PERFECT ACTIONS

Peter Hasenfratz*

Institute for Theoretical Physics
University of Bern, Sidlerstrasse 5
CH-3012 Bern, Switzerland

INTRODUCTION

This lecture note is a pedagogical introduction to the theoretical background and properties of perfect actions. It is intended to be largely self-contained. The general aspects of lattice regularization and Wilson's renormalization group[1] and in particular those which are closely related to our subject will be explicitly introduced and discussed. The reader might even consider the topic of perfect actions as a pretext to learn about these powerful theoretical and practical tools in quantum field theories. I hope, however, that the unexpected and amazing properties of the perfect actions will catch the readers' fantasy also.

The very definition of a quantum field theory requires a regularization. As it is well known, all the regularizations break some of the symmetries of the underlying classical theory. They introduce a new scale (the cut-off) which is actually reflected in the final predictions even after the regularization is removed: naive dimensional analysis breaks down, anomalous dimensions are created, dynamical mass generation might occur, etc. On the other hand, beyond these exciting physical effects, the new scale creates dirt, cut-off artifacts everywhere which have to be eliminated in a theoretically and practically difficult limiting process. On the tree level (in the classical theory) the new introduced scale creates dirt only, nothing else. If the classical theory is scale invariant (like the Yang-Mills theory or massless QCD) the cut-off artifacts destroy the most interesting and relevant classical properties of the model like the existence of scale invariant classical solutions or fermionic zero modes and index theorems.

Regularizations often break other symmetries also. The lattice, the only known non-perturbative regularization, breaks Euclidean rotation symmetry (Lorentz symmetry) which is less of a headache, and also chiral symmetry which creates a difficult tuning and renormalization problem. Chiral symmetry awakes deep theoretical issues (anomalies, no-go theorems) culminating in problems concerning chiral gauge theories.

On the other hand, perfect actions[2] are perfect. We shall call a lattice regularized local action *classically perfect* if its classical predictions (independently whether the lattice is fine or coarse, whether the resolution is good or bad) agree with those of the

*Work supported in part by Schweizerischer Nationalfonds

continuum. The *quantum perfect action* does the same for all the physical questions in the quantum theory. That such actions exist might seem to be surprising. Their existence is closely related to renormalization group theory. As we shall see, they have beautiful properties which one would not expect a lattice action can have.

The classically perfect action has scale invariant instanton solutions, has no topological artifacts, satisfies the index theorem concerning the fermionic zero modes on the lattice, preserves some (presumably, all) important physical consequences of chiral symmetry in the quantum theory and is expected to reduce the cut-off effects significantly even in quantum simulations.

In this note we shall mainly discuss the theoretical properties of the classically perfect actions. These actions are the fixed points of renormalization group transformations in asymptotically free theories and are determined by classical field theory equations. They are not abstract theoretical constructions only. Actually, most of the effort during the last four years was directed towards constructing and parametrizing them explicitly and testing their performance in numerical simulations[3]. We shall not discuss these developments here at all beyond the remark that the classically perfect action gives very good results in quantum simulations including questions related to topology. This way, this lecture note presents the first part of a longer story. Even this first part is a torso: no discussion will be given on the recent developments concerning chiral symmetry.

We close this introduction with a list of content. Unlike in the main text, the sections and subsections are numbered in the list below. These numbers will be used later for reference.

LATTICE REGULARIZATION AND THE CONTINUUM LIMIT

In this Section we summarize some of the basic notions concerning lattice regularization, continuum limit and the relation between Quantum Field Theories (QFTs) and critical phenomena in classical statistical mechanics. Readers, who are familiar with these basics are advised to move directly to Section 2.4 on locality.

QFTs defined by path integrals in Euclidean space

In these lectures we shall consider QFTs in their Euclidean formulation and use path integrals. Using path integrals to describe quantum theories not only provides an appealing physical picture for the quantum evolution as a sum over classical paths (in quantum mechanics) or sum over classical configurations (in QFT), but gives a powerful framework for analytic manipulations and opens the way for numerical calculations also. In addition, it helps to understand the deep relation between QFTs and critical phenomena in classical statistical mechanics[4, 5].

The formal relation between QFTs and classical statistical physics is easy to see. Consider an n-point function in a scalar QFT in Minkowski space

$$\langle 0|T\Big(\hat{\phi}(x_1)\hat{\phi}(x_2)\ldots\hat{\phi}(x_n)\Big)|0\rangle\,, \tag{1}$$

where $\hat{\Phi}(x)$ is the field operator. In the path integral language this expression is given by

$$\frac{1}{Z}\int D\phi\,\phi(x_1)\phi(x_2)\ldots\phi(x_n)\exp\Big(i\int dtd\mathbf{x}\;\mathcal{L}(\phi,\partial_\mu\phi)\Big), \tag{2}$$

where $Z=\int D\phi\exp(i\int dtd\mathbf{x}\,\mathcal{L})$ provides for the correct normalization, $x=(t,\mathbf{x})=(t,x_1,\ldots,x_{d_s})$, and the Lagrangian is defined as

$$\mathcal{L}_{Mink}=\frac{1}{2}\partial_0\phi\partial_0\phi-\sum_{i=1}^{d_s}\frac{1}{2}\partial_i\phi\partial_i\phi-V(\phi)\,. \tag{3}$$

The relation we are looking for becomes clear if we go to Euclidean space by rotating time to imaginary time

$$t=x_0\to -ix_d \tag{4}$$

It can be shown[6] that in every order of perturbation theory the analytic continuation in eq. (4) (Wick rotation) is possible due to the specific pole structure of the Feynman propagators. It is an assumption that this remains true beyond perturbation theory. Using

$$i\int dtd\mathbf{x}\to\int d^dx, \tag{5}$$

and

$$\mathcal{L}_{Mink} \to -\Big(\sum_{\mu=1}^{d} \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + V(\phi)\Big) = -\mathcal{L}_{Eucl}, \tag{6}$$

we get

$$Z = \int D\phi \exp\Big(-\int d^d x\, \mathcal{L}_{Eucl}(\phi, \partial\phi)\Big), \tag{7}$$

$$\langle\phi(x_1)\dots\phi(x_n)\rangle = \frac{1}{Z}\int D\phi\, \phi(x_1)\dots\phi(x_n)\exp(-\int d^d x\, \mathcal{L}_{Eucl}). \tag{8}$$

These equations can be interpreted as the partition function and the correlation function of a system in classical statistical mechanics. The quantum theory of fields in d_s space dimensions is transformed into classical statistical mechanics of fields in $d = d_s + 1$ Euclidean dimensions. The Euclidean action plays the role of $E/k_B T$, where E is the classical energy of the d dimensional configuration.

Actually, the relation between QFTs and classical statistical systems is deeper and more specific than the observation above. We shall return to this question in Sect.2.3

Lattice regularization

Some of the mathematical operations which enter the definition of a QFT require a careful limiting procedure. In field theory, the variables are associated with space-time points. As can be seen from the form of Lagrangian's of different classical FT's (see, for example eq. (3)), these variables have some kind of self-interaction, whereas the elementary interaction between different degrees of freedom is over infinitesimal distances as expressed by derivatives. Already in the classical theory, the definition of a derivative requires the temporary introduction of a finite increment (of the argument of the function considered) which disappears at the end by some limiting procedure

$$f'(x) = \lim_{a\to 0}\frac{f(x+a) - f(x)}{a}\,. \tag{9}$$

Defining the derivative, which has the important role of providing communication between different field variables, is a much less trivial problem in QFTs. Like in eq. (9), the very definition of a QFT requires the temporary introduction of a defining framework called regularization, which disappears from the theory by a limiting process. The way to introduce and remove the regularization is a highly non-trivial problem.

We shall use lattice regularization in these lectures. A hypercubic lattice is introduced in the $d = d_s + 1$ dimensional Euclidean space. Scalar and fermion field variables live on the lattice points, vector fields on the connecting links. Derivatives are replaced by some kind of finite difference operation.

It will be convenient to work with dimensionless quantities. In the following, all the fields, momenta, masses, couplings and other possible parameters will be defined dimensionless by absorbing appropriate powers of the lattice unit a in their definition. For any quantity Q_{df} with dimension s (measured in mass units) we define

$$Q_{df} \to Q = a^{-s} Q_{df}\,. \tag{10}$$

For a scalar field $\phi(x)$ we write

$$\phi(x) \to \phi_n \tag{11}$$

where n is a d-dimensional vector of integers. The simple choice for the finite difference operation

$$\partial_\mu \phi(x) \to \nabla_\mu \phi_n = \phi_{n+\hat{\mu}} - \phi_n \tag{12}$$

leads to the following lattice regularized form of the classical Euclidean Lagrangian

$$\mathcal{A} = \int d^d x \, \mathcal{L} \to \sum_n \sum_\mu \frac{1}{2} \nabla_\mu \phi_n \nabla_\mu \phi_n + \sum_n V(\phi_n)\,. \tag{13}$$

Having a finite lattice unit a the lattice defines a short distance regularization. Going over to momentum space, in the Fourier integral $\exp(ipn)$ enters which is periodic under $p_\mu \to p_\mu + 2\pi$. This constrains the momentum to the Brillouin zone $-\pi \geq p_\mu < \pi$. Therefore, the lattice provides a momentum cut-off also.

The lattice regularization has several advantages over other conventional regularizations used in perturbation theory. In general, in problems which require numerical analysis (e.g. integrals, differential equations,...), the standard procedure is to introduce meshes. Also the path integral can be defined in a natural way by introducing meshes. The path integral regularized this way becomes a set of integrals and is ready for numerical procedures in case of non-perturbative problems. Actually, lattice regularization is the only known non-perturbative regularization. It is a very natural regularization also for theories with gauge invariance or with constrained variables (like the non-linear σ-model).

Undoubtedly, lattice regularization has some disadvantages also. It breaks certain symmetries, as every regularization does. But the lattice has special problems (although not unsurmountable) with chiral symmetry and tough problems with chiral gauge theories. The discretization of space-time breaks Lorentz symmetry (O(d) symmetry in Euclidean space), but, as we shall see later (Sect. 8.1), the remaining cubic symmetry is sufficient to have it restored automatically as the regularization is removed (at least in asymptotically free theories). The lack of Poincaré symmetry creates problems, however, in theories with supersymmetry.

Although the lattice is used, in general, to investigate non-perturbative problems, for certain questions (renormalization of operators, finding improved actions or comparing lattice results with those obtained in other regularizations) it is unavoidable to use lattice perturbation theory. Unfortunately, perturbation theory on the lattice is technically somewhat cumbersome.

Some aspects of lattice regularization will be discussed in Sect. 5. For further reading we refer to textbooks[7].

The continuum limit of QFTs versus critical phenomena in statistical mechanics

Assume, we can solve our lattice regularized QFT at some set of (bare) couplings and masses in the Lagrangian. The predicted spectrum will contain dimensionless numbers, since all our quantities were made dimensionless by absorbing appropriate powers of the lattice unit a (Sect. 2.2). The mass of an excitation is given by a number M, the corresponding dimensionful mass M_{df}, $[M_{df}] = g$ and correlation length ξ_{df}, $[\xi_{df}] = cm$ are given by

$$M_{df} = M\frac{1}{a} \qquad \xi_{df} = \frac{1}{M_{df}} = \frac{1}{M}a = \xi a\,. \tag{14}$$

The correlation length defined above is the length scale over which the massive particle can propagate with a significant amplitude. For a generic value of the couplings and masses in the Lagrangian the predicted M will be an $O(1)$ number, the mass of the particle is of the order of the cut-off $1/a$, the correlation length is $O(a)$. This corresponds to a very poor resolution, the situation is far from the continuum limit. It requires a careful tuning of a certain number of parameters in the Lagrangian to make the predicted M's in the low lying spectrum to become very small numbers which correspond to physical length scales much larger than the lattice unit a. In this limit the lattice becomes very fine and the presence of the regularization will not distort the physical results (like mass ratios) anymore. This is the process of removing the lattice regularization which leads to the continuum limit. Theories in which this can be achieved by tuning a finite number of parameters (and by fixing the normalization of fields appropriately) are called renormalizable. This criterion is identical to that introduced in perturbation theory.

In the continuum limit the correlation length of physical excitations is much larger than the lattice unit a. But how large is it in Fermi? This can not be predicted, the absolute scale should come from observations. If the theory is expected to describe the physics of hadrons, for example, then the lowest lying particle with spin=1/2, electric charge=1, baryon number=1, strangeness=0,... should be identified with the proton having a mass of 940 MeV. This fixes the physical length scale and relative to that unit the lattice constant goes to zero in the continuum limit.

In Sect. 2.2 we observed a formal relation between QFTs and classical statistical systems. We see now that the continuum limit of a QFT corresponds to a statistical system with a correlation length which is much larger than the lattice unit a. This is the case for critical statistical systems. In critical solid state problems the lattice unit is fixed (typically $O(\text{Å})$) and the correlation length becomes macroscopical. This is a matter of choosing the absolute scale, however, and it does not reduce the extent of the deep analogy between the two subjects.

Locality

A recurrent issue in our discussion will be the locality of the action density (Lagrangian density). The laws of classical physics (from Newton through Maxwell to Einstein) are expressed in terms of (partial) differential equations (rather than, say, integro-differential equations). They correspond to actions, where the interaction between variables in different space-time points extends over infinitesimal distances as expressed by derivatives. This continues to be true in quantum systems also. No symmetry principles would prevent us to add a term to the Lagrangian in eq. (3) like

$$\mathcal{L}(x) = \ldots + \int d^d y \; f(x-y)\phi(x+\frac{y}{2})\phi(x-\frac{y}{2}) \,, \tag{15}$$

where $f(z)$ is an even function of z and has an extension λ, for example $f(z) \sim \exp(-z^2/2\lambda^2)$, where λ is finite when measured in Fermi. One can imagine functions also which decay only as a power in z^2. Nevertheless, such models found their place neither theoretically, nor experimentally.

In a QFT with non-local interactions renormalizability (in the sense discussed in Sect. 2.3) will be lost. In addition, a vital and beautiful property of local QFTs (shared by critical statistical systems), called universality will be lost also. Universality means that the physical predictions become independent of the microscopic details of the Lagrangian, i.e. they are not sensitive to the detailed form of the interaction at the cut-off scale. Since our knowledge on interactions much above the scales of present experiments is very limited, universality is vital to preserve the predictive power of a QFT.

On the lattice we shall call an action density local, if it has an extension of $O(a)$. Typically, it is an exponential function of the distance $|n-n'|$ between the variables at n and n': $\sim \exp(-\gamma|n-n'|)$, $\gamma = O(1)$. In the continuum limit ($a \to 0$), the extension of the action density measured in physical units goes to zero.

A Lagrangian which has nearest-neighbour interactions only or interactions which are identically zero beyond a few lattice units is certainly local, but no physical principle requires to have this extreme case. If this were necessary to assure the above mentioned nice properties of a QFT (or of a critical statistical system), then no experiment on ferromagnets close to the Curie temperature would observe universality. Really, the elementary interaction between the magnetic moments in the crystal decays rapidly, but certainly does not become identically zero beyond a few lattice spacings.

In the following we shall call an action where the couplings become identically zero beyond a certain $O(a)$ range ultralocal. We shall also use the expression 'short ranged' for local actions where the exponential decay goes with a large γ.

RENORMALIZATION GROUP

A QFT is defined over a large span of scales from low (relative to the cut-off) physical scales up to the cut-off which goes to infinity in the continuum limit. Although field variables associated with very high scales do influence the physical predictions through a complicated cascade process, no physical question involves them directly. Their presence and indirect influence makes it difficult to establish an intuitive connection between the form of the interaction and the final expected predictions. The presence of a large number of degrees of freedom makes the problem technically difficult also. It is, therefore a natural idea to integrate them out in the path integral. This process, which reduces the number of degrees of freedom, taking into account their effect on the remaining variables exactly, is called a renormalization group transformation[1, 8] (RGT).

In this Section we introduce some of the basic notions related to RG theory and set the notations. Since these lectures are mainly concerned with asymptotically free (AF) QFTs, our discussion will be biased by this goal.

Renormalization group transformation

For simplicity consider a scalar field theory regularized on the lattice in d Euclidean dimensions as discussed in Section 2.2. The RGT averages out the short distance fluctuations, i.e. the fluctuations over distances $O(a)$. The lattice is indexed by n, the field associated with the point n is denoted by ϕ_n. We introduce a blocked lattice with lattice unit $a' = 2a$ whose points are labeled by n_B and the associated block field is denoted by χ_{n_B}. The block variable χ_{n_B} is an average of the original ϕ fields in the neighbourhood of the point n_B:

$$\chi_{n_B} = b \sum_n \omega(2n_B - n)\phi_n \,, \tag{16}$$

where ω specifies the weights of the averaging whose normalization is chosen as

$$\sum_n \omega(2n_B - n) = 1 \,. \tag{17}$$

The significance of the scale factor b in eq. (16) will become clear later. A trivial choice for ω might be to take $\omega(2n_B - n) = 2^{-d}$ if n is in the hypercube whose center is indexed by n_B and zero otherwise.

We integrate out now the original ϕ_n variables keeping the block averages fixed:

$$\exp\Big(-\mathcal{A}'(\chi)\Big) = \prod_n \int d\phi_n \prod_{n_B} \delta(\, \chi_{n_B} - b \sum_n \omega(2n_B - n)\phi_n \,) \exp\Big(-\mathcal{A}(\phi)\Big) \,. \tag{18}$$

The new action $\mathcal{A}'(\chi)$ describes the interaction between the block variables. The partition function remains unchanged

$$Z' = \prod_{n_B} \int d\chi_{n_B} \exp\Big(-\mathcal{A}'(\chi)\Big) = \prod_n \int d\phi_n \exp\Big(-\mathcal{A}(\phi)\Big) = Z \,. \tag{19}$$

The long-distance behaviour† of Green's functions, and so the spectrum and other low energy properties of the system is expected to remain unchanged as well. On the other hand, the lattice unit is increased by a factor of 2: $a' = 2a$. Since the dimensions are carried by the lattice unit (Sect. 2.2), the dimensionless correlation length, which is measured in the actual lattice distance, is reduced by a factor of 2:

$$\xi \to \frac{\xi}{2} \,. \tag{20}$$

By eliminating the fluctuations on the shortest scales we reduced the number of variables by a factor of 2^d, while taking their effect into account by changing the action $\mathcal{A} \to \mathcal{A}'$ appropriately. Although the lattice unit is increased by a factor of 2, and so the resolution becomes worse, the long-distance behaviour remains unchanged. In particular, no new cut-off effects are generated in the predictions for physical quantities. Iterating this RGT, the goal formulated at the beginning of Sect. 3 is achieved.

We close this subsection by generalizing the RGT of eq. (18) slightly[9] (a step which will be very useful later):

$$\exp\Big(-\mathcal{A}'(\chi)\Big) = \\ \mathcal{N}(\kappa) \prod_n \int d\phi_n \exp\Big[-\mathcal{A}(\phi) - \kappa \sum_{n_B} \Big(\chi_{n_B} - b \sum_n \omega(2n_B - n)\phi_n\Big)^2\Big], \tag{21}$$

where κ is a free parameter. For $\kappa \to \infty$ eq. (21) goes over to eq. (18), for finite κ the block average is allowed to fluctuate around χ (slightly, if κ is not too small). Therefore, the parameter κ specifies the stiffness of the RG averaging. $\mathcal{N}$ is a normalization factor (which is field independent, trivial in this case) introduced to keep eq. (19) valid.

Constraints on the block transformation from symmetries

It is largely arbitrary how the block averages in a RGT are constructed, but the procedure should conform with the intuitive goal of a RGT: it should lead to the elimination of the short distance fluctuations. Identifying, for example, the block variable with one of the original fine variables in the block ('decimation') is not really an averaging, and although it is a legal transformation of the path integral, it will not lead to useful results in general.

Beyond this general requirement, symmetries put further constraints on the form of the block transformation. Denote the fields on the fine and coarse lattice by ϕ and χ, respectively and write the RGT in the form

$$\exp\Big(-\mathcal{A}'(\chi)\Big) = \int D\phi \prod_{blocks} f(\chi_{block}; \phi) \exp\Big(-\mathcal{A}(\phi)\Big) \,. \tag{22}$$

†'Long-distance' is meant here and elsewhere in this work as 'many lattice units', and it might mean short, or long physical distances.

Here ϕ, χ can be scalar, vector or fermion fields and f defines how the coarse field in the block χ_{block} is constructed from the (neighbouring) ϕ fields.

Assume, there is a symmetry transformation $\phi \to R\phi$ under which the action and the measure are invariant: $\mathcal{A}(R\phi) = \mathcal{A}(\phi)$, $D(R\phi) = D\phi$. The blocked action $\mathcal{A}'(\chi)$ will inherit this symmetry, $\mathcal{A}'(R\chi) = \mathcal{A}'(\chi)$, if the averaging function satisfies

$$f(R\chi_{block}; \phi) = f(\chi_{block}; R^{-1}\phi)\,, \tag{23}$$

as it is seen easily by changing integration variables in eq. (22).

In case of gauge symmetries we shall require somewhat more: we shall not only require that the effect of a gauge transformation on the coarse lattice on f is equivalent to a gauge transformation on the fine lattice, eq. (23), but also the other way around. This constraint implies that gauge equivalent fine configurations contribute equally to the path integral in eq. (22).

A basic assumption of the RG theory

Consider a step of the RGT (eq. (18) or eq. (21)) for the case where the parameters of $\mathcal{A}(\phi)$ were chosen so that the original system was close to the continuum (or, in the language of statistical physics, close to criticality). As eq. (21) shows, performing a RGT on a system with action $\mathcal{A}(\phi)$ leads to a path integral where $\mathcal{A}$ is replaced by:

$$\mathcal{A}(\phi) + \kappa \sum_{n_B} \left(\chi_{n_B} - b \sum_{n} \omega(2n_B - n)\phi_n\right)^2 . \tag{24}$$

For this path integral the field χ enters as an external field. Even if $\mathcal{A}(\phi)$ was carefully tuned to criticality, the system in eq. (24) is not expected to be critical. Actually, this is a basic assumption of the RG theory: the path integral entering a step of the RGT defines a non-critical problem with short-range interactions only. This is what one expects intuitively: the χ fields constrain the ϕ-averages in their neighbourhood and disrupt the long-range fluctuations. This assumption leads to the important conclusion that the new action $\mathcal{A}'(\chi)$ will be local. Really, if the r.h.s. of eq. (21) describes a system with short-range fluctuations only, the generated interaction between distant χ fields is expected to be negligibly small.

Another consequence of this basic assumption is that the path integral in eq. (18) or eq. (21) is a technically much simpler problem than the path integral of the original system. Saying differently: to perform a RG step is not an easy problem in general, but it is much simpler than to solve the original (near) critical theory. This is what makes the RG theory a practically useful idea.

Fixed Point (FP) and the behaviour in the vicinity of a FP

In general, the transformed action $\mathcal{A}'(\chi)$ in eq. (21) will contain all kinds of interactions, even if the original action $\mathcal{A}(\phi)$ had a simple form. It is useful to introduce a sufficiently general interaction space to describe the actions generated by the RGT, and write

$$\mathcal{A}(\phi) = \sum_{\alpha} K_\alpha \theta_\alpha(\phi), \tag{25}$$

where $\theta_\alpha(\phi)$, $\alpha = 1, 2, ...$ denotes the different interaction terms like (for scalar fields)

$$\theta_1 = \sum_{n,\mu} \nabla_\mu\phi_n \nabla_\mu\phi_n\,, \qquad \theta_2 = \sum_{n} \phi_n^2\,, \ldots,\; \theta_i = \sum_{n,\mu} \phi_n^4 \nabla_\mu\phi_n\nabla_\mu\phi_n, \ldots \tag{26}$$

and K_α are the corresponding dimensionless couplings. The transformed action $\mathcal{A}'(\chi)$ is also expanded in terms of these interaction terms

$$\mathcal{A}'(\chi) = \sum_\alpha K'_\alpha \theta_\alpha(\chi)\,. \tag{27}$$

The RGT induces a motion in the coupling constant space:$\{K_\alpha\} \to \{K'_\alpha\}$. Under repeated RGTs a coupling constant flow is generated

$$\{K_\alpha\} \to \{K'_\alpha\} \to \{K''_\alpha\} \to \dots\,, \tag{28}$$

while the (dimensionless) correlation length is reduced with every step (eq. (20))

$$\xi \to \frac{1}{2}\xi \to \frac{1}{4}\xi \to \dots\,. \tag{29}$$

It might happen that certain points in the coupling constant space are reproduced by the RGT

$$\{K^*_\alpha\} \to \{K^*_\alpha\}\,. \tag{30}$$

A point with this property is called a fixed point (FP) of the RGT and the corresponding action

$$\mathcal{A}^{FP}(\phi) = \sum_\alpha K^*_\alpha \theta_\alpha(\phi) \tag{31}$$

is the FP action. FP actions play an important role in QFTs in general and will play an important role in our later discussion of the perfect actions.

Eq. (28) and eq. (29) imply that at the FP, $\xi = \infty$ or $\xi = 0$. We will be interested in FPs with $\xi = \infty$. The set of points in the coupling constant space where $\xi = \infty$, forms a hypersurface, which is called the critical surface. As eq. (29) shows, an RGT drives the point $\{K_\alpha\}$ away from the critical surface, except when $\{K_\alpha\}$ is on the critical surface.

Let us consider now the behaviour of the flow under the RGT in the vicinity[10] of a FP. Take a point $\{K_\alpha\}$ with $\triangle K_\alpha = K_\alpha - K^*_\alpha$ small. Under the RGT we have

$$\triangle K_\alpha \to \triangle K'_\alpha = K'_\alpha - K^*_\alpha\,. \tag{32}$$

Expanding $\triangle K'_\alpha = \triangle K'_\alpha(\{K\})$ around $\{K^*\}$

$$\triangle K'_\alpha = \triangle K'_\alpha(\{K^*\}) + \sum_\beta \frac{\partial}{\partial K_\beta} \triangle K'_\alpha(\{K\})_{|K=K^*} \triangle K_\beta + O\Big((\triangle K)^2\Big)\,, \tag{33}$$

and observing that the first term on the r.h.s. is zero, we get the following linearized RG equation

$$\triangle K'_\alpha = \sum_\beta T_{\alpha\beta} \triangle K_\beta, \tag{34}$$

with

$$T_{\alpha\beta} = \frac{\partial}{\partial K_\beta} \triangle K'_\alpha(\{K\})_{|K=K^*}\,. \tag{35}$$

Let us denote the eigenvectors and eigenvalues of the matrix T by h_a and λ_a, respectively:

$$\sum_\beta T_{\alpha\beta} h^a_\beta = \lambda^a h^a_\alpha\,,\ a = 1, 2, \dots\,. \tag{36}$$

The eigenvectors h^a define the eigenoperators

$$h^a(\phi) = \sum_\alpha h^a_\alpha \theta_\alpha(\phi)\,. \tag{37}$$

These operators define a new basis in terms of which the action can be expanded

$$\mathcal{A}(\phi) = \mathcal{A}^{FP}(\phi) + \sum_a c^a h^a(\phi), \tag{38}$$

where c^a are the corresponding expansion coefficients (couplings) which are small if $\mathcal{A}$ is close to $\mathcal{A}^{FP}$. Repeated application of the RGT gives

$$\mathcal{A}(\phi) \to \mathcal{A}^{FP}(\phi) + \sum_a \lambda^a c^a h^a(\phi) \to \mathcal{A}^{FP}(\phi) + \sum_a (\lambda^a)^2 c^a h^a(\phi) \to \dots\,, \tag{39}$$

i.e., the coupling c^a goes over to $(\lambda^a)^n c^a$ after n RGT steps. For $|\lambda^a| > 1$ ($|\lambda^a| < 1$) the coupling $|c^a|$ is increasing (decreasing) under repeated RG steps. The corresponding interaction h^a is called relevant (irrelevant). For $|\lambda^a| = 1$ the fate of the operator (called marginal) is decided by the higher order corrections suppressed in eq. (33).

RG flows of a Yang-Mills theory

The following discussion applies also for other AF theories, like the $O(N)$ non-linear σ-model or the CP^n model in $d = 2$.

The classical action of an $SU(N)$ Yang-Mills theory has the form

$$\beta\mathcal{A}_{cont} = \frac{\beta}{2N}\int_x F^a_{\mu\nu}(x)F^a_{\mu\nu}(x)\,,\ a = 1,\dots, N^2-1\,, \tag{40}$$

where $\beta = 2N/g^2$ and $F^a_{\mu\nu}$ is the colour field strength tensor. Any lattice representation of this action should go over to the form in eq. (40) for slowly changing smooth fields. Even if we use a gauge symmetric discretization (as we always do), there is a large freedom in writing down a local action with this property on the lattice. As we discussed in Sect. 3.4, we have to introduce an infinite dimensional coupling constant space if we want to follow the change of the action under RGTs. We write

$$\beta\mathcal{A}_{latt} = \beta\mathcal{A}_{latt}(K_1, K_2, \dots), \tag{41}$$

where we indicated the coupling constant dependence of the action only. Under a RGT we have: $\{\beta, K_1, K_2, ...\} \to \{\beta', K'_1, K'_2, ...\}$.

The expected flow diagram is sketched in fig. 1. In the $\beta = \infty$ hyperplane, which is the critical surface, there is a FP with coordinates $K^*_1, K^*_2, ...$. We shall suppress the index 'latt' and introduce the notation $\mathcal{A}(K^*_1, K^*_2, ...) = \mathcal{A}^{FP}$. All the eigenoperators which lie in the $\beta = \infty$ hypersurface are irrelevant. There is one marginal direction which is pointing out of this surface. Actually, this direction is marginal in the linear approximation only, it becomes weakly relevant by the higher order corrections to eq. (33).

Consider the point $\{\beta, K_1, K_2, ...\}$, β very large. Under RGTs this point runs rapidly towards the FP (the largest irrelevant eigenvalue is 1/4), then slowly moves away from it along the weakly relevant direction tracing a trajectory, called the renormalized trajectory.

Consider the interaction corresponding to the FP, $\mathcal{A}^{FP}$, multiply it by β: $\beta\mathcal{A}^{FP}$ and allow β to move away from ∞. This is a straight line (the dashed line in fig. 1.)

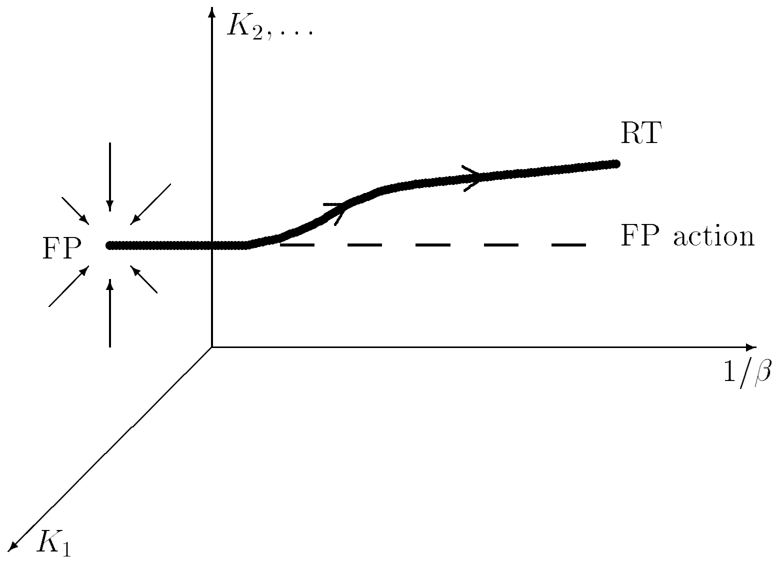

Figure 1. Flow of the couplings under RG transformations in the O(N) non–linear σ–model.

which is not a RG flow, but defines an action for every value of β. We shall call this action the FP action.

We postpone until Sects. 7 and 8 the discussion on the arguments supporting the structure of the flow diagram in fig. 1. We mention here only that theories with this flow diagram can be renormalized by tuning a single coupling constant and the resulting renormalized theory is universal (independent of the value of the other couplings in the (bare) action) . The continuum renormalized theory is obtained as $g \to 0$ ($\beta \to \infty$) in which limit the correlation length goes to infinity, the lattice unit $a \to 0$, the resolution becomes infinitely good and the cut-off artifacts disappear from the predictions.

PERFECT ACTIONS IN AF THEORIES

We shall call a lattice regularized local action *classically perfect* if its classical predictions (independently whether the lattice is fine or coarse, whether the resolution is good or bad) agree with those of the continuum. The *quantum perfect action* does the same for all the physical questions in the quantum theory. That such actions exist might seem to be surprising. As we shall see, they do exist and have beautiful properties which one would not expect a lattice action can have[2, 3, 11, 12, 13, 14].

We shall argue now that the actions defined by the points of the renormalized trajectory (RT) in fig. 1 define quantum perfect actions. It means that by taking an arbitrary point $(\beta, K_1, K_2, ...) \in RT$, where $g^2 = 2N/\beta$ need not be small, the corresponding action $\mathcal{A}(\beta, K_1, K_2, ...)$ will produce quantum results for physical questions (mass ratios, for example) which are exactly the same as in the continuum limit. This is surprising, since in the point $(\beta, K_1, K_2, ...)$ the correlation length is not large, the lattice unit a is not small, the resolution is not fine. Nevertheless, the statement is true.

The argument goes as follows[2]. At any given β, the point of the RT is connected to the infinitesimal neighbourhood of the FP by (infinitely many steps of) exact RG transformations. Since each step increases the lattice unit by a factor of 2, *any* distance at the given β (even 1 lattice unit) corresponds to a long distance close to the FP. The infinitesimal neighbourhood of the FP is in the continuum limit, there are no cut-off effects there at long distances. On the other hand, for all the questions which can be formulated in terms of the degrees of freedom after the transformation we get the same answer as before the transformation. Thus, there are no lattice artifacts at the given β on the RT at any distances.

We shall also argue that the FP action, as defined in Sect. 3.5 (fig. 1) is a classically perfect action. We are not able to demonstrate this statement so compactly as that for the quantum perfect action before. In the following sections we shall go over the different classical properties of continuum Yang-Mills, QCD and other AF theories and show one by one that they are reproduced by the FP action independently of the lattice resolution.

The detailed form of the FP action and the RT depend on the form and parameters of the block transformation. The theoretical properties of the FP action and RT are, however, independent of these details. In this sense, any RGT which has a local FP is suitable. On the other hand, different local FP actions (belonging to different block transformations) have different extensions and this has an important practical significance when it comes to parametrization and simulation. In order to find short ranged actions (Sect. 2.4), different block transformations were tested and optimized[3].

LATTICE REGULARIZATION AND EXPLICIT RGTs IN DIFFERENT AF THEORIES

It is time now to come down from generalities to concrete examples. We shall discuss how the lattice regularization is set up and give explicit RGTs in different AF theories. First, the $O(N)$ non-linear σ-model and the $SU(N)$ Yang-Mills theory will be considered. Then, as a preparation for QCD, the special difficulties of lattice regularized Dirac fermions will be illustrated by treating the case of free spin-1/2 fermions. We close this section with QCD.

In all these theories we write down the 'standard' lattice action which is the simplest discretization of the continuum action satisfying the basic requirements. These are the actions which - due to their technical simplicity - were used in most of the numerical simulations. These actions are simple, but they represent poorly most of the properties even of the classical field theory if the resolution is not sufficiently fine. As we discussed in Sect. 4, in the framework of the RG theory actions can be defined, which have beautiful properties. For later use we define explicit RGTs in all the theories considered in this section.

O(N) non-linear σ-model in d=2

The classical action in the continuum has the form

$$\beta\mathcal{A}_{cont}(\mathbf{S}) = \frac{\beta}{2}\int_x \partial_\mu \mathbf{S}(x)\partial_\mu \mathbf{S}(x)\,, \tag{42}$$

where $\mathbf{S}$ is an N-component vector satisfying the constraint $\mathbf{S}^2(x) = 1$, $\beta = 1/g$. Writing

$$\mathbf{S}(x) = \begin{pmatrix} \sqrt{1-\boldsymbol{\pi}(x)^2} \\ \boldsymbol{\pi}(x) \end{pmatrix} \tag{43}$$

and expanding in the fluctuations $\boldsymbol{\pi}(x)$ (it has $N-1$ components) a systematic perturbation theory can be set up. The theory is AF in the coupling g for $N \geq 3$. It is believed (and this is strongly supported by different theoretical and numerical results) that a non-zero mass is generated dynamically, the spectrum contains a massive $O(N)$ multiplet. The σ-model is discussed in different modern textbooks on QFT and statistical systems[5, 15, 16]

Looking for classical solutions with finite action, the spin vectors $\mathbf{S}$ should be parallel at large distances. For $N=3$ this leads to an $S_2 \to S_2$ mapping between the coordinate and group spaces. The $O(3)$ model has a non-trivial topology, there exist scale invariant instanton solutions with an action $\mathcal{A}=4\pi|Q|$, where Q is the topological charge of the solution.

The $O(3)$ model shows many analogies[17, 16] with the Yang-Mills theory in $d=4$. It offers the possibility to test non-perturbative ideas and numerical methods.

On the lattice, the scalar field $\mathbf{S}$ lives on the lattice points. The simplest realization of the action has the form ('standard action')

$$\beta \mathcal{A}_{st}(\mathbf{S}) = \frac{\beta}{2} \sum_{n,\mu} \nabla_\mu \mathbf{S}_n \nabla_\mu \mathbf{S}_n \,, \tag{44}$$

where ∇_μ is defined in eq. (12) and $\mathbf{S}_n^2 = 1$. On smooth configurations $\mathcal{A}_{st}$ goes over to $\mathcal{A}_{cont}$ – a property we shall require for any lattice action.

Consider RGTs with a scale factor of 2, forming averages out of the four fine spins in a block. The block spin will be denoted by $\mathbf{R}_{n_B}$, $\mathbf{R}_{n_B}^2 = 1$. The RGT has the form

$$\exp\Big(-\beta' \mathcal{A}'(\mathbf{R})\Big) = \int D\mathbf{S} \exp\Big[-\beta\Big(\mathcal{A}(\mathbf{S}) + T_\sigma(\mathbf{R},\mathbf{S})\Big)\Big] \,, \tag{45}$$

where T_σ defines the averaging process and we used the notation

$$\int D\mathbf{S} = \prod_n \int d\mathbf{S}_n \delta(\mathbf{S}^2 - 1) \,. \tag{46}$$

We shall give two explicit examples for the averaging which were studied in the context of perfect actions[2, 19]. Both RGTs are $O(N)$ invariant, satisfying eq. (23) for global $O(N)$ transformations. In the first case:

$$T_{\sigma 1}(\mathbf{R},\mathbf{S}) = \sum_{n_B} \Big(-\kappa_\sigma \mathbf{R}_{n_B} \mathbf{\Sigma}_{n_B} + \mathcal{N}(|\mathbf{\Sigma}_{n_B}|)\Big) \,, \tag{47}$$

where $\mathbf{\Sigma}_{n_B}$ is the sum of the four $\mathbf{S}$ spins in the block n_B

$$\mathbf{\Sigma}_{n_B} = \sum_{n \in n_B} \mathbf{S}_n \,, \tag{48}$$

and $\mathcal{N}$ assures the correct normalization, eq. (19). We get

$$\mathcal{N}(|\mathbf{\Sigma}_{n_B}|) = \frac{1}{\beta} \ln Y_N(\beta\kappa |\mathbf{\Sigma}_{n_B}|) \,, \tag{49}$$

where $Y_N(z)$ is related to the modified Bessel function (some of its properties are summarized in appendix F in[20], for example), specifically $Y_3(z) \sim \sin(z)/z$. For $\kappa \to \infty$, the block transformation goes over to a δ-function constraining the average to lie parallel to the coarse spin

$$\frac{\mathbf{\Sigma}_{n_B}}{|\mathbf{\Sigma}_{n_B}|} = \mathbf{R}_{n_B}, \qquad (\kappa = \infty) \,. \tag{50}$$

It will be useful, however, to keep κ finite and optimize its value to get a short ranged FP action.

Modifying slightly the averaging function T

$$T_{\sigma 2} = 2\kappa_\sigma \sum_{n_B} \Big(\mathbf{R}_{n_B} - \frac{\mathbf{\Sigma}_{n_B}}{|\mathbf{\Sigma}_{n_B}|} \Big)^2 , \tag{51}$$

we can make the technically cumbersome normalizing function $\mathcal{N}$ constant. Really, the integral

$$\int d\mathbf{R}\, \delta(\mathbf{R}^2 - 1) \exp(4\beta\kappa_\sigma \mathbf{R}\mathbf{e}) , \tag{52}$$

with $\mathbf{e} = \mathbf{\Sigma}/|\mathbf{\Sigma}|$, depends on the length of $\mathbf{e}$ only, therefore it is an $\mathbf{S}$-independent constant.

SU(N) Yang-Mills theory in d=4

The classical action in the continuum has the form

$$\beta \mathcal{A}^{YM}_{cont}(A_\mu) = \frac{\beta}{4N} \int_x Tr(F_{\mu\nu}F_{\mu\nu}) , \tag{53}$$

where $\beta = 2N/g^2$, $F_{\mu\nu} = F^a_{\mu\nu}T^a, a = 1, ..., N^2 - 1$ and T^a are the generators of the colour group $SU(N)$, $Tr(T^aT^b) = 1/2\delta_{ab}$.
The classical theory has scale invariant instanton solutions[18] with an action $\mathcal{A} = 8\pi^2|Q|$.

The quantum theory is AF in the coupling g. It is believed (and this is supported by many, but non-rigorous results) that non-perturbative effects create a confining potential between static sources in the fundamental colour representation (static quarks) and the spectrum contains massive colour singlet excitations (glueballs) only.

On the lattice, the vector gauge field lives on the links of the hypercubic lattice[21, 7]. The convenient variables, in terms of which gauge invariance is easily kept on the lattice are not the vector potentials themselves, but their exponentiated forms. Consider a static quark and antiquark in the points x and y, respectively (in the continuum). The form

$$\bar\psi(x)\, \mathcal{P} \exp\Big(i \int_x^y A_\mu(z) dz_\mu \Big)\, \psi(y) , \tag{54}$$

where the integral runs along some path between x and y and $\mathcal{P}$ denotes path ordering, is gauge invariant. The gauge part in eq. (54), which leads the colour flux between the sources, is an element of the $SU(N)$ group rather than that of the algebra. Going to the lattice and placing x, y in the endpoints of the link $(n, n+\hat\mu)$, we get

$$\bar\psi_n\, U_\mu(n)\, \psi_{n+\hat\mu} , \tag{55}$$

where $U_\mu(n)$ is related to the lattice vector potential as

$$U_\mu(n) = \exp(iaA_\mu(n)) . \tag{56}$$

The variables of the lattice regularized Yang-Mills theory are the link matrices $U_\mu(n)$. The link matrix associated with the same link but directed oppositely: $(n+\hat\mu, n)$ is $U^\dagger_\mu(n)$. Under a gauge transformation the link matrix transforms as

$$U_\mu(n) \to V(n)U_\mu(n)V^\dagger(n+\hat\mu) , \tag{57}$$

where $V(n) \in SU(N)$ is the gauge transformation in the point n. Consequently, the trace of the product of link matrices along a closed path on the lattice is gauge invariant.

The simplest lattice realization of the Yang-Mills action has the form ('Wilson action')

$$\beta \mathcal{A}_W(U) = \frac{\beta}{N} \sum_{plaq} (N - Re Tr\, U_p)\,, \tag{58}$$

where the sum is over the plaquettes and U_p is the product of four directed link matrices around the plaquette p. For smooth gauge fields eq. (58) goes over to eq. (53), as it should.

We shall consider RGTs with a scale factor of 2. The blocked link variable $V_\mu(n_B) \in SU(N)$, which lives on the coarse lattice with lattice unit $a' = 2a$, is coupled to a local average of the original link variables. The blocking kernel $T(V, U)$ enters the RGT as in the σ-model

$$\exp\Big(-\beta' \mathcal{A}'(V)\Big) = \int DU \exp\Big[-\beta\Big(\mathcal{A}(U) + T_g(V, U)\Big)\Big]\,, \tag{59}$$

where

$$\int DU = \prod_{n,\mu} \int dU_\mu(n)\,. \tag{60}$$

The kernel T_g is taken in the form

$$T_g(V, U) = \sum_{n_B,\mu} \Big[-\frac{\kappa_g}{N} Re Tr\, \Big(V_\mu(n_B) Q_\mu^\dagger(n_B)\Big) + \mathcal{N}\Big(Q_\mu(n_B)\Big)\Big]\,. \tag{61}$$

The $N \times N$ complex matrix $Q_\mu(n_B)$ is an average of the fine link variables in the neighbourhood of the coarse link $l_B = (n_B, \mu)$. $\mathcal{N}(Q_\mu(n_B))$ is fixed by the normalization condition, eq. (19):

$$\exp\Big(\mathcal{N}(Q)\Big) = \int dW \exp\Big(\frac{\beta \kappa_g}{N} Re Tr\, (W Q^\dagger)\Big), \qquad W \in SU(N)\,. \tag{62}$$

We shall give two examples which were studied in detail before[13, 22]:

RGT type A (fig. 2):

$$\begin{aligned} Q_\mu(n_B) = (1-6c) U_\mu(n) U_\mu(n+\hat\mu) + & \\ c \sum_{\nu \neq \mu} \Big[U_\nu(n) U_\mu(n+\hat\nu) U_\mu(n+\hat\mu+\hat\nu) U_\nu^\dagger(n+2\hat\mu) + & \\ U_\nu^\dagger(n-\hat\nu) U_\mu(n-\hat\nu) U_\mu(n+\hat\mu-\hat\nu) U_\nu(n+2\hat\mu-\hat\nu)\Big]\,, & \end{aligned} \tag{63}$$

where c, the relative weight of the staples versus the central link, is a tunable parameter (see, last paragraph of Sect. 4).

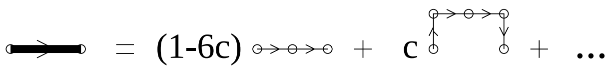

Figure 2. The renormalization group transformation of type A.

The averaging process in eq. (63) is not a very good one: there are many link variables on the fine lattice which do not contribute to any of the Q's. Experience shows that poor averaging might lead to a FP which is local, but not sufficiently short ranged or might not even have a FP.

RGT type B:

The second example provides for a better averaging. Instead of using just simple staples in constructing $Q_\mu(n_B)$ it builds also 'diagonal staples'. Let us define first the matrices $W^m(n,n')$ connecting the sites n and n', where $n'_\mu - n_\mu = 0,\ |n'_\nu - n_\nu| \le 1$ (for any ν):

$$W^{(0)}(n,n) = 1\,, \qquad W^{(1)}(n,n+\hat\nu) = U_\nu(n)\,,$$
$$W^{(2)}(n,n+\hat\nu+\hat\rho) = \frac{1}{2}\left(U_\nu(n)U_\rho(n+\hat\nu) + U_\rho(n)U_\nu(n+\hat\rho)\right)$$
$$W^{(3)}(n,n+\hat\nu+\hat\rho+\hat\lambda) = \frac{1}{6}\left(U_\nu(n)U_\rho(n+\hat\nu)U_\lambda(n+\hat\nu+\hat\rho) + \text{permutations}\right)\,. \tag{64}$$

Here ν,ρ and λ go over all (positive and negative) directions different from μ and from each other (with $U_{-\nu}(n) = U_\nu^\dagger(n-\hat\nu)$). Values of $W^{(m)}(n,n')$ not indicated are taken to be zero. $W^{(2)}(n,n')$ represents the 'planar diagonal link', $W^{(3)}(n,n')$ the spatial one. In eq. (64) the sums are taken over all shortest paths leading to the endpoint n' of the corresponding diagonal. A fuzzy link operator is constructed then as

$$\mathcal{W}_\mu(n) = \sum_{m=0}^{3}\sum_{n'} c_m W^{(m)}(n,n')U_\mu(n')W^{(m)}(n'+\hat\mu,n+\hat\mu). \tag{65}$$

The coefficients c_m are free parameters subject to the constraint

$$c_0 + 6\,c_1 + 12\,c_2 + 8\,c_3 = 1. \tag{66}$$

assuring that for a trivial configuration $Q_\mu(n_B)$ is equal to the unit matrix. Finally, the matrix $Q_\mu(n_B)$ is the product of two fuzzy link operators connecting the points $2n_B$ and $2n_B + 2\hat\mu$ on the fine lattice

$$Q_\mu(n_B) = \mathcal{W}_\mu(2n_B)\mathcal{W}_\mu(2n_B+\hat\mu)\,. \tag{67}$$

Free fermion fields in d=4

The action of spin 1/2 Dirac fermions in the continuum has the form

$$\mathcal{A}_{cont}(\bar\psi,\psi) = \int_x \bar\psi(x)(\gamma_\mu\partial_\mu + m)\psi(x)\,, \tag{68}$$

where $\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}$.

On the lattice, the fermion field lives on the lattice points. The most general action which is invariant under cubic rotation, reflection, permutation of the coordinate axes and discrete translations, reads

$$\mathcal{A}(\bar\psi,\psi) = \sum_{n,n'} \bar\psi_n\Big(f^\mu(n-n')\gamma_\mu + f(n-n')\Big)\psi_{n'}\,, \tag{69}$$

where the local functions f^μ and f have the following reflection properties

$$f^\mu(-r) = -f^\mu(r), \qquad f(-r) = f(r). \tag{70}$$

A naive discretization of eq. (68) leads to

$$\mathcal{A}_{naive}(\bar\psi,\psi) = \frac{1}{2}\sum_{n,\mu}\left(\bar\psi_n\gamma_\mu\psi_{n+\hat\mu} - \bar\psi_{n+\hat\mu}\gamma_\mu\psi_n\right) + \sum_n m\bar\psi_n\psi_n\,, \tag{71}$$

corresponding to

$$f^{\mu}(r) = 1/2(\delta_{r+\hat{\mu},0} - \delta_{r-\hat{\mu},0}), \qquad f(r) = m\delta_{r,0} \,, \tag{72}$$

or, in Fourier space

$$f^{\mu}(k) = i\sin(k_{\mu}), \qquad f(k) = m \,. \tag{73}$$

The momentum space propagator has the form

$$D(k) = \frac{-i\sum_{\mu}\sin(k_{\mu})\gamma_{\mu} + m}{\sum_{\mu}\sin^2(k_{\mu}) + m^2} \,. \tag{74}$$

The poles of the denominator determine the spectrum. Since $\sin(k_{\mu}) = 0$ at $k_{\mu} = 0$ and $k_{\mu} = \pi$, the propagator in eq. (74) has 16 poles defining 16 fermion species with the correct relativistic dispersion relation $E = -ik_4 = (\vec{k}^2 + m^2)^{1/2}$ in the continuum limit. This is the species doubling problem specific to fermions on the lattice.

For $m = 0$, the naive action in eq. (71) is chiral invariant, i.e. it remains unchanged under the transformation

$$\psi_n \to \exp(i\alpha\gamma_5)\,\psi_n \,, \qquad \bar{\psi}_n \to \bar{\psi}_n \exp(i\alpha\gamma_5) \,. \tag{75}$$

This symmetry requires $f = 0$ in eq. (69) and leads to a propagator $(\sum_{\mu} f^{\mu}\gamma_{\mu})^{-1}$. For small k, $f^{\mu}(k) \sim ik_{\mu}$ since the lattice action should reproduce the continuum action in this limit. In addition, $f^{\mu}(k)$ is a periodic function (Sect. 2.2). It follows then that $-if^{\mu}(k) > 0$ for k_{μ} small and positive and < 0 for k_{μ} close to, but less than 2π. If $f^{\mu}(k)$ is a continuous function, it should have a zero in between. This intuitive argument implies a close relation between the chiral symmetry of the action and the doubling problem: if the action is chiral symmetric, we have species doubling[23].

The arguments above use certain assumptions and it is useful to think it over, which of them one would be willing to give up. The periodicity in k might be avoided using a random lattice. The related theoretical and technical difficulties do not make this possibility attractive. We assumed that $-if^{\mu}(k)$ is a real function, which corresponds to hermiticity of the action in Minkowski space. If we do not want to see ghosts in the spectrum it is better to keep this. We used translation symmetry (f_{μ} depends only on $n - n'$, eq. (69)) related to momentum conservation - we want to keep this also[‡]. Finally, we might drop the assumption that $f^{\mu}(k)$ is a continuous function. This would imply loosing the locality of $f^{\mu}(n - n')$ and of the action. This is the most horrible of all the possibilities.

There remains to violate the chiral symmetry of the action. It is well known in perturbation theory in models without chiral symmetry (scalar QFT) or when non-chiral invariant regularization is used (QED, for example, with Pauli-Villars regularization) that radiative corrections induce an additive mass renormalization and so, a tuning problem. This will be the generic situation in our case also. However, as we mentioned in the introduction, the FP action offers an elegant solution to the tuning and other related problems[3].

A simple way to kill the doublers is to add to the action an appropriate non-chiral invariant term which disappears in the continuum limit. Wilson suggested[24] ('Wilson fermion action') to add to eq. (71)

$$\frac{r}{2}\sum_{n,\mu}\left(2\bar{\psi}_n\psi_n - \bar{\psi}_n\psi_{n+\hat{\mu}} - \bar{\psi}_{n+\hat{\mu}}\psi_n\right) \tag{76}$$

[‡] I do not know whether it is possible to break hermiticity or translation symmetry so that the symmetry comes back in the continuum limit but the doublers are killed

with $r \neq 0$ a free parameter. This term gives a contribution to $f(k)$

$$f(k) = m + 2r \sum_{\mu} \sin^2(\frac{k_\mu}{2}) \,. \tag{77}$$

The extra term gives an $O(1/a)$ mass to the doublers, but it is suppressed around the $k \sim 0$ pole. This procedure can be generalized to interactive theories also and this is the method which is used in most of the QCD calculations.

Consider now RGTs on a d-dimensional hypercubic lattice with a scale factor of 2. Write the RGT in the form

$$\exp\Big(-\mathcal{A}'(\bar{\chi},\chi)\Big) = \int D\bar{\psi}D\psi \exp\Big[-\Big(\mathcal{A}(\bar{\psi},\psi) + T_f(\bar{\chi},\chi;\bar{\psi},\psi)\Big)\Big] \tag{78}$$

with

$$\int D\bar{\psi}D\psi = \prod_n \int d\bar{\psi}_n d\psi_n \,, \tag{79}$$

where $\bar{\psi}, \psi$ are Grassmann variables. Consider blocking kernels of the form

$$T_f(\bar{\chi},\chi;\bar{\psi},\psi) = \kappa_f \sum_{n_B} \Big(\bar{\chi}_{n_B} - b_f \sum_n \bar{\psi}_n \omega(2n_B - n)\Big) \times \Big(\chi_{n_B} - b_f \sum_n \omega(2n_B - n)\psi_n\Big) \,. \tag{80}$$

We shall assume that the averaging function ω is diagonal in Dirac space and real. Observe that T_f is not chiral invariant.

We shall consider two different block transformations[12, 25, 26]. The first is just the blocking introduced for a scalar field in Sect. 3.1 and also for the σ-model in Sect. 5.1. We define $\omega(2n_B - n) = 2^{-d}$ if n is in the hypercube whose center in indexed by n_B and zero otherwise.

Keeping in mind QCD, the blocking in the second example has a form which is easy to make gauge invariant in the presence of gauge interactions. Place the block fields in the even points of the fine lattice (n is even if all its coordinates are even). A simple choice for the averaging function ω in this case is

$$\omega(0,0,\ldots,0) = c_0, \qquad \omega(\pm1,0,\ldots,0) = \ldots = \omega(0,0,\ldots,\pm1) = c_1, \ldots, \omega(\pm1,\pm1,\ldots,\pm1) = c_d \,. \tag{81}$$

The parameters $c_0, c_1, \ldots, c_d$ satisfy the normalization condition, eq. (17),

$$c_0 + 2d\, c_1 + 2d(d-1)\, c_2 + \ldots + 2^d\, c_d = 1 \,. \tag{82}$$

Observe that the fine field ψ_n contributes to several block averages, in general. We shall call a blocking with this property 'overlapping'. The block transformations in eqs. (47,51) and the first example above are 'non-overlapping', while those in eqs. (63,67, 81) are overlapping.

There is a natural symmetric choice for the c_i parameters in eq. (81):

$$c_i = 2^{-d-i}, \qquad i = 0, 1, \ldots, d \,. \tag{83}$$

In this case, the sum of contributions of ψ_n to all the block fields χ is independent of n, the averaging is 'flat'.

QCD

The action is the sum of the Yang-Mills action and the action of quarks in interaction with the colour gauge field

$$\mathcal{A}^{QCD}_{cont}(\bar\psi,\psi,A_\mu)=\beta\mathcal{A}^{FP}_{cont}(A_\mu)+\int_x\bar\psi(x)\Big[\Big(\gamma_\mu(\partial_\mu-iA_\mu(x)\Big)+m\Big]\psi(x)\,. \qquad (84)$$

The quark fields carry Dirac, colour and flavour indices, the gauge field does not know about flavour, the mass (matrix) does not know about colour. For $N=3$, this theory is believed to be the theory of the hadrons. It should explain the masses, widths and other static properties of hadrons, scattering events, all kinds of hadronic matrix elements, the behaviour of hadronic matter at finite temperature and density, spontaneous chiral symmetry breaking, the η' problem, etc. Most of these problems are non-perturbative.

The simplest lattice regularized QCD action, which we shall call the 'Wilson action' is the sum of the gauge action eq. (58) and the fermionic action, eq. (71) plus eq. (76), made gauge invariant. Take $r=1$, which is the preferred value (for technical reasons) in simulations

$$\begin{aligned}\mathcal{A}^{QCD}_W(\bar\psi,\psi,U)&=\frac{\beta}{N}\sum_{plaq}(N-ReTr\,U_p)+\\ &\frac{1}{2}\sum_{n,\mu}\Big(\bar\psi_n(\gamma_\mu-1)U_\mu(n)\psi_{n+\hat\mu}-\bar\psi_{n+\hat\mu}(\gamma_\mu+1)U^\dagger_\mu(n)\psi_n\Big)+\sum_n\bar\psi_n(m+4)\psi_n\,. \qquad (85)\end{aligned}$$

We write the RGT in the form

$$\begin{aligned}&\exp\Big[-\Big(\beta'\mathcal{A}'_g(V)+\mathcal{A}'_f(\bar\chi,\chi,V)\Big)\Big]=\\ &\int D\bar\psi D\psi DU\exp\Big[-\beta\Big(\mathcal{A}_g(U)+T_g(V,U)\Big)+\Big(\mathcal{A}_f(\bar\psi,\psi,U)+T_f(\bar\chi,\chi;\bar\psi,\psi,U)\Big)\Big]\,. \qquad (86)\end{aligned}$$

The gauge kernel T_g is defined in eq. (61), while the fermion kernel is the gauge invariant version of eq. (80)

$$\begin{aligned}T_f(\bar\chi,\chi;\bar\psi,\psi,U)=\kappa_f\sum_{n_B}&\Big(\bar\chi_{n_B}-b_f\sum_n\bar\psi_n\omega^\dagger(U)_{n,n_B}\Big)\\ &\times\Big(\chi_{n_B}-b_f\sum_n\omega(U)_{n_B,n}\psi_n\Big)\,. \qquad (87)\end{aligned}$$

In eq. (87), the averaging function ω is made gauge invariant by connecting n_B with the point n by a product of U matrices. This can be done without problems if n_B overlaps with one of the fine lattice points as in the second example in Sect. 5.3, eq. (81). Beyond hypercubic symmetry the choice of these paths is part of the freedom we have in defining an RGT. A simple choice is to take the shortest connecting paths. For $U\to 1$, $\omega(U)_{n_B,n}\to\omega(2n_B-n)$ defined in eq. (80), which satisfies the normalization condition, eq. (17). This condition gives a meaning to b_f in eq. (87). As we shall see (Sect. 8), in a free field theory b_f is determined by the engineering dimension of the field. This should be the case also in the classical theory even in the presence of interaction, since the corrections to b_f are related to the anomalous dimension of the field generated by quantum fluctuations[1, 8, 5]. We shall return to this point later.

THE SADDLE POINT EQUATION FOR THE FIXED POINT

For free field theories with a quadratic blocking kernel, like those in eq. (80) or in eq. (21), the RGT is reduced to Gaussian integrals. The integrals can be performed exactly and the problem can be investigated in every detail (Sect. 8). In general, however, the path integral of a RGT is a highly non-trivial problem.

A basic observation to proceed is that in AF theories the FP lies at $\beta = \infty$ and in this limit the path integral can be calculated in the saddle point approximation[2]. This leads to an equation in classical field theory which determines the FP action. We shall derive this equation first in the non-linear σ-model and then in QCD.

The FP of the O(N) non-linear σ-model

As eqs. (47,49) or eq. (51) show, for $\beta \to \infty$ the blocking kernel is proportional to β. For a given fixed coarse field configuration $\mathbf{R}$ we can perform the path integral

$$\int D\mathbf{S} \exp\Big[-\beta\Big(\mathcal{A}(\mathbf{S}) + T_\sigma(\mathbf{R},\mathbf{S})\Big)\Big] \tag{88}$$

in the saddle point approximation leading to the classical equation

$$\mathcal{A}'(\mathbf{R}) = \min_{\{\mathbf{S}\}}\Big[\mathcal{A}(\mathbf{S}) + T_\sigma^\infty(\mathbf{R},\mathbf{S})\Big], \tag{89}$$

where T_σ^∞ is the $\beta = \infty$ limit of the blocking kernel. The FP of the transformation is determined by the equation

$$\mathcal{A}^{FP}(\mathbf{R}) = \min_{\{\mathbf{S}\}}\Big[\mathcal{A}^{FP}(\mathbf{S}) + T_\sigma^\infty(\mathbf{R},\mathbf{S})\Big]. \tag{90}$$

Write down this equation explicitly for the block transformation defined in eqs. (47,49)

$$\mathcal{A}^{FP}(\mathbf{R}) = \min_{\{\mathbf{S}\}}\Big[\mathcal{A}^{FP}(\mathbf{S}) - \kappa_\sigma \sum_{n_B}\Big(\mathbf{R}_{n_B}\mathbf{\Sigma}_{n_B} - |\mathbf{\Sigma}_{n_B}|\Big)\Big], \tag{91}$$

where we have used $\ln Y_N(z) = z(1 + O(\ln z/z))$ for large z and $\mathbf{\Sigma}_{n_B}$ is defined in eq. (48).

Observe that eq. (88) reduces to the saddle point equation, eq. (89), for any configuration $\mathbf{R}$. If the configuration $\mathbf{R}$ is strongly fluctuating then the minimizing configuration $\mathbf{S}_{min}$ will not be smooth either. In general, eqs. (89,90) and their solutions have nothing to do with perturbation theory. A starting condition for the FP equation, eq. (90), is that $\mathcal{A}^{FP}$ goes over to the classical action $\mathcal{A}_{cont}(\mathbf{R})$, eq. (42), for very smooth $\mathbf{R}$ configurations.

The FP of QCD

In the $\beta \to \infty$ limit ($V, \bar{\chi}, \chi$ fixed), the Boltzmann factor on the r.h.s. of eq. (86) is dominated by the gauge part. The integral over the U field is saturated by the minimizing configuration $U_{min} = U_{min}(V)$ leading to the equation

$$\mathcal{A}_g'(V) = \min_{\{U\}}\Big[\mathcal{A}_g(U) + T_g^\infty(V,U)\Big]. \tag{92}$$

The gauge part of the FP is determined by

$$\mathcal{A}_g^{FP}(V) = \min_{\{U\}}\Big[\mathcal{A}_g^{FP}(U) + T_g^\infty(V,U)\Big]. \tag{93}$$

Using the explicit form of the blocking kernel T_g in eq. (61) we obtain

$$\mathcal{A}_g^{FP}(V) = \min_{\{U\}}\left(\mathcal{A}_g^{FP}(U) - \frac{\kappa_g}{N}\sum_{n_B,\mu}\Big[ReTr\Big(V_\mu(n_B)Q_\mu^\dagger(n_B)\Big) - f\Big(Q_\mu(n_B)\Big)\Big]\right), \tag{94}$$

where

$$f(Q) = \max_W \Big[ReTr(WQ^\dagger) \Big], \qquad W \in SU(N)\,. \tag{95}$$

Eq. (95) follows from eq. (62) in the $\beta \to \infty$ limit.

The remaining fermionic integral has the form

$$\exp\Big(-\mathcal{A}'_f(\bar\chi,\chi,V)\Big) = \int D\bar\psi D\psi \exp\Big[-\Big(\mathcal{A}_f(\bar\psi,\psi,U_{min}) + T_f(\bar\chi,\chi;\bar\psi,\psi,U_{min})\Big)\Big], \tag{96}$$

where $U_{min}(V)$ is the minimizing configuration from eq. (92). If the action $\mathcal{A}_f$ is quadratic in the fermion fields, then the blocked action $\mathcal{A}'_f$ is quadratic also, since the kernel T_f is quadratic (eq. (87)) and the integral in eq. (96) is Gaussian. Write the fermion action in the following general form

$$\mathcal{A}_f(\bar\psi,\psi,U) = \sum_{n,n'} \bar\psi_n h_{n,n'}(U)\psi'_n\,, \tag{97}$$

where $h(U)_{n,n'}$ carries Dirac, colour and flavour indices which are not indicated explicitly. Using eq. (87) and performing the Gaussian integral in eq. (96) we get

$$h'(V)^{-1}_{n_B,n'_B} = \frac{1}{\kappa_f}\delta_{n_B,n'_B} + b_f^2 \sum_{n,n'} \omega(U)_{n_B,n} h(U)^{-1}_{n,n'} \omega(U)^\dagger_{n',n'_B}\,. \tag{98}$$

The FP satisfies the equation

$$h^{FP}(V)^{-1}_{n_B,n'_B} = \frac{1}{\kappa_f}\delta_{n_B,n'_B} + b_f^2 \sum_{n,n'} \omega(U)_{n_B,n} h^{FP}(U)^{-1}_{n,n'} \omega(U)^\dagger_{n',n'_B}\,. \tag{99}$$

Eqs. (94,99) determine the FP action of QCD. Gaussian integrals of c-number fields are equivalent to minimization. This can be generalized to Gaussian Grassmann integrals to express eq. (99) as

$$\mathcal{A}_f^{FP}(\bar\chi,\chi,V) = \Big[\mathcal{A}_f^{FP}(\bar\psi,\psi,U) + T_f(\bar\chi,\chi;\bar\psi,\psi,U)\Big]_{|\bar\psi_{st},\psi_{st}} \tag{100}$$

where $\psi_{st},\bar\psi_{st}$ make the r.h.s. stationary, $\delta/\delta\psi$ [...]=0, $\delta/\delta\bar\psi$ [...]=0. In eq. (100), $U = U(V)$ and the fields $\bar\chi,\chi,\bar\psi,\psi$ are c-number fields.

THE FP ACTION FOR WEAK FIELDS

We have claimed in Sect. 4 that the FP action is classically perfect: it reproduces all the essential physical properties of the continuum classical action on the lattice even if the resolution of the lattice is poor. The proof of this statement follows directly from the FP equations (eq. (91) for the σ-model or eqs. (94,99) for QCD) without solving them explicitly. Before presenting these arguments let us study the FP equations in a limit where analytic results can be obtained.

Consider first the FP equation in the $O(N)$ non-linear σ-model, eq. (91). Take a configuration where the spins fluctuate around the first axis in $O(N)$ space

$$\mathbf{R}_{n_B} = \begin{pmatrix} \sqrt{1-\boldsymbol{\chi}^2_{n_B}} \\ \boldsymbol{\chi}_{n_B} \end{pmatrix}\,, \tag{101}$$

where $\boldsymbol{\chi}_{n_B}$ has $(N-1)$ components. Assume, these fluctuations are 'weak', $|\boldsymbol{\chi}_{n_B}| \ll 1$. In this case, the minimizing field $\mathbf{S}_{min}(\mathbf{R})$ will also fluctuate around the first axis, eq. (43), and these fluctuations will also be weak, $|\boldsymbol{\pi}_{n_B}| \ll 1$. For weak fields eq. (91) can be expanded in powers of the fluctuations and one can study the solution order by order.

There is a difference between the 'weak field' introduced here and the 'smooth field' used repeatedly earlier. A smooth field changes slowly in configuration space and is dominated by small k values in Fourier space. In a weak field, the size of deviations from the classical vacuum configuration is small, but the field might change rapidly in configuration space, so higher momentum components might be important in Fourier space.

In leading order of the weak field expansion, the FP equation is quadratic in the fluctuations. Using translation symmetry we can write

$$\mathcal{A}^{FP}(\mathbf{R}) = \frac{1}{2}\sum_{n_B,r_B} \rho(r_B)\boldsymbol{\chi}_{n_B}\boldsymbol{\chi}_{n_B+r_B} + O(\chi^4)\,, \tag{102}$$

where the unknown couplings, to be determined by the FP equation, are denoted by $\rho(r_B)$. Expanding also the blocking kernel in eq. (91), we get at the quadratic level

$$\frac{1}{2}\sum_{n_B,r_B} \rho(r_B)\boldsymbol{\chi}_{n_B}\boldsymbol{\chi}_{n_B+r_B} = \min_{\{\pi\}}\Big[\frac{1}{2}\sum_{n,r}\rho(r)\boldsymbol{\pi}_n\boldsymbol{\pi}_{n+r} + 2\,\kappa_\sigma\sum_{n_B}(\boldsymbol{\chi}_{n_B} - \frac{1}{4}\sum_{n\in n_B}\boldsymbol{\pi}_n)^2\Big]\,. \tag{103}$$

Using the trivial relation between Gaussian integrals and minimization we observe that eq. (103) is just the FP equation of a free scalar theory:

$$c\exp\Big(-\frac{1}{2}\sum_{n_B,r_B}\rho(r_B)\boldsymbol{\chi}_{n_B}\boldsymbol{\chi}_{n_B+r_B}\Big) = \prod_n\int d\boldsymbol{\pi}_n\exp\Big[-\frac{1}{2}\sum_{n,r}\rho(r)\boldsymbol{\pi}_n\boldsymbol{\pi}_{n+r} - 2\,\kappa_\sigma\sum_{n_B}(\boldsymbol{\chi}_{n_B} - \frac{1}{4}\sum_{n\in n_B}\boldsymbol{\pi}_n)^2\Big]\,. \tag{104}$$

The same steps can be used for the Yang-Mills or QCD FP actions. In quadratic order the Yang-Mills FP equation, eq. (94), goes over to the FP problem of a free (Abelian) gauge theory, while the fermion part of the QCD FP action, eq. (99), becomes equivalent to the FP equation of free Dirac fermions. In the next section we give a general solution of the free field FP problem.

THE FREE FIELD FP PROBLEM

As we discussed in Sect. 7, in leading order of the weak field expansion the FP equations of AF theories are reduced to that of free fields. We shall study the free field problem in this section.

In all our previous considerations we applied lattice regularization. For free field theories without gauge symmetry we might also use a simple momentum space cut-off, which leads quickly to some results we already referred to in the previous sections[1, 8, 27]. We present these results first, and then turn to the problem of FP actions on the lattice.

Scalar field with momentum space cut-off

We work in Fourier space and constrain all the momenta to be smaller than Λ^{cut}. As always, we use dimensionless quantities, the dimensions are carried by the cut-off. We write the quadratic action in the general form

$$\mathcal{A}(\phi) = \frac{1}{2}\int_{|q|\leq 1} \phi(-q)\,\rho(q)\,\phi(q)\,, \tag{105}$$

and perform a RGT by integrating out the fields in the momentum range $1/2 < |q| \leq 1$. We write

$$\phi(q) = \phi_0(q) + \phi_1(q)\,, \tag{106}$$

where $\phi_0(q)$ and $\phi_1(q)$ are non-zero only for $|q| \leq 1/2$ and $1/2 < |q| \leq 1$, respectively. In the RGT we have to integrate out $\phi_1(q)$, i.e. perform the path integral

$$\exp\Big(-\tilde{\mathcal{A}}(\phi_0)\Big) = \prod_{\frac{1}{2}<|q|\leq 1} \int d\phi_1(q)\ \exp\Big(\mathcal{A}(\phi_0+\phi_1)\Big)\,. \tag{107}$$

Using the quadratic action in eq. (105) we get

$$\mathcal{A}(\phi_0+\phi_1) = \frac{1}{2}\int_{|q|\leq\frac{1}{2}} \phi(-q)\rho(q)\phi(q) + \frac{1}{2}\int_{\frac{1}{2}<|q|\leq 1} \phi(-q)\rho(q)\phi(q)\,. \tag{108}$$

Writing eq. (108) into eq. (105), the ϕ_0-dependent part can be brought out of the integral, while the ϕ_1-integral gives a field independent, irrelevant constant. By rescaling and relabeling the ϕ_0 field as

$$\phi'(q') = \zeta^{-1}\phi_0(q) \qquad q' = 2\,q\,, \qquad 0 \leq |q'| \leq 1\,, \tag{109}$$

we get

$$\tilde{\mathcal{A}}(\phi_0) = \zeta^2\frac{1}{2}\int_{|q|\leq\frac{1}{2}} \phi'(-2\,q)\,\rho(q)\,\phi'(2\,q) =$$

$$\zeta^2 2^{-d}\frac{1}{2}\int_{|q|\leq 1} \phi'(-q)\,\rho(q/2)\,\phi'(q) = \mathcal{A}'(\phi')\,. \tag{110}$$

The role of the rescaling factor ζ in eq. (109) is the same as that of b in eqs. (16,18, …), whose significance we shall see in a moment. As eq. (105) and eq. (110) show, under the RGT ρ transforms as

$$\rho(q) \to \zeta^2 2^{-d}\rho(q/2)\,. \tag{111}$$

Expand ρ in powers of momenta

$$\rho(q) = \alpha_0 + \alpha_2\sum_\mu q_\mu q_\mu + \alpha_4\,(\sum_\mu q_\mu q_\mu)^2 + \alpha_6\,(\sum_\mu q_\mu q_\mu)^3 \ldots\,. \tag{112}$$

For α_0 small, the pole of the propagator $1/\rho(q)$ is close to $q=0$ and $\alpha_0 = m^2$, where m is the mass of the free particle. We shall require that the action density before and after the transformation goes over into the form of classical field theory for small q-values, i.e. we fix the coefficient of $\sum_\mu q_\mu q_\mu$ in ρ to be 1. This condition fixes ζ

$$\zeta^2 2^{-d}\sum_\mu \frac{q_\mu}{2}\frac{q_\mu}{2} = \sum_\mu q_\mu q_\mu \to \zeta = 2^{\frac{d+2}{2}}\,. \tag{113}$$

Eq. (111) gives then the transformation rules for the other couplings

$$\alpha_0 \to 2^2\,\alpha_0\,, \qquad \alpha_4 \to 2^{-2}\,\alpha_4\,, \qquad \alpha_6 \to 2^{-3}\,\alpha_6\,, \ldots\,. \tag{114}$$

Eq. (114) shows that the operator ϕ^2 (the coupling α_0 or m^2) is relevant with an eigenvalue $\lambda_0 = 4$, all the other operators (couplings) which contain four or more derivatives are irrelevant. The largest irrelevant eigenvalue is $1/4$.

Replacing the region of allowed momenta of a cut-off sphere by a cut-off hypercube, the regularization will violate rotation symmetry. Only hypercubic symmetry remains, like on the lattice. In this case we have to allow in eq. (112) terms like $\beta_4 \sum_\mu (q_\mu q_\mu q_\mu q_\mu)$, which is not $O(d)$ invariant, but hypercubic symmetric. The coupling β_4 is also irrelevant, $\beta_4 \to 1/4\beta_4$ under the RGT. That means that rotational symmetry is restored as the RGT is iterated, as we go towards the infrared.

If we start with $\alpha_0 = 0$, we are and remain on the critical surface and run rapidly towards the FP which, in this case, has the simple form

$$\mathcal{A}^{FP}(\phi) = \frac{1}{2}\int_{|q|\le 1} \phi(-q)q^2\phi(q)\,, \tag{115}$$

describing a free massless scalar particle. As we mentioned before, the form of the FP depends on the block transformation (typically, it will look more complicated on the lattice), but the conclusions concerning the relevant, marginal and irrelevant eigenvalues is independent of that.

Consider now small perturbations around the FP, eq. (115), like $\lambda_4\phi^4$, $\lambda_6\phi^6, \cdots$, with λ_4, $\lambda_6, \cdots$ small, as discussed in Sect. 3.4

$$\begin{aligned}\mathcal{A}(\phi) = \mathcal{A}^{FP}(\phi) + \frac{\alpha_0}{2}\int_q \phi(-q)\phi(q) + \lambda_4 \int_{q_1,\cdots,q_4} \phi(q_1)\ldots\phi(q_4)\delta(q_1+\ldots+q_4) + \\ \lambda_6 \int_{q_1,\cdots,q_6} \phi(q_1)\ldots\phi(q_6)\delta(q_1+\ldots+q_6) + \ldots\,,\end{aligned} \tag{116}$$

where only three of the perturbations are written out explicitly. Using this action $\mathcal{A}$ in eq. (107), expanding the exponent in $\alpha_0, \lambda_4, \lambda_6\ldots$ up to linear order, performing the Gaussian ϕ_0-integrals and rescaling and relabeling as in eq. (107), we get

$$\begin{aligned}\alpha_0 &\to (\alpha_0 + 6\,c\lambda_4 + 15\,c^2\lambda_6 + \ldots)\zeta^2 2^{-d}\,,\\ \lambda_4 &\to (\lambda_4 + 15\,c^2\lambda_6 + \ldots)\zeta^4 2^{-3d}\,,\\ \lambda_6 &\to (\lambda_6 + \ldots)\zeta^6 2^{-5d}\,,\end{aligned} \tag{117}$$

with

$$c = \int_{|q|=\frac{1}{2}}^{|q|=1} \frac{1}{q^2}\,. \tag{118}$$

The corresponding T-matrix (eq. (35)) is a triangular matrix, the eigenvalues λ are just the diagonal elements

$$\begin{aligned}\lambda_1 &= 2^2\,, & h^{(1)} &= \int_x \phi^2(x)\,,\\ \lambda_2 &= 2^{4-d}\,, & h^{(2)} &= \int_x \left[\phi^4(x) + \alpha\phi^2(x)\right]\,,\\ \lambda_3 &= 2^{6-2d}\,, & h^{(3)} &= \int_x \left[\phi^6(x) + \beta_1\phi^4(x) + \beta_2\phi^2(x)\right]\,.\end{aligned} \tag{119}$$

As eq. (119) shows, there is a simple relation between the eigenvalue λ_θ and the dimension d_θ of the operator $\theta(x)$ which has the highest dimension in the corresponding eigenoperator h:

$$\lambda_\theta = 2^{d-d_\theta} \tag{120}$$

In the eigenvector h the highest dimensional operator $\theta(x)$ is mixed with lower dimensional operators.

In $d=4$, operators with dimension less than, equal, larger than 4 are relevant, marginal and irrelevant, respectively. This result is also valid in gauge and fermion theories. In $SU(N)$ Yang-Mills theory the lowest dimensional gauge invariant operator is $\mathrm{Tr}(F_{\mu\nu}F_{\mu\nu})$, which is marginal.

Lattice FP actions in different free field theories

Consider first a real scalar field. Denote the fields on the coarse and on the fine lattice by χ and ϕ, respectively. The lattice units are a_0 and $a_1 = a_0/2$. Assume, we are in the FP, and perform a RGT:

$$c\exp\Big(-\frac{1}{2}\sum_{n_B,r_B}\rho(r_B)\chi_{n_B}\chi_{n_B+r_B}\Big) = \prod_n\int d\phi_n \exp\Big[-\frac{1}{2}\sum_{n,r}\rho(r)\phi_n\phi_{n+r} - \frac{1}{2}\kappa\sum_{n_B}\big(\chi_{n_B} - b\sum_n\omega(2n_B-n)\phi_n\big)^2\Big]. \tag{121}$$

Multiply this equation by $\chi_{n'_B}\chi_{n''_B}$ and integrate over the χ fields. Using eq. (19) we obtain

$$\langle\chi_{n'_B}\chi_{n''_B}\rangle = b^2\sum_{n',n''}\omega(2n'_B-n')\omega(2n''_B-n'')\langle\phi_{n'}\phi_{n''}\rangle + \frac{1}{\kappa}\delta_{n'_B,n''_B}\,. \tag{122}$$

In Fourier space eq. (122) has the form

$$\frac{1}{\tilde\rho(q_B)} = b^2\frac{1}{2^d}\sum_l\tilde\omega(\frac{q_B}{2}+\pi l)\frac{1}{\tilde\rho(\frac{q_B}{2}+\pi l)}\tilde\omega^*(\frac{q_B}{2}+\pi l) + \frac{1}{\kappa},$$
$$l = (l_1,\dots,l_d)\,,l_\mu = 0,1\,. \tag{123}$$

The Fourier transformation is defined as

$$f(n) = \int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi}\frac{d^dq}{(2\pi)^d}\exp(iqn)\tilde f(q)\,. \tag{124}$$

We demand, as discussed in Sect. 8.1, that any action should go over into the classical continuum action for smooth fields. That requires that $\rho(q_B)\to q_B^2 + O(q_B^4)$ for small q_B. (The FP action is on the critical surface, the mass parameter α_0 in eq. (112) is zero.) Considering eq. (123) in the $q_B\to 0$ limit, the $l\neq 0$ contribution on the r.h.s. can be neglected and we obtain

$$\frac{1}{q_B^2} = b^2\frac{1}{2^d}|\tilde\omega(0)|^2\frac{1}{(q_B/2)^2}\,. \tag{125}$$

The normalization condition, eq. (17), gives $\tilde\omega(0)=1$ and so

$$b = 2^{-\frac{d-2}{2}}\,. \tag{126}$$

We shall use the relation of eq. (122) recursively, i.e. we connect the propagator $\langle\phi_{n'}\phi_{n''}\rangle$ on the r.h.s. of eq. (122) with the propagator on the lattice with unit $a_2 = a_0/4$. Denote the field on this lattice by η and the lattice points by n_f:

$$\langle\chi_{n'_B}\chi_{n''_B}\rangle = b^4 \sum_{n',n'_f} \omega(2n'_B - n')\omega(2n' - n'_f) \times \sum_{n'',n''_f} \omega(2n''_B - n'')\omega(2n'' - n''_f)\langle\eta_{n'_f}\eta_{n''_f}\rangle + \frac{1}{\kappa}\Big[\delta_{n'_B,n''_B} + b^2\sum_n \omega(2n'_B - n)\omega(2n''_B - n)\Big]. \quad (127)$$

Iterating this equation further, after j steps one arrives at a lattice with lattice unit $a_j = a_0/2^j$. For $j \to \infty$, the lattice will be infinitely fine and the propagator for this lattice can be replaced by the propagator in the continuum. This way, the FP propagator $\langle\chi_{n'_B}\chi_{n''_B}\rangle$ will be expressed explicitly in terms of the propagator in the continuum and of the product and sum of the averaging function ω. The inverse propagator gives the FP action.

If the block transformation is non-overlapping (see the paragraph after eq. (82)), the sum in the last term of eq. (127) is $\sim 1/\kappa\delta_{n'_B,n''_B}$. This remains true in every order of the iteration. Although for overlapping transformations this is not true anymore, the $\sim 1/\kappa$ contribution to the propagator remains local. The non-locality of the propagator is carried completely by the first term in eq. (127).

Without going through the detailed derivation, we quote the final results only. Let us start with an example: take the simple non-overlapping transformation, where $\omega(2n_B - n) = 2^{-d}$ if n is in the hypercube whose center is indexed by n_B and zero otherwise (Sect. 3.1). The FP propagator in Fourier space has the form[9, 28, 29]

$$\frac{1}{\tilde{\rho}(q)} = \sum_{l\in Z^d} \frac{1}{(q+2\pi l)^2} \prod_\mu \frac{\sin^2(\frac{q_\mu}{2} + \pi l_\mu)}{(\frac{q_\mu}{2} + \pi l_\mu)^2} + \frac{4}{3\kappa}, \quad (128)$$

where the summation is over integer vectors $l = (l_1, \cdots, l_d)$ and $(q+2\pi l)^2 = \sum_\mu (q_\mu + 2\pi l_\mu)^2$. The $\sim 1/\kappa$ term in eq. (128) is a constant in Fourier space, hence $\sim \delta_{n,n'}$ in configuration space, which is the general result for non-overlapping transformations. The second factor under the sum in eq. (128) is regular, it is the Fourier transform of the product of the local averaging function ω. The pole singularities come entirely from the $(q+2\pi l)^{-2}$ piece. These poles define the spectrum of the FP theory on the lattice. Fix the spatial part of the momentum to be $\mathbf{q} = \mathbf{p}, p_i \in (-\pi, \pi)$, and find the pole singularities in the complex q_d-plane. There is an infinite sequence of poles on the imaginary axis

$$q_d = \pm i\,|\mathbf{p} + 2\pi\mathbf{l}|, \qquad \mathbf{l} \in Z^{d-1}. \quad (129)$$

Denoting $\mathbf{p} + 2\pi\mathbf{l}$ by $\mathbf{k}, k_i \in (-\infty, \infty)$, we get the exact relativistic dispersion relation of a massless particle

$$-i\,q_d = E(\mathbf{k}) = |\mathbf{k}|. \quad (130)$$

Through the l-summation in eq. (128), the full exact continuum spectrum is reproduced (the momentum $\mathbf{k}$ is not constrained to the Brillouin zone), even though we are on the lattice, which resolves the large k waves very poorly. This result is to be compared with the dispersion relation of the standard nearest-neighbour action (eq. (13) with $V = 0$). If k is close to the end of the Brillouin zone, the dispersion relation is wrong by a factor of 2 or more.

Table 1. The couplings of the free field FP action at a distance $r=(r_0,r_1)$ for the optimal choice of the block transformation with $\kappa=4$. Note that in our convention, for the standard action the only non–vanishing entry in this list would be $\rho_{ST}(1,0)=-1$.

r	$\rho(r)$	r	$\rho(r)$
(1,0)	-0.61802	(4,0)	$-2.632\cdot 10^{-6}$
(1,1)	-0.19033	(4,1)	$7.064\cdot 10^{-7}$
(2,0)	$-1.998\cdot 10^{-3}$	(4,2)	$1.327\cdot 10^{-6}$
(2,1)	$-6.793\cdot 10^{-4}$	(4,3)	$-7.953\cdot 10^{-7}$
(2,2)	$1.625\cdot 10^{-3}$	(4,4)	$6.895\cdot 10^{-8}$
(3,0)	$-1.173\cdot 10^{-4}$	(5,0)	$-8.831\cdot 10^{-8}$
(3,1)	$1.942\cdot 10^{-5}$	(5,1)	$3.457\cdot 10^{-8}$
(3,2)	$5.232\cdot 10^{-5}$	(5,2)	$3.491\cdot 10^{-8}$
(3,3)	$-1.226\cdot 10^{-5}$	(5,3)	$-3.349\cdot 10^{-8}$
		(5,4)	$8.408\cdot 10^{-9}$
		(5,5)	$-1.657\cdot 10^{-10}$

Performing the sum in eq. (128) and calculating $\rho(r)$ by Fourier transformation is an easy numerical exercise. The function $\rho(r)$ and hence the FP action is local, as expected: it decays exponentially $\sim\exp(-\gamma|r|)$, as discussed in Sect. 2.4. The size of γ which defines how short ranged the action is, depends on the free parameter κ. The value $\kappa\approx 4$ is optimal. For $d=2$, we have $\gamma\approx 3.4$ for $\kappa=4$. In this case $\rho(r)$ is strongly dominated by the nearest neighbour and diagonal couplings as table 1 shows[2].

Actually, $\rho(r)$ is a perfect discretization of the continuum Laplacian. The question raised by this observation is, whether the FP idea can be used in solving partial differential equations, like the Navier-Stokes equation, numerically[30] .

Let us present now the general result. The fields can be scalars, fermions or vectors. For the blocking kernel we write

$$\kappa\sum_{n_B}\Big(\bar\chi^\alpha_{n_B}-b\sum_{n,\beta}\bar\phi^\beta_n\omega^\dagger_{\beta,\alpha}(2n_B-n)\Big)\Big(\chi^\alpha_{n_B}-b\sum_{n,\beta}\omega_{\alpha,\beta}(2n_B-n)\phi^\beta_n\Big)\,, \tag{131}$$

where the bar above the fields denotes complex or Dirac conjugation[§], the index α can be a Lorentz or Dirac index[¶]. The FP propagator is given by

$$\tilde D^{FP}(q)=\lim_{n\to\infty}\tilde D^{(n)}(q)\,, \tag{132}$$

where

$$\tilde D^{(n)}(q)=(2^{-d}b^2)^n\sum_{l_\mu=0}^{2^n-1}\tilde\Omega^{(n)}(\frac{q+2\pi l}{2^n})\tilde D^{(0)}(\frac{q+2\pi l}{2^n})\tilde\Omega^{\dagger(n)}(\frac{q+2\pi l}{2^n})+\frac{1}{\kappa}\sum_{j=0}^{n-1}\tilde Q^{(j)}(q)\,, \tag{133}$$

where Ω and Q are defined as

$$\tilde\Omega^{(j)}=\tilde\omega(2^{-1})\tilde\omega(2^{-2})\ldots\tilde\omega(2^{-j})\,, \tag{134}$$

$$\tilde Q^{(j)}(q)=(2^{-d}b^2)^j\sum_{l_\mu=0}^{2^j-1}\tilde\Omega^{(j)}(\frac{q+2\pi l}{2^j})\tilde\Omega^{\dagger(j)}(\frac{q+2\pi l}{2^j})\,, \tag{135}$$

§For a real scalar or vector field $\bar\phi=\phi$ and we include a factor of 1/2 both in the action and in the kernel, as in eq. (121).
¶We shall consider the block transformation only where ω is diagonal in the Dirac indices.

and the rescaling factor b is given by

$$b = 2^{-d_\phi}\,, \tag{136}$$

where d_ϕ is the dimension of the field ϕ. For non-overlapping transformations $\tilde{Q}^{(j)}(q)$ is independent of q, in the overlapping case it is an analytic function of q. For illustration, consider free Dirac fermions in $d = 4$ using the overlapping block transformation defined by the eqs. (81,83). In this case[26]

$$\begin{aligned}
\tilde{\omega}(q) &= \prod_\mu \cos^2(\frac{q_\mu}{2})\,, \\
\tilde{\Omega}^{(j)}(q) &= \prod_\mu 2^{-2j}\frac{\sin^2(\frac{q_\mu}{2})}{\sin^2(\frac{q_\mu}{2^{j+1}})}\,, \\
\tilde{Q}^{(j)}(q) &= \frac{2^{-j}}{81}\prod_\mu\Big[(1+4^{-j})\cos(q_\mu)+(2+4^{-j})\Big]\,.
\end{aligned} \tag{137}$$

For the initial fermion propagator $D^{(0)}$ one might take the Wilson fermion propagator (Sect. 5.3). Eq. (133) and eq. (137) gives the FP fermion propagator

$$D^{FP}(q) = -i\alpha^\mu(q)\gamma_\mu + \beta(q)\,, \tag{138}$$

where

$$\alpha^\mu(q) = \sum_{l\in Z^d}\frac{q_\mu+2\pi l_\mu}{(q+2\pi l)^2}\prod_\nu\frac{\sin^4(\frac{q_\nu}{2})}{(\frac{q_\nu}{2}+\pi l_\nu)^4} \tag{139}$$

and

$$\begin{aligned}
\beta(q) = \frac{1}{81\kappa}\Big[c_1\prod_\mu\cos(q_\mu) &+ c_2\Big(\cos(q_1)\cos(q_2)\cos(q_3)+\ldots \\
+\left(\cos(q_2)\cos(q_3)\cos(q_4)\right)\Big) &+ c_3\Big(\cos(q_1)\cos(q_2)+\ldots+\cos(q_3)\cos(q_4)\Big) + \\
& c_4\sum_\mu\cos(q_\mu)+c_5\Big]\,.
\end{aligned} \tag{140}$$

The constants $c_1,\ldots,c_5$ result from different geometric sums and have the value $c_1 = 0.592581$, $c_2 = 1.388406$, $c_3 = 3.349502$, $c_4 = 8.336553$, $c_5 = 102.410571$. This FP propagator has all the features we expected. The spectrum is given by the $\sim\alpha^\mu\gamma_\mu$ part and it is exact. The $\sim 1/\kappa$ part is analytic in q-space and ultralocal (Sect. 2.4) in configuration space. The propagator is not chiral symmetric. The chiral symmetry breaking term is $\sim 1/\kappa$ comes entirely from the non-chiral invariant block transformation.

INSTANTON SOLUTIONS AND THE TOPOLOGICAL CHARGE ON THE LATTICE

In this section we shall consider the $O(3)$ non-linear σ-model in $d = 2$. The arguments and results can be immediately generalized to Yang-Mills theories in $d = 4$.

As we discussed in Sect. 5.1, the configurations of the $O(3)$ model fall into different topological sectors characterized by an integer number Q, the topological charge of the configuration. While configurations from the same topological sector can be continuously deformed into each other, this is not possible for configurations with a different charge Q. The charge Q is the number of times the internal sphere is wrapped as the coordinate sphere is traversed. It may be defined as

$$Q = \frac{1}{8\pi}\int d^2x \epsilon_{\mu\nu}\mathbf{S}(\partial_\mu \mathbf{S}\times\partial_\nu \mathbf{S})\,, \tag{141}$$

and it is related to the action by the inequality

$$\mathcal{A} \geq 4\,\pi |Q|\,. \tag{142}$$

If, for a given configuration the equality is satisfied, the configuration minimizes the action for the given topological charge Q and is therefore a solution of the equations of motion.

Instanton solutions of the FP action

Replacing the continuous Euclidean space by a discrete set of points, the notion of a 'continuous deformation of the configuration' is lost and the definition of the topological charge becomes problematic. In addition, since the lattice introduces a scale (lattice unit a), classical scale invariance is broken and no scale invariant (instanton) solutions are expected to exist.

Of coarse, if the radius ρ of the instanton is much larger than the lattice unit a, the values of the continuum solution in the lattice points define a configuration which is 'almost' a solution of the lattice Euler-Lagrange equations. The action of this quasi-solution will be

$$\mathcal{A} = 4\,\pi|Q|\left(1 + c\,\frac{a^2}{\rho^2} + O(\frac{a^4}{\rho^4})\right)\,. \tag{143}$$

The size and sign of c depends on the details of the form of the lattice action. Depending on the sign of c, large instantons shrink or grow to get a smaller action.

An even more serious problem is the possible existence of dislocations, topological artifacts, which are small, $O(a)$, objects but contribute to the topological charge and have an action below $4\,\pi|Q|$.

There are many high energy physicists who believe that fluctuating instanton configurations play an important role in QCD, especially for questions where chiral symmetry is relevant. If a discretized action describes the classical topological properties of the theory poorly, then it might happen that continuum quantum physics will be reproduced on very fine lattices only. This would increase the numerical difficulties which are not small anyhow.

The FP actions offer a solution to this problem[2, 13, 31, 32, 33, 34]. We show now that the Euler-Lagrange equations of the FP action have scale invariant instanton solutions. We demonstrate first the following statement[2]:

If the configuration $\mathbf{R}$ on the coarse lattice satisfies the FP classical equations and it is a local minimum of $\mathcal{A}^{FP}(\mathbf{R})$, then the configuration $\mathbf{S}_{min}(\mathbf{R})$ on the finer lattice which minimizes the r.h.s. of the FP equation eq. (91) satisfies the FP equations as well. In addition, the value of the action remains unchanged, $\mathcal{A}^{FP}(\mathbf{S}_{min}) = \mathcal{A}^{FP}(\mathbf{R})$.

This statement is easy to show. Since $\mathbf{R}$ is a solution of the classical equations of motion $\delta\mathcal{A}^{FP}/\delta\mathbf{R} = 0$ and the configuration $\mathbf{S} = \mathbf{S}_{min}(\mathbf{R})$ should satisfy the equation

$$\mathbf{\Sigma}_{n_B} = \lambda_{n_B}\mathbf{R}_{n_B}\,, \tag{144}$$

for any n_B. Here $\mathbf{\Sigma}_{n_B}$ is the sum of the four $\mathbf{S}_n$ spins in the block, eq. (48). Eq. (144) follows by taking the variation $\delta/\delta\mathbf{R}$ of the FP equation, eq. (91). Since we assumed that the configuration $\mathbf{R}$ is a local minimum of $\mathcal{A}^{FP}$, we have $\lambda_{n_B} \geq 0$. On the other hand, the $\sim \kappa_\sigma$ term in eq. (91)

$$-\kappa_\sigma \sum_{n_B} \Big(\mathbf{R}_{n_B} \mathbf{\Sigma}_{n_B} - |\mathbf{\Sigma}_{n_B}| \Big) , \tag{145}$$

takes its absolute minimum (zero) on the configuration $\mathbf{S}_{min}(\mathbf{R})$ satisfying eq. (144). Since $\mathbf{S}_{min}(\mathbf{R})$ is the minimum of the sum of eq. (145) and $\mathcal{A}^{FP}(\mathbf{S})$, it follows that $\mathbf{S}_{min}(\mathbf{R})$ is a stationary point of $\mathcal{A}^{FP}$:

$$\left(\frac{\delta \mathcal{A}^{FP}(\mathbf{S})}{\delta \mathbf{S}} \right)_{|\mathbf{S}=\mathbf{S}_{min}} = 0 \, . \tag{146}$$

Since eq. (145) is zero if eq. (144) is satisfied, we get $\mathcal{A}^{FP}(\mathbf{R}) = \mathcal{A}^{FP}(\mathbf{S}_{min})$, as we wanted to show.

According to this statement, if $\mathcal{A}^{FP}$ has an instanton solution of size ρ, then there exists an instanton solution of size $2\rho, 2^2\rho, \ldots$ with the value of the action being exactly 4π for all these instantons (scale invariance). The value 4π follows from the fact that very large instantons are smooth on the lattice and then any valid lattice action gives the continuum value.

It is important to observe that the reverse of the statement is not true: if the configuration $\mathbf{S}$ is a solution, then the configuration $\mathbf{R}$, where $\mathbf{R}_{n_B} = \mathbf{\Sigma}_{n_B}/|\mathbf{\Sigma}_{n_B}|$, is not necessarily a solution. The proof fails because for this configuration $\mathbf{R}$ the minimizing configuration is not necessarily equal to $\mathbf{S}$ itself. One can show that $\mathbf{S}$ is a minimum, but not always the absolute minimum. Actually, this is the mechanism which prevents the existence of arbitrarily small instanton solutions on the lattice.

To prove the statement above we only had to show that the blocking kernel $T_\sigma^{(\infty)}(\mathbf{R}, \mathbf{S})$ in the FP equation, eq. (90), is zero at its absolute minimum in $\mathbf{R}$. This is true in all the cases we considered (σ-model, Yang-Mills theory) simply by construction: the normalization condition of the kernel reads

$$\int D\mathbf{R} \exp\Big(-\beta T_\sigma(\mathbf{R}, \mathbf{S}) \Big) = 1 \, , \tag{147}$$

which implies in the $\beta \to \infty$ limit that $\min_{\mathbf{R}} T_\sigma^{(\infty)}(\mathbf{R}, \mathbf{S}) = 0$. This can be seen explicitly in eq. (91) and eq. (51) for two different RGTs in the σ-model, and in eqs. (94,95) in the Yang-Mills theory.

The way this theorem works for the instanton solutions in the $O(3)$ σ-model and in the $SU(2)$ Yang-Mills theory has been tested in detail numerically. We shall return to the results after discussing the construction of the topological charge Q.

Fixed point operators

Until now we have discussed the construction of the FP action only. We can add to the action different operators O_i multiplied by infinitesimal sources c_i, and consider the RGT

$$\begin{aligned} &\exp\Big[-\beta' \mathcal{A}'(\mathbf{R}) - c_1' \, O_1'(\mathbf{R}) - c_2' \, O_2'(\mathbf{R}) - \ldots \Big] = \\ &\qquad \int D\mathbf{S} \exp\Big[-\beta \Big(\mathcal{A}(\mathbf{S}) + T_\sigma(\mathbf{R}, \mathbf{S}) \Big) - c_1 \, O_1(\mathbf{S}) - c_2 \, O_2(\mathbf{S}) - \ldots \Big] \, . \end{aligned} \tag{148}$$

Start with the FP action $\mathcal{A}^{FP}(\mathbf{S})$ and take the $\beta \to \infty$ limit. Using eq. (90) we obtain

$$c_1' \, O_1'(\mathbf{R}) = c_1 O_1(\mathbf{S}_{min}(\mathbf{R})), \qquad c_2' \, O_2'(\mathbf{R}) = c_2 O_2(\mathbf{S}_{min}(\mathbf{R})), \ldots \, . \tag{149}$$

The FP form of the operator is reproduced by the RGT:

$$\lambda_1 O_1^{FP}(\mathbf{R}) = O_1^{FP}(\mathbf{S}_{min}(\mathbf{R}))\,, \qquad \lambda_2 O_2^{FP}(\mathbf{R}) = O_2^{FP}(\mathbf{S}_{min}(\mathbf{R}))\,, \ldots\,, \tag{150}$$

with $\lambda_i = c_i'/c_i$. The analysis above is similar to the general discussion in Sect. 3.4 or Sect. 8.1, except that here we also allow operators which one can not consider as part of the action (for example local operators which are not summed over the lattice points). As we discussed in Sect. 8.1, the eigenvalues λ are determined by simple dimensional analysis if the FP is the Gaussian FP (which is the case for free fields and for AF theories).

Let us consider examples. Denote the field variables of a free scalar theory in d dimension by ϕ_n and χ_{n_B} on the fine and coarse lattice, respectively. Consider a RGT with scale factor 2. The FP field at the point n_B is constructed out of the χ fields in the local neighbourhood of n_B and has the transformation law

$$2^{-\frac{d-2}{2}}\varphi^{FP}(\chi; n_B) = \varphi^{FP}(\phi_{min}(\chi); n = 2n_B)\,, \tag{151}$$

where $\phi_{min}(\chi)$ is the minimizing configuration of the quadratic FP equation of the free field theory, Sect. 7.

As a second example, consider a current $J_\mu(n)$ in QCD. The FP current is a local combination of the fermion fields connected by the product of gauge matrices to assure gauge invariance. The details are determined by the equation

$$2^{-3} J_\mu(\bar{\chi}, \chi, V; n_B) = J_\mu\Big(\bar{\psi}_{st}(\bar{\chi}, U_{min}), \psi_{st}(\chi, U_{min}), U_{min}(V); n = 2n_B\Big)\,, \tag{152}$$

where U_{min} is the minimizing configuration of the FP equation, eq. (94), while $\bar{\psi}_{st}, \psi_{st}$ are the configurations which make the r.h.s. of eq. (100) stationary.

Only a few cases were treated explicitly until now: the FP free field eq. (151), the FP Polyakov loop[13] and the FP topological charge [31, 32], which will be discussed in the next subsection.

The FP topological charge

There are many ways to discretize the continuum definition of the topological charge, eq. (141). One might replace the derivatives by finite differences taking care of the basic symmetries of the expression. This is the so called 'field theoretical' definition. The corresponding topological charge on the lattice will not be an integer number. This creates immediately a complicated renormalization problem since there will be contributions to this operator even in perturbation theory. An alternative possibility is to construct an operator on the lattice which gives an integer number on any configuration ('geometric definition') and so it is protected in perturbation theory. This definition assigns, however, a non-zero topological charge to different topological dislocations of size $O(a)$ which have nothing to do with instantons and which heavily distort the results for quantities related to topology.

The FP topological charge used together with the FP action avoids these difficulties.
Since Q is a dimensionless quantity it satisfies the FP equation:

$$Q^{FP}(\mathbf{R}) = Q^{FP}(\mathbf{S}_{min}(\mathbf{R}))\,. \tag{153}$$

One can solve this equation the following way. Let us denote the minimizing configuration associated with $\mathbf{R}$ by $\mathbf{S}^{(1)}_{min}(\mathbf{R})$ ($\equiv \mathbf{S}_{min}(\mathbf{R})$ in eq. (153)). Consider now the FP equation, eq. (90), with $\mathbf{S}_{min}^{(1)}$ on the l.h.s . The corresponding minimizing configuration $\mathbf{S}_{min}^{(2)} = \mathbf{S}_{min}^{(2)}(\mathbf{S}_{min}(\mathbf{R}))$ lives on a lattice whose lattice unit is a factor of 2^2 smaller than that of the lattice of $\mathbf{R}$. Iterating eq. (153) this way, we get

$$Q^{FP}(\mathbf{R}) = Q^{FP}(\mathbf{S}_{min}^{(j)}(\mathbf{R}))\,, \tag{154}$$

where $\mathbf{S}_{min}^{(j)}(\mathbf{R})$ lives on a very fine lattice (containing big instantons with small fluctuations) if j is large. On such configurations any lattice definition Q_{latt} gives the correct result

$$Q^{FP}(\mathbf{R}) = \lim_{j\to\infty} Q_{latt}(\mathbf{S}_{min}^{(j)}(\mathbf{R}))\,. \tag{155}$$

In practice the $j = 2$ value agrees with the $j = \infty$ limit if the geometrical definition is used[31, 33] for Q_{latt}. The construction of Q^{FP} goes the same way in Yang-Mills theories[32, 35, 36].

There are no topological artifacts if FP operators are used

We show now that using the FP action and the FP topological charge, the action of *any* configuration (classical solution or not) satisfies

$$\mathcal{A}^{FP}(\mathbf{R}) \geq 4\,\pi|Q|\,, \tag{156}$$

like in the continuum classical theory. There are no configurations with topological charge whose action is below that of the instanton solutions.

In the equation $\mathcal{A}^{FP}(\mathbf{R}) = \mathcal{A}^{FP}(\mathbf{S}_{min}) + T_\sigma^{(\infty)}(\mathbf{R}, \mathbf{S}_{min})$ we can express $\mathcal{A}^{FP}(\mathbf{S}_{min})$ again in terms of the sum of the action and the kernel on the next finer lattice, etc. Iterating further we get arbitrarily close to the continuum, where the inequality eq. (156) is certainly satisfied. Since the kernels are non-negative (Sect. 9.1), eq. (156) follows.

Numerical tests on the classical solutions

As we discussed in Sect. 9.1, if the FP action has an instanton solution with size ρ then there is certainly a solution with size $2\,\rho$. The opposite is not always true: if there is a solution $\mathbf{S}$ with size ρ, then the coarse configuration $\mathbf{R}_{n_B} = \mathbf{\Sigma}_{n_B}/|\mathbf{\Sigma}_{n_B}|$ does not necessarily satisfy the equations of motion. This happens if in the minimization problem, eq. (90), more than one local minima occurs and the absolute minimum does not agree with the starting $\mathbf{S}$. It is expected that this happens if $\rho = O(a)$ and the would be instanton with $\rho' = \rho/2$ 'falls through the lattice'. The theorem leaves open when this happens.

Detailed numerical studies of classical solutions show[31, 33, 35, 36] that the radius ρ_{min} of the smallest instanton which exists on the lattice is less than the lattice unit a both for spin and for gauge models (for the O(3) model with the kernel eq. (47), for example, $\rho_{min} \approx 0.7a$). For instantons with $\rho_{min} < \rho < \infty$, the value the action is exactly $4\,\pi|Q|$, where Q is the topological charge given by the operator Q^{FP}.

FERMIONIC ZERO MODES AND THE INDEX THEOREM ON THE LATTICE

Consider the continuum eigenvalue problem of the Dirac operator of massless fermions in the presence of a background gauge field

$$\gamma_\mu\Big(\partial_\mu - i\,A_\mu(x)\Big)\psi^{(\lambda)}(x) = \lambda\psi^{(\lambda)}(x)\,. \tag{157}$$

If the background gauge field has a non-zero topological charge Q, then there necessarily exist zero eigenvalues. The corresponding eigenvectors are γ_5 eigenstates: $\gamma_5\psi^{(0)}_{L,R} = \pm\psi^{(0)}_{L,R}$. If we denote by n_L and n_R the number of left and right handed eigenvalues, we have

$$n_R - n_L = Q\,, \tag{158}$$

which is a special case of the Atiyah-Singer index theorem[37]. It is believed by many that eq. (158) has important consequences on the low energy properties of QCD. A possible intuitive picture of a typical gauge configuration in QCD is that of a gas or liquid of instantons and anti-instantons with quantum fluctuations. Over such a configuration, the Dirac operator is expected to have a large number of quasi-zero modes which could be responsible for spontaneous chiral symmetry breaking.

Standard lattice formulations violate the conditions of the index theorem in all possible ways: the topology of the gauge sector is in a bad shape, the Dirac operator breaks chiral symmetry (Sect. 5.3, 5.4) and the discretization errors distort the spectrum.

We shall show now that using the FP action the index theorem for the zero modes of the Dirac operator remains valid even for coarse lattices[38]. This statement is based on the FP equations for the gauge action (eq. (93)) and for the Dirac operator (eq. (100).

Consider a gauge configuration V on a lattice with lattice unit a_0. Assume that V has a topological charge Q. As discussed in Sect. 9.3, the FP topological charge is defined through a sequence of minimizing configurations $U^{(1)}_{min}(V), U^{(2)}_{min}(U^{(1)}_{min}(V)), \ldots$. Here $U^{(j)}_{min}$ lives on a lattice with lattice unit $a_j = a_0/2^j$. By construction all these configurations have the same topological charge Q, which is the charge of $U^{(\infty)}_{min}$ in the continuum, at the end of the sequence.

Assume that the FP Dirac operator h^{FP} has a zero eigenvalue for the background field V

$$\sum_{n'_B} h^{FP}(V)_{n_B,n'_B}\chi_{n'_B} = 0\,. \tag{159}$$

Eq. (159) is the Dirac equation, i.e. the classical Euler-Lagrange equation of motion. We proceed now as in the case of the classical solutions of the Yang-Mills theory, Sect. 9. Taking the variation of eq. (100) with respect to $\bar\chi_{n_B}$ and using eqs. (97,159,87) we find that ψ^{st}_n, which makes the r.h.s. of eq. (100) stationary, satisfies the Dirac equation for the background field $U^{(1)}_{min}(V)$. Since eq. (100) is quadratic in the fermion field, the stationary configuration is unique. Hence the opposite statement is also true: if ψ_n is a solution of the Dirac equation for the background field $U^{(1)}_{min}(V)$ then $\chi_{n_B} = b_f\sum_n \omega(U)_{n_B,n}\psi_n$ is a solution on the coarse lattice for the configuration V.

This result can be iterated towards finer lattices. Having a zero mode eigenfunction ψ^{st}_n for $U^{(1)}_{min}$ there is a zero mode on the next finer lattice for the configuration $U^{(2)}_{min}$, etc. At every step we can also move in the opposite direction. Going up, we finally get to a very smooth configuration for which the Atiyah-Singer theorem is applicable. This way we get that for each gauge configuration of the sequence, the FP Dirac eigenvalue problem has the same number of zero modes and with the same helicities as in the continuum. The statement for the helicities follows from the relation between the eigenvectors of the i-th and $(i-1)$-th step

$$\psi^{st}_{n_{(i-1)}} = b_f \sum_{n_{(i)}} \omega(U^{(i)}_{min})_{n_{(i-1)},n_{(i)}}\psi^{st}_{n_{(i)}}\,, \tag{160}$$

and from the fact that ω is diagonal in Dirac space (Sect. 5.3).

REFERENCES

1. K. G. Wilson and J. B. Kogut, Phys. Rep. C12 (1974) 75;
   K. G. Wilson, Rev. Mod. Phys. 47 (1975) 773, *ibid.*, 55 (1983) 583;
   K. G. Wilson, Adv. Math. 16 (1975) 176, 444.
2. P. Hasenfratz and F. Niedermayer, Nucl. Phys. B414 (1994) 785;
3. For a recent summary on the perfect actions, see P. Hasenfratz, hep-lat/9709110.
4. V. N. Gribov and A. A. Migdal, Sov. Phys. JETP 28 (1968) 784;
   A. A. Migdal, Sov. Phys. JETP 32 (1971) 552;
   A. M. Polyakov, Sov. Phys. JETP 29 (1969) 533;
   M. A. Moore, Nuovo Cim. 3 (1972) 275.
5. G. Parisi, Statistical Field Theory, Benjamin/Cummings (1988);
   J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, Oxford Univ. Press, (1993).
6. J. Schwinger, Phys. Rev. 115 (1959) 721;
   G. C. Wick, Phys. Rev. 96 (1954) 1124.
7. M. Creutz, Quarks, Gluons and Lattices, Cambridge Univ. Press (1983);
   I. Montvay and G. Münster, Quantum Fields on the Lattice, Cambridge Univ. Press (1994).
8. S. K. Ma, Rev. Mod. Phys. 45 (1973) 589;
   S. K. Ma, Modern Theory of Critical Phenomena, Benjamin Reading, MA (1976);
   L. L. Kadanoff, Rev. Mod. Phys. 49 (1977) 267;
   Th. Niemeyer and J. M. J. van Leeuwen, Phys. Rev. Lett. 31 (1973) 1411.
9. T. L. Bell and K. G. Wilson, Phys. Rev. B10 (1974) 3935.
10. F. Wegner, Phys. Rev. B5 (1972) 4529.
11. P. Hasenfratz, Nucl. Phys. B(Proc.Suppl.) 34 (1994) 3;
    T. DeGrand, A. Hasenfratz, P. Hasenfratz, F. Niedermayer and U.-J. Wiese,
    Nucl. Phys. B(Proc.Suppl.) 42 (1995) 67;
    F. Niedermayer, Nucl. Phys. B(Proc.Suppl.) 53 (1997) 56.
12. U.-J. Wiese, Phys. Lett. B315 (1993) 417;
    W. Bietenholz and U.-J. Wiese, Nucl. Phys. B(Proc.Suppl.) 34 (1994) 516.
13. T. DeGrand, A. Hasenfratz, P. Hasenfratz and F. Niedermayer,
    Nucl. Phys. B454 (1995) 587.
14. W. Bietenholz, R. Brower, S. Chandrasekharan and U.-J. Wiese,
    Nucl. Phys. B(Proc.Suppl.) 53 (1997) 921.
15. M. E. Peskin and D. V. Schroeder, An Introduction to Quantum Field Theory,
    Addison-Wesley (1995).
16. A. M. Polyakov, Gauge Fields and Strings, Harwood (1987).
17. A. A. Migdal, Sov. Phys. JETP 42 (1975) 413; 42 (1976) 743;
    L. Kadanoff, Ann. Phys.(NY) 100 (1976) 359; Rev. Mod. Phys. 49 (1977) 267.
18. R. Rajaraman, Solitons and Instantons, North-Holland, Amsterdam (1982).
19. P. Hasenfratz and F. Niedermayer, hep-latt/9706002.
20. P. Hasenfratz and H. Leutwyler, Nucl. Phys. B343 (1990) 241.
21. K. G. Wilson, Phys. Rev. D10 (1974) 2445;
    R. Balian, J.-M. Drouffe and C. I. Itzykson, Phys. Rev. D10 (1974) 3376.
22. M. Blatter and F. Niedermayer, Nucl. Phys. B482 (1996) 286.
23. N. B. Nielsen and M. Ninomiya, Phys. Lett. B105 (1981) 219; Nucl. Phys. B185 (1981) 20.
24. K. G. Wilson, *in* New Phenomena in Subnuclear Physics, ed. A. Zichichi,
    Plenum Press, New York (1997).
25. T. DeGrand, A. Hasenfratz, P. Hasenfratz, P. Kunszt and F. Niedermayer,
    Nucl. Phys. B(Proc.Suppl.) 53 (1997) 942.
26. P. Kunszt, hep-lat/9706019.
27. S. H. Shenker, *in* Recent Development in Field Theory and Statistical Mechanics,
    eds. J. B. Zuber and R. Stora, North Holland (1984).
28. Y. Iwasaki, Nucl. Phys. B314 (1989) 63.
29. K. Kawedzki and A. Kupiainen, Comm. Math. Phys. 77 (1980) 31.
30. E. Katz and U.-J. Wiese, comp-gas/9709001.
31. F. Niedermayer, Nucl. Phys. B(Proc.Suppl.) 34 (1994) 513;
    M. Blatter, R. Burkhalter, P. Hasenfratz and F. Niedermayer, Phys. Rev. D53 (1996) 923.
32. T. DeGrand, A. Hasenfratz and D. Zhu, Nucl. Phys. B475 (1996) 321;
    Nucl. Phys. B478 (1996) 349; Nucl. Phys. B(Proc.Suppl.) 53 (1997) 945.
33. R. Burkhalter, Phys. Rev. D54 (1996) 4121.

34. W. Bietenholz, R. Brower, S. Chandrasekharan and U.-J. Wiese, hep-lat/9704015.
35. T. DeGrand, A. Hasenfratz and T. Kovacs, hep-lat/9705009.
36. F. Farchioni and A. Papa, hep-lat/9709012 .
37. M. Atiyah and I. M. Singer, Ann. Math. 93 (1971) 139.
38. P. Hasenfratz and F. Niedermayer, unpublished.