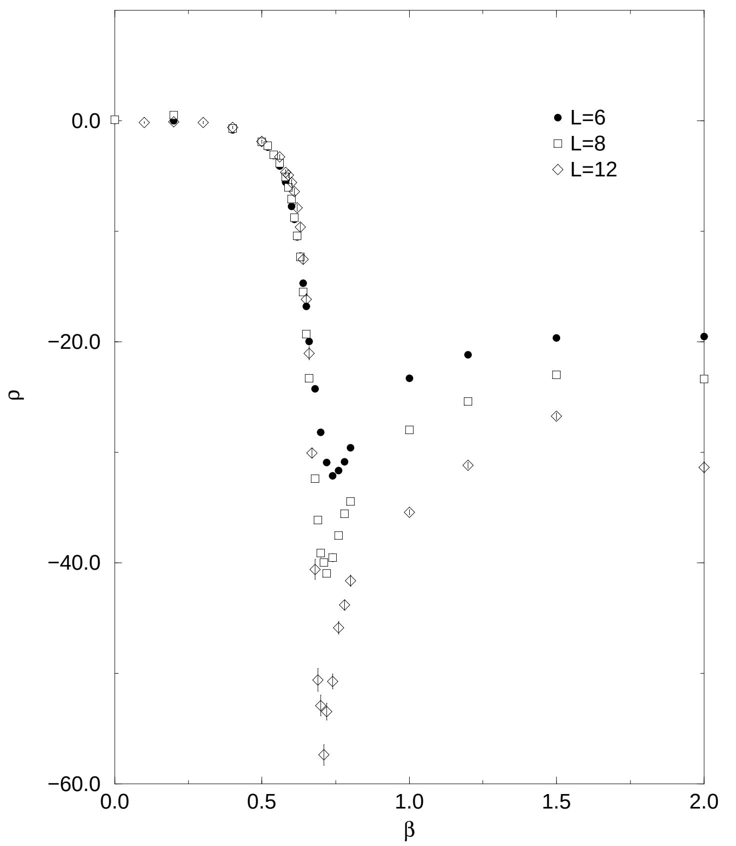

Figure 8. ρ v.s. β. Heisenberg model.

$\vec{j}_\mu$ can also be written as

$$\vec{j}_\mu = \frac{\vec{n}}{8\pi} \left[\varepsilon_{\mu\alpha\beta} \left(\vec{\omega}_\alpha \wedge \vec{\omega}_\beta\right) \cdot \vec{n}\right] \tag{97}$$

Except for the singularities corresponding to the locations of the instantons $\vec{\omega}_1 \wedge \vec{\omega}_2 = \partial_1 \vec{\omega}_2 - \partial_2 \vec{\omega}_1$ and

$$Q = \frac{1}{4\pi} \oint \vec{\omega}_\alpha \cdot \vec{n}\ dx_\alpha \tag{98}$$

A nontrivial connection is created by the presence of solitons. A direct calculation shows that the field $\vec{\omega}_\mu$ has Dirac string singularities propagating in time from the center of the soliton. The transition from a magnetized phase to a disordered phase can be seen as a condensation of solitons.

The system undergoes a second order phase transition at $\beta = \beta_c \simeq 0.7$ from the magnetized phase to a disordered phase. We have investigated if the solitons described above condense in the disordered phase. A disorder parameter can be constructed, on the same line as in the compact $U(1)$ and in 3d $X-Y$ model, as the vev of the creation operator of a soliton, as follows

$$\langle \mu(x) \rangle = \frac{Z[S+\Delta S]}{Z[S]} \tag{99}$$

where $S+\Delta S$ is obtained from S by the change

$$(\Delta_0 \vec{n}(\vec{x},x_0))^2 \equiv (\vec{n}(\vec{x},x_0+1) - \vec{n}(\vec{x},x_0))^2 \to (R_q^{-1}\vec{n}(\vec{x},x_0+1) - \vec{n}(\vec{x},x_0))^2 \tag{100}$$

R_q is a time independent transformation which adds a soliton of charge q to a configuration.

Numerical simulations show that in the thermodynamical limit $\langle \mu(x) \rangle$ vanishes in the ordered (magnetized) phase, is different from zero in the disordered phase, and at the phase transition obeys a finite size scaling law from which the critical indices and the transition temperature can be extracted. A typical form of $\rho = d \ln\langle \mu \rangle / d\beta$ is shown in fig.8.

The results are still preliminary but agree with the values known from other methods[31]. We get

$$\beta_c = 0.69 \pm 0.01 \tag{101}$$

and

$$\nu = 0.7 \pm 0.2 \tag{102}$$

This shows that the phase transition to disorder in the Heisenberg magnet can be viewed as condensation of solitons.

The string structure of the singularities of the field $\vec{\omega}_\mu$ is similar to Georgi Glashow model, and the field strength tensor is generated by the topology of the field $\vec{n}$, even in the absence of gauge fields. Indeed going back to the previous section, the monopoles only depend on the Higgs field. Monopoles exposed by the abelian projection are not lattice artifacts, but reflect the dynamics of the field $\vec{\Phi}$.

DUAL SUPERCONDUCTIVITY IN QCD

I will present the first results of a systematic exploration of monopole condensation below and above the deconfining transition in different abelian projections[33, 32].

Besides the disorder parameter we also measure, at $T = 0$, the monopole anti-monopole correlation function to extract a lower limit to the effective Higgs mass, as well as the penetration depth of the electric field, i.e. the mass of the photon which produces the (dual) Meissner effect. We use for that APE QUADRIX machines.

The creation operator for a monopole is constructed in analogy with the $U(1)$ operator as follows: the Π^{0i} plaquettes in the action at, say the time $n_0 = 0$, when the monopole is created are modified as follows

$$\begin{aligned} \Pi^{0i}(\vec{n},n_0) &= \mathrm{Tr}\left\{U_i(\vec{n},0)U_0(\vec{n}+\hat{i},0)U_i^\dagger(\vec{n},1)U_0^\dagger(\vec{n},0)\right\} \\ &\to \mathrm{Tr}\left\{U_i'(\vec{n},0)U_0(\vec{n}+\hat{i},0)U_i^\dagger(\vec{n},1)U_0^\dagger(\vec{n},0)\right\} \end{aligned} \tag{103}$$

In Eq.(103)

$$U_i'(\vec{n},0) = \mathrm{e}^{\mathrm{i}\Lambda(\vec{n})}\exp(\mathrm{i}\hat{\Phi}\cdot\vec{\sigma}\frac{b_i^{\perp}(\vec{n})}{2})U_i(\vec{n},0)\exp(\mathrm{i}\hat{\Phi}\cdot\vec{\sigma}\frac{b_i^{\perp}(\vec{n})}{2})\mathrm{e}^{-\mathrm{i}\Lambda(\vec{n}+\hat{\imath})} \tag{104}$$

We call $S+\Delta S$ the resulting action. Then

$$\langle\mu\rangle = \frac{Z[S+\Delta S]}{Z[S]} \tag{105}$$

The vector potential describing the field of the monopole b_i has been split in a transverse part $b_i^{\perp}$ with $\partial_i b_i = 0$ and a gauge part $\partial_i\Lambda$. The gauge dependence in Eq.(105) can be reabsorbed in a redefinition of the temporal links, which leaves the measure invariant.

A redefinition of U_i to $\tilde{U}_i = \mathrm{e}^{\mathrm{i}\sigma_3\frac{b_i^{\perp}(n)}{2}}U_i\mathrm{e}^{\mathrm{i}\sigma_3\frac{b_i^{\perp}(n)}{2}}$ in the abelian projected gauge can be reabsorbed in a change of variables which leaves the measure invariant. The space plaquettes however acquire in each link factors

$$\mathrm{e}^{\mathrm{i}\vec{\sigma}\cdot\hat{\Phi}\frac{b_i^{\perp}(n)}{2}}\tilde{U}_i\mathrm{e}^{\mathrm{i}\vec{\sigma}\cdot\hat{\Phi}\frac{b_i^{\perp}(n)}{2}} \tag{106}$$

In the abelian projected gauge the generic $\tilde{U}_i$ can be written as

$$\tilde{U}_i = \mathrm{e}^{\mathrm{i}\sigma_3\alpha_i}\mathrm{e}^{\mathrm{i}\sigma_2\gamma_i}\mathrm{e}^{\mathrm{i}\sigma_3\beta_i} = \mathrm{e}^{\mathrm{i}\sigma_3\alpha_i}\mathrm{e}^{\mathrm{i}\sigma_2\gamma_i}\mathrm{e}^{-\mathrm{i}\sigma_3\alpha_i}\mathrm{e}^{\mathrm{i}\sigma_3(\alpha_i+\beta_i)} = V\mathrm{e}^{\mathrm{i}\sigma_3\theta_i} \tag{107}$$

with $\exp(\mathrm{i}\sigma_3\theta_i)$ the abelian link and $\theta_i = \alpha_i + \beta_i$. The transformation Eq.(104) adds b_i to θ_i or adds the magnetic field of the monopole to the abelian magnetic field

$$\Delta_i\theta_j - \Delta_j\theta_i \to \Delta_i\theta_j - \Delta_j\theta_i + \Delta_i b_j - \Delta_j b_i \tag{108}$$

An abelian projected monopole has been created. $\langle\mu\rangle$, or better $\rho = d(\ln\langle\mu\rangle)/d\beta$, is measured across the deconfining phase transition. Typical behaviours are shown in fig.9 and fig.10 for the monopole defined by the Polyakov loop.

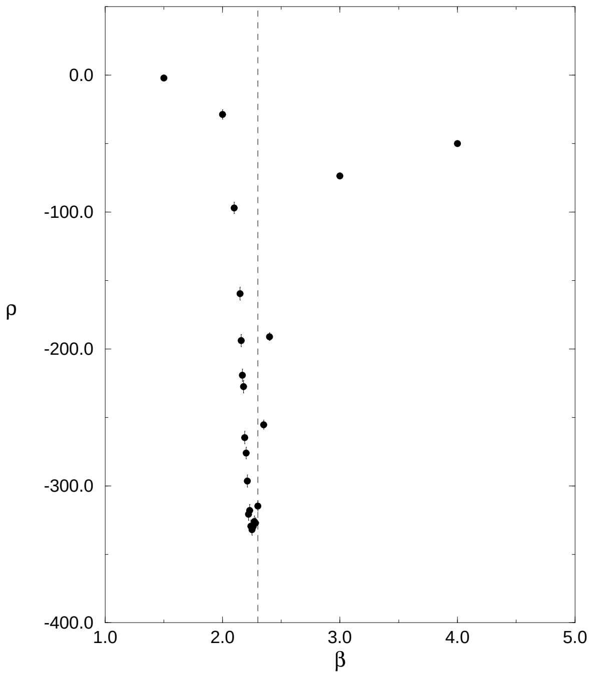

Figure 9. ρ v.s. β $SU(2)$ gauge theory. (Lattice $12^3 \times 4$.)

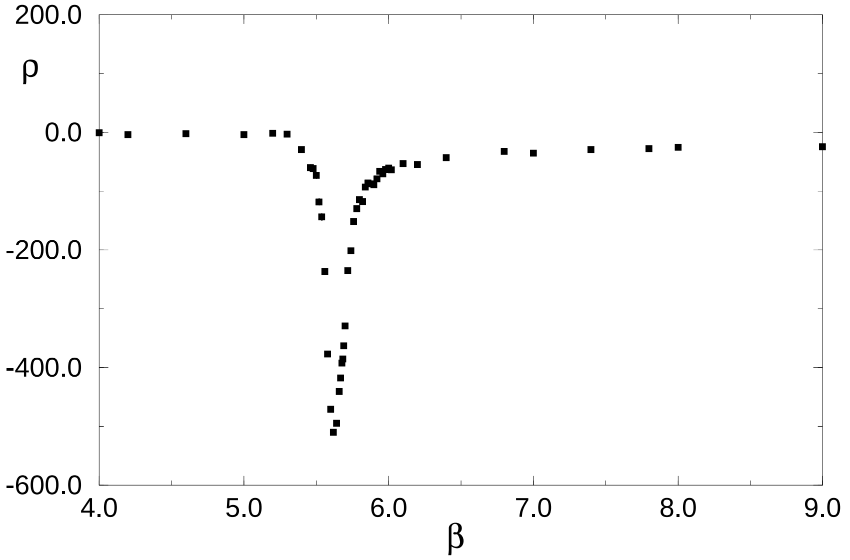

Figure 10. ρ v.s. β $SU(3)$ gauge theory. (Lattice $12^3 \times 4$.)

A careful analysis of the thermodynamical limit produces evidence of dual superconductivity below the deconfining temperature T_c for different choices of $\hat{\Phi}$: Polyakov line, $F_{\mu\nu}$, maximal abelian.

We need more work and more statistics to determine the critical indices and the type of superconductor.

DISCUSSION

Most of the success of the dual superconductivity mechanism for confinement is presently based on the abelian dominance and on the monopole dominance, numerically observed in the maximal abelian projection[34, 35, 36]. In $SU(2)$ gauge theory after maximal abelian projection, which amounts to maximizing the quantity

$$A = \sum_{n,\mu} \mathrm{Tr}\left\{\sigma_3 U_\mu(n) \sigma_3 U_\mu^\dagger(n)\right\}$$

with respect to gauge transformations, all the links are on the average parallel to the 3 axis within $80 - 90\%$. The string tension computed from the abelian part of the Wilson loops accounts for $80 - 90\%$ of the full string tension. Similar behaviour is found for many other quantities. This is called abelian dominance. In addition, the abelian plaquettes can be split as in Eq.(29) in a monopole part $n_{\mu\nu}$ and the residual angle $\bar{\theta}_{\mu\nu}$, usually called the Coulomb field. The likewise empirical observation is that the contribution of the Coulomb part to the abelian quantities is a small factor, so that for example the string tension is dominated in fact by the contribution of the abelian monopoles (monopole dominance).

Apparently such dominance is not observed in other abelian projections, except maybe in the Polyakov line projection where, if the string tension is measured from the correlation between Polyakov lines, it is 100% abelian dominated by construction.

The more or less explicitly expressed idea is then that some abelian projection (the maximal abelian) is better than others and is "the abelian projection", identifying the degrees of freedom relevant for confinement. On the one hand the fact that in this projection links are in the abelian direction at $80 - 90\%$ implies dominance as a kinematical fact: on the other hand maybe it is nontrivial that such a gauge exists.

The attitude presented in this paper is somewhat complementary: dual superconductivity is related to symmetry, and the way to detect it is to look for symmetry. From this point of view, independent of possible abelian dominances, what we find is more similar to the idea of 't Hooft that all abelian projections are physically equivalent. We find condensation of max abelian, Polyakov line, field strength monopoles.

There are a few aspects of the mechanism which need further understanding.

1) Any abelian projection implies that the gluons corresponding to the residual $U(1)$'s have no electric charge with respect to them, and hence are not confined. This contrasts with the observation about the string tension in the adjoint representation reported in the introduction.

2) If the mechanism of confinement were superconductivity produced by condensation of the monopoles defined by some abelian projection then the electromagnetic field in the flux tubes joining $q - \bar{q}$ pairs should belong to the projected $U(1)$. Lattice exploration show that it is isotropically distributed in color space[6, 37].

3) There are infinitely many abelian projections which can be obtained from each other by continuous transformations, e.g. by shifting the zero of $\vec{\Phi}(x)$. If one of them were privileged, it is hard to understand how others, which differ by small continuous changes, could be so different.

Most probably a more complicated and new mechanism is at work, a kind of non-abelian dual superconductivity, which manifests itself as abelian superconductivity in different abelian projected gauges. Our aim is to understand this.

Acknowledgements

It is a special pleasure to thank all the collaborators who contributed to this research, in particular Giampiero Paffuti, Manu Mathur and Luigi Del Debbio.

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