ABELIAN PROJECTIONS AND MONOPOLES

M.N. Chernodub and M.I. Polikarpov*

Institute of Theoretical and Experimental Physics,
B. Cheremushkinskaya 25, Moscow, 117259, Russia

INTRODUCTION

In these lectures we give an introduction to the theory of the confinement of color in lattice gauge theories. For the sake of self-consistency, we explain all definitions. The reader is supposed to be familiar with basic notions of lattice gauge theory.

Lattice field theories were originally formulated[1] in order to explain the confinement of color in nonabelian gauge theories. The leading term of the strong coupling $(1/g^2)$ expansion in lattice gauge theories yields the area law for the Wilson loop. The D dimensional theory is reduced to the 2–dimensional one, the field strength tensors are independent on different plaquettes and the confinement has a stochastic nature. The expectation value of the Wilson loop is simply $< P >^S$, where $< P >$ is the expectation value of the plaquette, S is the area of the minimal surface spanned on the loop. Therefore, the string tension is $\sigma = -\ln < P >$.

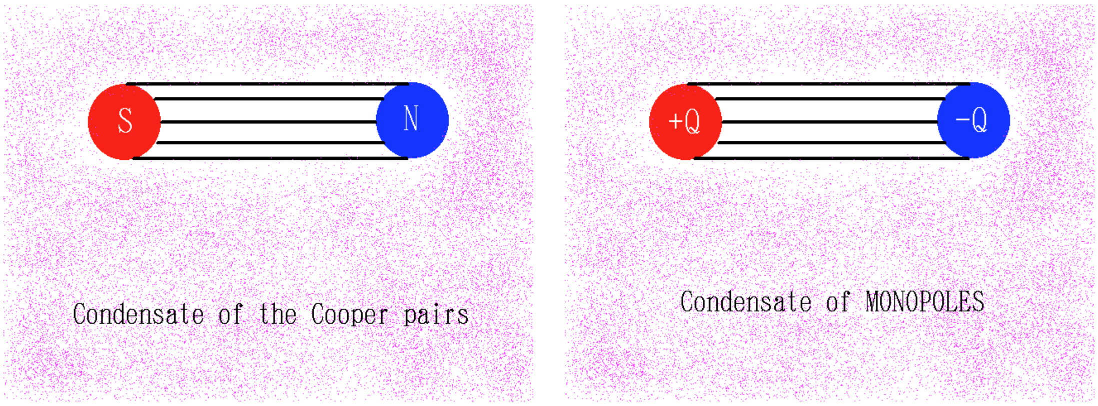
The numerical study of lattice gauge theories initiated by Creutz[2] shows that the strong coupling expansion of lattice gauge theory has no relation to continuum physics: the effects of lattice regularization are very strong, the results are not rotationally invariant, and there is no scaling for hadron masses[†].

In the weak coupling region (where the existence of the continuum limit was found in numerical calculations), the confinement mechanism has no stochastic origin. There are several approaches to explain color confinement. We will describe the most traditional one, namely, confinement caused by the dual Meissner effect. The linear confining potential can be explained by the formation of a string (flux tube) connecting a quark and an anti-quark. A well-known example of the string–like solution of the classical equations of motion is given in Ref.[3]. If we have a medium of condensed electric charges (a superconductor), then between the monopole and anti-monopole an Abrikosov string is formed, see Figure 1(a). To explain the confinement of electric charges, we need a condensate of magnetic monopoles, Figure 1(b). This simple qualitative idea was suggested by 't Hooft and Mandelstam[4]. There is a long way from this picture to real QCD. First, we have to explain how the abelian gauge field with monopoles is obtained from

*Lectures presented by Mikhail Polikarpov

†Here, scaling means independence of the ratios of the hadron masses m_k/m_i on the parameters of the theory such as the bare coupling and the cutoff parameter.

the non-abelian gauge field. Secondly, we have to explain why a dual superconductor is involved here. We discuss these two questions in these lectures.



(a) (b)

Figure 1. The Abrikosov string between the monopoles in the superconductor (a) and an analogue of the Abrikosov string between the electrically charged particles in the dual superconductor (b).

At present, we have no analytic proof of the existence of the condensate of abelian magnetic monopoles in gluodynamics and in chromodynamics. However, in all theories allowing for an analytical proof of confinement, the latter is due to the condensation of monopoles. These analytical examples are: compact electrodynamics[5], the Georgi–Glashow model[6], and super-symmetric Yang–Mills theory[7].

On the other hand, many numerical facts (some of these are discussed in these lectures) suggest that the vacuum in $SU(2)$ and $SU(3)$ lattice gauge theories behaves as a dual superconductor. As an illustration, we give two figures obtained by numerical calculations in $SU(2)$ lattice gluodynamics.

In Figure 2, taken from Ref.[8], the action density (vertical axis) of the $SU(2)$ fields is shown. The two peaks correspond to the quark–anti-quark pair, the formation of the flux tube is clearly seen. In Figure 3, taken from Ref.[9], the abelian monopole currents near the center of the flux tube formed by the quark–anti-quark pair are shown. It is seen that the monopoles wind around the center of the flux tube just as the Cooper pairs wind around the center of the Abrikosov string.

We first explain how to get the abelian fields and monopoles from the non-abelian fields. Then we present the results of numerical studies of the confinement mechanism in lattice gluodynamics.

All technical details such as the formalism of differential forms on the lattice are given in the Appendices.

ABELIAN MONOPOLES FROM NON-ABELIAN GAUGE FIELDS

In this Section we discuss the question how to obtain the abelian monopoles from non-abelian gauge fields.

The Method of Abelian Projection

The abelian monopoles arise from non-abelian gauge fields as a result of the abelian projection suggested by 't Hooft[10]. The abelian projection is a partial gauge fixing under which the abelian degrees of freedom remain unfixed. For example, the abelian

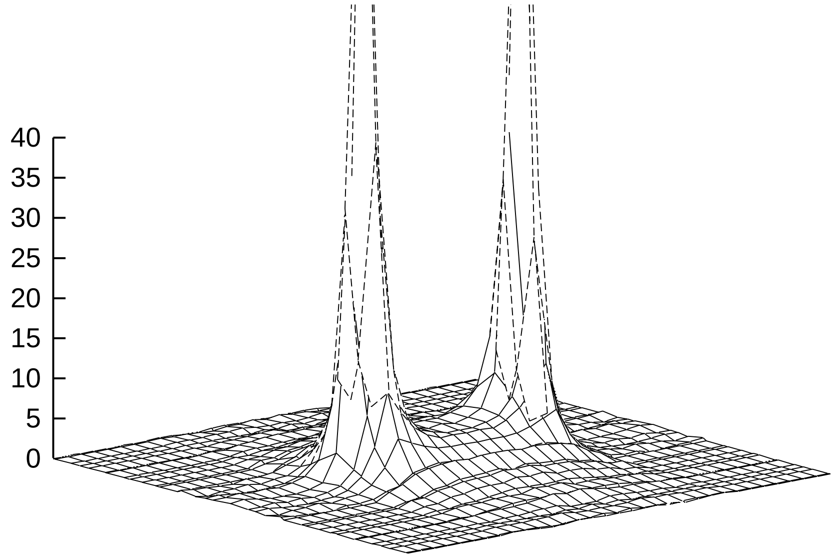


Figure 2. The action density distribution[8] at $\beta = 2.635$, the distance between quark and anti-quark sources is $\approx 1.35 fm$. Two horizontal axes correspond to two spatial lattice axes.

projection of a theory with $SU(N)$ gauge symmetry leads to a theory with $[U(1)]^{N-1}$ gauge symmetry.

Since the original $SU(N)$ gauge symmetry group is compact, the remaining abelian gauge group is also compact. But the abelian gauge theories with compact gauge symmetry group possess abelian monopoles. Therefore $SU(N)$ gauge theory in the abelian gauge has abelian monopoles.

First, consider the simplest example of F_{12} *abelian projection* for $SU(2)$ gauge theory. This gauge is defined by the following condition:

$$\hat{F}_{12}(x) = \text{diagonal matrix}\,. \tag{1}$$

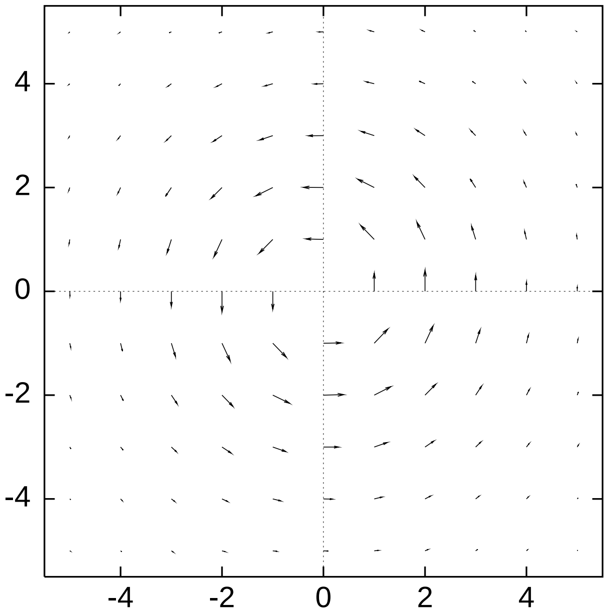


Figure 3. The monopole currents around the string tube which is formed between the static quark and anti-quark in gluodynamics, Ref.[9].

It is easy to fix to this gauge, since under gauge transformations the field strength tensor $\hat{F}_{12}$ transforms as

$$\hat{F}_{12}(x) \to \hat{F}'_{12}(x) = \Omega^{+}(x)\hat{F}_{12}(x)\Omega(x)\,. \tag{2}$$

If we fix to the F_{12} gauge, then the field strength tensor $\hat{F}'_{12}(x)$ is invariant under $U(1)$ gauge transformations:

$$\hat{F}'_{12}(x) = \Omega^{+}_{U(1)}(x)\hat{F}'_{12}(x)\Omega_{U(1)}(x)\,, \tag{3}$$

where

$$\Omega_{U(1)}(x) = \begin{pmatrix} e^{i\alpha(x)} & 0 \\ 0 & e^{-i\alpha(x)} \end{pmatrix}, \quad \alpha \in [0, 2\pi)\,. \tag{4}$$

Therefore, the gauge condition (1) fixes the $SU(2)$ gauge group up to the diagonal $U(1)$ subgroup.

The $SU(2)$ gauge field

$$\hat{A}_{\mu} = \begin{pmatrix} A^{3}_{\mu} & A^{+}_{\mu} \\ A^{-}_{\mu} & -A^{3}_{\mu} \end{pmatrix} \tag{5}$$

transforms under the gauge transformations as

$$\hat{A}_{\mu} \to \Omega^{+}\hat{A}_{\mu}\Omega - \frac{i}{g}\Omega^{+}\partial_{\mu}\Omega\,. \tag{6}$$

If we fix to the F_{12} gauge, then under the remaining $U(1)$ gauge transformation (4) the components of the nonabelian gauge field $\hat{A}$ transform as

$$a_{\mu} \to a_{\mu} - \frac{1}{g}\partial_{\mu}\alpha\,, \quad a_{\mu} = A^{3}_{\mu}\,; \qquad A^{+}_{\mu} \to e^{2i\alpha}A^{+}_{\mu}\,, \quad A^{+}_{\mu} = A^{1}_{\mu} + iA^{2}_{\mu}\,. \tag{7}$$

Thus, in the abelian gauge the field a_{μ} plays the role of the abelian gauge field and the field A^{+}_{μ} is a charge 2 abelian vector matter field.

We thus obtain abelian fields from non-abelian ones. It occurs that abelian projection is also responsible for the appearance of abelian monopoles. If a is a regular abelian field, then $\mathrm{div}\vec{H} = 0$, since $\vec{H} = \mathrm{curl}\,\vec{a}$. But the abelian gauge field may be non-regular, since the matrix of the gauge transformation Ω may contain singularities. The nonabelian field strength tensor $\hat{F}_{\mu\nu}$ is *not invariant* under singular gauge transformations:

$$\hat{F}_{\mu\nu}[A] \to \hat{F}_{\mu\nu}[A^{(\Omega)}] = \Omega^{+}\hat{F}_{\mu\nu}[A]\Omega + \hat{F}^{sing}_{\mu\nu}[\Omega]\,, \tag{8}$$

where

$$\hat{F}^{sing}_{\mu\nu}[\Omega] = -\frac{i}{g}\Omega^{+}(x)[\partial_{\mu}\partial_{\nu} - \partial_{\nu}\partial_{\mu}]\Omega(x)\,. \tag{9}$$

Therefore, if we fix to the abelian gauge (1), using the singular gauge rotation matrices, the *abelian* field strength tensor

$$f_{\mu\nu} = \partial_{\mu}a_{\nu} - \partial_{\nu}a_{\mu} \tag{10}$$

may contain singularities (the Dirac strings)

$$f_{\mu\nu} = f^{r}_{\mu\nu} + f^{s}_{\mu\nu}\,, \qquad \varepsilon_{\mu\nu\alpha\beta}\partial_{\nu}f^{r}_{\alpha\beta} = 0 \quad \text{and} \quad \varepsilon_{\mu\nu\alpha\beta}\partial_{\nu}f^{s}_{\alpha\beta} \neq 0\,, \tag{11}$$

and therefore, $\mathrm{div}\vec{H} \neq 0$.

The charge of the monopole can be calculated[11] by means of the Gauss law. Let us choose an abelian monopole in a certain time slice and surround it by an infinitesimally small sphere S. The monopole charge is

$$m = \frac{1}{4\pi} \oint \vec{H}\, d\vec{\sigma} = \frac{1}{8\pi} \int d\sigma_{\mu\nu}\, \varepsilon_{\mu\nu\alpha\beta} f_{\alpha\beta} = \frac{i}{4\pi g} \int d\sigma_{\mu\nu}\, \varepsilon_{\mu\nu\alpha\beta} \mathrm{Tr}[\Omega^+ \partial_\alpha \Omega \Omega^+ \partial_\beta \Omega]$$
$$= -\frac{i}{4\pi g} \int d\sigma_{\mu\nu}\, \varepsilon_{\mu\nu\alpha\beta} \partial_\alpha \mathrm{Tr}[\Omega^+ \partial_\beta \Omega] = 0, \pm\frac{1}{2g}, \pm\frac{1}{g}, \dots \,. \tag{12}$$

The quantization of the abelian monopole charge is due to topological reasons: the surface integral in eq.(12) is equal to the winding number of $SU(2)$ over the sphere S surrounding the monopole. The physics of quantization is simple: the electric charge is fixed by the gauge transformation (7), and magnetic charge should obey the Dirac quantization condition. The last integral can be seen as the magnetic flux through the Dirac string that arises as a consequence of the gauge singularity.

Various Abelian Projections

There is an infinite number of abelian projections. In the previous section we have considered the $\hat{F}_{12}$ abelian gauge. Instead of the diagonalization of the tensor component $\hat{F}_{12}$ by the gauge transformation, we can diagonalize any operator X which transforms under the gauge rotation as follows: $X \to \Omega^+ X \Omega$. Each operator X defines an abelian projection. At finite temperature one can consider the so-called *Polyakov abelian gauge* which is defined by the diagonalization of the Polyakov line.

The most interesting numerical results are those obtained in the Maximal Abelian (MaA) gauge. This gauge is defined by the maximization of the functional

$$\max_{\Omega} R[\hat{A}^\Omega]\,, \qquad R[\hat{A}] = -\int d^4x\, [(A^1_\mu)^2 + (A^2_\mu)^2]\,, \tag{13}$$

The condition of a local extremum of the functional R is

$$(\partial_\mu \pm igA^3_\mu) A^\pm = 0. \tag{14}$$

Clearly, this condition (as well as the functional $R[A]$) is invariant under the $U(1)$ gauge transformations (7). The meaning of the MaA gauge is simple: by gauge transformations we make the field $\hat{A}_\mu$ as diagonal as possible.

Abelian Projection on the Lattice

The $SU(2)$ gauge fields U_l on the lattice are defined by $SU(2)$ matrices attached to the links l. These lattice fields are related to the continuum $SU(2)$ fields $\hat{A}$: $U_{x,\mu} = e^{iag\hat{A}_\mu(x)}$, here a is the lattice spacing. Under the gauge transformation, the field U_l transforms as $U^\Omega_{x,\mu} = \Omega^+_x\, U_{x,\mu}\, \Omega_{x+\hat{\mu}}$, the matrices of the gauge transformation are attached to the sites x of the lattice.

The Maximal Abelian gauge is defined on the lattice by the following condition[11]:

$$\max_{\Omega} R[\hat{U}^\Omega_l]\,, \qquad R[U_l] = \sum_l Tr[\sigma_3 U_l^+ \sigma_3 U_l]\,, \quad l = \{x, \mu\}\,. \tag{15}$$

This gauge condition corresponds to an abelian gauge, since R is invariant under the gauge transformations defined by the matrices (4).

Let us parametrize the link matrix U in the standard way

$$U_l = \begin{pmatrix} \cos\varphi_l e^{i\theta_l} & \sin\varphi_l e^{i\chi_l} \\ -\sin\varphi_l e^{-i\chi_l} & \cos\varphi_l e^{-i\theta_l} \end{pmatrix}, \tag{16}$$

where $\theta, \chi \in [-\pi, +\pi)$ and $\varphi \in [0, \pi)$. In this parameterization,

$$R[U_l] = \sum_l \cos 2\varphi_l \,. \tag{17}$$

Thus, the maximization of R corresponds to the maximization of the diagonal elements of the link matrix (16).

Under the $U(1)$ gauge transformations, the components of the gauge field (16) are transformed as

$$\theta_{x,\mu} \to \theta_{x,\mu} + \alpha_x - \alpha_{x+\hat\mu}\,, \quad \chi_{x,\mu} \to \chi_{x,\mu} + \alpha_x + \alpha_{x+\hat\mu}\,, \quad \varphi_{x,\mu} \to \varphi_{x,\mu} \tag{18}$$

Therefore, in the MaA the gauge, the field θ is the $U(1)$ gauge field, the field χ is the abelian charge 2 vector matter field, the field φ is the non–charged vector matter field.

Monopoles on the Lattice

A configuration of abelian gauge fields θ_l can contain monopoles. The position of the monopoles is defined by the lattice analogue of the Gauss theorem. Consider the elementary three–dimensional cube C (Figure 4(a)) on the lattice.

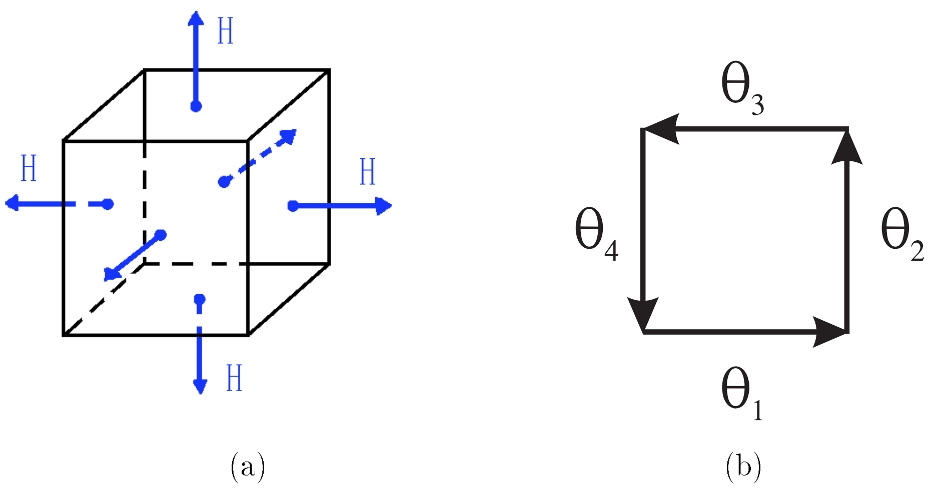


Figure 4. The magnetic flux through the boundary of the cube C (a) and lattice field strength tensor (b).

The abelian magnetic flux $\vec H$ through the surface of the cube C is given by the formula

$$m = \frac{1}{2\pi} \sum_{P \in \partial C} \bar\theta_P \,, \tag{19}$$

where $\bar\theta_P$ is the magnetic field defined as follows. Consider the plaquette angle $\theta_P = \theta_1 + \theta_2 - \theta_3 - \theta_4 \equiv \mathrm{d}\theta$, the θ_i's are attached to the links i which form the boundary of the plaquette P, Figure 4(b). The definition of $\bar\theta_P$ is $\bar\theta_P = \theta_P + 2\pi k$, where the integer k is such that $-\pi < \bar\theta_P \leq \pi$. The restriction of $\bar\theta_P$ to the interval $(-\pi, \pi]$ is natural since (as we point out in Appendix A) the abelian action for the compact fields θ_l is a periodic function of $\bar{\theta}_P$. Equation (19) is the lattice analogue of the continuum formula $m = \oint \vec{H}\, \mathrm{d}\vec{S}$. Due to the compactness of the lattice field θ ($-\pi < \theta \leq \pi$) there exist singularities (Dirac strings), and therefore, $\mathrm{div}\vec{H} \neq 0$.

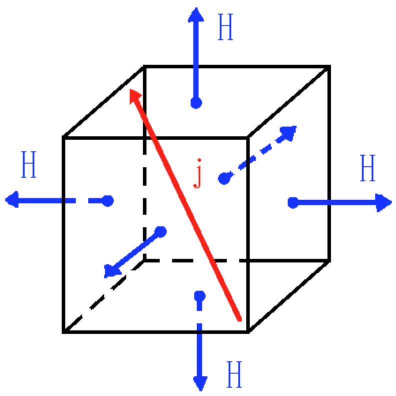


Figure 5. The definition of the magnetic monopole current.

The magnetic charge m defined by eq. (19) has the following properties:

1. m is quantized: $m = 0, \pm 1, \pm 2;$
2. If $m \neq 0$, then there exists a magnetic current j. This current is attached to the link dual to cube C, Figure 5.
3. Monopole currents j are conserved: $\delta j = 0$, the currents form closed loops on the $4D$ lattice. The proof is given in Example 3 of Appendix A.

VACCUM OF GLUODYNAMICS AS A (DUAL) SUPERCONDUCTOR

As we have just shown, the non-abelian gauge field in the abelian projection is reduced to abelian fields and abelian monopoles. The static quark–anti-quark pair in the abelian projection becomes the electric charge–anti-charge pair, and if the monopoles are condensed, then the quark and anti-quark are connected to each other by an analogue of the Abrikosov string. A more detailed discussion of the confinement in the abelian projection is given in Refs.[10]. Thus, in order to justify the monopole confinement mechanism we have to prove the existence of the monopole condensate. For lattice gluodynamics we have a lot of numerical facts which confirm the monopole confinement mechanism. We discuss these at the end of this section. But first we give an analytical example which shows how the monopole condensate appears in compact electrodynamics.

Compact $U(1)$ Lattice Gauge Theory

We show that lattice compact electrodynamics can be represented as a dual Abelian Higgs model, with the Higgs particles being monopoles. The partition function of the compact $U(1)$ gauge theory can be written as

$$\mathcal{Z} = \int_{-\pi}^{+\pi} \mathcal{D}\theta_l \, \exp\{-S(\mathrm{d}\theta)\}\,, \tag{20}$$

where, as below, $\int_{-\pi}^{+\pi} \mathcal{D}\theta_l = \prod_l \int_{-\pi}^{+\pi} \mathrm{d}\theta_l$ is the integral over all link variables. In the continuum limit ($a \to 0$), $S(\theta_P) \propto f^2_{\mu\nu}\, a^4$.

By means of the duality transformation (see Appendix B), the theory (20) can be rewritten in the form

$$\mathcal{Z} = \mathrm{const.} \cdot \sum_{\substack{{}^*n({}^*\mathrm{C}_1)\in \mathbb{Z} \\ \delta^* n=0}} \exp\{-S^d(\mathrm{d}^* n)\}\,, \tag{21}$$

where $\mathrm{d}^* n = {}^*n_1 + {}^*n_2 - {}^*n_3 - {}^*n_4$ is the plaquette constructed from the integers n (see Figure 6). The dual action is defined by eq.(B.3).

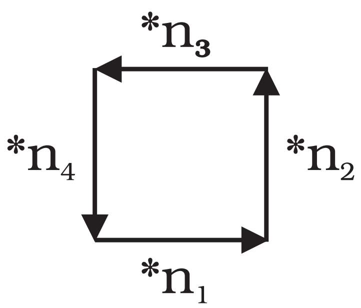


Figure 6. Field strength tensor on the dual lattice.

The integer valued variable $n_{x,\mu}$ is dual to the field $\theta_{x,\mu}$. In the continuum limit, $\theta_{x,\mu} \propto A_\mu(x)$ is the ordinary $U(1)$ gauge field and $n_{x,\mu} \propto B_\mu(x)$ is the dual $U(1)$ gauge field[‡], $\partial_\mu A_\nu - \partial_\nu A_\mu = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\partial_\alpha B_\beta$.

The partition function (21) can be represented as that of the (dual) Abelian Higgs model.

$$\mathcal{Z} = \mathrm{const.} \cdot \lim_{\gamma\to\infty} \lim_{\lambda\to\infty} \int_{-\infty}^{+\infty} \mathcal{D}^* B \int \mathcal{D}^*\Phi \, \exp\{-S^{AH}({}^*B, {}^*\Phi)\}\,, \tag{22}$$

where the action of the (dual) Abelian Higgs model is

$$S^{AH}({}^*B, {}^*\Phi) = \sum_P S^d(\mathrm{d}^* B) + \frac{1}{2}\sum_x \sum_{\mu=1}^{4} \left|{}^*\Phi_x - e^{i^* B_{x,\mu}}\, {}^*\Phi_{x+\hat\mu}\right|^2 + \lambda \sum (|{}^*\Phi_x|^2 - \gamma)^2 \,. \tag{23}$$

Here $\mathrm{d}^* B$ is the plaquette variable constructed from the link variable ${}^*B_l$, where ${}^*B_l$ is the dual gauge field. In the continuum limit, we have $S(\mathrm{d}^* B) \to \beta(\partial_\mu B_\nu - \partial_\nu B_\mu)^2$. The second term in eq.(23) has the following continuum limit: $|\Phi_x - e^{i^* B_{x,\mu}}\, {}^*\Phi_{x+\hat\mu}|^2 \to |(\partial_\mu - iB_\mu)\Phi|^2$. The action S^{AH} is invariant under the following gauge transformations: ${}^*B \to {}^*B - \mathrm{d}^*\alpha$, ${}^*\Phi \to {}^*\Phi e^{i^*\alpha}$.

In the London limit, $\lambda \to \infty$, the mass of the Higgs particle tends to infinity. The radius of the Higgs field $|{}^*\Phi|$ is fixed and only the phase ${}^*\varphi$ of the Higgs field ${}^*\Phi = |{}^*\Phi| e^{i^*\varphi}$ is a physical degree of freedom. In the unitary gauge ${}^*\varphi = 0$, the action of the Abelian Higgs model (23) is

$$S^{AH} = \sum_P S^d(\mathrm{d}^* B) + \frac{\gamma}{2}\sum_x \sum_{\mu=4}^{4} \left|1 - e^{i^* B_{x,\mu}}\right|^2 . \tag{24}$$

‡In the continuum notations we do not use the symbol "*" to denote the dual fields.

In the limit $\gamma \to \infty$, the photon mass $m^2_{ph} = \gamma/\beta$ becomes infinite and the field $B_{x,\mu}$ is equal to zero modulo 2π: ${}^*B_{x,\mu} = 2\pi {}^*n_{x,\mu}$. In this limit, the partition function (24) reduces to

$$\mathcal{Z} = \mathrm{const.} \cdot \sum_{\substack{{}^*n({}^*\mathrm{C}_1)\in \mathbb{Z} \\ \delta {}^*n=0}} e^{-S^d(\mathrm{d}^*n)}\,, \qquad S(\mathrm{d}^*n) = 4\pi^2\,\beta\,(\mathrm{d}^*n)^2\,. \tag{25}$$

Therefore, the compact $U(1)$ gauge theory is equivalent to the dual abelian Higgs model in the double limit

$$\mathrm{Compact}\,U(1)\,\mathrm{gauge\,theory} \;=\; \lim_{\gamma\to\infty}\lim_{\lambda\to\infty}\,(\mathrm{Abelian\,Higgs\,model}) \tag{26}$$

The gauge fields B_l in eq.(23) are dual to the original gauge fields θ_l, and these interact via the covariant derivative with the Higgs field Φ. Therefore, the field Φ carries the magnetic charge, and due to the Higgs potential in eq.(23), these monopoles are condensed at the classical level. It is well known that in the quantum $4D$ compact electrodynamics there exists a confinement–deconfinement phase transition. It can be shown by numerical calculations[12, 13] that in the confinement phase the monopoles are condensed and that in the deconfinement phase the monopoles are not condensed.

What is the Theory Dual to Gluodynamics?

In any abelian projection, lattice gluodynamics corresponds to some abelian gauge theory. This abelian theory, in general, is very complicated and non-local. Nevertheless, in the Maximal Abelian projection, numerical calculations show that the abelian monopoles are important degrees of freedom, and that they are responsible for the confinement. Moreover, as we show in the next section (see also Refs.[14, 15]), the distribution of monopole currents indicates that, at large distances, gluodynamics is equivalent to the dual Abelian Higgs model, the Higgs particles are abelian monopoles and these are condensed in the confinement phase.

NUMERICAL FACTS IN $4D$ $SU(2)$ GLUODYNAMICS

The standard scheme of numerical calculations can be described as follows.

1. Generate the lattice Yang–Mills fields U_l using the standard Monte-Carlo method. Thus we obtain the configurations of fields distributed according to the Boltzmann factor $e^{-S^{YM}}$;

2. Perform the abelian gauge fixing and extract abelian gauge fields from the non-abelian ones. In the case of the MaA projection, the abelian gauge fixing is a rather time-consuming problem.

3. Extract abelian monopole currents from the abelian gauge fields. As mentioned above, the monopole currents form closed paths on the dual lattice. In Figure 7 we show the abelian monopole currents for the confinement (a) and the deconfinement (b) phases. It is seen that in the confinement phase the monopoles form a dense cluster, and there is a number of small mutually disjoint clusters. In the deconfinement phase the monopole currents are dilute.

4. Calculate expectation values of various operators using the monopole currents. Below we discuss a number of numerical facts which show that abelian monopoles in the MaA gauge are the appropriate degrees of freedom to describe confinement.

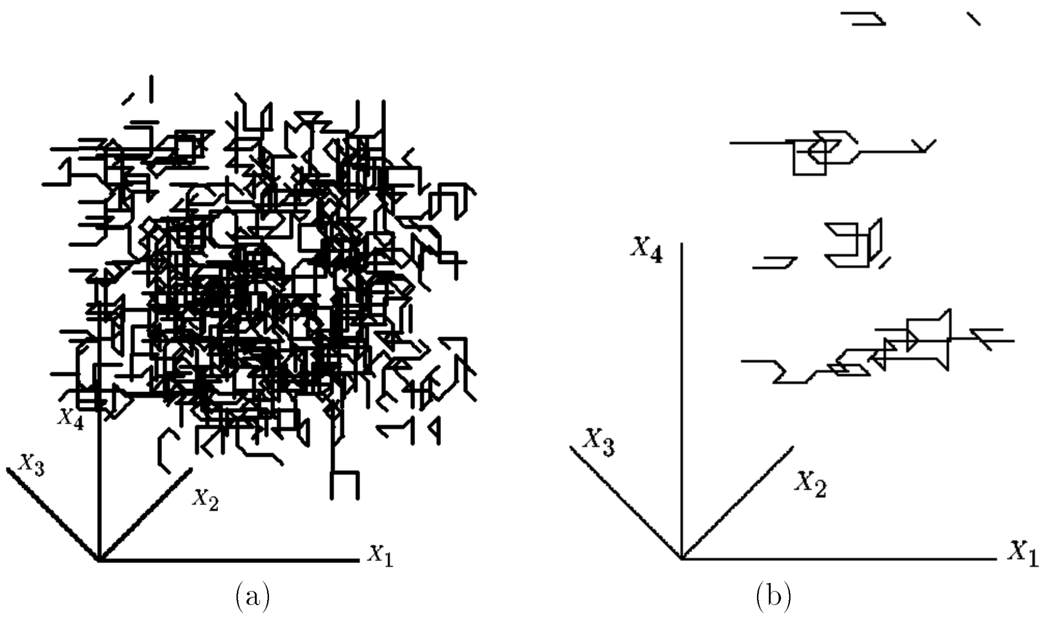


Figure 7. The abelian monopole currents for the confinement (a) ($\beta = 2.4$, 10^4 lattice) and the deconfinement (b) phases ($\beta = 2.8$, $12^3 \cdot 4$ lattice).

Fact 1: Abelian and Monopole Dominance

The notion of the "abelian dominance" introduced in Ref.[16] means that the expectation value of a physical quantity $< \mathcal{X} >$ in the nonabelian theory coincides with (or is very close to) the expectation value of the corresponding abelian operator in the abelian theory obtained by abelian projection. Monopole dominance means that the same quantity can be calculated in terms of the monopole currents extracted from the abelian fields. If we have N configurations of nonabelian fields on the lattice, the abelian and monopole dominance means that

$$\frac{1}{N} \sum_{conf} \mathcal{X}(\hat{U}_{nonabelian}) = \frac{1}{N} \sum_{conf} \mathcal{X}'(U_{abelian}) = \frac{1}{N} \sum_{conf} \mathcal{X}''(j)\,. \qquad (27)$$

Here each sum is taken over all configurations; $U_{abelian} = e^{i\theta_l}$ is the abelian part of the nonabelian field $\hat{U}_{nonabelian}$, j is the monopole current extracted from $U_{abelian}$. It is clear that $\frac{1}{N}\sum_{conf} \mathcal{X}(\hat{U}_{nonabelian})$ is a gauge invariant quantity, while the abelian and the monopole contributions depend on the type of abelian projection. In numerical calculations the equalities (27) can be satisfied only approximately.

Among the well-studied problems is that of the abelian and the monopole dominance for the string tension [16, 17, 18, 19]. In this case, $\mathcal{X}(\hat{U}_{nonabelian}) = \sigma_{SU(2)}$, $\mathcal{X}(U_{abelian}) = \sigma_{U(1)}$ and the string tension $\sigma_{SU(2)}$ ($\sigma_{U(1)}$) is calculated by means of the nonabelian (abelian) Wilson loops, $Tr \prod_{l\in C} \hat{U}_l$ ($\prod_{l\in C} e^{i\theta_l}$). An accurate numerical study of the MA projection of $SU(2)$ gluodynamics on the 32^4 lattice at $\beta = 2.5115$ is performed in Ref.[19].

The corresponding string tension can be taken from the potential between a heavy quark and antiquark: $V(r) = V_0 + \sigma r - \frac{e}{r}$, where r is the distance between the quark and antiquark. The abelian and the nonabelian potentials are shown in Figure 8. The contribution of the photon and the monopole parts to the abelian potential is shown in Figure 9.

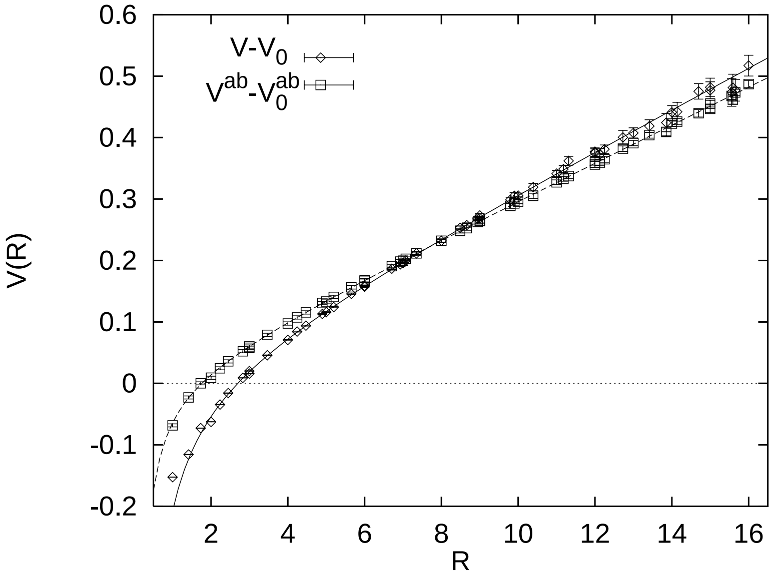


Figure 8. Abelian and nonabelian potentials (with the self energy V_0 subtracted), Ref.[19].

The differences in the slopes of the linear part of the potentials in Figure 8 and Figure 9 yield the following relations: $\sigma_{U(1)} \approx 92\% \, \sigma_{SU(2)}$, $\sigma_j \approx 95\% \, \sigma_{U(1)}$, where σ_j is the monopole current contribution to the string tension. It is important to study a widely discussed idea that in the continuum limit ($\beta \to \infty$), the abelian and the monopole dominance is exact (27): $\sigma_{SU(2)} = \sigma_{U(1)} = \sigma_j$.

There are many examples of abelian and monopole dominance in the MA projection. The monopole dominance for the string tension has been found for the $SU(2)$ positive plaquette model in which Z_2 monopoles are suppressed[20], and also for the $SU(2)$ string tension at finite temperature[21], and for the string tension in $SU(3)$ gluodynamics[22]. Abelian and monopole dominance for $SU(2)$ gluodynamics has been found in[23, 24] for the Polyakov line and for critical exponents for the Polyakov line, for the value of the quark condensate, for the topological susceptibility and also for the hadron masses in quenched $SU(3)$ QCD with Wilson fermions[25].

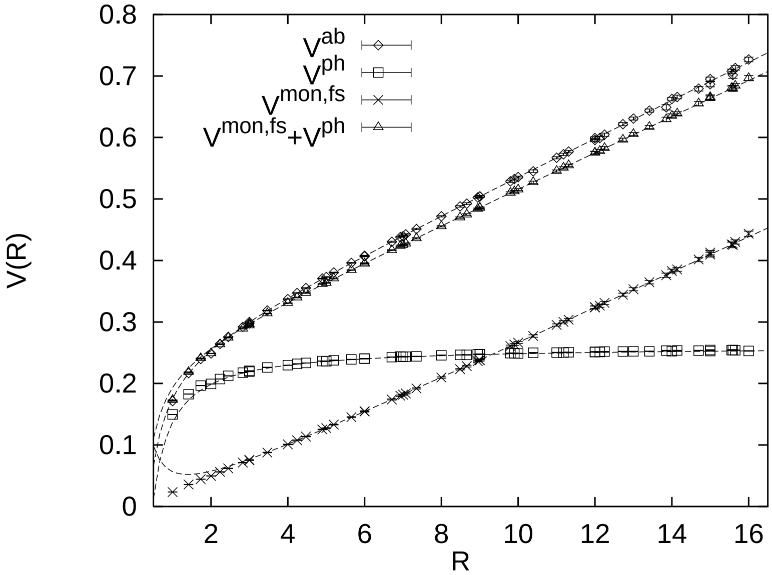


Figure 9. The abelian potential (diamonds) in comparison with the photon contribution (squares), the monopole contribution (crosses) and the sum of these two parts (triangles), Ref.[19].

Fact 2: London Equation for Monopole Currents

In the ordinary superconductor the current of the Cooper pairs satisfies the London equation. If the vacuum of gluodynamics behaves as a dual superconductor, then it is natural to assume that the abelian monopole currents should satisfy the dual London equation in the presence of the dual string

$$\vec{E} - \delta^2 \,\mathrm{curl}\, \vec{j}_m = \Phi_m \delta(x_x)\delta(x_y)\vec{n}_z \,, \tag{28}$$

where $\vec{E}$ is the abelian electric field, $\delta^{-1} = m_{ph}$ is the dual photon mass, $\Phi_m = \frac{2\pi}{g_m}$ is the electric flux of the dual string and g_m is the magnetic charge of the monopole. The string is placed along the oz axis, the vector $\vec{n}_z$ is parallel to the direction of the string and we assume for simplicity that the core of the string is a delta–function.

Indeed, as shown numerically in Ref.[26], the dual London equation is satisfied in the MaA projection of $SU(2)$ gluodynamics. A recent detailed investigation[27, 9] of the electric field profiles and the distribution of monopole currents around the string shows that the structure of the chromo-electric string in the MaA projection is very similar to that of the Abrikosov string in the superconductor.

The following physical question is relevant: what kind of superconductor do we have. If $m_{ph} < m_{mon}$, then the superconductor is of the second type, the Abrikosov vortices are attracted to each other. If $m_{ph} > m_{mon}$, then the superconductor is of the first type and the vortices repel each other. The computation[26] of the dual photon mass from the dual London equation shows that $m_{ph} \approx m_{mon}$. This means that the vacuum of gluodynamics is close to the border between type-I and type-II dual superconductors. The same conclusion is also obtained in Ref.[28]. A recent detailed study of the abelian flux tube[9] shows that the dual photon mass is definitely smaller then the mass of the monopole, and therefore the vacuum of $SU(2)$ is the type-II dual superconductor (see also Ref.[14]).

Fact 3: Monopole Condensate

If the vacuum of $SU(2)$ gluodynamics in the abelian projection is similar to the dual superconductor, then the value of the monopole condensate should depend on the temperature as a disorder parameter: at low temperatures it should be nonzero, and it should vanish above the deconfinement phase transition.

The behaviour of the monopole condensate can be studied with the help of the monopole creation operator. In gluodynamics this operator can be derived by means of the Fröhlich and Marchetti construction [29] for the compact electrodynamics. For $SU(2)$ lattice gluodynamics in the MA projection, it is convenient to study the effective constraint potential for the monopole creation operator (similar calculations were performed for compact electrodynamics in Ref.[13]):

$$V_{eff}(\Phi) = -\ln(< \delta(\Phi - \frac{1}{V}\sum_x \Phi_{mon}(x)) >) \tag{29}$$

where φ is the monopole field. If this potential has a Higgs-like form

$$V_{eff}(\Phi) \propto \lambda(|\Phi|^2 - \Phi_c^2)^2 \,, \qquad \lambda, \Phi_c^2 > 0 \,, \tag{30}$$

then a monopole condensate exists. The dual superconductor picture predicts this behaviour of the effective constraint potential in the confinement phase. In the deconfinement phase the following form of the potential is expected:

$$V_{eff}(\Phi) \propto m^2|\Phi|^2 + \lambda|\Phi|^4 + \dots \,, \qquad m^2, \lambda > 0 \,; \tag{31}$$

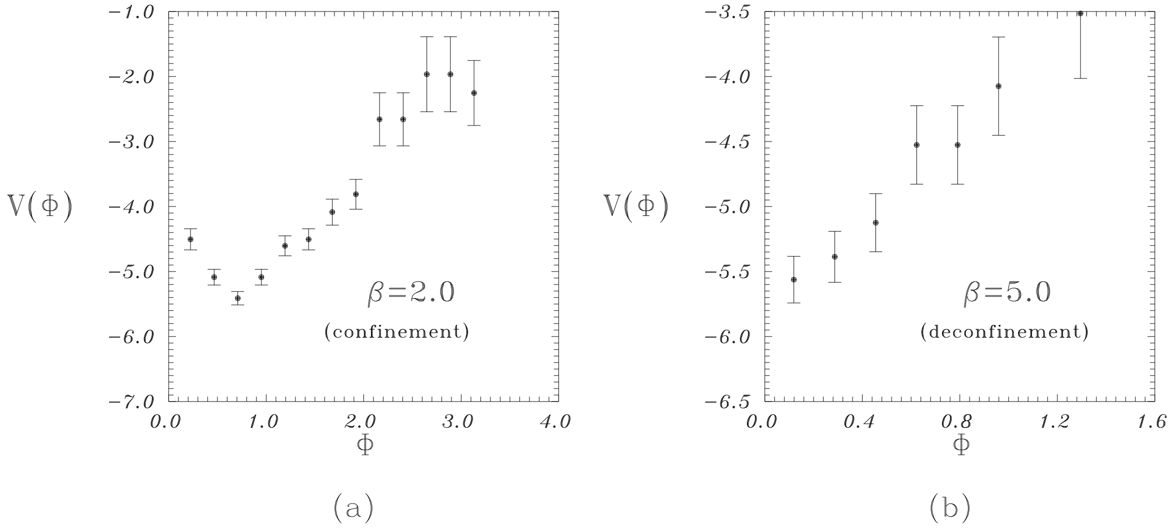


Figure 10. $V(\Phi)$ for the confinement (a) and the deconfinement (b) phases, Ref.[31].

(no monopole condensate).

The numerical calculation of the effective constraint potential (29) is very time-consuming, and therefore, in Refs.[30, 31] the following quantity $V(\Phi)$ has been studied:

$$V(\Phi) = -\ln(< \delta(\Phi - \Phi_{mon}(x)) >) \,. \tag{32}$$

Numerical calculations of this quantity were performed on lattices of size $4 \cdot L^3$, for $L = 8, 10, 12, 14, 16$ with anti–periodic boundary conditions in space directions for the abelian fields. Periodic boundary conditions are forbidden, since a *single* magnetic charge cannot physically exist in a closed volume with periodic boundary conditions due to the Gauss law. The results on finite lattices have been extrapolated to the infinite volume, since near the phase transition finite volume effects are very strong.

In Figures 10 (a,b) the (right-hand side of the) effective potential (32) is shown for the confinement and the deconfinement phases, the calculations being performed on a $4 \cdot 12^3$ lattice.

In the confinement phase (Figure 10 (a)), the minimum of $V(\Phi)$ is shifted from zero, while in the deconfinement phase, the minimum is at the zero value of the monopole field Φ. The value Φ_c of the monopole field at which the potential has a minimum is equal to the value of the monopole condensate.

The potential shown in Figure 10 (b) corresponds to a trivial potential with a minimum at a zero value of the field: $\Phi_c = 0$. The dependence of Φ_c, the minimum of the potential, on the spatial size L of the lattice is shown in Figure 11(a).

We fit the data for Φ_c by the formula $\Phi_c = AL^\alpha + \Phi_c^{inf}$, where A, α and Φ_c^{inf} are the fitting parameters. It occurs that $\alpha = -1$ within statistical errors. Figure 11(b) shows the dependence on β of the value of the monopole condensate extrapolated to the infinite spatial volume Φ_c^{inf}. It is clearly seen that Φ_c^{inf} vanishes at the point of the phase transition and it plays the role of the order parameter.

The monopole creation operator[32] in the monopole current representation is studied in Ref.[33]. First the monopole action is reconstructed from the monopole currents in the MA projection, and after that the expectation value of the monopole creation operator is calculated in the quantum theory of monopole currents. Again, the monopole creation operator depends on the temperature as the disorder parameter. A slightly different monopole creation operator was studied in Refs.[34, 35].

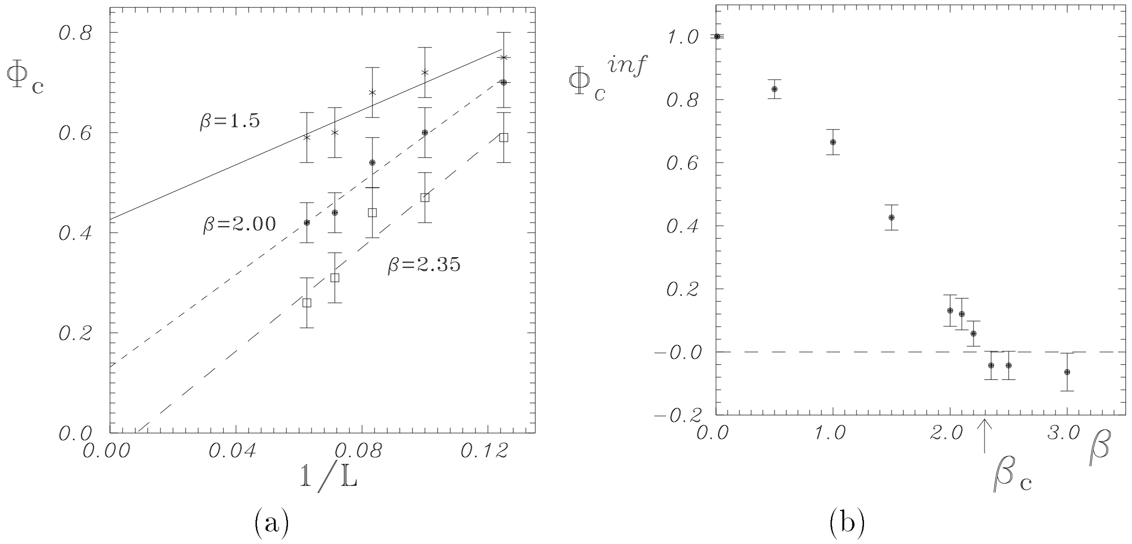


Figure 11. The dependence of (a) Φ_c on the spatial size of the lattice for three values of β and (b) of Φ_c^{inf} on β, Ref.[31].

Abelian Monopole Action: Analytical examples

In the next section we explain how to study the action of the monopoles extracted from $SU(2)$ fields in the MaA projection. Now we give several examples of monopole actions corresponding to abelian gauge theories.

Example 1: $4D$ compact abelian electrodynamics

We have already discussed the duality transformation of $4D$ compact electrodynamics. There is another exact transformation which represents the partition function as a sum over monopole currents. This transformation was initially performed by Berezinsky [36] and by Kosterlitz and Thouless[37] for the $2D$ XY model. Accordingly, we call it the BKT transformation. For compact electrodynamics with the Villain action this transformation was found by Banks, Myerson and Kogut[38]. The partition function for compact electrodynamics with the Villain action is

$$\mathcal{Z}^{Villain} = \int\limits_{-\pi}^{+\pi} \mathcal{D}\theta \sum_{m(\mathrm{C}_2)\in \mathbb{Z}} \exp\Bigl\{-||\mathrm{d}\theta + 2\pi m||^2\Bigr\}\,. \tag{33}$$

As explained in Appendix C, the BKT transformation of this partition function has the form

$$\mathcal{Z} = [\int\limits_{-\infty}^{+\infty} \mathcal{D}A \exp\{-\beta||\mathrm{d}A||^2\}] \times [\sum_{\substack{^*j(^*\mathrm{C}_1)\in \mathbb{Z} \\ \delta^*j=0}} \exp\{-4\pi^2\beta\,(^*j, \Delta^{-1\,*}j)\}]\,. \tag{34}$$

Here the partition function for the non–compact gauge field A is inessential because it is Gaussian. All the dynamics is in the partition function for the monopole currents $^*j$ which are lying on the dual lattice and form closed loops ($\delta^*j = 0$).

It is possible to find the BKT transformation for compact electrodynamics with a general form for the action[31]. In this general case the monopole action is non-local (see Appendix C).

Example 2: Abelian Higgs theory in the London limit

The partition function for the Abelian Higgs model is (cf., eqs.(22,23[§])):

$$\mathcal{Z}^{AH} = \int\limits_{-\infty}^{+\infty} \mathcal{D}B \int\limits_{-\pi}^{+\pi} \mathcal{D}\varphi \sum_{m(C_2)\in \mathbb{Z}} \exp\Bigl\{-\beta ||\mathrm{d}B||^2 - \gamma ||\mathrm{d}\varphi + B + 2\pi m||^2\Bigr\}\,, \qquad (35)$$

where B is the non-compact gauge field and φ is the phase of the Higgs field (the radial part of the Higgs field is frozen, since we consider the London limit).

After the BKT transformation [39, 40], the partition function takes the form

$$\mathcal{Z}^{AH} = \mathrm{const.} \cdot \sum_{\substack{j(C_1)\in \mathbb{Z} \\ \delta j = 0}} \exp\Bigl\{-S^{AH}_{mon}(j)\Bigr\}\,, \quad S^{AH}_{mon}(j) = \frac{1}{4\beta}(j, \Delta^{-1} j) + \frac{1}{4\gamma}||j||^2\,, \qquad (36)$$

where Δ^{-1} is the inverse lattice Laplacian and j is the current of the Higgs particles.

Example 3: Abelian Higgs theory near the London limit

As pointed out by T. Suzuki[14], the numerical data show that in the MaA projection the currents in lattice $SU(2)$ gluodynamics behave as the currents of the Higgs particles in the Abelian Higgs model near the London limit. This means that the coefficient λ of the Higgs potential $\lambda(|\Phi|^2 - 1)^2$, eq.(23), is large but finite. In this case it is impossible to get an explicit expression for the monopole action, but there exists a $1/\lambda$ expansion of this action

$$\begin{aligned} S_{mon}({}^*j) = & \frac{1}{4\beta}({}^*j, \Delta^{-1} {}^*j) + \Bigl(\frac{1}{4\gamma} + a_1\Bigr) ||{}^*j||^2 + a_2 \sum_x \Bigl(\sum_{\mu=-4}^{4} {}^*j^2_{x,\mu}\Bigr)^2 \\ & + a_3 \sum_x \sum_{\mu=1}^{4} {}^*j^4_{x,\mu} + \dots\,, \end{aligned} \qquad (37)$$

where

$$a_k = \sum_{n=1}^{\infty} \frac{f_k^{(n)}}{\lambda^n}\,,$$

$f_k^{(n)}$ are the coefficients which can be calculated[14]. In the limit $\lambda \to +\infty$, this action is reduced to $S^{AH}_{mon}({}^*j)$, eq.(36).

Monopole Action from "Inverse Monte–Carlo"

The $1/\lambda$ expansion of the monopole action has an important application in lattice gluodynamics. It is possible to show that the monopole currents in the MaA projection of $SU(2)$ gluodynamics are in a sense equivalent to the currents generated by the theory with the action (37). This means that (at least at large distances) gluodynamics is equivalent to the Abelian Higgs model. The details can be found in the lecture of T. Suzuki [14]. Below we briefly describe some of these results.

It occurs that from a given distribution of currents on the $4D$ lattice it is possible to find the action of the currents. This can be done by the Swendsen ("inverse Monte–Carlo") [41, 42] method. By the usual Monte–Carlo method we get an ensemble of the

§We assume the Villain form for the interaction of the gauge and Higgs fields.

fields $\{\Phi\}$ with the probability distribution $e^{-S(\Phi)}$. The "inverse Monte–Carlo" allows us to reconstruct the action $S(\Phi)$ from the distribution of the fields $\{\Phi\}$.

The procedure of the reconstruction of the monopole action for lattice gluodynamics is the following. First the $SU(2)$ gauge fields are generated by the usual Monte–Carlo method. Then, the MaA gauge fixing is performed and the abelian monopole currents are extracted from the abelian gauge fields. Finally, the inverse Monte–Carlo method is applied to the ensemble of these currents and the coefficients $\beta, \gamma, a_1, a_2, a_3, \dots$ of the action (37) are determined. It occurs that at large distances the distribution of the currents is well described by the action (37). In that sense, the effective action of gluodynamics is the Abelian Higgs model, the monopoles play the role of the Higgs particles. In contrast to compact electrodynamics, the monopole potential is not infinitely deep, see Figure 12(a,b).

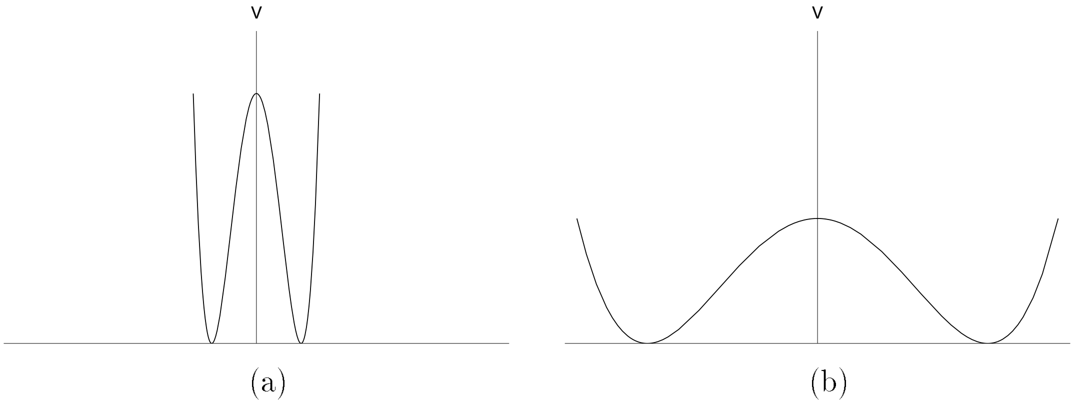


Figure 12. Schematic representation of the monopole potential $\lambda(|\Phi|^2 - 1)^2$ for compact electrodynamics (a), $\lambda \to +\infty$; and for lattice $SU(2)$ gluodynamics (b), λ is finite.

Various Representations for the Partition Function

There are several equivalent representations of the partition function:

$$\mathcal{Z}^j = \sum_{\delta j=0} \exp\{-S_{mon}(j)\}\,, \tag{38}$$

where S_{mon} is given by (37).

Representation 1 as the dual abelian Higgs model:

As we have already said, the model (38) is related to the (dual) Abelian Higgs theory by the inverse BKT transformation

$$\mathcal{Z}^j \propto \mathcal{Z}^{AH} = \int \mathcal{D}B\, \mathcal{D}\Phi \exp\{-S^{AH}(B,\Phi)\}\,, \tag{39}$$

where S^{AH} is given by eq.(23).

Representation 2 in terms of the Nielsen-Olesen strings:

Using the BKT transformation for the partition function (39), one can show that model (38) is equivalent[43] to the string theory:

$$\mathcal{Z}^j \propto \mathcal{Z}^{str} = \sum_{\substack{\sigma(C_2)\in \mathbb{Z} \\ \delta\sigma=0}} e^{-S^{AH}_{string}(\sigma)} \tag{40}$$

$$S^{AH}_{string}(\sigma) \,=\, 4\pi\gamma(\sigma,(\Delta + m^2)^{-1}\sigma) + O(\frac{1}{\lambda})\,, \quad m^2 = \frac{\gamma}{\beta}\,. \tag{41}$$

Here σ is the closed world sheet of the Nielsen-Olesen string. In the continuum limit[44] we have

$$S^{AH}_{string} \propto \int d\sigma_{\mu\nu}(\tilde{x})\mathcal{D}^{(4)}_m(\tilde{x} - \tilde{x}')d\sigma_{\mu\nu}(\tilde{x}')\,, \tag{42}$$

where $\mathcal{D}^{(4)}_m$ is the free scalar propagator, $(\Delta + m^2)\mathcal{D}^{(4)}_m(x) = \delta^{(4)}(x)$.

Representation 3 in terms of the gauge field θ:

Using the inverse dual transformation for the partition function (39), one can get the representation of the monopole partition function in terms of the *compact* gauge field θ which is dual to the gauge field B, (23), (39):

$$\mathcal{Z}^j \propto \mathcal{Z}^\theta = \int\limits_{-\pi}^{+\pi} \sum_{n(\mathrm{C}_2)\in \mathbb{Z}} \mathcal{D}\theta e^{-S^o(\theta,n)}\,; \tag{43}$$

$$S^o = \frac{1}{16\pi^2\beta}||\mathrm{d}\theta + 2\pi n||^2 - \frac{1}{16\pi^2\gamma}\,(\mathrm{d}\theta + 2\pi n, \Delta(\mathrm{d}\theta + 2\pi n)) + O(\frac{1}{\lambda})\,. \tag{44}$$

Representation 4 in terms of hypergauge fields:

Applying the duality transformation to the partition function (43), we get the representation in terms of the hypergauge (Kalb–Ramond) field

$$\mathcal{Z}^j \propto \mathcal{Z}^{HG} = \sum_{h(\mathrm{C}_2)\in \mathbb{Z}} \int \mathcal{D}A \exp\{-S(A,h)\}$$

$$S(A,h) \,=\, \frac{1}{4\gamma}\|dh\|^2 + \frac{1}{4\beta}\|dA + h\|^2 + O(\frac{1}{\lambda})\,, \tag{45}$$

where h is the hypergauge field $(h_{\mu\nu}(x))$ interacting with the gauge field A $(A_\mu(x))$; the field h is dual to the monopole field Φ, and the field A is dual to the field B.

The action $S(A,h)$ is invariant under gauge transformations: $A \to A + \mathrm{d}\alpha$, $h \to h$, and under *hypergauge* transformations $A \to A - \gamma$, $h \to h + \mathrm{d}\gamma$.

Representation 5 by fourier transformation of the string world sheets:

This representation is intermediate between the representations (40) and (39):

$$\mathcal{Z} \propto \mathcal{Z}^h = \int \mathcal{D}\xi \int \mathcal{D}\zeta \sum_{\sigma(\mathrm{C}_2)\in \mathbb{Z}} \exp\left\{2\pi i(\xi,\sigma) \,-\, \tilde{S}[\xi,\zeta] + O(\frac{1}{\lambda})\right\}\,, \tag{46}$$

$$S[\xi,\zeta] \,=\, \frac{1}{4\beta}||\xi + \mathrm{d}\zeta||^2 + \frac{1}{4\gamma}||\mathrm{d}\xi||^2\,, \tag{47}$$

where σ are the world sheets of the Abrikosov-Nielsen-Olesen strings. Again, there exists a hypergauge symmetry ($\xi \to \xi + \mathrm{d}\gamma$, $\zeta \to \zeta - \gamma$), which reflects the fact that the string worlds sheets are closed (or equivalently, conservation of the magnetic flux).

It is possible to derive a lot of physical consequences from these representations[14]. One of the most interesting is that the classical string tension which is calculated from the string action (41) coincides, within statistical errors, with the string tension in $SU(2)$ lattice gauge theory. The coefficients β and γ in eq.(41) are found by the inverse Monte–Carlo method, as we have just explained.

Abelian Monopoles as Physical Objects

The abelian monopoles arise in the continuum theory[10] from singular gauge transformation (8)-(12) and it is not clear whether these monopoles are "real" objects. A physical object is something which carries action and below we only discuss the question if there are any correlations between abelian monopole currents and the $SU(2)$ action. In Ref.[42] it was found that the total action of the $SU(2)$ fields is correlated with the total length of the monopole currents, so there exists a global correlation. We now discuss the local correlations between the action density and the monopole currents[45].

In lattice calculations, the monopole current $j_\mu(x)$ lies on the dual lattice and it is natural to consider the correlator of the current and the dual action density¶:

$$C_1 = < \frac{1}{2} Tr \left(j_\mu(x) \, \frac{1}{2} \varepsilon_{\nu\mu\alpha\beta} F_{\alpha\beta}(x) \right)^2 > - < j_\mu^2(x) >< \frac{1}{2} Tr F_{\alpha\beta}^2(x) > . \tag{48}$$

The density of $j_\mu(x)$ strongly depends on $\beta = \frac{4}{g^2}$, therefore it is convenient to normalize C_1:

$$C_2 = \frac{< \frac{1}{2}\mathrm{Tr} \left(j_\mu(x) \, \frac{1}{2} \varepsilon_{\nu\mu\alpha\beta} F_{\alpha\beta}(x) \right)^2 >}{< j_\mu^2(x) >< \frac{1}{2}\mathrm{Tr} F_{\alpha\beta}^2(x) >} - 1 \, . \tag{49}$$

For the static monopole we have $j_0(x) \neq 0$, $j_i(x) = 0$, $i = 1, 2, 3$, and the fact that $C_2 \neq 0$ means that the magnetic part S_m of the $SU(2)$ action is correlated with $j_\mu(x)$.

The correlator C_2 is related to the relative excess of the action carried by the monopole current. The expectation value of S_m on the monopole current is

$$S_m = < \frac{1}{24} \mathrm{Tr} \left(n_\mu(x) \, \frac{1}{2} \varepsilon_{\nu\mu\alpha\beta} F_{\alpha\beta}(x) \right)^2 > , \tag{50}$$

where $n_\mu(x) = j_\mu(x)/|j_\mu(x)|$. The relative excess of the action on the monopole current is

$$\eta = \frac{S_m - S}{S} , \quad S = < \frac{1}{24} Tr F_{\alpha\beta}^2(x) > . \tag{51}$$

where $S = < \frac{1}{24} Tr F_{\alpha\beta}^2(x) >$ is the expectation value of the action, the coefficient $\frac{1}{24}$ corresponds to the lattice definition of the "plaquette action" $< S >$. If $j_\mu(x) = 0, \pm 1$, then $\eta = C_2$. Since in lattice calculations at sufficiently large values of β the probability of $j_\mu(x) = \pm 2$ is small in the MaA projection, we have $\eta \approx C_2$ at large values of β. From numerical calculations we have found that $\eta = C_2$ with 5% accuracy for $\beta > 1.5$ on lattices of sizes 10^4 and $12^3 \cdot 4$. The lattice definition of S_m is

$$S_m = < \sum_{\nu=1}^{4} n_\nu^2(x) \cdot \frac{1}{6} \sum_{P \in \partial C_\nu(x)} \left(1 - \frac{1}{2} Tr \, U_P \right) > , \tag{52}$$

where the summation is over the plaquettes P which are the faces of the cube $C_\nu(x)$; the cube $C_\nu(x)$ is dual to $n_\nu(x)$; U_P is the plaquette matrix. Thus, S_m is the average action on the plaquettes closest to the magnetic current (see Figure 5(b)). The lattice definition of S is standard: $S = < \left(1 - \frac{1}{2} Tr \, U_P \right) >$. We use these definitions of S and S_m in our lattice calculations.

It is interesting to study the quantity (51) in various gauges. In Figure 13(a) the relative excess of the magnetic action density near the monopole current η is shown for

¶Here and below, formulae correspond to the continuum limit implied by the lattice regularization.

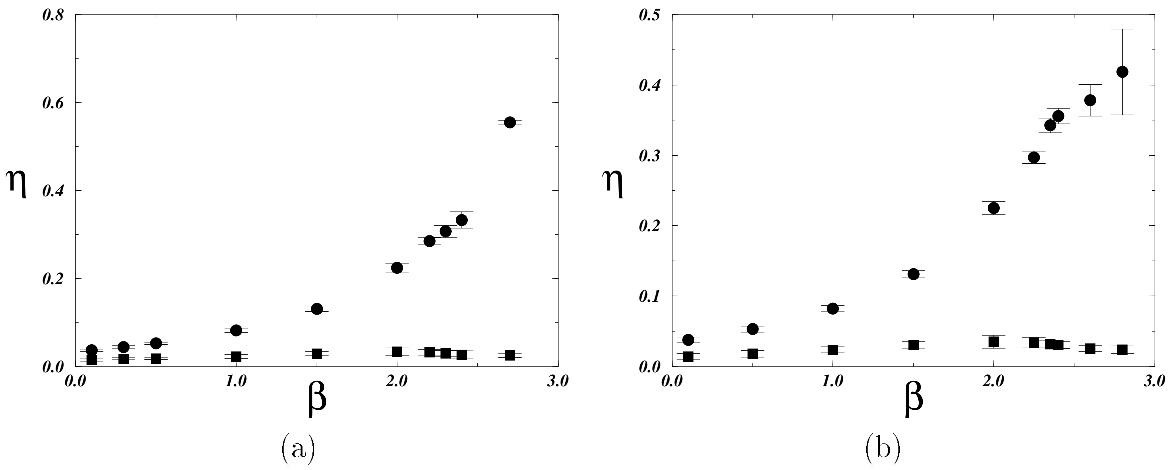


Figure 13. The relative excess η, eq.(51), for 10^4 lattice (a) and for $12^3 \cdot 4$ lattice (b) in the MA (circles) and the Polyakov abelian (squares) gauges, Ref.[45].

a 10^4 lattice. Circles correspond to the MaA projection, and squares to the Polyakov gauge. The data for the F_{12} projection coincide, within statistical errors, with those for the Polyakov gauge.

The quantity (51) at finite temperatures has also been studied. In Figure 13(b) the relative excess of the magnetic action density near the monopole current η is shown for $12^3 \cdot 4$ lattice. Again, the circles correspond to the MaA projection and the squares to the Polyakov gauge.

Thus we have shown that in the MaA projection the *abelian* monopole currents are surrounded by regions with a high *nonabelian* action. This fact presumably means that the monopoles in the MaA projection are physical objects. It does not mean that they have to be real objects in the Minkowsky space. What we have found is that these currents carry $SU(2)$ action in Euclidean space. It is important to understand what is the general class of configurations of $SU(2)$ fields which generate the monopole currents. Some specific examples are known. These are instantons[46] and the BPS–monopoles (periodic instantons)[40].

Monopoles are Dyons

A dyon is an object which has both electric and magnetic charge. In the field of a single instanton the monopole currents in the MaA projection are accompanied by electric currents[47]. The qualitative explanation of this fact is simple. Consider the (anti)self-dual configuration

$$F_{\mu\nu}(A) = \pm {}^*F_{\mu\nu}(A) \; . \tag{53}$$

The MaA projection is defined[11] by the minimization of the off-diagonal components of the non-abelian gauge field, so that in the MaA gauge one can expect the abelian component of the commutator term $1/2Tr(\sigma^3[A_\mu, A_\nu]) = \varepsilon^{3ab} A^a_\mu A^b_\nu$ to be small compared with the abelian field-strength $f_{\mu\nu}(A) = \partial_{[\mu} A^3_{\nu]}$. Therefore, in the MaA projection eq.(53) yields

$$f_{\mu\nu}(A) \approx \pm {}^* f_{\mu\nu}(A) \, . \tag{54}$$

Due to eq.(54), the monopole currents have to be correlated with electric ones, since

$$j^e_\mu = \partial_\nu f_{\mu\nu}(A) \approx \pm \partial_\nu {}^* f_{\mu\nu}(A) = j^m_\mu \; . \tag{55}$$

It is known that the (anti-) instantons produce abelian monopole currents[46]. The abelian monopole trajectories may go through the center of the instanton (Figure 14(a)) or form a circle around it (Figure 14(b)).

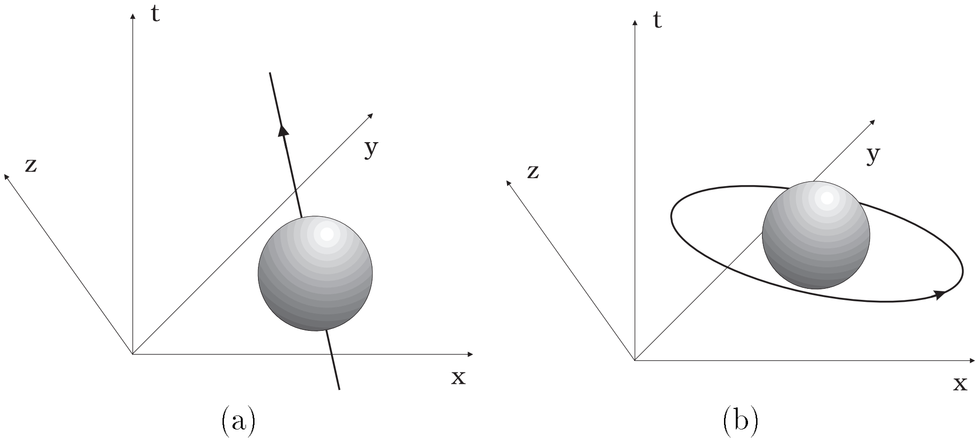


Figure 14. The instantons (spheres) inducing monopoles (thin lines).

Due to eq.(55), the monopole currents are accompanied by electric currents in the self-dual fields. Therefore, the monopoles are dyons for an instanton background. However, the real vacuum is not an ensemble of instantons, and below we discuss the correlation of j^e and j^m in the vacuum of lattice gluodynamics.

There is a lot of vacuum models: the instanton gas, instanton liquid, toron liquid, *etc.* Suppose that the vacuum is a "topological medium", and there are self-dual and anti-self-dual regions (domains). In this vacuum the electric and the magnetic currents should correlate with each other. But the sign of the correlator depends on the topological charge of the domain.

We define the electric current as[47]

$$J^e_\mu(x) = \frac{1}{2\pi}\sum_\nu[\bar\theta_{\mu\nu}(x) - \bar\theta_{\mu\nu}(x-\hat\nu)]. \qquad (56)$$

In the continuum limit, this definition corresponds to the usual one: $J^e_\mu = \partial_\nu f_{\mu\nu}$. The electric currents are conserved ($\partial_\mu J^e_\mu = 0$) and are attached to the links of the original lattice. Electric currents are not quantized.

In order to calculate the correlators of the type $<J^e_\mu(x) J^m_\mu(x)...>$, one has to define the electric current on the dual lattice or the magnetic current on the original lattice. We define the electric current on the dual lattice in the following way:

$$J^e_\mu(y) = \frac{1}{16}\sum_{x\in {}^*C(y,\mu)}\left[J^e_\mu(x) + J^e_\mu(x-\hat\mu)\right]\,. \qquad (57)$$

Here the summation on the r.h.s. is over eight vertices x of the 3-dimensional cube ${}^*C(y,\mu)$ to which the current $J^e_\mu(y)$ is dual. The point y lies on the dual lattice and the points x on the original one.

For the topological charge density operator we use the simplest definition:

$$Q(x) = \frac{1}{2^9\pi^2}\sum_{\mu_1,...,\mu_4=-4}^{4}\varepsilon^{\mu_1,...,\mu_4}Tr[U_{\mu_1,\mu_2}(x)U_{\mu_3,\mu_4}(x)], \qquad (58)$$

where U_{μ_1,μ_2} is the plaquette matrix. On the dual lattice the topological charge density corresponding to the monopole current $J^m_\mu(y)$ is defined by taking the average over the eight sites nearest to the current $J^m_\mu(y)$: $Q(y) = \frac{1}{8}\sum_x Q(x)$.

The simplest (connected) correlator of electric and magnetic currents is given by $< J^m_\mu J^e_\mu >_c \equiv < J^m_\mu J^e_\mu > - < J^m_\mu >< J^e_\mu >=< J^m_\mu J^e_\mu >$, here we used the fact that $< J^m_\mu >=< J^e_\mu >= 0$ due to the Lorentz invariance. The connected correlator $< J^m_\mu J^e_\mu >_c$ is equal to zero, since J^m_μ and J^e_μ have opposite parities. A scalar quantity can be constructed if we multiply $J^m_\mu J^e_\mu$ with the topological charge density. The corresponding connected correlator

$$< J^m_\mu J^e_\mu Q >_c =< J^m_\mu J^e_\mu Q > \tag{59}$$

is nonzero for the vacuum consisting of (anti-)self-dual domains (*cf.* eq. (55)). Note that we derived eq.(59) using the equalities $< J^m_\mu >=< J^e_\mu >=< Q >= 0$.

We consider $SU(2)$ lattice gauge theory on a 8^4 lattice with the Wilson action. We calculate[48] the correlator $<J^m_\mu(y) J^e_\mu(y) Q(y)>$, using 100 statistically independent configurations at each value of β.

This correlator strongly depends on β and it is convenient to normalize it by dividing by $\rho^m \rho^e$. Here ρ^m and ρ^e are the monopole and the electric current densities:

$$\rho_{m(e)} = \frac{1}{4V} \sum_l |J_l^{m(e)}|, \tag{60}$$

V is the lattice volume (the total number of sites).

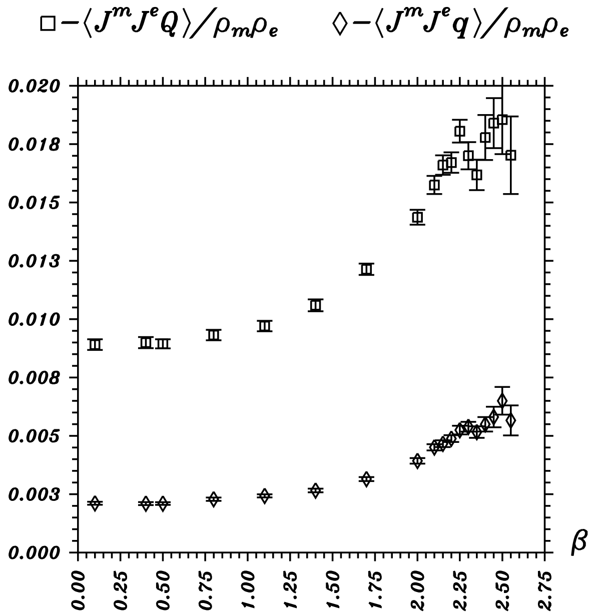


Figure 15. Correlators $<J^m J^e Q>/\rho_m \rho_e$ and $<J^m J^e q>/\rho_m \rho_e$ as functions of β, Ref.[48].

The correlators $<J^m J^e Q>/\rho_m \rho_e$ and $<J^m J^e q>/\rho_m \rho_e$ $\quad (q(y) = Q(y)/|Q(y)|)$ are represented in Figure 15. As one can see from Figure 15, the product of the electric and the magnetic currents is correlated with the topological charge.

Thus our results show that in the vacuum of lattice gluodynamics the magnetic current is correlated with the electric current, the abelian monopoles have an electric charge. The sign of the electric charge depends on the sign of the topological charge density.

CONCLUSIONS

Now we briefly summarize the properties of the abelian monopole currents in the MaA projection of lattice $SU(2)$ gluodynamics:

- Monopoles are responsible for $\approx 90\%$ of the $SU(2)$ string tension.
- Monopole currents satisfy the London equation for a superconductor.
- Monopoles are condensed in the confinement phase.
- The effective monopole Lagrangian is similar to the Lagrangian of the dual Abelian–Higgs model.
- Monopoles carry the $SU(2)$ action.
- Monopoles are dyons.

The main conclusion which can be obtained from these facts is that the vacuum of lattice gluodynamics behaves as a dual superconductor: the monopole currents are condensed and they are responsible for confinement of color.

We have to note that the approach discussed above is not unique and there are several other approaches to the confinement problem in non-abelian gauge theories. We cannot discuss all these here, but mention the description of confinement in terms of Z_2 vortices[49], the description of the QCD vacuum in terms of the non-abelian dual lagrangians[50], and the study of QCD by means of the cumulant expansion[51].

ACKNOWLEDGMENTS

Authors are grateful to E.T. Akhmedov, P. van Baal, F.V. Gubarev, T.L. Ivanenko, Yu.A. Simonov and T. Suzuki for useful discussions. M.N.Ch and M.I.P. acknowledge the kind hospitality of the Theoretical Department of the Kanazawa University. This work has been supported by the JSPS Program on Japan – FSU scientists collaboration, and also by the Grants: INTAS-94-0840, INTAS-94-2851, INTAS-RFBR-95-0681, and Grant No. 96-02-17230a of the Russian Foundation for Fundamental Sciences.

Appendix A: Differential Forms on the Lattice

Here we briefly summarize the main notions from the theory of differential forms on the lattice. The calculus of differential forms was developed for field theories on the lattice in ref.[52]. Since all lattice formulas have a direct continuum interpretation, the standard formalism of differential geometry can have wide applications in lattice theories. The advantage of the calculus of differential forms consists in the general character of the obtained expressions. Most of the transformations depend neither on the space–time dimension nor on the rank of the fields. With minor modifications, the transformations are valid for lattices of any form (triangular, hypercubic, random, etc).

A differential form of rank k on the lattice is a skew-symmetrical function ϕ_k defined on k-dimensional cells C_k of the lattice. The scalar lattice field is a 0–form. The $U(1)$ gauge field is a 1–form. The exterior differential operator d is defined as follows:

$$(\mathrm{d}\phi)(C_{k+1}) = \sum_{\mathrm{C}'_k \in \partial \mathrm{C}_{k+1}} \phi(C'_k) \tag{A.1}$$

Here ∂C_k is the boundary of the k-cell C_k. Thus, the operator d increases the rank of the form by one. For instance, in Wilson's $U(1)$ theory with the dynamic variables $U = e^{i\theta}$, the action $\cos(\mathrm{d}\theta)$ depends on the plaquette $\mathrm{d}\theta(C_2)$ constructed from the links $\theta(C_1)$.

The dual lattice is defined as follows. The sites of the dual lattice are placed at the centers of the D–dimensional cells of the original lattice. The object dual to an oriented k-cell (C_k) is an oriented $(D-k)$–cell $({}^*C_k)$, which lies on the dual lattice and has an intersection with C_k. The dual of the cubic lattice is also a cubic lattice obtained by shifting the lattice along each axis by $1/2$ of the lattice spacing. Every k-form on the lattice corresponds to a $(D-k)$–form on the dual lattice: ${}^*\phi({}^*C_k) = \phi(C_k)$. The co-derivative is defined as follows:

$$\delta = {}^*\mathrm{d}^*. \tag{A.2}$$

This operator decreases the rank of the form

$$(\delta\phi)(C_{k-1}) = \sum_{all\ \mathrm{C}_k:\mathrm{C}_{k-1}\in\partial\mathrm{C}_k} \phi(C_k). \tag{A.3}$$

The square of d and δ is equal to zero:

$$\begin{aligned} \mathrm{d}^2 &= 0, \\ \delta^2 &= 0. \end{aligned} \tag{A.4}$$

The first equality is a consequence of the well-known geometrical fact: "the boundary of the boundary is the empty set"; the second equality follows from the first one and the definition of δ (A.2).

Now we discuss the lattice version of the Laplace operator, which is defined as

$$\Delta = \delta\mathrm{d} + \mathrm{d}\delta. \tag{A.5}$$

This operator acts on 0–forms (scalar fields) in the same way as the usual finite-difference version of the continuous Laplacian. For the forms of non-zero rank the relation (A.5) is the generalization of the usual Laplacian. The obvious properties of δ are

$$\begin{aligned} \delta\Delta &= \Delta\delta \\ \mathrm{d}\Delta &= \Delta\mathrm{d} \\ 1 &= \delta\Delta^{-1}\mathrm{d} + \mathrm{d}\Delta^{-1}\delta \end{aligned} \tag{A.6}$$

The last relation, called the Hodge identity, implies the widely used decomposition formula for an arbitrary k–form:

$$\phi = \delta\Delta^{-1}(\mathrm{d}\phi) + \mathrm{d}\Delta^{-1}(\delta\phi) + \phi_h, \tag{A.7}$$

where ϕ_h is a harmonic form, $\delta\phi_h = \mathrm{d}\phi_h = 0$. The number of the harmonic forms depends on the topology of the space-time; for instance, the number of the harmonic 1–forms is equal to 0,1 and D respectively, for the space–time topology R^D, $R^{D-1}\otimes S^1$ and S^D.

For two forms of the same rank the scalar product is introduced in a natural way:

$$(\phi,\psi) = \sum_{all\ \mathrm{C}_k} \phi(C_k)\psi(C_k) \tag{A.8}$$

This scalar product agrees with the definitions of the d and δ operators in the sense that the following formula of "integration by parts" is valid:

$$(\mathrm{d}\phi, \psi) = (\phi, \delta\psi). \tag{A.9}$$

The norm of a k-form is defined as usual by

$$\|\phi\|^2 = \sum_{\mathrm{C}_k} \phi(C_k)^2 = (\phi, \phi) \tag{A.10}$$

We illustrate the above definitions by four simple examples.

Example 1. Let us show that the general action for the compact gauge field is a periodic function of the angle $\bar{\theta}_P$ corresponding to the plaquette. By definition, $\bar{\theta}_P = \mathrm{d}\theta + 2\pi k$, where the integer valued 2–form k is defined in such a way that $-\pi < \bar{\theta}_P \leq \pi$. Under a gauge transformation we have $\theta' = \theta + \mathrm{d}\chi$ and $\bar{\theta}'_P = \mathrm{d}\theta + \mathrm{d}^2\chi + 2\pi k' = \mathrm{d}\theta + 2\pi k'$, where k' is an integer chosen in such a way that $-\pi < \theta' \leq \pi$. Therefore, the gauge invariance requires the periodicity condition on the action: $S(\bar{\theta}_P) = S(\bar{\theta}_P + 2\pi k)$.

Example 2. Note, that the definition of the norm (A.10) allows us to write in concise form the action of the lattice theory, e.g., $S = \sum_{\mathrm{C}_2}[\mathrm{d}(A(C_1))]^2 = \|\mathrm{d}A\|^2$ for noncompact electrodynamics. We often use a similar notation for the Villain action.

Example 3. Let us show that in standard compact lattice electrodynamics there exists a conservation law. As we have mentioned before, the compact character of the fields implies the existence of monopoles. The monopole charge inside an elementary 3-dimensional cube is defined by $m = \frac{1}{4\pi}\mathrm{d}\bar{\theta}_P = \frac{1}{2}\mathrm{d}k$, where $\bar{\theta}_P = \mathrm{d}\theta + 2\pi k$, as in Example 1. In the continuum limit, the above definition corresponds to the Gauss law: $m = \frac{1}{4\pi}\oint_S \vec{H}\mathrm{d}\vec{S}$, where S is the surface of the elementary cube. If $j = {}^*m$, then $\delta j = {}^*\mathrm{d}^{**}m = \frac{1}{2}{}^*\mathrm{d}^2 k = 0$. Since m is a rank 3 form, it follows that j is a rank 1 form and the equation

$$\delta j = 0 \tag{A.11}$$

means that for each site the incoming current j is equal to the outgoing one. Therefore, the monopole current j is conserved. It is easy to prove that j is gauge invariant. This result is well known for 4-dimensional lattice QED[38]. Note that all transformations considered above are valid for any space-time dimension D and for the field θ of any rank k; the current j in this case is a $(D-k-2)$–form. For the XY model in $D = 3$ we have $k = 0$ and the conservation law (A.11) for the 1-form j implies that the excitations, called vortices, form closed loops. In the 4-dimensional XY model the conserved quantity j is represented by a 2-form, which means that there exist excitations forming closed surfaces. These objects are related to "global" cosmic strings, forming closed surfaces[53] in 4-dimensional space–time.

Example 4. To perform the BKT transformation (see Appendix C) we have to solve the so-called cohomological equation

$$\mathrm{d}l = n, \tag{A.12}$$

where n is a given k–form and l is a $(k-1)$-form to be found. Using the Hodge decomposition, we can easily show that $l(n) = \Delta^{-1}\delta n$ is a particular solution which, being added to the general solution of the homogeneous equation, yields the general solution of (A.12): $l = \Delta^{-1}\delta n + \mathrm{d}m$ (m is an arbitrary $(k-2)$-form).

If n is an integer valued form, then the solution l of (A.12) can also be found in terms of integer numbers. We explicitly construct such a solution for $k = 2$. The generalization for arbitrary values of k is obvious. Let us assign the value 0 to the links which belong to a given maximal tree on the lattice (a maximal tree is a maximal set of links which do not contain closed loops). To each link that does not belong to the tree we attach the value equal to the value of the plaquette formed by the link and the tree (there exists one and only one such plaquette). It is easy to see that we have thus constructed the required particular solution $l(n)$. The general solution has the form $l = l(n) + \mathrm{d}m\ ,\ m \in \mathbb{Z}$. For simplicity, we have neglected finite volume effects (harmonic forms) in the above construction.

Appendix B: Duality transformation

In this Appendix we perform the duality transformation for $U(1)$ compact gauge theory with an arbitrary form of the action. Initially the duality transformation was discussed by Kramers and Wannier[54] for the 2–dimensional Ising model.

Let us consider the $U(1)$ gauge theory with an arbitrary periodic action $S[\mathrm{d}\theta]$, $S[\dots, X_P + 2\pi, \dots] = S[\dots, X_P, \dots]$, where θ_l is the compact $U(1)$ link field and P denotes any plaquette of the original lattice. We use the formalism of differential forms on the lattice (see Appendix A). We start from the partition function of the compact $U(1)$ theory:

$$\mathcal{Z} = \int\limits_{-\pi}^{+\pi} \mathcal{D}\theta \exp\Bigl\{-S[\mathrm{d}\theta]\Bigr\}\,. \tag{B.1}$$

The Fourier expansion of the function $e^{-S(X)}$ yields

$$\mathcal{Z} = \mathrm{const.} \cdot \int\limits_{-\pi}^{+\pi} \mathcal{D}\theta \sum_{k(\mathrm{C}_2)\in\mathbb{Z}} e^{-S^d(k)} e^{i(k,\mathrm{d}\theta)}\,, \tag{B.2}$$

where k is an integer valued two-form, and the action S^d is

$$e^{-S^d(k)} = \mathrm{const.} \cdot \int\limits_{-\pi}^{+\pi} \mathcal{D}X \exp\Bigl\{-S[X] - i(k,X)\Bigr\}\,. \tag{B.3}$$

The integration over the field θ in eq.(B.2) gives the constraint $\delta k = 0$, which can be resolved by introducing the new 3-form n: $k = \delta n$. For simplicity, it is assumed here that we are dealing with a lattice having trivial topology (e.g. $\mathcal{R}^4$). In the case of a lattice with a nontrivial topology, arbitrary harmonic forms must be added to the r.h.s. of this equation. Finally, changing the summation $\sum_{k(\mathrm{C}_2)\in\mathbb{Z}} \to \sum_{{}^*n({}^*\mathrm{C}_1)\in\mathbb{Z}}$, we get the dual representation of compact $U(1)$ gauge theory:

$$\mathcal{Z} \propto \mathcal{Z}^d = \sum_{{}^*n({}^*\mathrm{C}_1)\in\mathbb{Z}} e^{-S^d({}^*\mathrm{d}n)}\,. \tag{B.4}$$

Consider now compact $U(1)$ gauge theory with the action in the Villain form eq.(33). Now the integral is Gaussian and the dual action is $S^d({}^*\mathrm{d}n) = ||{}^*\mathrm{d}n||^2/4\beta$.

Appendix C: The BKT transformation

In this Appendix we show how to perform the BKT transformation for compact $U(1)$ gauge theory with an arbitrary form for the action (B.1). Let us insert the unity

$$1 = \int\limits_{-\infty}^{+\infty} \mathcal{D}G\, \delta(G - k)$$

(here G is a real-valued two-form) into the sum (B.2)

$$\mathcal{Z} = \int\limits_{-\pi}^{+\pi} \mathcal{D}\theta \exp\Bigl\{-S_P(\mathrm{d}\theta)\Bigr\} = \mathrm{const.} \cdot \int\limits_{-\pi}^{+\pi} \mathcal{D}\theta \int\limits_{-\infty}^{+\infty} \mathcal{D}G \sum_{k(\mathrm{C}_2)\in \mathbb{Z}} \delta(G-k) e^{-S^d(G)} e^{i(G,\mathrm{d}\theta)} . \quad \text{(C.1)}$$

Using the Poisson summation formula

$$\sum_{k(\mathrm{C}_2)\in \mathbb{Z}} \delta(G-k) = \sum_{k(\mathrm{C}_2)\in \mathbb{Z}} e^{2\pi i(G,k)} , \quad \text{(C.2)}$$

we get

$$\mathcal{Z} = \mathrm{const.} \cdot \int\limits_{-\pi}^{+\pi} \mathcal{D}\theta \int\limits_{-\infty}^{+\infty} \mathcal{D}G \sum_{k(\mathrm{C}_2)\in \mathbb{Z}} e^{-S^d(G)}\, e^{i(\mathrm{d}\theta + 2\pi k, G)} . \quad \text{(C.3)}$$

Now we perform the BKT transformation with respect to the integer valued 2-form k:

$$k = m[j] + \mathrm{d}q\,, \quad \mathrm{d}m[j] = j\,, \quad \mathrm{d}j = 0\,, \quad \text{(C.4)}$$

where q and j are one- and three-forms, respectively. First, we change the summation variable, $\sum\limits_{k(\mathrm{C}_2)\in \mathbb{Z}} = \sum\limits_{\substack{j(\mathrm{C}_3)\in \mathbb{Z}\\ \mathrm{d}j=0}} \sum\limits_{q(\mathrm{C}_1)\in \mathbb{Z}}$. Using the Hodge–de–Rahm decomposition, we adsorb the d–closed part of the 2–form k into the compact variable θ:

$$\mathrm{d}\theta + 2\pi k = \mathrm{d}\theta_{n.c.} + 2\pi\delta\Delta^{-1} j\,, \quad \theta_{n.c.} = \theta + 2\pi\Delta^{-1}\delta m[j] + 2\pi q\,. \quad \text{(C.5)}$$

Substituting eq.(C.5) in eq.(C.3) and integrating over the noncompact field $\theta_{n.c.}$, we get the following representation of the partition function:

$$\mathcal{Z} = \mathrm{const.} \cdot \int\limits_{-\infty}^{+\infty} \mathcal{D}G \sum_{\substack{{}^*j({}^*\mathrm{C}_1)\in \mathbb{Z}\\ \delta^* j=0}} e^{-S^d(G)} \exp\Bigl\{2\pi i(G, \delta\Delta^{-1} j)\Bigr\}\, \delta(\delta G)\,. \quad \text{(C.6)}$$

The constraint $\delta G = 0$ can be solved by $G = \delta H$, where H is a real valued 3–form. Substituting this solution into eq.(C.6) we obtain the final expression for the BKT–transformed action on the dual lattice:

$$\mathcal{Z} = \mathrm{const.} \cdot \sum_{\substack{{}^*j({}^*\mathrm{C}_1)\in \mathbb{Z}\\ \delta^* j=0}} e^{-S_{mon}({}^*j)}\,. \quad \text{(C.7)}$$

where

$$S_{mon}({}^*j) = -\ln\left(\int\limits_{-\infty}^{+\infty} \mathcal{D}H e^{-S^d(\delta H)} \exp\Bigl\{2\pi i({}^*H, {}^*j)\Bigr\} \right)\,. \quad \text{(C.8)}$$

We have used the relation $\mathrm{d}\delta\Delta^{-1} j \equiv j$, $\forall j : \mathrm{d}j = 0$. Therefore, for the general $U(1)$ action $S[\mathrm{d}\theta]$ the monopole action (C.8) is nonlocal and it is expressed through the integral over the entire lattice ($\int\limits_{-\infty}^{+\infty} \mathcal{D}H$).

As an example we consider the Villain form of the $U(1)$ action (33). Repeating all the steps we get the following monopole action:

$$S_{mon}({}^*j) = 4\pi^2\beta({}^*j, \Delta^{-1\,*}j)\,. \quad \text{(C.9)}$$

REFERENCES

1. K. Wilson, *Phys. Rev.* **D10** (1974) 2445.
2. Creutz, *Phys. Rev. Lett.* **43** (1979) 553; *Phys. Rev.* **D21** (1980) 1006, 2308; *Phys. Rev. Lett.* **45** (1980) 313;
3. A.A. Abrikosov, *JETP* **32** (1957) 1442.
4. S. Mandelstam, Phys. Rep., 23C (1976) 245; G. 't Hooft, "High Energy Physics", Zichichi, Editrice Compositori, Bolognia, 1976.
5. A.M. Polyakov, *Phys.Lett.* **59B** (1975) 82.
6. A.M. Polyakov *Nucl.Phys.* **B120** (1977) 429.
7. N. Sieberg and E. Witten: *Nucl.Phys.* **B341** (1994) 484.
8. G.S. Bali K. Schilling and Ch. Schlichter, *Phys.Rev.* **D51** (1995) 5165.
9. Ch. Schlichter, G.S. Bali and K. Schilling, to be published.
10. G. 't Hooft, Nucl. Phys., **B190** [FS3] (1981) 455; G. 't Hooft, these proceedings.
11. A.S. Kronfeld, M.L. Laursen, G. Schierholz and U.J. Wiese, *Phys.Lett.* **198B** (1987) 516; A.S. Kronfeld, G. Schierholz and U.J. Wiese, *Nucl.Phys.* **B293** (1987) 461.
12. L. Polley and U.J. Wiese, *Nucl.Phys.* **B356** (1991) 629.
13. M.I. Polikarpov, L. Polley and U.J. Wiese, *Phys.Lett.* **B253** (1991) 212.
14. T. Suzuki *et al.*, these proceedings.
15. S. Kato *et al.*, preprint KANAZAWA 97-15, `hep-lat/9709092`.
16. T. Suzuki and I. Yotsuyanagi, Phys. Rev.**D42** (1990) 4257.
17. S. Hioki *et. al.*, *Phys.Lett.* **B272** (1991) 326; *ERRATUM-ibid.* **B281** (1992) 416.
18. J.D. Stack, S.D. Neiman and R.J. Wensley, *Phys.Rev.* **D50** (1994) 3399.
19. G. Bali, V. G. Bornyakov, M. Müller-Preussker and K. Schilling, *Phys. Rev.* **D54** (1996) 2863.
20. J.D. Stack, S.D. Neiman, *Phys.Lett.* **B377** (1996) 113.
21. J.D. Stack, R.J. Wensley and S.D. Neiman, *Phys.Lett.* **B385** (1996) 261; J.D. Stack, *Nucl.Phys.Proc.Suppl.* **53** (1997) 524, `hep-lat/9607064`.
22. K. Yee, *Nucl.Phys.* **B** *(Proc. Suppl.)* **34** (1994) 189.
23. S. Kitahara, *preprint KANAZAWA-96-12*.
24. S. Ejiri, *Nucl.Phys.Proc.Suppl.* **53** (1997) 491, `hep-lat/9608001`.
25. T. Suzuki, S. Kitahara, T. Okude, F. Shoji, K. Moroda and O. Miyamura, *Nucl. Phys.* **B** *(Proc. Suppl.)* **47** (1996) 374, `hep-lat/9509016`; T. Suzuki, *preprint KANAZAWA-96-08*, `hep-lat/9606015`.
26. V. Singh, R.W. Haymaker and D.A. Browne *Phys.Rev.* **D47** (1993) 1715.
27. Ch. Schlichter, G.S. Bali and K. Schilling, `hep-lat/9709114`.
28. Y. Matsubara, S. Ejiri and T. Suzuki *Nucl.Phys.* **B** *Proc.Suppl.* **34** (1994) 176, `hep-lat/9311061`.
29. J. Fröhlich and P.A. Marchetti, *Europhys. Lett.* **2** (1986) 933.
30. M.N. Chernodub, M.I. Polikarpov and A.I. Veselov, *Nuclear Physics* **B** *(Proc. Suppl.)* **49** (1996) 307; `hep-lat/9512030`.
31. M.N. Chernodub, M.I. Polikarpov and A.I. Veselov, *Phys.Lett.* **B399** (1997) 267.
32. T. Kennedy and C. King, *Comm.Math.Phys.* **104** (1986) 327.
33. N. Nakamura, V. Bornyakov, S. Ejiri, S. Kitahara, Y. Matsubara and T. Suzuki, *preprint KANAZAWA-96-16*, `hep-lat/9608004`.
34. A. Di Giacomo, these proceedings
35. L. Del Debbio, A. Di Giacomo, G. Paffuti and P. Pieri, *Phys.Lett.* **B355** (1995) 255; also talk by A. Di Giacomo, these proceedings.
36. V.L. Berezinskii, *Sov. Journal JETP*, **32** (1971) 493.
37. J.M. Kosterlitz, D.J. Thouless, *J.Phys.* **C6** (1973) 1181.
38. T. Banks, R. Myerson and J. Kogut, *Nucl.Phys.* **B129** (1977) 493.
39. T.L. Ivanenko and M.I. Polikarpov, *preprint ITEP 49-91* (1991).
40. J. Smit and A. van der Sijs, *Nucl.Phys.* **B355** (1991) 603.
41. R.H. Swendsen, *Phys. Rev. Lett.* **52** (1984) 1165; *Phys. Rev.* **D30** (1984) 3866,3875.
42. H. Shiba and T. Suzuki, *Phys. Lett.* **B343** (1995) 315; *Phys. Lett. B351* (1995) 519.
43. M.I. Polikarpov, U.-J. Wiese and M.A. Zubkov, *Phys. Lett.*, **272B** (1991) 326.
44. P. Orland, *Nucl.Phys.* **B428** (1994) 221; E.T. Akhmedov *et al.*, *Phys.Rev.* **D53** (1996) 2087.
45. B.L.G. Bakker, M.N. Chernodub and M.I. Polikarpov, *preprint KANAZAWA-97-10*, `hep-lat/9706007`, *preprint ITEP-TH-43-97*, `hep-lat/9709038`.
46. O.Miyamura, S.Origuchi, *'QCD monopoles and Chiral Symmetry Breaking in* $SU(2)$ *Lattice Gauge Theory'*, RCNP Confinement 1995, Osaka, Japan, Mar 22-26, 1995, p.137;

M.N.Chernodub, F.V.Gubarev, *JETP Lett.* **62** (1995) 100; R.C.Brower, K.N.Orginos, Chung-I Tan, *Phys. Rev.***D55** (1997) 6313; A.Hart, M.Teper, *Phys. Lett.* **371** (1996) 261; M. Feurstein, H. Markum and S. Thurner, *Phys.Lett.* **B396** (1997) 203; M.Fukushima *et al.*, *Phys. Lett.* **B399** (1997) 141.

47. V. Bornyakov and G. Schierholz, *Phys.Lett.* **B384** (1996) 190.
48. M.N. Chernodub, F.V. Gubarev and M.I. Polikarpov, *preprint ITEP-TH-44-97*, `hep-lat/9709039`.
49. M. Faber, J. Greensite, S. Olejnik, `hep-lat/9710039`; L. Del Debbio, M. Faber, J. Greensite, S. Olejnik, `hep-lat/9709032`; `hep-lat/9708023`.
50. M. Baker, J.S. Ball, N. Brambilla, G.M. Prosperi and F. Zachariasen, *Phys.Rev.* **D54** (1996) 2829; *Erratum-ibid.* **D56** (1997) 2475; M. Baker, J.S. Ball and F. Zachariasen, *J.Mod.Phys.* **A11** (1996) 343.
51. Yu.A. Simonov, *Phys.Usp.* **39** (1996) 313; `hep-ph 9709344`; H.G. Dosh and Yu.A. Simonov *Phys.Lett.* **205B** (1988) 339.
52. P. Becher and H. Joos, *Z.Phys.* **C15** (1982) 343; A.H. Guth, *Phys. Rev.* **D21** (1980) 2291
53. M.I. Polikarpov and U.J. Wiese, *preprint HLRZ-90-78* (1990); A.K. Bukenov, M.I. Polikarpov, A.V. Pochinskii and U.J. Wiese, *Phys.Atom.Nucl.* **56** (1993) :122.
54. H.A. Kramers and G.H. Wannier, *Phys.Rev.* **60** (1941) 252.