NON-PERTURBATIVE GAUGE DYNAMICS IN SUPERSYMMETRIC THEORIES. A PRIMER

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ABSTRACT

I give an introductory review of recent, fascinating developments in supersymmetric gauge theories. I explain pedagogically the miraculous properties of supersymmetric gauge dynamics allowing one to obtain exact solutions in many instances. Various dynamical regimes emerging in supersymmetric Quantum Chromodynamics and its generalizations are discussed. I emphasize those features that have a chance of survival in QCD and those which are drastically different in supersymmetric and non-supersymmetric gauge theories. Unlike most of the recent reviews focusing almost entirely on the progress in extended supersymmetries (the Seiberg-Witten solution of $N = 2$ models), these lectures are mainly devoted to $N = 1$ theories. Developments "after Seiberg" (domain walls in supersymmetric gluodynamics) are briefly discussed.

PREFACE

All fundamental interactions established in nature are described by non-Abelian gauge theories. The standard model of the electroweak interactions belongs to this class. In this model, the coupling constant is weak, and its dynamics is fully controlled (with the possible exception of a few, rather exotic problems, like baryon number violation at high energies).

Another important example of non-Abelian gauge theories is Quantum Chromodynamics (QCD). This theory has been under intense scrutiny for over two decades, yet remains mysterious. Interaction in QCD becomes strong at large distances. What is even worse, the degrees of freedom appearing in the Lagrangian (microscopic variables – colored quarks and gluons in the case at hand) are not those degrees of freedom that show up as physical asymptotic states (macroscopic degrees of freedom – colorless hadrons). Color is permanently confined. What are the dynamical reasons of this phenomenon?

Color confinement is believed to take place even in pure gluodynamics, i.e. with no dynamical quarks. Adding massless quarks produces another surprise. The chiral symmetry of the quark sector, present at the Lagrangian level, is spontaneously broken (realized nonlinearly) in the physical amplitudes. Massless pions are the remnants of the spontaneously broken chiral symmetry. What can be said, theoretically, about the pattern of the spontaneous breaking of the chiral symmetry?

Color confinement and the spontaneous breaking of the chiral symmetry are the two most sacred questions of strong non-Abelian dynamics; and the progress of understanding them is painfully slow. At the end of the 1970's Polyakov showed that in 3-dimensional compact electrodynamics (the so called Georgi-Glashow model, a primitive relative of QCD) color confinement does indeed take place[1]. Approximately at the same time a qualitative picture of how this phenomenon could actually happen in 4-dimensional QCD was suggested by Mandelstam[2] and 't Hooft[3]. Some insights, though quite limited, were provided by models of the various degree of fundamentality, and by numerical studies on lattices. This is, basically, all we had before 1994, when a significant breakthrough was achieved in understanding both issues in supersymmetric (SUSY) gauge theories.

Unlike the Georgi-Glashow model in three dimensions mentioned above, which is quite a distant relative of QCD, four-dimensional supersymmetric gluodynamics and supersymmetric gauge theories with matter come much closer to genuine QCD. Moreover, the dynamics of these theories is rich and interesting by itself, which accounts for the attention they have attracted in the last three years when we have been witnessing an unprecedented progress. It turns out that supersymmetry helps unravel several intriguing and extremely elegant properties which shed light on subtle aspects of the gauge theories in general. These developments can, eventually, lead us to a breakthrough in QCD. Today we are definitely one step closer to the dream of every QCD practitioner: understanding of two most salient properties of QCD, color confinement and spontaneous breaking of the chiral symmetry, in analytic terms.

Strongly coupled supersymmetric gauge theories is the topic of this lecture course. My task, as I see it, is educational rather than providing a comprehensive coverage. I will try to be as pedagogical as possible focusing on basic ideas and approaches and avoiding, whenever possible, more technical and involved aspects.

BASICS OF SUPERSYMMETRIC GAUGE THEORIES

In this Lecture we will start our excursion in supersymmetric gauge theories. There is a long way to go before we will be able to discuss a variety of fascinating results obtained in this field recently. As a first step let me briefly review some basic features of these theories and key elements of the formalism we will need below.

Introducing supersymmetry

Supersymmetry relates bosonic and fermionic degrees of freedom[4, 5]. A necessary condition for any theory to be supersymmetric is the balance between the number of the bosonic and fermionic degrees of freedom, having the same mass and the same "external" quantum numbers, e.g. color. Let us consider several simplest examples of practical importance.

A scalar complex field ϕ has two degrees of freedom (a particle plus antiparticle). Correspondingly, its fermion superpartner is the Weyl (two-component) spinor, which also has two degrees of freedom – say, the left-handed particle and the right-handed antiparticle. Alternatively, instead of working with the complex fields, one can introduce real fields, with the same physical content: two real scalar fields ϕ_1 and ϕ_2 describing two "neutral" spin-0 particles, plus the Majorana (real four-component) spinor describing a "neutral" spin-1/2 particle with two polarizations. (By neutral I mean that the corresponding antiparticles are identical to their particles). This family has a balanced number of the degrees of freedom both in the massless and massive cases. Below we will see that in the superfield formalism it is described, in a concise form, by one *chiral superfield.*

When we speak of the quark flavors in QCD we count the Dirac spinors. Each Dirac spinor is equivalent to two Weyl spinors. Therefore, in supersymmetric QCD (SQCD) each flavor requires two chiral superfields. Sometimes, the superfields from this chiral pair are referred to as subflavors. Two subflavors comprise one flavor.

Another important example is vector particles, gauge bosons (gluons in QCD, W bosons in the Higgs phase). Each gauge boson carries two physical degrees of freedom (two transverse polarizations). The appropriate superpartner is the Majorana spinor. Unlike the previous example the balance is achieved only for massless particles, since the massive vector boson has three, not two, physical degrees of freedom. The superpartner to the massless gauge boson is called gaugino. Notice that the mass still can be introduced through the (super)Higgs mechanism. We will discuss the Higgs mechanism in supersymmetric gauge theories later on.

In counting the degrees of freedom above, the external quantum numbers were left aside. Certainly, they should be the same for each member of the superfamily. For instance, if the gauge group is $SU(2)$, the gauge bosons are "color" triplets, and so are gauginos. In other words, the Majorana fields describing gauginos are provided by the "color" index a taking three different values, $a = 1, 2, 3$.

If we consider the free field theory with the balanced number of degrees of freedom, the vacuum energy vanishes. Indeed, the vacuum energy is the sum of the zero-point oscillation frequencies for each mode of the theory,

$$E_{\rm vac} = \sum_{\vec{k}} \left\{ \omega_{\vec{k}}^{\rm boson} - \omega_{\vec{k}}^{\rm ferm} \right\} . \tag{1.1}$$

I remind that the modes are labeled by the three-momentum $\vec{k}$; say, for massive particles

$$\omega_{\vec{k}} = \sqrt{m^2 + \vec{k}^2} .$$

It is important that the boson and fermion terms enter with the opposite signs and cancel each other, term by term. This observation, which can be considered as a precursor to supersymmetry, was made by Pauli[6] in 1950! If interactions are introduced in such a way that supersymmetry remains unbroken, the vanishing of the vacuum energy is preserved in dynamically nontrivial theories.

Balancing the number of degrees of freedom is the necessary but not sufficient condition for supersymmetry in dynamically nontrivial theories, of course. All vertices must be supersymmetric too. This means that each line can be substituted by that of a superpartner. Let us consider, for instance, QED, the simplest gauge theory. We start from the electron-electron-photon coupling (Fig. 1a). Now, in SQED the electron is accompanied by two selectrons (two, because the electron is described by the four-component Dirac spinor rather than the Weyl spinor). Thus, supersymmetry requires the selectron-selectron-photon vertices, (Fig. 1b), with the same coupling constant. Moreover, the photon can be substituted by its superpartner, photino, which generates the electron-selectron-photino vertex (Fig. 1c), with the same coupling. In the old-fashioned language of the pre-SUSY era we would call this vertex the Yukawa coupling. In the supersymmetric language this is the gauge interaction since it generalizes the gauge interaction coupling of the photon to the electron.

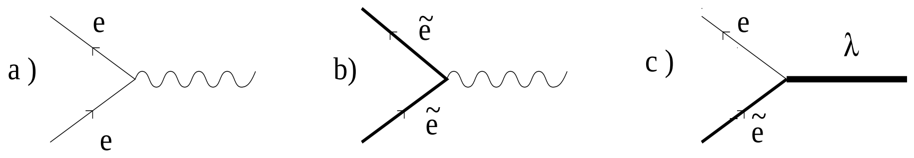

Figure 1. Interaction vertices in QED and its supergeneralization, SQED. (a) $\bar{e}e\gamma$ vertex; (b) selectron coupling to photon; (c) electron-selectron-photino vertex. All vertices have the same coupling constant. The quartic self-interaction of selectrons is also present, but not shown.

With the above set of vertices one can show that the theory is supersymmetric at the level of trilinear interactions, provided that the electrons and the selectrons are degenerate in mass, while the photon and photino fields are both massless. To make it fully supersymmetric one should also add some quartic terms, describing self-interactions of the selectron fields, as we will see shortly.

Now, the theory is dynamically nontrivial, the particles – bosons and fermions – are not free and still $E_{vac} = 0$. This is the *first miracle* of supersymmetry.

The above pedestrian (or step-by-step) approach to supersymmetrizing the gauge theories is quite possible, in principle. Moreover, historically the first supersymmetric model derived by Golfand and Likhtman, SQED, was obtained in this way[4]. This is a painfully slow method, however, which is totally out of use at the present stage of the theoretical development. The modern efficient approach is based on the superfield formalism, introduced in 1974 by Salam and Strathdee[7] who replaced conventional four-dimensional space by superspace.

Superfield formalism: bird's eye view

I will be unable to explain this formalism, even briefly. The reader is referred to the text-books and numerous excellent reviews, see the list of recommended literature at the end. Below some elements are listed mostly with the purpose of introducing relevant notations, to be used throughout the entire lecture course. (A summary of our notation and conventions is given in the Appendix.)

If conventional space-time is parametrized by the coordinate four-vector x_μ, superspace is parametrized by x_μ and two Grassmann variables, θ and $\bar\theta$. The Grassmann numbers obey all standard rules of arithmetic except that they anticommute rather than commute with each other. In particular, the product of a Grassmann number with itself is zero, for this reason.

With respect to the Lorentz properties, θ and $\bar\theta$ are spinors. As well known, the four-dimensional Lorenz group is equivalent to $SU(2)\times SU(2)$ and, therefore, there exist two types of spinors, left-handed and right-handed, denoted by undotted and dotted indices, respectively; θ_α is the left-handed spinor while $\bar\theta_{\dot\alpha}$ is the right-handed one ($\alpha, \dot\alpha = 1,2$). The indices of the right-handed spinors are supplied by dots to emphasize the fact that their transformation law does not coincide with that of the left-handed spinors.

The Lorentz scalars can be formed as a convolution of two dotted or two undotted spinors, $\theta^\alpha\theta_\alpha$ or $\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}$, with one lower and one upper index. Raising and lowering of indices is realized by virtue of the antisymmetric (Levi-Civita) symbol,

$$\bar\theta_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\theta^{\dot\beta}\,, \quad \theta_\alpha = \epsilon_{\alpha\beta}\theta^\beta\,,$$

where

$$\epsilon^{12} = \epsilon^{\dot1\dot2} = 1\,; \quad \epsilon_{12} = \epsilon_{\dot1\dot2} = -1$$

so that $\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^{\alpha}_{\gamma}$. When one raises or lowers the index of θ the ϵ symbol must be placed to the left of θ.

A shorthand notation when the indices of the spinors are implicit is widely used, for instance,

$$\theta^2 \equiv \theta^\alpha \theta_\alpha$$

and

$$\bar\theta^2 \equiv \bar\theta_{\dot\alpha}\theta^{\dot\alpha}\,.$$

Notice that in convoluting the undotted indices one writes first the spinor with the upper index while for the dotted indices the first spinor has the lower index. The ordering is important since the elements of the spinors are anticommuting Grassmann numbers.

It remains to be added that the vector quantities can be obtained from two spinors – one dotted and one undotted. Thus, $\theta_\alpha\bar\theta_{\dot\alpha}$ transforms as a Lorentz vector.

Now, we can introduce the notion of supertranslations in the superspace $\{x, \theta, \bar\theta\}$. The generic (infinitesimal) supertransformation has the form

$$\theta \to \theta + \varepsilon\,, \quad \bar\theta \to \bar\theta + \bar\varepsilon\,,$$

$$x_{\alpha\dot\beta} \to x_{\alpha\dot\beta} + 2i\varepsilon_\alpha\bar\theta_{\dot\beta} - 2i\theta_\alpha\bar\varepsilon_{\dot\beta}\,. \tag{1.2}$$

The supertranslations generalize conventional translations in ordinary space.

One can also consider the so called chiral and antichiral superspaces (chiral realizations of the supergroup); the first one does not explicitly contain $\bar\theta$ while the second does not contain θ. It is not difficult to see that a point from the chiral superspace is parametrized by $\{x_L, \theta\}$, and that from the antichiral superspace is parametrized by $\{x_R, \bar\theta\}$. Here

$$(x_L)_{\alpha\dot\alpha} = x_{\alpha\dot\alpha} - 2i\theta_\alpha\bar\theta_{\dot\alpha}\,, \quad (x_R)_{\alpha\dot\alpha} = x_{\alpha\dot\alpha} + 2i\theta_\alpha\bar\theta_{\dot\alpha}\,. \tag{1.3}$$

Under this definition the supertransformations corresponding to the shifts in θ and $\bar\theta$, respectively, leave us inside the corresponding superspace. Indeed, if $\theta \to \theta + \varepsilon$ and $\bar\theta \to \bar\theta + \bar\varepsilon$, then

$$(x_L)_{\alpha\dot\beta} \to (x_L)_{\alpha\dot\beta} - 4i\theta_\alpha\bar\varepsilon_{\dot\beta} \;\text{ and }\; (x_R)_{\alpha\dot\beta} \to (x_R)_{\alpha\dot\beta} + 4i\varepsilon_\alpha\bar\theta_{\dot\beta}\,. \tag{1.4}$$

Superfields provide a very concise description of supersymmetry representations. They are very natural generalizations of conventional fields. Say, the scalar field $\phi(x)$ in the $\lambda\phi^4$ theory is a function of x. Correspondingly, superfields are functions of x and θ's. For instance, the chiral superfield $\Phi(x_L, \theta)$ depends on θ and x_L (and has no explicit $\bar\theta$ dependence). If we Taylor-expand it in powers of θ we get the following formula:

$$\Phi(x_L, \theta) = \phi(x_L) + \sqrt{2}\psi(x_L)\theta + \theta^2 F(x)\,. \tag{1.5}$$

There are no higher-order terms in the expansion since higher powers of θ vanish due to the Grassmannian nature of this parameter. For the same reason the argument of the last component of the chiral superfield, F, is set equal to x. The distinction between x and x_L is not important in this term. The last component of the chiral superfield is always called F. F terms of the chiral superfields are non-dynamical, they appear in the Lagrangian without derivatives. We will see later that F terms play a distinguished role.

The lowest component of the chiral superfield is a complex scalar field ϕ, and the middle component is a Weyl spinor ψ. Each of these fields describes two degrees of freedom, so the appropriate balance is achieved automatically. Thus, we see that the superfield is a concise form of representing a set of components. The transformation law of the components follows immediately from Eq. (1.4), for instance, $\delta\phi(x) = \sqrt{2}\psi(x)\varepsilon$, and so on.

The antichiral superfields depend on x_R and $\bar\theta$. The chiral and antichiral superfields describe the matter sectors of the theories to be studied below. The gauge field appears from the so called vector superfield V which depends on both, θ and $\bar\theta$ and satisfies the condition $V = V^\dagger$. The component expansion of the vector superfield has the form

$$V(x,\theta,\bar\theta) = C(x) + i\theta\chi(x) - i\bar\theta\bar\chi(x) + \frac{i}{2}\theta^2[M(x)+iN(x)] - \frac{i}{2}\bar\theta^2[M(x)-iN(x)] -$$

$$2\theta_\alpha\bar\theta_{\dot\beta}v^{\alpha\dot\beta}(x) + 2\left\{i\theta^2\bar\theta_{\dot\beta}\left[\bar\lambda^{\dot\beta} - \frac{i}{4}\partial^{\alpha\dot\beta}\chi_\alpha\right] + \text{h.c.}\right\} + \theta^2\bar\theta^2\left[D(x) - \frac{1}{4}\Box C(x)\right]. \qquad (1.6)$$

The components C, D, M, N and $v^{\alpha\dot\beta}$ must be real to satisfy the condition $V = V^\dagger$. The vector field $v^{\alpha\dot\beta}$ gives its name to the entire superfield.

The last component of the vector superfield, apart from a full derivative, is called the " D term". D terms also play a special role.

Let me say a few words about the gauge transformations. For simplicity I will consider the case of the Abelian (U(1)) gauge group. In the non-Abelian case the corresponding formulae become more bulky, but the essence stays the same.

As is well known, in nonsupersymmetric gauge theories the matter fields transform under the gauge transformations as

$$\phi(x) \to e^{i\alpha(x)}\phi(x)\,,\quad \phi(x)^\dagger \to e^{-i\alpha(x)}\phi(x)^\dagger\,, \qquad (1.7)$$

while the gauge field

$$v_\mu(x) \to v_\mu(x) + \partial_\mu\alpha(x)\,, \qquad (1.8)$$

where $\alpha(x)$ is an arbitrary function of x. Equations (1.7) and (1.8) prompt the supersymmetric version of the gauge transformations,

$$\Phi(x_L,\theta) \to e^{i\Lambda}\Phi(x_L,\theta)\,,\quad \bar\Phi(x_R,\bar\theta) \to e^{-i\bar\Lambda}\Phi(x_R,\bar\theta) \qquad (1.9)$$

and

$$V \to V - i(\Lambda - \bar\Lambda) \qquad (1.10)$$

where Λ is an arbitrary chiral superfield, $\bar\Lambda$ is its antichiral partner. $\bar\Phi e^V\Phi$ is then a gauge invariant combination playing the same role as $\mathcal{D}_\mu\phi^\dagger\mathcal{D}_\mu\phi$ in non-supersymmetric theories. Let me parenthetically note that supersymmetrization of the gauge transformations, Eqs. (1.9), (1.10), was the path which led Wess and Zumino[5] to the discovery of the supersymmetric theories (independently of Golfand and Likhtman).

In components

$$C \to C - i(\phi - \phi^\dagger)\,,\quad \chi \to \chi - \sqrt{2}\psi\,,\quad M + iN \to M + iN + 2F$$

$$v_{\alpha\dot\beta} \to v_{\alpha\dot\beta} + \frac{1}{2}\partial_{\alpha\dot\beta}(\phi + \phi^\dagger)\,,\quad \lambda \to \lambda\,,\quad D \to D\,. \qquad (1.11)$$

We see that the C, χ, M and N components of the vector superfield can be gauged away. This is what is routinely done when the component formalism is used. This gauge bears the name of its inventors – it is called the *Wess-Zumino* gauge. Imposing the Wess-Zumino gauge condition in supersymmetric theory one actually does not fix the gauge completely. The component Lagrangian one arrives at in the Wess-Zumino gauge still possesses the gauge freedom with respect to non-supersymmetric (old-fashioned) gauge transformations.

It remains to introduce spinorial derivatives. They will be denoted by capital D and $\bar{D}$,

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\partial_{\alpha\dot\beta}\bar\theta^{\dot\beta}\,, \quad \bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\partial_{\beta\dot\alpha}\theta^\beta\,. \tag{1.12}$$

The relative signs in Eq. (1.12) are fixed by the requirements $D_\alpha(x_R)_{\beta\dot\gamma} = 0$ and $\bar D_{\dot\alpha}(x_L)_{\beta\dot\gamma} = 0$.

To make the spinorial derivatives distinct from the regular covariant derivative the latter will be denoted by the script $\mathcal{D}$. The supergeneralization of the field strength tensor of the gauge field has the form

$$W_\alpha(x_L,\theta) = \frac{1}{8}\bar D^2(e^{-V}D_\alpha e^V) =$$

$$i\lambda_\alpha(x_L) - \theta_\alpha D(x_L) - i\theta^\beta G_{\alpha\beta}(x_L) + \theta^2\mathcal{D}_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}(x_L)\,, \tag{1.13}$$

where $G_{\alpha\beta}$ is the gauge field strength tensor in the spinorial form.

This brief excursion in the formalism, however boring it might seem, is necessary for understanding physical results to be discussed below. I will try to limit such excursions to absolute minimum, but we will not be able to avoid them completely. Now, the stage is set, and we are ready to submerge in the intricacies of the supersymmetric gauge dynamics.

Simplest supersymmetric models

In this section we will discuss some simple models. Our basic task is to reveal general features playing the key role in various unusual dynamical scenarios realized in supersymmetric gauge theories. One should keep in mind that all theories with matter can be divided in two distinct classes: chiral and non-chiral matter. The second class includes supersymmetric generalization of QCD, and all other models where each matter multiplet is accompanied by the corresponding conjugate representation. In other words, a mass term is possible for all matter fields. Even if the massless limit is considered, the very possibility of adding the mass term is very important for dynamics. In particular, dynamical SUSY breaking cannot happen in the non-chiral models.

Models with chiral matter are those where the mass term is impossible. The matter sector in such models is severely constrained by the absence of internal anomalies in the theory. The most well-known example of this type is the $SU(5)$ model with equal number of chiral quintets and (anti)decuplets. Each quintet and anti-decuplet, together, is called a *generation*; when the number of generations is three this is nothing but the most popular grand unified theory of electroweak interactions. The chiral models are singled out by the fact that dynamical SUSY breaking is possible, in principle, only in this class. In the present lecture course dynamical SUSY breaking is not our prime concern. Rather, we will focus on various non-trivial dynamical regimes. Most of the regimes to be discussed below manifest themselves in the non-chiral models, which are simpler. Therefore, the emphasis will be put on the non-chiral models, digression to the chiral models will be made occasionally.

Supersymmetric gluodynamics

To begin with we will consider supersymmetric generalization of pure gluodynamics – i.e. the theory of gluons and gluinos. The Lagrangian has the form[8]

$$\mathcal{L} = -\frac{1}{4g^2}G^a_{\mu\nu}G^a_{\mu\nu} + \frac{\vartheta}{32\pi^2}G^a_{\mu\nu}\tilde G^a_{\mu\nu} + \frac{i}{2g^2}\bar\lambda^a\mathcal{D}_\mu\gamma^\mu\lambda^a \tag{1.14}$$

where $G^a_{\mu\nu}$ is the gluon field strength tensor, $\tilde{G}^a_{\mu\nu}$ is the dual tensor, g is the gauge coupling constant, ϑ is the vacuum angle, and $\mathcal{D}_\mu$ is the covariant derivative. Moreover, λ^a is the gluino field, which can be described either by four-component Majorana (real) fields or two-component Weyl (complex) fields.

In terms of superfields

$$\mathcal{L} = \mathrm{Re}\left\{\frac{1}{2g_0^2}\,\mathrm{Tr}\int d^2\theta W^2\right\}, \tag{1.15}$$

where the superfield W is a color matrix,

$$W = W^a T^a\,,$$

T^a are the generators of the gauge group (in the fundamental representation), $\mathrm{Tr}(T^aT^b) = (1/2)\delta^{ab}$. It is very important that the gauge constant $1/g_0^2$ in Eq. (1.15) can be treated as a complex parameter. The subscript 0 emphasizes the fact that the gauge couplings in Eqs. (1.15) and (1.14) are different,

$$\frac{1}{g_0^2} = \frac{1}{g^2} - i\frac{\vartheta}{8\pi^2}\,, \tag{1.16}$$

its real part is the conventional gauge coupling while the imaginary part is proportional to the vacuum angle. Thus, the gauge coupling becomes complexified in SUSY theories. This fact has far-reaching consequences.

Equivalence between Eqs. (1.15) and (1.14) is clear from Eq. (1.13). The F component of W^2 includes the kinetic term of the gaugino field (or gluino, I will use these terms indiscriminately),

$$\bar{\lambda}\mathcal{D}_\mu\gamma^\mu\lambda\,,$$

and that of the gauge field,

$$G^a_{\mu\nu}G^a_{\mu\nu} + iG^a_{\mu\nu}\tilde{G}^a_{\mu\nu}\,.$$

Superficially the model looks very similar to conventional QCD; the only difference is that the quark fields belonging to the fundamental representation of the gauge group in QCD are replaced by the gluino field belonging to the adjoint representation in supersymmetric gluodynamics. Like QCD, supersymmetric gluodynamics is a strong coupling non-Abelian theory. Therefore, it is usually believed that

- only colorless asymptotic states exist;
- the Wilson loop (in the *fundamental* representation) is subject to the area law (confinement);
- a mass gap is dynamically generated; all particles in the spectrum are massive.

I would like to stress the word "believe" since the above features are hypothetical. Although the theory does indeed look pretty similar to QCD, supersymmetry brings in remarkable distinctions – some quantities turn out to be exactly calculable. Namely, we know that the gluino condensate develops,

$$\langle\lambda^{a\alpha}\lambda^a_\alpha\rangle = \text{const.}\times\Lambda^3 e^{2\pi ik/N_c} \tag{1.17}$$

where N_c is the number of colors (an $SU(N_c)$ gauge group is assumed and the vacuum angle ϑ is set equal to zero), Λ is the scale parameter of supersymmetric gluodynamics, k is an integer ($k = 0, 1, ..., N_c - 1$), and the constant in Eq. (1.17) is *exactly* calculable[9, 10]. A discrete Z_{2N_c} symmetry of the model, a remnant of the anomalous

$U(1)$, is spontaneously broken by the gluino condensate* down to Z_2. Correspondingly, there are N_c degenerate vacua, counted by the integer parameter k. Supersymmetry is unbroken – all vacua have vanishing energy density.

Moreover, the Gell-Mann–Low function of the model, governing the running of the gauge coupling constant, is also exactly calculable[13],

$$\beta(\alpha) = -\frac{3N_c\alpha^2}{2\pi}\,\frac{1}{1-N_c\alpha/(2\pi)}\,. \tag{1.18}$$

By "exactly" I mean that all orders of perturbation theory are known, and one can additionally show that in the case at hand there are no nonperturbative contributions.

Equations (1.17) and (1.18) historically were the first examples of non-trivial (i.e. non-vanishing) quantities exactly calculated in four-dimensional field theories in the strong coupling regime. These examples, alone, show that the supersymmetric gauge dynamics is full of hidden miracles. We will encounter many more examples in what follows. Eventually, after learning more about supersymmetric theories, you will be able to understand how Eqs. (1.17) and (1.18) are derived. But this will take some time. Here I would like only to add an explanatory remark regarding the vacuum degeneracy in supersymmetric gluodynamics. At the classical level Lagrangian (1.14) has a $U(1)$ symmetry corresponding to the phase rotations of the gluino fields,

$$\lambda \to e^{i\alpha}\lambda\,. \tag{1.19}$$

The corresponding current is sometimes called the R_0 current; it is a superpartner of the energy-momentum tensor and the supercurrent. The R_0 current exists in any supersymmetric theory. Moreover, in conformally invariant theories – and supersymmetric gluodynamics is conformally invariant at the classical level – it is conserved[14]. In the spinor notation the R_0 current has the form $J_{\alpha\dot\alpha} = (1/g^2)\bar\lambda_{\dot\alpha}\lambda_\alpha$, while in the Majorana notation the very same current takes the form $J_\mu = (1/g^2)\bar\lambda\gamma_\mu\gamma_5\lambda$. (Let me parenthetically note that the vector current of the Majorana gluino identically vanishes. The proof of this fact is left as an exercise.) The conservation of the axial current above is broken by the triangle anomaly,

$$\partial^\mu J_\mu = \frac{N_c}{16\pi^2} G^a_{\mu\nu}\tilde G^a_{\mu\nu}\,.$$

So, there is no continuous $U(1)$ symmetry in the model. By the same token, the conformal invariance is ruined by the anomaly in the trace of the energy-momentum tensor. As a matter of fact, the divergence of the R_0 current and the trace of the energy-momentum tensor can be combined in one superfield[15] (see the Appendix).

However, a remnant of the would-be symmetry remains, in the form of the discrete phase transformations of the type (1.19) with $\alpha = \pi k/N_c$. The gluino condensate further breaks this symmetry to Z_2 corresponding to $\lambda \to -\lambda$. The number of degenerate vacuum states, N_c, coincides with Witten's index for the $SU(N_c)$ theory[16], an invariant

*I hasten to add that it was argued recently[11] that supersymmetric gluodynamics actually has two phases: one with the spontaneously broken Z_{2N_c} invariance, and another, unconventional, phase where the chiral Z_{2N_c} symmetry is unbroken and the gluino condensate does not develop. Dynamics of the chirally symmetric phase is drastically different from what we got used to in QCD. In particular, although no invariance is spontaneously broken, massless particles appear, and no mass gap is generated. This development is too fresh, however, to be included in this lecture course.
The existence of the gluino condensate was anticipated[12] from the analysis of the so called Veneziano-Yankielowicz effective Lagrangian, even prior to the first dynamical calculation[9]. The Veneziano-Yankielowicz Lagrangian, very useful for orientation, is not a genuinely Wilsonian construction, and one must deal with it extremely cautiously in extracting consequences. For a recent discussion see Ref.[11].

which counts the number of the boson zero energy states minus the number of the fermion zero energy states. If Witten's index is non-vanishing supersymmetry cannot be spontaneously broken, of course.

An interesting aspect, related to the discrete degeneracy of the vacuum states, is the ϑ dependence. What happens with the vacua if $\vartheta \neq 0$? The question was answered in Ref.[10]. The ϑ dependence of the gluino condensate is

$$\langle\lambda\lambda\rangle_\vartheta = \langle\lambda\lambda\rangle_{\vartheta=0} e^{\frac{i\vartheta}{N_c}} . \tag{1.20}$$

This shows that the N_c vacua are intertwined as far as the ϑ evolution is concerned. When ϑ changes continuously from 0 to 2π the first vacuum becomes second, the second becomes third, and so on, in a cyclic way.

SU(2) SQCD with one flavor

As the next step on a long road leading us to understanding of supersymmetric gauge dynamics we will consider SUSY generalization of $SU(2)$ QCD with the matter sector consisting of one flavor. This model will serve us as a reference point in all further constructions.

Since the gauge group is $SU(2)$ we have three gluons and three superpartners – gluinos. As far as the matter sector is concerned, let us remember that one quark flavor in QCD is described by a Dirac field, a doublet with respect to the gauge group. One Dirac field is equivalent to two chiral fields: a left-handed and a right-handed, both transforming according to the fundamental representation of $SU(2)$. Moreover, the right-handed doublet is equivalent to the left-handed anti-doublet, which in turn is equivalent to a doublet. The latter fact is specific to the $SU(2)$ group, all whose representations are (pseudo)real. Thus, the Dirac quark reduces to two left-handed Weyl doublet fields.

Correspondingly, in SQCD each of them will acquire a scalar partner. Thus, the matter sector will be built from two superfields, S_1 and S_2. In what follows we will use the notation S^α_f where $\alpha = 1, 2$ is the color index, and $f = 1, 2$ is a "subflavor" index. Two subflavors comprise one flavor. The chiral superfield has the usual form, see Eq. (1.5).

In the superfield language the Lagrangian of the model can be represented in a very concise form

$$\mathcal{L} = \frac{1}{2g_0^2} \operatorname{Tr} \int d^2\theta W^2 + \frac{1}{4} \int d^2\theta d^2\bar\theta \bar S_f e^V S_f + \left(\frac{m_0}{4} \int d^2\theta S^{\alpha f} S_{\alpha f} + \text{H.c.} \right) , \tag{1.21}$$

where the superfields V and W_α are matrices in the color space, for instance, $V \equiv V^a \tau^a/2$, with τ^a denoting the Pauli matrices. The subscript 0 indicates that the mass parameter and the gauge coupling constant are bare parameters, defined at the ultraviolet cut off. In what follows we will omit this subscript to ease the notation in several instances where it is unimportant.

If we take into account the rules of integration over the Grassmann numbers we immediately see that the integral over $d^2\theta$ singles out the θ^2 component of the chiral superfields W^2 and S^2, i.e. the F terms. Moreover, the integral over $d^2\theta d^2\bar\theta$ singles out the $\theta^2\bar\theta^2$ component of the real superfield $\bar S e^V S$, i.e. the D term.

Note that the $SU(2)$ model under consideration, with *one flavor* possesses a global $SU(2)$ ("subflavor") invariance allowing one to freely rotate the superfields $S_1 \leftrightarrow S_2$. This symmetry holds even in the presence of the mass term, see Eq. (1.21), and is specific for $SU(2)$ gauge group, with its pseudoreal representations. All indices corresponding to the $SU(2)$ groups (gauge, Lorentz and subflavor) can be lowered and raised by means of the ε symbol, according to the general rules.

The Lagrangian presented in Eq. (1.21) is not generic. Renormalizable models with a richer matter sector usually allow for one more type of F terms, namely

$$\int d^2\theta S^3 \,.$$

These terms are called the Yukawa interactions, since one of the vertices they include corresponds to a coupling of two spinors to a scalar. Strictly speaking, they should be called the super-Yukawa terms, since spinor-spinor-scalar vertices arise also in the (super)gauge parts of the Lagrangian. This jargon is widely spread, however; eventually you will get used to it and learn how to avoid confusion. The combination of the F terms S^2+S^3 is generically referred to as superpotential. The conventional potential of self-interaction of the scalar fields stemming from the given superpotential is referred to as *scalar potential*.

It is instructive to pass from the superfield notations to components. We will do this exercise now in some detail, putting emphasis on those features which are instrumental in the solutions to be discussed below. Once the experience is accumulated the need for the component notation will subside.

Let us start from W^2. The corresponding F term was already discussed below Eq. (1.16). There is one new important point, however. We omitted the square of the D term present in $W^2|_F$, see Eq. (1.13),

$$\Delta\mathcal{L}_D^{(W)} = \frac{1}{2g^2} D^a D^a \,. \tag{1.22}$$

If the matter sector of the theory is empty, this term is unimportant. Indeed, the D field enters with no derivatives, and, hence, can be eliminated from the Lagrangian by virtue of the equations of motion. With no matter fields $D=0$. In the presence of the matter fields, however, eliminating D we get a non-trivial term constructed from the scalar fields, which is of a paramount importance. This point will be discussed later; here let me only note that the sign of D^2 in the Lagrangian, Eq. (1.22), is unusual, it is positive.

The next term to be considered is $\int d^2\theta d^2\bar\theta \bar S e^V S$. Calculation of the D component of $\bar S e^V S$ is a more time-consuming exercise since we must take into account the fact that S depends on x_L while $\bar S$ depends on x_R; both arguments differ from x. Therefore, one has to expand in this difference. The factor e^V sandwiched between $\bar S$ and S covariantizes all derivatives. Needless to say that the field V is treated in the Wess-Zumino gauge. It is not difficult to check that

$$\frac{1}{4}\int d^2\theta d^2\bar\theta \bar S_f e^V S_f = \bar\psi_f i \not{\mathcal{D}} \psi_f - \phi_f^\dagger \mathcal{D}^2 \phi_f +$$

$$\left[(\psi_f\lambda)\phi_f^\dagger + \text{H.c.}\right] + D^a \phi_f^\dagger T^a \phi_f \tag{1.23}$$

where T^a are the matrices of the color generators. In the $SU(2)$ theory $T^a = \tau^a/2$. Now we see why the D^2 term is so important in the presence of matter; D^a does not vanish anymore. Moreover, using the equation of motion we can express D^a in terms of the squark fields, generating in this way a quartic self-interaction of the scalar fields,

$$V_D = \frac{1}{2g^2} D^a D^a \,, \quad D^a = -\frac{g^2}{2}\left(\phi_1^\dagger \tau^a \phi_1 + \phi_2^\dagger \tau^a \phi_2\right) \,. \tag{1.24}$$

In the old-fashioned language of the pre-SUSY era one would call the term $(\psi_f\lambda)\phi_f^\dagger$ from Eq. (1.23) the Yukawa interaction. The SUSY practitioner would refer to this term as to the gauge coupling since it is merely a supersymmetric generalization of the quark-quark-gluon coupling. I mention these terms here because later on their analysis will help us establish the form of the conserved R currents.

Vacuum valleys

Let us examine the D potential V_D more carefully, neglecting for the time being F terms altogether. As is well-known, the energy of any state in any supersymmetric theory is positive-definite. The minimal energy state, the vacuum, has energy exactly at zero. Thus, in determining the classical vacuum we must find all field configurations corresponding to vanishing energy. From Eq. (1.24) it is clear that in the Wess-Zumino gauge the *classical space of vacua* (sometimes called the moduli space of vacua) is defined by the *D-flatness condition*

$$D^a = 0 \text{ for all } a\,. \tag{1.25}$$

More exactly, Eq. (1.25) is called the Wess-Zumino gauge D flatness condition. Since this gauge is always implied, we will omit the reference to the Wess-Zumino gauge.

The D potential V_D represents a quartic self-interaction of the scalar fields, of a very peculiar form. Typically in the ϕ^4 theory the potential has one – at most several – minima. In other words, the space of the vacuum fields corresponding to minimal energy, is a set of isolated points. The only example with a continuous manifold of points of minimal energy which was well studied previously is the spontaneous breaking of a global continuous symmetry, say, $U(1)$. In this case all points belonging to this vacuum manifold are physically equivalent. The D potential (1.24) has a specific structure – minimal (zero) energy is achieved along entire directions corresponding to the solution of Eq. (1.25). It is instructive to think of the potential as of a mountain ridge; the D flat directions then present the flat bottom of the valleys. Sometimes, for transparency, I will call the D flat directions *the vacuum valleys*. Their existence was first noted in Ref.[17]. As we will see, different points belonging to the bottom of the valleys are physically inequivalent. This is a remarkable feature of the supersymmetric gauge theories.

In the case of the $SU(2)$ theory with one flavor it is not difficult to find the D flat direction explicitly. Indeed, consider the scalar fields of the form

$$\phi_1 = v\begin{pmatrix}1\\0\end{pmatrix}\,,\quad \phi_2 = v\begin{pmatrix}0\\1\end{pmatrix}\,, \tag{1.26}$$

where v is an arbitrary complex constant. It is obvious that for any value of v all D^a's vanish. D^1 and D^2 vanish because $\tau^{1,2}$ are off-diagonal matrices; D^3 vanishes after summation over two subflavors.

It is quite obvious that if $v \neq 0$ the original gauge symmetry $SU(2)$ is totally spontaneously broken. Indeed, under the condition (1.26) all three gauge bosons acquire masses $\sim gv$. Thus, we deal here with the supersymmetric generalization of the Higgs phenomenon. Needless to say that supersymmetry is not broken. It is instructive to trace the reshuffling of the degrees of freedom before and after the Higgs phenomenon. In the unbroken phase, corresponding to $v = 0$, we have three massless gauge bosons (6 degrees of freedom), three massless gaugino (6 degrees of freedom), four matter fermions (the Weyl fermions, 8 degrees of freedom), and four matter scalars (complex scalars, 8 degrees of freedom). In the broken phase three matter fermions combine with the gauginos to form three massive Dirac fermions (12 degrees of freedom). Moreover, three matter scalars combine with the gauge fields to form three *massive* vector fields

(9 degrees of freedom) plus three massive (real) scalars. What remains massless? One complex scalar field, corresponding to the motion along the bottom of the valley, v, and its fermion superpartner, one Weyl fermion. The balance between the fermion and boson degrees of freedom is explicit.

A gauge invariant description of the system of the vacuum valleys was suggested in Refs.[18, 19] (see also[17]). In these works it was noted that the set of proper coordinates parametrizing the space of the classical vacua is nothing else but the set of all independent (local) products of the chiral matter fields existing in the theory. Since this point is very important let me stress once more that the variables to be included in the set are polynomials built from the fields of one and the same chirality *only*. These variables are clearly gauge invariant.

At the intuitive level this assertion is almost obvious. Indeed, if there is a D flat direction, the motion along the degenerate bottom of the valley must be described by some effective (i.e. composite) chiral superfield, which is gauge invariant and has no superpotential. The opposite is also true. If we are able to build some chiral gauge invariant (i.e. colorless) superfield Φ, as a local product of the chiral matter superfields of the theory at hand, then the energy is guaranteed to vanish, since (in the absence of the F terms) all terms which might appear in the effective Lagrangian for ϕ necessarily contain derivatives. Here ϕ is the lowest component of the above superfield Φ. In other words, then, changing the value of ϕ we will be moving along the bottom of the valley.

A formal proof of the fact that the classical vacua are fully described by the set of local (gauge invariant) products of the chiral fields comprising the matter sector is given in the recent work[20] which combines and extends results scattered in the literature[17, 18, 21, 22].

The approach based on the chiral polynomials is very convenient for establishing the fact of the existence (non-existence) of the moduli space of the classical vacua, and in counting the dimensionality of this space. For instance, in the $SU(2)$ model with one flavor there exists only one invariant, $S^2 \equiv S_{\alpha f} S_{\beta g} \epsilon^{\alpha\beta}\epsilon^{fg} = 2v^2$. Correspondingly, there is only one vacuum valley – a one-dimensional complex manifold. The remaining three (out of four) complex scalar fields are eaten up in the super-Higgs phenomenon by the vector fields, which immediately tells us that a generic point from the bottom of the valley corresponds to fully broken gauge symmetry.

In other cases we will have a richer structure of the moduli space of the classical vacua. In some instances no chiral invariants can be built at all. Then the D flat directions are absent.

If the D flat directions exist, and the gauge symmetry is spontaneously broken, then the constraints of the type of Eq. (1.26) can be viewed as a gauge fixing condition. This is nothing else but the unitary gauge in SUSY. Those components of the matter superfields which are set equal to zero are actually eaten up by the vector particles which acquire the longitudinal components through the super-Higgs mechanism.

Although constructing the set of the chiral invariants is helpful in the studies of the general properties of the space of the classical vacua, sometimes it is still necessary to explicitly parametrize the vacuum valleys, just in the same way as it is done in Eq. (1.26). As we have seen, this problem is trivially solvable in the $SU(2)$ model with one flavor. For higher groups and representations the general situation is much more complicated, and the generic solution is not found. Many useful tricks for finding explicit parametrization of the vacuum valleys in particular examples were suggested in Refs.[17, 18, 19]. A few simplest examples are considered below. More complicated instances are considered in the literature. For instance, a parametrization of the valleys in the $SU(5)$ model with two quintets and two antidecuplets was given in Ref.[23] and in the E(6) model with the 27-plet in Ref.[24]. The correspondence between the explicit parametrization of the D flat directions and the chiral polynomials was discussed recently more than once, see e.g.[25-29]. I would like to single out Ref.[30] where a catalog of the flat directions in the minimal supersymmetric standard model (MSSM) was obtained by analyzing all possible chiral polynomials and eliminating those of them which are redundant.

Once the existence of the D-flat directions is established at the classical level one may be sure that a manifold of the degenerate vacua will survive at the quantum level, provided no F terms appear in the action which might lift the degeneracy. Indeed, in this case the only impact of the quantum corrections is providing an overall Z factor in front of the kinetic term, which certainly does not affect the vanishing of the D terms. The F terms which could lift the degeneracy must be either added in the action by hand (e.g. mass terms), or generated nonperturbatively. A remarkable non-renormalization theorem[31] guarantees that no F terms can be generated perturbatively. We will return to the discussion of this *second miracle* of SUSY further on.

In this respect the supersymmetric theories are fundamentally different from the non-supersymmetric ones. Say, in the good old ϕ^4 theory with the Yukawa interaction

$$\mathcal{L} = |\partial_\mu \phi|^2 + \bar{\psi} i \partial_\mu \gamma^\mu \psi + (g\phi\bar{\psi}\psi + \text{H.c.})$$

we could also assume that that the mass and self-interaction of the scalar field vanish at the classical level. Then, classically, we will have a flat direction – any constant value of ϕ corresponds to the vanishing vacuum energy. However, this vacuum valley does not survive inclusion of the quantum corrections. Already at the one-loop level both the mass term of the scalar field, and its self-interaction, will be generated, and the continuous vacuum degeneracy will inevitably disappear.

In search of the valleys

Although our excursion in the $SU(2)$ model with one flavor is not yet complete, the issue of the D flat directions is so important in this range of problems that we pause here to do, with pedagogical purposes, a few simple exercises. If you choose to skip this section in the first reading it will be necessary to return to it later.

$SU(N_c)$ model with N_f flavors ($N_f < N_c$)[18]

The matter sector includes $2N_f$ subflavors – N_f chiral fields in the fundamental representation of $SU(N_c)$, $S^{\alpha f}$, and N_f chiral fields in the antifundamental representation, $\tilde{S}_{\alpha f}$, where $\alpha = 1, ..., N_c$ and $f = 1, ...N_f$. It is quite obvious that one can form N_f^2 chiral products of the type

$$\tilde{S}^f_\alpha S^{\alpha g}\,, \quad f, g = 1, ..., N_f\,. \tag{1.27}$$

All these chiral invariants are independent. Thus, the moduli space of the classical vacua (the vacuum valley) is a complex manifold of dimensionality N_f^2, parametrized by the coordinates (1.27). A generic point from the vacuum valley corresponds to spontaneous breaking of $SU(N_c) \to SU(N_c - N_f)$ (except for the case when $N_f = N_c - 1$, when the original gauge group is completely broken). The number of the broken generators is $2N_cN_f - N_f^2$; hence, the same amount of the complex scalar fields are eaten up in the super-Higgs mechanism. The original number of the complex scalar fields was $2N_cN_f$. The remaining N_f^2 degrees of freedom are the moduli (1.27) corresponding to the motion along the bottom of the valley.

In this particular problem it is not difficult to indicate a concrete parametrization of the vacuum field configurations. Indeed, consider a set

$$S_1 = \tilde{S}_1^\dagger = v_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \ldots \\ 0 \end{pmatrix}, S_2 = \tilde{S}_2^\dagger = v_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \ldots \\ 0 \end{pmatrix}, \ \ldots, S_{N_f} = \tilde{S}_{N_f}^\dagger = v_{N_f} \begin{pmatrix} 0 \\ 0 \\ \ldots \\ 1 \\ \ldots \\ 0 \end{pmatrix} \tag{1.28}$$

where the unity in S_{N_f} occupies the N_f-th line and $v_{1,2,\ldots,N_f}$ are N_f arbitrary complex numbers. It is rather obvious that for this particular set all D terms vanish. In verifying this assertion it is convenient to consider first those D^a's which lie outside the Cartan subalgebra of $SU(N_c)$. Since the corresponding T^a matrices are off-diagonal each term in the sum $\sum_f (S^\dagger T^a S + \tilde{S}^\dagger T^a \tilde{S})$ vanishes individually. For the generators from the Cartan subalgebra the fundamentals and anti-fundamentals cancel each other.

The point (1.28) is *not* a generic point from the bottom of the valley. This is clear from the fact that it is parametrized by only N_f complex numbers. To get a generic solution one observes that the theory is invariant under the global $SU(N_f) \times SU(N_f)$ flavor rotations (the fundamentals and antifundamentals can be rotated separately). On the other hand, the solution (1.28) is not invariant. Therefore, we can apply a general $SU(N_f) \times SU(N_f)$ rotation to Eq. (1.28) without destroying the condition $D^a = 0$. It is quite obvious that the generators belonging to the Cartan subalgebra of $SU(N_f) \times SU(N_f)$ do not introduce new parameters. The remaining rotations introduce $N_f^2 - N_f$ complex parameters, to be added to $v_1, \ldots, v_{N_f}$, altogether N_f^2 parameters, as it was anticipated from counting the number of chiral invariants.

$SU(N_c)$ model with N_f flavors ($N_f = N_c + 1$)

The vacuum valley is parametrized by N_f^2 complex parameters, although the number of the chiral invariants is larger, $N_f^2 + 2N_f$. Not all chiral invariants are independent. For further details see Eq. (2.8) and further.

$SU(5)$ model with one quintet and one (anti)decuplet

This gauge model describes Grand Unification, with one generation of quarks and leptons. This is our first example of non-chiral matter; it is singled out historically – the instanton-induced dynamical supersymmetry breaking was first found in this model[32]. The quintet field is V^α, the (anti)decuplet field is antisymmetric $X_{\alpha\beta}$. It is quite obvious that there are no chiral invariants at all. Indeed, the only candidate, VVX, vanishes due to antisymmetricity of $X_{\alpha\beta}$. This means that no D flat directions exist. The same conclusion can be reached by explicitly parametrizing V and X; inspecting then the D-flatness conditions one can conclude that they have no solutions, see e.g. Appendix A in Ref.[33].

$SU(5)$ model with two quintets and two (anti)decuplets and no superpotential

This model (with a small tree-level superpotential term) was the first example of the instanton-induced supersymmetry breaking in the weak coupling regime[34]. It presents another example of the anomaly-free chiral matter sector. Unlike the one-family model (one quintet and one antidecuplet) flat directions do exist (in the absence of a superpotential). The system of the vacuum valleys in the two-family $SU(5)$ model was analyzed in Ref.[23]. Generically, the gauge $SU(5)$ symmetry is completely broken, so that 24 out of 30 chiral matter superfields are eaten up in the super-Higgs mechanism. Therefore, the vacuum valley should be parametrized by six complex moduli.

Denote the two quintets present in the model as V_f^α ($f = 1, 2$), and the two antidecuplets as $(X_{\bar{g}})_{\alpha\beta}$ where $\bar{g} = 1, 2$ and the matrices $X_{\bar{g}}$ are antisymmetric in the color indices α, β. Indices f and $\bar{g}$ reflect the $SU(2)_X \times SU(2)_V$ flavor symmetry of the model. Six independent chiral invariants are

$$M_{\bar{g}} = V_k X_{\bar{g}} V_l \epsilon^{kl} ,$$

$$B_{\bar{g}f} = X_{\bar{g}} X_{\bar{k}} X_{\bar{l}} V_f \epsilon^{\bar{k}\bar{l}} ,$$

where the gauge indices in the first line are convoluted in a straightforward manner $V^\alpha X_{\alpha\beta} V^\beta$, while in the second line one uses the ϵ symbol,

$$X_{\bar{g}} X_{\bar{k}} X_{\bar{l}} V_f = \epsilon^{\alpha\beta\gamma\delta\rho} (X_{\bar{g}})_{\alpha\beta} (X_{\bar{k}})_{\gamma\delta} (X_{\bar{l}})_{\rho\kappa} (V_f)^\kappa .$$

The choice of invariants above implies that there are no moduli transforming as $\{4,\ 2\}$ under the flavor group (such moduli vanish).

In this model the explicit parametrization of the valley is far from being obvious, to put it mildly. The most convenient strategy for the search is analyzing the five-by-five matrix

$$D^\alpha_\beta = V^\alpha_f \bar{V}^f_\beta - 2(\bar{X}^{\bar{g}})^{\alpha\gamma} (X_{\bar{g}})_{\gamma\beta} \tag{1.29}$$

where $\bar{V} = V^\dagger$ and $\bar{X} = X^\dagger$. If this matrix is proportional to the unit one, the vanishing of the D terms is guaranteed. (Similar strategy based on analyzing analogs of Eq. (1.29) is applicable in other cases as well).

A solution of the D-flatness condition which contains 7 real parameters looks as follows:

$$V_1 = \begin{pmatrix} a_1 \\ 0 \\ 0 \\ a_4 \\ 0 \end{pmatrix} , \quad V_2 = \begin{pmatrix} 0 \\ a_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} ,$$

$$X_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & s & 0 \\ 0 & 0 & b & 0 & 0 \\ 0 & -b & 0 & 0 & f \\ -s & 0 & 0 & 0 & 0 \\ 0 & 0 & -f & 0 & 0 \end{pmatrix} , \quad X_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & d & 0 & 0 & g \\ -d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h \\ -g & 0 & 0 & -h & 0 \end{pmatrix} , \tag{1.30}$$

where

$$|a_1|^2 = r^2 \cos^2\theta\, , \quad |a_4|^2 = r^2 \cos^2\alpha \tan^2\theta\, , \quad |a_2|^2 = r^2 \tan^2\theta \frac{\cos^2\theta - cos^2\alpha}{\sin^2\theta + \cos^2\alpha} ,$$

and

$$|s|^2 = \frac{r^2 \cos^2\alpha}{\cos^2\theta \sin^2\theta + \cos^2\alpha} , \quad |b|^2 = r^2 \sin^2\theta + \cos^2\alpha\, , \quad |f|^2 = r^2 \frac{\cos^2\alpha}{\cos^2\theta} \frac{\cos^2\theta - cos^2\alpha}{\sin^2\theta + \cos^2\alpha} ,$$

$$|d|^2 = \frac{r^2}{\cos^2\theta} (cos^2\theta - cos^2\alpha)\, , \quad |g|^2 = r\cos^2\alpha\, , \quad |h|^2 = r^2 \sin^2\theta\, . \tag{1.31}$$

Thus, the absolute values of matrix elements are parametrized by three real parameters r, α and θ. Additional 4 parameters appear *via* phases δ of 9 elements a_i, s, b, f, d, g, h. Three phases, out of nine, are related to gauge rotations and are not observable in the gauge singlet sector. Additionally, there are two constraints,

$$\delta_1 - \delta_4 = \delta_h - \delta_g ,$$

$$\delta_f - \delta_b = \delta_g - \delta_d ,$$

which are readily derived from vanishing of the off-diagonal terms in D^α_β.

Substituting the above expressions it is easy to check that the invariants

$$B_{\bar{g}_1\bar{g}_2\bar{g}_3 f} = X_{\{\bar{g}_1}X_{\bar{g}_2}X_{\bar{g}_3\}}V_f$$

symmetrized over g_1, g_2, g_3 (i.e. the $\{4, 2\}$ representation of $SU(2)_X \times SU(2)_V$) do vanish, indeed.

The most general valley parametrization depends on 12 real parameters, while so far we have only 7. The remaining five parameters are provided by $SU(2)_X \times SU(2)_V$ flavor rotations of the configuration (1.30).

Back to the SU(2) model – dynamics of the flat direction

After this rather lengthy digression into the general theory of the vacuum valleys we return to our simplest toy model, $SU(2)$ with one flavor. The vacuum valley in this case is parametrized by one complex number, v, which can be chosen at will since for any value of v the vacuum energy vanishes. One can quantize the theory near any value of v. If $v \neq 0$, the theory splits into two sectors – one containing massive particles which form SU(2) triplets, and another sector which includes only one massless Weyl fermion and one massless complex scalar field. These massless particles are singlets with respect to both $SU(2)$ groups – color and subflavor.

So far we totally disregarded the mass term in the action, assuming $m = 0$. If $m \neq 0$ the corresponding term in the superpotential lifts the vacuum degeneracy, making the bottom of the valley non-flat. Indeed,

$$F = mv\,,$$

the corresponding contribution in the scalar potential is

$$\Delta V = |mv|^2\,,$$

which makes the theory "slide down" towards the origin of the valley. Since the perturbative corrections do not renormalize the F terms, this type of behavior – sliding down to the origin of the former valley – is preserved to any finite order in perturbation theory.

What happens if one switches on nonperturbative effects?

The non-renormalization theorem[31], forbidding the occurrence of the F terms, does not apply to nonperturbative effects, which, thus, may or may not generate relevant F terms. The possibility of getting a superpotential can be almost completely investigated by analyzing the general properties of the model at hand, with no explicit calculations. Apart from the overall numerical constant, the functional form of the superpotential, if it is generated, turns out to be fixed.

Let me elucidate this point in more detail. First, on what variables can the superpotential depend? The vector fields are massive and are integrated over. Thus, we are left with the matter fields only, and the only chiral invariant is

$$I = S^{\alpha f}S_{\alpha f}\,. \tag{1.32}$$

The superpotential, if it exists, must have the form

$$\mathcal{L}_F = \int d^2\theta f(I(x_L,\theta)) + \text{H.c.} \tag{1.33}$$

where f is some function. Notice that the mass term has just this structure, with $f(z) = z$. We will discuss the possible impact of the mass term later, assuming at the beginning that $m = 0$.

Now, our task is to find the function f exploiting the symmetry properties of the model. At the classical level there exist two conserved currents. One of them, the R_0 current, is the superpartner of the energy-momentum tensor and the supercurrent[35]. The divergence of the R_0 current and the trace of the energy-momentum tensor can be combined in one superfield. The R_0 current exists in any supersymmetric theory, and, moreover, in conformally invariant theories it is conserved. Indeed, since the trace of the energy-momentum tensor vanishes in conformally invariant theories the divergence of R_0 vanishes as well. In our present model the R_0 current corresponds to the following rotations of the fields

$$\lambda_\alpha \to e^{i\beta}\lambda_\alpha\,, \quad \psi^f_\alpha \to e^{-(i/3)\beta}\psi^f_\alpha\,, \quad \phi^f_\alpha \to e^{(2i/3)\beta}\phi^f_\alpha\,. \tag{1.34}$$

If we denote this current by R^0_μ, then

$$R^0_\mu = \frac{1}{g^2}\bar{\lambda}\gamma_\mu\gamma_5\lambda - \frac{1}{3}\sum_f\left(\bar{\psi}^f\gamma_\mu\gamma_5\psi^f - 2\phi^{f\dagger}\overleftrightarrow{D}_\mu\,\phi^f\right)\,. \tag{1.35}$$

The relative phase between ψ and ϕ is established in the following way. Let us try to add the S^3 term to the superpotential. In the model at hand it is actually forbidden by the color gauge invariance, but we ignore this circumstance, since the form of the R_0 current is general, and in other models the S^3 term is perfectly allowed. It violates neither supersymmetry, nor the conformal invariance (at the classical level), which is obvious from the fact that its dimension is 3. The S^3 term in the superpotential produces a $\phi\psi^2$ term in the Lagrangian. In this way we arrive at the relative phase between ψ and ϕ indicated in Eq. (1.34). The relative phase between λ and ψ is fixed by requiring the term $\lambda\psi\phi^\dagger$ in the Lagrangian (the supergeneralization of the gauge coupling) to be invariant under the $U(1)$ transformation at hand. In the superfield language the last two transformations in Eq. (1.34) can be concisely written as

$$S^f \to \exp\left(\frac{2i}{3}\beta\right)S^f\,, \quad \theta_\alpha \to \exp(i\beta)\theta_\alpha\,. \tag{1.36}$$

The second (classically) conserved current is built from the matter fields,

$$J_\mu = \sum_f\left(\bar{\psi}^f\gamma_\mu\gamma_5\psi^f + \phi^{f\dagger}\overleftrightarrow{D}_\mu\,\phi^f\right)\,. \tag{1.37}$$

The corresponding $U(1)$ transformation is

$$\psi^f_\alpha \to e^{i\gamma}\psi^f_\alpha\,, \quad \phi^f_\alpha \to e^{i\gamma}\phi^f_\alpha\,, \tag{1.38}$$

or, in the superfield notation, $S^f(x_L,\theta) \to \exp(i\gamma)S^f(x_L,\theta)$.

The conservation of both axial currents is destroyed by the quantum anomalies. At one-loop level

$$\partial_\mu R^0_\mu = \frac{5}{3}\frac{\alpha}{4\pi}G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}\,, \quad \partial_\mu J_\mu = \frac{\alpha}{4\pi}G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}\,. \tag{1.39}$$

One can form, however, one linear combination of these two currents,

$$R_\mu = R^0_\mu - \frac{5}{3}J_\mu \tag{1.40}$$

which is anomaly free and is conserved even at the quantum level. The occurrence of a strictly conserved axial current, the so called R current, is a characteristic feature of many supersymmetric models. In what follows we will have multiple encounters with the R currents in various models. The one presented in Eq. (1.40) was given in Ref.[18].

Combining both transformations, Eqs. (1.34) and (1.38), in the appropriate proportion, see Eq. (1.40), we conclude that the $SU(2)$ model under consideration is strictly invariant under the following transformation

$$S^f \to e^{-i\beta} S^f \,, \quad \theta_\alpha \to e^{i\beta}\theta_\alpha \,. \tag{1.41}$$

This R invariance leaves us with a unique possible choice for the superpotential

$$\mathcal{L}_F = \text{const.} \times \int d^2\theta \frac{1}{I(x_L, \theta)} \to \text{const.} \times \int d^2\theta \frac{\Lambda^5}{S^{\alpha f} S_{\alpha f}} \tag{1.42}$$

where Λ is a scale parameter of the model, and the factor Λ^5 has been written out on the basis of dimensional arguments.

Whether the superpotential is actually generated, depends on the value of the numerical constant above. In principle, it could have happened that the constant vanished. However, since no general principle forbids the F term (1.42), the vanishing seems highly improbable. And indeed, the direct one-instanton calculation in the weak coupling regime (i.e. $v^2 \gg \Lambda^2$) shows[18, 23] that this term is generated.

The impact of the term (1.42) is obvious.The corresponding extra contribution to the self-interaction energy of the scalar field is

$$\Delta V = |F|^2 = 4 \frac{\Lambda^{10}}{|v|^6} \tag{1.43}$$

where the numerical constant is included in the definition of Λ. Thus, we see that the instanton-generated contribution ruins the indefinite equilibrium along the bottom of the valley, pushing the theory away from the origin. As a matter of fact, in the absence of the mass term, $m = 0$, the theory does not have any stable vacuum at all since the minimal (zero) energy is achieved only at $|v| \to \infty$. We encounter here an example of the *run-away vacuum* situation.

Switching on the mass term blocks the exits from the valley. Indeed, now

$$F = mv - 2\frac{\Lambda^5}{v^3} \,, \tag{1.44}$$

and the lowest energy state shifts to a finite value of v. It is easy to see that now there are two points at the bottom of the former valley where the energy vanishes, namely

$$v^2 = \sqrt{2} \frac{\Lambda^{5/2}}{m^{1/2}} \text{ and } v^2 = -\sqrt{2}\frac{\Lambda^{5/2}}{m^{1/2}} \,. \tag{1.45}$$

In other words, the continuous vacuum degeneracy is lifted, and only two-fold degeneracy survives; the theory has two vacuum states. The number of vacuum states could have been anticipated from a general argument based on Witten's index[16].

I pause here to make a few remarks. First, we observe that the supersymmetric version of QCD dynamically has very little in common with QCD. Indeed, the chiral limit of QCD, when all quark masses are set equal to zero, is non-singular – nothing spectacular happens in this limit except that the pions become strictly massless. At the same time, in the supersymmetric $SU(2)$ model at hand the limit of the massless matter fields results in the run-away vacuum. This situation is quite general, and takes place in many models, although not in all.

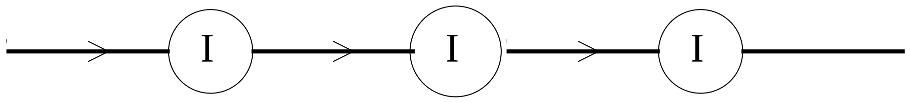

Figure 2. One-particle reducible graphs which might lead to $1/\sigma$ terms. This mechanism is not (and must not be) included in the superpotential.

Second, the analysis of the dynamics of the flat directions presented above is somewhat simplified. Two subtle points deserve mentioning. The general form of the superpotential compatible with the symmetry of the model was established in the massless limit. In this way we arrived at Eq. (1.42). The mass term was then introduced to avoid the run-away vacuum. If $m \neq 0$ the R current is not conserved any more, even at the classical level. To keep the invariance (1.41) alive one must simultaneously rotate the mass parameter,

$$m \to e^{4i\beta} m\,. \tag{1.46}$$

Let us call this invariance, supplemented by the phase rotation of the mass parameter, an *extended R symmetry*.

One could think of m as of a vacuum expectation value of some auxiliary chiral field, to be rotated in a concerted way in order to maintain the R invariance. (We will discuss this trick later on in more detail). It is clear then that multiplying Eq. (1.42) by any function of the dimensionless complex parameter $\sigma = mS^4/\Lambda^5$ is not forbidden by the extended symmetry. Extra arguments are needed to convince oneself that this additional function actually does not appear. Let us assume it does. Then it should be expandable in the Laurent series of the type

$$\sum_n \frac{C_n}{\sigma^n}\,.$$

If negative powers of n were present then the function would grow at large $|\sigma|$, a behavior one can immediately reject on physical grounds. The masses of the heavy particles, which we integrate over to obtain the superpotential, are proportional to gv; large values of $|v|$ imply heavier masses, which implies, in turn, that the impact on the superpotential should be weaker. Thus, all C_n's with negative n must vanish. Positive n are not acceptable as well. If positive powers of n were present then the function would blow up at fixed $|v|$ and $m \to 0$. At fixed $|v|$, however, no dynamically nontrivial singularity develops in the theory. The only mechanism which could provide powers of m in the denominator is a chain of instantons connected by one massless fermion line depicted on Fig. 2. The corresponding contribution, however, is one-particle reducible and should not be included in the effective Lagrangian. This concludes our proof of the fact that Eq. (1.42) is exact.

The second subtle point is related to the discussion of the anomalies in the R_0 and matter axial currents. The consideration presented above assumes that both anomalies are one-loop. Actually, the anomaly in the R_0 current is multiloop[36]. This fact slightly changes the form of the conserved R current. The very fact of existence of the R current remains intact. All expressions for the currents and charges presented above refer to the extreme ultraviolet where the gauge coupling (in asymptotically free theories) tends to zero. The final conclusion that the only superpotential compatible with the symmetry of the model is that of Eq. (1.42) is valid[37].

Thirdly, the consideration above (Eqs. (1.42), (1.45)) strictly speaking, does not tell us what happens at the origin of the valley, $v^2 = 0$, where all expressions become

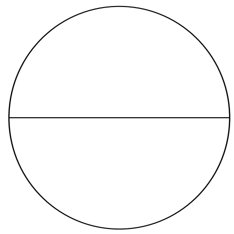

Figure 3. A typical two-loop supergraph for the vacuum energy.

inapplicable. Logically, it is possible to have an extra vacuum state characterized by $S^2 = 0$. This state would correspond to the strong coupling regime and will not be discussed here. The interested reader is referred to Ref.[11].

One last remark before concluding this section. Equation (1.43) illustrates why different points from the vacuum valley are physically inequivalent. In the conventional situation of the pre-SUSY era, the spontaneous breaking of a global symmetry, different vacua differ merely by a phase of v. Since physics depends on the ratio $|\Lambda/v|$, this phase is irrelevant. In supersymmetric theories the vacuum valleys are typically non-compact manifolds. Different points are marked not only by the phase of v, but by its absolute value as well. The dimensionless ratio above is different in different vacua. In particular, if $|\Lambda/v| \ll 1$ we are in the weak coupling regime; if $|\Lambda/v| \sim 1$ we are in the strong coupling regime.

Miracles of supersymmetry

Two of many miraculous dynamical properties of SUSY have been already mentioned – the vanishing of the vacuum energy and the non-renormalization theorem for F terms. It is instructive to see how these features emerge in perturbation theory.

Let us start from the vacuum energy. Consider a typical two-loop (super)graph shown on Fig. 3.

Each line on the graph represents the Green's function of some superfield. We do not even need to know what it is. The crucial point is that (if one works in the coordinate representation) each interaction vertex can be written as an integral over $d^4xd^2\theta d^2\bar{\theta}$. Assume that we substitute explicit expressions for Green's functions and vertices in the integrand, and carry out the integration over the second vertex keeping the first vertex fixed. As a result, we must arrive at an expression of the form

$$\int d^4xd^2\theta d^2\bar{\theta} \times \text{ a function of } x\,,\theta\,,\bar{\theta}\,. \tag{1.47}$$

Since the superspace is homogeneous (there are no points that are singled out, we can freely make translations, any point in the superspace is equivalent to any other point) the function in Eq. (1.47) can be only constant. If so, the result vanishes because of the integration over the Grassmann variables θ and $\bar{\theta}$.

What remains to be demonstrated is that the one-loop vacuum graphs, not representable in the form given on Fig. 3, also vanish. The one-loop (super)graph, however, is the same as for the free particles, and we know already that for free particles $E_{\rm vac} = 0$, see Eq. (1.1), thanks to the balance between the bosonic and fermionic degrees of freedom.

This concludes the proof of the fact that if the vacuum energy is zero at the classical level it remains there to any finite order – there is no renormalization. What changes if, instead of the vacuum energy, we would consider renormalizations of the F terms?

The proof presented above can be easily modified to include this case as well. Technically, instead of the vacuum loops, we will consider now loop (super)graphs in a background field.

The basic idea is straightforward. In any supersymmetric theory there are several – at least four – supercharge generators. In a generic background all supersymmetries are broken since the background field is generically not invariant under supertransformations. One can select such a background field, however, that leaves a part of the supertransformations as valid symmetries. For this specific background field some terms in the effective action will vanish, others will not. (Typically, F terms do not vanish while D terms do). The nonrenormalization theorems refer to those terms which do not vanish in the background field chosen.

Consider, for definiteness, the Wess-Zumino model[38],

$$\mathcal{S}^{\rm WZ} = \frac{1}{4}\int d^4x d^2\theta d^2\bar\theta \bar\phi\phi + \frac{1}{2}\left[\int d^4x d^2\theta \left(m\phi^2 + g\phi^3\right) + \text{H.c.}\right] . \tag{1.48}$$

An appropriate choice of the background field in this case is

$$\bar\phi_0 = 0\,, \quad \phi_0 = C_1 + C_2^\alpha\theta_\alpha + C_3\theta^2\,, \tag{1.49}$$

where $C_{1,2,3}$ are some constants and the subscript 0 marks the background field. This choice assumes that ϕ and $\bar\phi$ are treated as independent variables, not connected by the complex conjugation (i.e. we keep in mind a kind of analytic continuation). The x independent chiral field (1.49) is invariant under the action of $\bar Q_{\dot\alpha}$, i.e. under the transformations

$$\delta\theta_\alpha = 0\,, \quad \delta\bar\theta_{\dot\alpha} = \bar\varepsilon_{\dot\alpha}\,, \quad \delta x_{\alpha\dot\alpha} = -2i\theta_\alpha\bar\varepsilon_{\dot\alpha}\,.$$

Now, we proceed in the standard way – decompose the superfields

$$\phi = \phi_0 + \phi_{\rm qu}\,, \quad \bar\phi = \bar\phi_0 + \bar\phi_{\rm qu}\,,$$

where the subscript qu denotes the quantum part of the superfield, expand the action in $\phi_{\rm qu}, \bar\phi_{\rm qu}$, drop the linear terms and treat the remainder as the action for the quantum fields. We then integrate over the quantum fields order by order, keeping the background field fixed. The key element is the fact that in the problem we get for the quantum fields there still exists the exact symmetry under the transformations generated by $\bar Q$.

This means that the boson-fermion degeneracy holds, just as in the "empty" vacuum. All lines on the graphs of Fig. 3 have to be treated now as Green's functions in the background field (1.49). After substituting these Green's functions and integrating over all vertices except the first one we come to an expression of the type

$$\int d^4x d^2\bar\theta \times \text{ a } \bar\theta \text{ independent function} = 0\,.$$

The $\bar\theta$ independence follows from the fact that our superspace is homogeneous in the $\bar\theta$ direction even in the presence of the background field (1.49). This completes the proof of the non-renormalization theorem for the F terms. Note that the kinetic term (D terms) vanishes in the background (1.49), so nothing can be said about its renormalization (and it gets renormalized, of course). The above, somewhat non-standard, proof of the Grisaru-Roček-Siegel theorem was suggested in Ref.[36].

A word of caution is in order here. Our consideration tacitly assumes that there are no massless fields which can cause infrared singularities. Infrared singular contributions may lead to the so called *holomorphic anomalies*[39] invalidating the non-renormalization theorem. We will discuss the property of the holomorphy and the corresponding anomalies later, and now will illustrate how infrared singular D term renormalizations can effectively look as F terms. Consider the D term of the form

$$\int d^2\theta d^2\bar{\theta}\frac{D^2}{\Box}F(\phi)\,.$$

It can be rewritten as

$$\int d^2\theta F(\phi)$$

by using the property $[D^2,\bar{D}^2]\propto\Box$ and by integrating by parts in the superspace[21]. It is obvious that the $\Box^{-1}$ singularity can appear only due to massless poles. It was explicitly shown[40] that in the massless Wess-Zumino model such "fake" F terms appear at the two-loop level. The origin of the two-loop and all higher order terms in the Gell-Mann-Low function of supersymmetric gauge theories is the same – they emerge as a "fake" F term which is actually an infrared singular D term[36].

A recent discussion of the "fake" F terms is given in Ref.[41].

Holomorphy

At least some of the miracles of supersymmetry can be traced back to a remarkable property which goes under the name holomorphy. Some parameters in SUSY Lagrangians, usually associated with F terms, are complex rather than real numbers. The mass parameter in the superpotential is an obvious example. Another example mentioned in the section on supersymmetric gluodynamics is the inverse gauge coupling, $1/g_0^2$. Now, it is known for a long time, since the mid-eighties, that appropriately chosen quantities depend on these parameters analytically, with possible singularities in certain well-defined points. It is obvious that the statement that a function (analytically) depends on a complex variable is infinitely stronger than the statement that a function just depends on two real parameters. The power of holomorphy is such that one can obtain a variety of extremely non-trivial results ranging from non-renormalization theorems to exact β functions, the first time ever in dynamically non-trivial four-dimensional theories.

In this section we will outline the basic steps, keeping in mind that the corresponding technology will be of use more than once in what follows. Let us consider, as an example, $SU(2)$ SQCD with one flavor. I have already mentioned that m_0, the complex mass parameter in the action, can be viewed as a vacuum expectation value (of the lowest component) of an auxiliary chiral superfield, let us call it M. It is important that M is singlet with respect to the gauge group and thus, say, fermions from M do not contribute to the triangle anomalies. One can think of the corresponding degrees of freedom as of very heavy particles. Then the only role of M is to develop $\langle M\rangle\neq 0$ which obviously does not violate SUSY and provides the mass term.

The theory is strictly invariant under the following phase transformations

$$W_\alpha\to e^{i\beta}W_\alpha\,,\;\; S_f\to e^{-i\beta}S_f\,,\;\; M\to e^{4i\beta}M\,,\;\; \theta_\alpha\to e^{i\beta}\theta_\alpha\,. \tag{1.50}$$

This is an extended R invariance – extended, because it takes place in the extended theory with the chiral superfield M introduced by hand.

It is rather clear that one chiral superfield can depend only on the expectation value of another superfield of the same chirality – otherwise transformation properties under SUSY would be broken. Thus, the expectation value of W^2 can depend only on that of M; $\bar{M}$ cannot be involved in this relation. Equation (1.50) then tells us that

$$\langle \mathrm{Tr}\, W^2\rangle = \mathrm{const.}(M)^{1/2}. \tag{1.51}$$

In other words, the gluino condensate

$$\langle\lambda\lambda\rangle \propto \sqrt{m_0}\,, \tag{1.52}$$

and this relation is exact as far as the m_0 dependence is concerned. It holds for small $|m_0|$ when the theory is weakly coupled, as well as for large $|m_0|$, when we are in the strong coupling regime.

A similar assertion is valid regarding the vacuum expectation value of S^2. Now, Eq. (1.50) tells us that

$$\langle S^2\rangle = \mathrm{const.}(M)^{-1/2}\,, \tag{1.53}$$

implying, in turn, that

$$\langle\phi^2\rangle \propto 1/\sqrt{m_0}\,. \tag{1.54}$$

It is worth emphasizing that the exact dependence of the condensates on the mass parameter established above refers to the bare mass parameter. If we decided to eliminate the bare mass parameter m_0 in favor of the physical mass of the Higgs field m, we would have to introduce the corresponding Z factor which depends on m in a complicated non-holomorphic way.

Equations (1.52) and (1.54), first derived in Ref.[10] (a similar argument was also given in Ref.[33]), lead to far reaching consequences. Indeed, since the functional dependence of the condensates is fully established, we can calculate the relevant constant at small $|m_0|$, when $\langle\phi^2\rangle$ is large, which ensures weak coupling. I remind that the masses of the gauge bosons in this limit are proportional to

$$M_V^2 \propto |g^2\langle\phi^2\rangle|.$$

The result will still be valid for large $|m_0|$, in the strong coupling regime! This line of reasoning[10], based on holomorphy, lies behind many advances achieved recently in SUSY gauge dynamics.

Let me parenthetically note that a nice consistency check to Eqs. (1.52), (1.54) is provided by the so called Konishi anomaly[42]. In the model at hand the Konishi relation takes the form

$$\bar{D}^2\bar{S}^{\alpha f}e^V S^{\alpha f} = 4m_0 S_{\alpha f}S^{\alpha f} + \frac{1}{2\pi^2}\mathrm{Tr}\,W^2\,. \tag{1.55}$$

This expression, or more exactly, the second term on the right hand side, is nothing but a supergeneralization of a the triangle anomaly in the divergence of the axial current of the matter fermions, cf. Eq. (1.39). Now, if SUSY is unbroken, the expectation value of the left-hand side must vanish, since the left-hand side is a full superderivative. This fact implies that

$$4m_0\langle S_{\alpha f}S^{\alpha f}\rangle = -\frac{1}{2\pi^2}\langle\mathrm{Tr}\,W^2\rangle \tag{1.56}$$

which is consistent with Eqs. (1.52), (1.54). Another side remark: the exact proportionality of $\langle\phi^2\rangle$ to $1/\sqrt{m_0}$ presents a somewhat different proof of the fact that the instanton-generated superpotential (1.42) is exact even in the presence of the mass term.

One can take advantage of these observations in many ways. One direction is finding the exact β function of the theory. The idea is as follows. First we assume that $|m_0|$ is small and we are in the weak coupling regime. Then we are able to calculate the gluino condensate – in the weak coupling regime it is saturated by the one-instanton contribution. Since the functional dependence on m_0 is known we can then proceed to the limit of large $|m_0|$, or small ϕ^2. Moreover, the vacuum expectation value of $\lambda\lambda$ is, in principle, a physically measurable quantity. The operator $\lambda\lambda$ has strictly vanishing anomalous dimension since it is the lowest component of the superfield W^2, and the upper component of the same superfield contains the trace of the energy-momentum tensor. This means that if $\langle\lambda\lambda\rangle$ is expressed in terms of the gauge coupling g_0^2 and the ultraviolet cut-off M_0, when one changes the cut off, one should also change g_0^2 in a concerted way, to ensure that $\langle\lambda\lambda\rangle$ stays intact. In this way we obtain a relation between the bare coupling constant and M_0, which is equivalent to the knowledge of the β function.

More concretely, the one-instanton result for the gluino condensate is[43]

$$\langle\lambda\lambda\rangle = \text{const.} \times \frac{M_0^5 e^{-8\pi^2/g_0^2}}{g_0^4 v_0^2}\,. \tag{1.57}$$

This result is exact; only zero modes in the instanton background contribute in the calculation. It is worth emphasizing that the expectation value of the scalar field appearing in Eq. (1.57) refers to the bare field. The constant on the right-hand side is purely numerical; we will say more about this constant later on, but for the time being its value is inessential.

At the supersymmetric vacuum Eq. (1.56) must hold implying that

$$v_0^2 = \text{const.}\langle\lambda\lambda\rangle m_0^{-1}\,. \tag{1.58}$$

Combining Eqs. (1.58) and (1.57) we conclude that

$$\langle\lambda\lambda\rangle = \text{const.} M_0^{5/2} m_0^{1/2} e^{-4\pi^2/g_0^2}\frac{1}{g_0^2}\,. \tag{1.59}$$

When analyzing the response of g_0^2 with respect to the variations of M_0 one should keep in mind that m_0 also depends on M_0, implicitly. Indeed, the physical (low-energy) values of the parameters are kept fixed. This means that we fix the renormalized value of the mass, $m = m_0 Z^{-1}$, where Z is the Z factor renormalizing the kinetic term of the matter fields,

$$(\bar S^{\alpha f} e^V S^{\alpha f})_0 \to Z(\bar S^{\alpha f} e^V S^{\alpha f})_0\,.$$

In this way we arrive at the conclusion that the combination $M_0^{5/2} e^{-4\pi^2/g_0^2} Z^{1/2} g_0^{-2}$ is invariant. Differentiating it with respect to $\ln M_0$ we find the β function,

$$\beta(\alpha) = -\frac{\alpha^2}{2\pi}\,\frac{5+\gamma}{1-\alpha/\pi}\,, \tag{1.60}$$

where the β function is defined as

$$\beta(\alpha_0) = d\alpha_0/d\ln M_0\,, \quad \alpha_0 = g_0^2/4\pi\,,$$

and γ is the anomalous dimension of the matter fields,

$$\gamma = d\,\ln Z/d\,\ln M_0\,.$$

Note that due to the $SU(2)$ subflavor symmetry of the model at hand both matter fields, S_1 and S_2, have one and the same anomalous dimension.

Equation (1.60) is a particular case of the general β function, sometimes referred to as NSVZ β function,

$$\beta(\alpha) = -\frac{\alpha^2}{2\pi}\left(1-\frac{T(G)\alpha}{2\pi}\right)^{-1}\left[3T(G) - \sum_i T(R_i)(1-\gamma_i)\right]\,, \tag{1.61}$$

which can be derived in a similar manner[13]. Here $T(G)$ and $T(R)$ are the so called Dynkin indices defined as follows. Assume that the gauge group is G, and we have a field belonging to the representation R of the gauge group. If T^a is the generator matrix of the group G in the representation R then

$$\mathrm{Tr}\,(T^a T^b) = T(R)\delta^{ab}\,.$$

More exactly, $T(R)$ is one half of the Dynkin index. Moreover, $T(G)$ is $T(R)$ for the adjoint representation. Note that for the fundamental representation of the unitary groups $T(R) = 1/2$. The sum in Eq. (1.61) runs over all subflavors.

As is clear from its derivation, the NSVZ β function implies the Pauli-Villars regularization. It can also be derived purely perturbatively, with no reference to instantons, using only holomorphy properties of the gauge coupling[44]. The relation of this β function to that defined in other, more conventional regularization schemes is investigated in Ref.[45].

In some theories the NSVZ β function is exact – there are no corrections, either perturbative or nonperturbative, as is the case in the SU(2) model with one flavor. In other models it is exact only perturbatively – nonperturbative corrections do modify it. The most important example[46] of the latter kind is $N = 2$ supersymmetry.

The idea of using the analytic properties of chiral quantities for obtaining exact results was adapted for the case of superpotentials in Ref.[47]. As a matter of fact, I have already discussed some elements of the procedure suggested in Ref.[47], in analyzing possible mass dependence of nonperturbatively generated superpotential in the $SU(2)$ model. Let me summarize here the basic stages of the procedure in the general form, and give a few additional examples. Simultaneously, as a byproduct, we will obtain a different proof of some of the non-renormalization theorems considered in the section on miracles of supersymmetry. What is remarkable is that, unlike the proof presented in the section on miracles of supersymmetry, the one given below will be valid both perturbatively and nonperturbatively.

Thus, our task is establishing possible renormalizations of the superpotential $\mathcal{W}$ in a given model. All (complex) coupling constants of the model appearing as coefficients in front of F terms – denote them generically by h_i – are treated as vacuum expectation values of some (auxiliary) chiral superfields. The set of $\{h_i\}$ may include the mass parameters and/or Yukawa constants. Let us assume that if all $h_i = 0$, the model considered possesses a non-anomalous global symmetry group $\mathcal{G}$; however, the couplings $h_i \neq 0$ break this symmetry. Since h_i's are treated now as auxiliary chiral superfields one can always define transformations of these superfields in such a way as to restore the global symmetry $\mathcal{G}$. Then the calculated superpotential depending on the dynamical chiral superfields and on the auxiliary ones, should be invariant under this extended $\mathcal{G}$. This constraint becomes informative if we take into account the fact that the calculated superpotential $\mathcal{W}$ must be a holomorphic function of all chiral superfields. Thus, $\mathcal{W}$ can depend on h_i but cannot depend on h_i^*. A few additional rules apply. The effective superpotential may depend on the dynamically generated scale of the model Λ. It is clear that negative powers of Λ are forbidden since the result should be smooth in the limit when the interaction is switched off. Moreover, if we ensure that the theory is in the weak coupling regime, the possible powers of Λ are only those associated with one, two, three and so on instantons, i.e. $3T(G) - \sum_f T(R_f)$, $2(3T(G) - \sum_f T(R_f))$, and so on, since the instantons are the only source of the nonperturbative parameter Λ in the weak coupling regime. Finally, one more condition comes from analyzing the limit of the small bare couplings h_i. This limit can be often treated perturbatively. Sometimes additional massless fields appear in the limit $h_i = 0$, which are absent for $h_i \neq 0$. When these fields are integrated out and not included in the effective action, $\mathcal{W}$ may develop a singularity at $h_i = 0$.

To illustrate the power and elegance of this approach[47] let us turn again to the Wess-Zumino model, Eq. (1.48). If the bare mass and the coupling constant vanish, $m = g = 0$, then the model has two $U(1)$ global invariances – one associated with the rotations of the matter field, $\phi \to e^{i\alpha}\phi$, and another one is the R_0 symmetry, $\phi \to e^{2i\beta/3}\phi$, $\theta \to e^{i\beta}\theta$. To maintain both invariances with m and g switched on we demand that

$$m \to m e^{-2i\alpha}\,, \quad g \to g e^{-3i\alpha}$$

and

$$m \to m e^{2i\beta/3}\,, \quad g \to g$$

under $U(1)_\phi$ and $U(1)_R$, respectively. The most general renormalized superpotential compatible with these symmetries obviously has the form

$$\mathcal{W} = m\phi^2 f\left(\frac{g\phi}{m}\right)\,, \tag{1.62}$$

where f is an arbitrary function. Let us expand it in a power series and consider the coefficient in front of ϕ^n. From Eq. (1.62) it is clear that the corresponding term has the form

$$g^{n-2} m^{3-n} \phi^n\,.$$

The balance of powers of the coupling constant and m is such that this contribution could only be associated with the 1-particle reducible tree graphs, which should not be included in the effective action. Therefore, we conclude that there is no renormalization of the superpotential, $\mathcal{W} = \mathcal{W}_0$.

An example of a more sophisticated situation is provided by the $SU(N)$ theory with N_f flavors and the tree level superpotential[48]

$$\mathcal{W}_0 = hM\sum_f \tilde{Q}_f Q_f + h' M^3 \tag{1.63}$$

where Q_f is a chiral superfield belonging to the fundamental representation of $SU(N)$ while $\tilde{Q}_f$ is a chiral field in the anti-fundamental representation. The color indices are summed over; the additional chiral field M is color-singlet.

Now, if h and h' are set equal to zero, M obviously decouples, and the global symmetries of the model are those of the massless SQCD plus one extra global invariance associated with the rotations of the M field. Massless SQCD is invariant under

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_R\,.$$

The conserved R charge is established from consideration of the anomaly relations analogous to Eq. (1.39). Indeed, it is not difficult to obtain that

$$\partial_\mu R^0_\mu = \left(N - \frac{N_f}{3}\right)\frac{\alpha}{4\pi} G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}\,, \quad \partial_\mu J_\mu = N_f \frac{\alpha}{4\pi} G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}\,, \tag{1.64}$$

where the R_0 and J currents are defined in parallel to those in the $SU(2)$ model, see Eqs. (1.35) and (1.37). This means that the conserved R current has the form

$$R_\mu = R^0_\mu - \frac{N - (N_f/3)}{N_f} J_\mu\,. \tag{1.65}$$

Using this expression it is not difficult to calculate that the R charge of the matter field is $(N_f - N)/N_f$. The R charge of the M field can be set equal to zero.

Now, the superpotential (1.63) explicitly breaks both, $U(1)_R$ and $U(1)_M$. To restore the symmetry we must ascribe to h and h' the following charges

$$h \to \left(-1, 2\frac{N}{N_f}\right) , \quad h' \to (-3, 2) ,$$

where the first charge is with respect to $U(1)_M$ while the second charge is with respect to $U(1)_R$. If $h' = 0$ the model has a rich system of the vacuum valleys. Let us assume that we choose the one for which the expectation value of Q fields vanishes, but the expectation value of $M \neq 0$. Moreover, we will assume that $h\langle M\rangle$ is large. In this "corner" of the valley the matter fields are heavy, and can be integrated over. At very low energies the only surviving (massless) field is M. Our task is to find the effective Lagrangian for the M field. By inspecting the above charge assignments one easily establishes that the most general form of the effective superpotential compatible with all the assignments is

$$\mathcal{W} = h' M^3 f(x) , \quad x = \frac{\Lambda^{3-(N_f/N)} (hM)^{N_f/N}}{h' M^3} ,$$

where Λ is a dynamically generated scale of the (strongly coupled) SU(N) gauge theory. If $\Lambda \to 0$ there should be no singularities. This implies that $f(x)$ is expandable in positive powers of x. However, the behavior of the effective superpotential at $h' \to 0$ should also be smooth. These two requirements fix the function f up to a constant,

$$f(x) = 1 + \alpha x , \quad \alpha = \text{const.} ,$$

and

$$\mathcal{W} = h' M^3 + \alpha\, \Lambda^{3-(N_f/N)} (hM)^{N_f/N} . \tag{1.66}$$

The first term is the same as in the bare superpotential, the second term is generated nonperturbatively[49]. Note that the non-analytical behavior at $h = 0$ is due to the fact that at $h = 0$ there are massless matter fields, and we integrated over them assuming that they are massive. The superpotential (1.66) grows with M. This is natural since the interaction becomes stronger as M increases. The superpotential (1.66) leads to a supersymmetric minimum at $M \neq 0$.

Concluding this section I would like to return to one subtle and very important point for this range of questions, holomorphic anomalies. Consider Eq. (1.59) for the gluino condensate. So far we have studied the analytic dependence of this quantity on m_0. At the same time, however, $1/g_0^2$ is also a coefficient of the F term, which can be viewed as an expectation value of an auxiliary chiral superfield, dilaton/axion. One is tempted to conclude then that the dependence of $\langle\lambda\lambda\rangle$ on $1/g_0^2$ must be holomorphic, and its functional form must follow from consideration of the invariances of the theory. Is this the case?

The answer is yes and no. Let us examine the transformation properties of the SU(2) theory with respect to the matter $U(1)$ rotations, Eq. (1.38), supplemented by the rotation of the mass parameter $m_0 \to m_0 e^{-2i\gamma}$. At the classical level the theory is invariant. The invariance is broken, however, by the triangle anomaly. In order to restore the invariance we must simultaneously shift the vacuum angle ϑ (not to be confused with the supercoordinates θ_α!),

$$\vartheta \to \vartheta - \Delta \text{Arg}(m_0) = \vartheta + 2\gamma . \tag{1.67}$$

Taking into account the fact that $\vartheta = \mathrm{Im}(-8\pi^2/g_0^2)$ we conclude that if Eq. (1.59) contained no pre-exponential factor g_0^{-2} everything would be perfect – $\lambda\lambda$ would be a holomorphic function of g_0^{-2}, precisely the one needed for invariance of $\lambda\lambda$. The pre-exponential factor g_0^{-2} spoils the perfect picture. As a matter of fact, one can see that in the pre-exponential it is $\mathrm{Re}(g_0^{-2})$ which enters, and the holomorphy in g_0^{-2} is absent. The reason is the holomorphic anomaly associated with infrared effects. In Refs.[36, 39] it was first noted that all formal theorems regarding the holomorphic dependences on the gauge coupling constant are valid only provided we define the gauge coupling through the Wilsonian action which, by definition, contains no infrared contributions. What one usually deals with (and refers to as the action) is actually the generator of the one-particle irreducible vertices. In the absence of the infrared singularities these two notions coincide; generally speaking, they are different, however. In particular, the gauge coupling constant in the Wilsonian action, g_W^{-2} is related to g_0^{-2} by the following expression

$$\frac{8\pi^2}{g_W^2} = \frac{8\pi^2}{g_0^2} - T(G)\ln \mathrm{Re} g_0^{-2} \tag{1.68}$$

(in pure gluodynamics, without matter). The gluino condensate is holomorphic with respect to g_W^{-2}. The holomorphic anomaly in the gauge coupling due to massless matter fields was also observed in Ref.[50] in the stringy context, see also[51].

Supersymmetric instanton calculus

As was mentioned more than once, the instanton calculations, combined with specific features of supersymmetry, were instrumental in establishing various exact results in supersymmetric gluodynamics and other theories. We will continue to exploit them in further applications. Needless to say that I will be unable to present supersymmetric instanton calculus to the degree needed for practical uses. The interested reader is referred to Ref.[52]. Here I will limit myself to a few fragmentary remarks.

Technically, the most remarkable feature making the instanton calculations in supersymmetric theories by far more manageable than in non-supersymmetric ones is a residual supersymmetry in the instanton background field. It is clear that picking up a particular external field we typically break (spontaneously) supersymmetry: SUSY generators applied to this field act non-trivially. However, the self-dual (or anti-self-dual) Yang-Mills field, analytically continued to the Euclidean space, to which the instanton belongs, preserves a half of supersymmetry. Depending on the sign of the duality relation either Q_α or $\bar{Q}_{\dot\beta}$ act trivially, i.e. annihilate the background field[53, 54].

The fact that a part of supersymmetry remains unbroken in the instanton background leads to far-reaching consequences. Indeed, the spectrum of fluctuations around this background remains degenerate for bosons and fermions from one and the same superfield, and the form of the modes is in one-to-one correspondence[54], for all modes except the zero modes. An immediate consequence is vanishing of the one-loop quantum correction in the instanton background. Unsurprisingly, a more careful study shows[9] that all higher quantum corrections vanish as well.

Thus, the result of any instanton calculation is essentially determined by the zero modes alone. The problem reduces to quantum mechanics of the zero modes. The structure of the zero modes is governed by a set of relevant symmetries of the theory under consideration[43]. Therefore, all quantities that are saturated by instantons reflect the most general and profound geometrical properties of the theory. One of the examples, the gluino condensate, was already considered above. In the next part we will discuss another example – the instanton-induced modification of the quantum moduli space in SQCD with $N_f = N_c$.

Historically the first application of instantons in supersymmetric gluodynamics was the calculation of the gluino condensate[9] in the strong coupling regime. I mention this result here because although it is 15 years old, there is an intriguing mystery associated with it.

Let us consider for definiteness the $SU(2)$ gluodynamics. In this case there are four gluino zero modes in the instanton field and hence, there is no direct instanton contribution to the gluino condensate $\langle\lambda\lambda\rangle$. At the same time the instanton does contribute to the correlation function

$$\langle\lambda^a_\alpha(x)\lambda^{a\alpha}(x), \lambda^b_\beta(0)\lambda^{b\beta}(0)\rangle\,, \tag{1.69}$$

Here $a, b = 1, 2, 3$ are the color indices and $\alpha, \beta = 1, 2$ are the spinor ones. An explicit instanton calculation shows that the correlation function (1.69) is equal to a non-vanishing constant.

At first sight this result might seem supersymmetry-breaking since the instanton does not generate any bosonic analog of Eq. (1.69). Surprising though it is, supersymmetry does not forbid (1.69) provided that this two-point function is actually an x independent constant. For purposes which will become clear shortly let us sketch here the proof of the above assertion.

Three elements are of importance: (i) the supercharge $\bar{Q}^{\dot\beta}$ acting on the vacuum state annihilates it; (ii) $\bar{Q}^{\dot\beta}$ commutes with $\lambda\lambda$; (iii) the derivative $\partial_{\alpha\dot\beta}(\lambda\lambda)$ is representable as the anticommutator of $\bar{Q}^{\dot\beta}$ and $\lambda^\beta G_{\beta\alpha}$. (The spinor notations are used.) The second and the third point follow from the fact that $\lambda\lambda$ is the lowest component of the chiral superfield W^2, while $\lambda^\beta G_{\beta\alpha}$ is its middle component.

Now, we differentiate Eq. (1.69), substitute $\partial_{\alpha\dot\beta}(\lambda\lambda)$ by $\{\bar{Q}^{\dot\beta}, \lambda^\beta G_{\beta\alpha}\}$ and obtain zero. Thus, supersymmetry requires the x derivative of (1.69) to vanish[9]. This is exactly what happens if the correlator (1.69) is a constant.

If so, one can compute the result at short distances where it is presumably saturated by small-size instantons, and, then, the very same constant is predicted at large distances, $x \to \infty$. On the other hand, due to the cluster decomposition property which must be valid in any reasonable theory the correlation function (1.69) at $x \to \infty$ reduces to $\langle\lambda\lambda\rangle^2$. Extracting the square root we arrive at a (double-valued) prediction for the gluino condensate.

(The same line of reasoning is applicable in other similar problems, not only for the gluino condensate. The correlation function of the lowest components of any number of superfields of one and the same chirality, if non-vanishing, must be constant. By analyzing the instanton zero modes it is rather easy to catalog all such correlation functions, in which the instanton contribution does not vanish. Thus, for $SU(N)$ gluodynamics one ends up with the N-point function of $\lambda\lambda$. Inclusion of the matter fields, clearly, enriches the list of the instanton-induced "constant" correlators, but not too strongly[55]. The general strategy remains the same as above in all cases.)

Many questions immediately come to one's mind in connection with this argument. First, if the gluino condensate is non-vanishing and shows up in a roundabout instanton calculation through (1.69) why is it not seen in the direct instanton calculation of $\langle\lambda\lambda\rangle$? Second, the constancy of the two-point function (1.69) required by SUSY is ensured in the concrete calculation by the fact that the instanton size ρ turns out to be of order of x. The larger the value of x the larger ρ saturates the instanton contribution. For small x this is alright. At the same time at $x \to \infty$ we do not expect any coherent fields with the size of order x to survive in the vacuum; such coherent fields would contradict our current ideas of the infrared-strong confining theories like SUSY gluodynamics.

If there are no large-size coherent fields in the vacuum how can one guarantee the x independence of (1.69) at all distances?

A tentative answer to the first question might be found in the hypothesis put forward by Amati *et al.*[33]. It was assumed that, instead of providing us with the expectation value of $\lambda\lambda$ in the given vacuum, instantons in the strong coupling regime yield an average value of $\langle\lambda\lambda\rangle$ in all possible vacuum states. If there exist two vacua, with the opposite signs of $\langle\lambda\lambda\rangle$, the conjecture of Amati *et al.* would explain why instantons in the strong coupling regime do not generate $\lambda\lambda$ directly.

When we do the instanton calculation in the weak coupling regime (the Higgs phase) the averaging over distinct vacua does not take place. In the weak coupling regime, we have a marker: a large classical expectation value of the Higgs field tells us in what particular vacuum we do our instanton calculation. In the strong coupling regime, such a marker is absent, so that the recipe of Amati *et al.* seems plausible.

This is not the end of the story, however. One of the instanton computations which was done in the mid-eighties[43] remained a puzzle defying theoretical understanding for years. The result for $\langle\lambda\lambda\rangle$ obtained in the strong coupling regime (i.e. by following the program outlined after Eq. (1.69)) does *not* match $\langle\lambda\lambda\rangle$ calculated in an indirect way, as we did in the section on holomorphy – extending the theory by adding one flavor, doing the calculation in the weakly coupled Higgs phase, and then returning back to SUSY gluodynamics by exploiting the holomorphy of the condensate in the mass parameter. In Ref.[43] it was shown that

$$\langle\lambda\lambda\rangle^2_{\rm scr} = \frac{4}{5}\langle\lambda\lambda\rangle^2_{\rm wcr}\,, \tag{1.70}$$

where the subscripts scr and wcr mark the strong and weak coupling regime calculations.

The hypothesis of Amati *et al.*, by itself, does not explain the discrepancy (1.70). If there are only two vacua characterized by $\langle\lambda\lambda\rangle = \pm\Lambda^3$, the gluino condensate is not affected by the averaging over these two vacuum states, since the contributions of these two vacua to Eq. (1.69) are equal. If, however, there exist an *extra* zero-energy state with $\langle\lambda\lambda\rangle = 0$ involved in the averaging, the final result in the strong coupling regime is naturally different from that obtained in the weak coupling regime in the *given* vacuum. Moreover, the value of the condensate calculated in the strong coupling approach should be smaller, consistently with Eq. (1.70). At the moment there seems to be no other way out of the dilemma[11]. The conclusion of the existence of the extra vacuum with $\langle\lambda\lambda\rangle = 0$ is quite radical, and, perhaps, requires further verification, in particular, in connection with the Witten index counting. What is beyond any doubt, however is that the combination of instanton calculus with holomorphy and other specific features of supersymmetry provides us with the most powerful tool we have ever had in four-dimensional field theories.

It remains to be added that the interest in technical aspects of supersymmetric instanton calculus[43] was revived recently in connection with the Seiberg-Witten solution of the $N = 2$ theory. The solution was obtained[56] from indirect arguments, and it was tempting to verify it by direct instanton calculations[57]. Such calculations require extension of supersymmetric instanton calculus to $N = 2$, which was carried out, in a very elegant way, in Refs.[57, 58].

* * * * * * *

Concluding this part of the Lecture let me briefly summarize the main lessons. First, the most remarkable feature of the structure of SUSY gauge theories with matter is the existence of the vacuum valleys – classically flat directions along which the energy vanishes. This degeneracy may or may not be lifted dynamically, at the quantum level.

The $SU(2)$ model with one flavor is an example of the theory where the continuous degeneracy is lifted, and the quantum vacuum has only discrete (two-fold) degeneracy. If this does not happen, the classically flat directions give rise to *quantum moduli space of supersymmetric vacua*. This feature is the key element of the recent developments pioneered by Seiberg.

Second, holomorphic dependences of various chiral quantities enforced by supersymmetry lie behind numerous miracles occurring in SUSY gauge theories – from specific non-renormalization theorems to the exact β functions. This is also an important element of dynamical scenarios to be discussed below. Now the stage is set and we are ready for more adventures and surprises in supersymmetric dynamics.

VARIOUS DYNAMICAL REGIMES IN SUSY GAUGE THEORIES

In the first part I summarized what was known (or assumed) about the intricacies of the gauge dynamics in the eighties. In the following we will discuss the discoveries and exciting results of the recent years. I should say that the current stage of development was opened up by Seiberg, and many ideas and insights to be discussed today I learned from him or extracted from works of his collaborators.

Remarkable facets of the gauge dynamics will be revealed to us. First of all, we will encounter non-conventional patterns of the chiral symmetry breaking. Chiral symmetry breaking is one of the most important phenomena of which very little was known, beyond some empiric facts referring to QCD. In the eighties, when our knowledge of the gauge dynamics was less mature than it is now, it was believed that the massless fermion condensation obeys the so called maximum attraction channel (MAC) hypothesis[59]. In short, one was supposed to consider the one-gluon exchange between fermions, find a channel with such quantum numbers that the attraction was maximal, and then assume the condensation of the fermion pairs in this particular channel. The concrete quantum numbers of the fermion condensates imply a very specific pattern of the chiral symmetry breaking.

In SQCD we will find patterns *contradicting the MAC hypothesis*. This means that the chiral condensates are not governed by the one-gluon exchange, even qualitatively. The basic tool for exploring the chiral condensates is the 't Hooft matching condition. It was exploited for this purpose previously many times, in the context most relevant to us in Ref.[60]. Combining supersymmetry (the fact of the existence of the vacuum manifold) with the matching condition drastically enhances the method.

The second remarkable finding is the observation of conformally invariant theories in four dimensions in the strong coupling regime. The crucial instrument in revealing such theories is Seiberg's "electric-magnetic" duality in the infrared domain, connecting with each other two distinct gauge theories – one of them is strongly coupled while the other is weakly coupled. One can view the gluons and quarks of the weakly coupled theory as bound states of the gluons and quarks of its dual partner. If so, composite gauge bosons can exist! The arguments in favor of the "electric-magnetic" duality are again based on the 't Hooft matching condition (combined with supersymmetry) and some additional indirect consistency checks.

$SU(N_c)$ QCD with N_f flavors – preliminaries

The $SU(2)$ model considered previously is somewhat special since all representations of $SU(2)$ are (pseudo)real. For this reason the flavor sector of this model possesses an enlarged symmetry. Thus, for one flavor we observe the flavor $SU(2)$ symmetry, which is absent if the gauge group is, say, $SU(3)$. Now we will consider a more generic situation. The gauge group is assumed to be $SU(N_c)$ with $N_c > 2$. In accordance with Witten's index, if the matter sector consists of non-chiral matter allowing (at least, in principle) for a mass term for all matter fields, supersymmetry is unbroken.

To describe N_f flavors one has to introduce $2N_f$ chiral superfields, Q^i in the representation N_c and $\tilde{Q}_j$ in the representation $\bar{N}_c$. To distinguish between the fundamental and anti-fundamental representations the flavor indices used are superscripts and subscripts, respectively.

The Lagrangian is very similar to that of the $SU(2)$ model,

$$\mathcal{L} = \frac{1}{2g_0^2}\,\mathrm{Tr}\int d^2\theta W^2 + \frac{1}{4}\int d^2\theta d^2\bar{\theta}\left(Q^\dagger e^V Q + \tilde{Q}^\dagger e^{-V}\tilde{Q}\right) + \left(\frac{1}{2}\int d^2\theta\, \mathcal{W}(Q,\tilde{Q}) + \text{H.c.}\right), \tag{2.1}$$

where $\mathcal{W}(Q,\tilde{Q})$ is a superpotential which may or may not be present. Then the scalar potential has the form

$$V = \frac{1}{2g^2}D^a D^a + \sum_Q\left|\frac{\partial\mathcal{W}(Q,\tilde{Q})}{\partial Q}\right|^2 + \sum_{\tilde{Q}}\left|\frac{\partial\mathcal{W}(Q,\tilde{Q})}{\partial\tilde{Q}}\right|^2 . \tag{2.2}$$

An example of a possible superpotential is a generalized mass term,

$$\mathcal{W} = m_i^j \tilde{Q}_j Q^i$$

where m_i^j is a mass matrix. Most often we will work under conditions of vanishing superpotential, $\mathcal{W} = 0$.

It is convenient to introduce two $N_f \times N_c$ matrices of the form

$$q = \left\{Q^1, Q^2, ..., Q^{N_f}\right\}, \quad \tilde{q} = \left\{\tilde{Q}_1, \tilde{Q}_2, ..., \tilde{Q}_{N_f}\right\}. \tag{2.3}$$

The rows of these matrices correspond to different values of the color index. Thus, in the first row the color index is 1, in the second row 2, *etc.*, N_c rows altogether. Both matrices can be globally rotated in the color and flavor spaces. Let us assume first that $N_f < N_c$. Then, by applying these rotations one can always reduce the matrix q to the form

$$q = \begin{pmatrix} a_1 & 0 & ... & 0 \\ 0 & a_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & a_{N_f} \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \end{pmatrix}. \tag{2.4}$$

If $\tilde{q} = q$ we are at the bottom of the vacuum valley – the corresponding energy vanishes. The gauge invariant description is provided by the composite chiral superfield,

$$M_j^i = \tilde{Q}_j Q^i . \tag{2.5}$$

The points belonging to the bottom of the valley are parametrized by the expectation value of M_j^i. Generically, if we are away from the origin the $SU(N_c)$ gauge group is broken down to $SU(N_c - N_f)$. The first group has $N_c^2 - 1$ generators, the second one has $(N_c - N_f)^2 - 1$ generators. Thus, the number of the chiral fields eaten up in the super-Higgs mechanism is $2N_cN_f - N_f^2$. Originally we started from $2N_cN_f$ chiral superfields; N_f^2 remain massless – exactly the number of degrees of freedom in M_j^i. There are exceptional points. When $\det M = 0$ the unbroken gauge subgroup is larger than $SU(N_c - N_f)$ and, correspondingly, we have more than N_f^2 massless particles. At the origin of the vacuum valley the original gauge group $SU(N_c)$ remains unbroken.

The situation changes if the number of flavors is equal to or larger than the number of colors. Indeed, if $N_f > N_c$ the generic form of the matrix q, after an appropriate rotation in the flavor and color space, is

$$q = \begin{pmatrix} a_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{N_c} & 0 & \dots & 0 \end{pmatrix} . \tag{2.6}$$

The condition defining the bottom of the valley (the vanishing of the energy) is

$$|a_i|^2 - |\tilde{a}_i|^2 = \text{ constant independent of } i \tag{2.7}$$

where $i = 1, 2, ..., N_c$. At a generic point of the bottom of the valley the gauge group is completely broken. The gauge invariant chiral variables parametrizing the bottom of the valley (the moduli space) now are

$$M_j^i = \tilde{Q}_j Q^i \,, \quad B = Q^{[i_1} ... Q^{i_{N_c}]} \,, \quad \tilde{B} = \tilde{Q}_{[j_1} ... \tilde{Q}_{j_{N_c}]} \,, \tag{2.8}$$

where the color indices in B and $\tilde{B}$ (they are not written out explicitly) are contracted with the help of the ε symbol; the flavor indices $i_1, ..., i_{N_c}$ and $j_1, ..., j_{N_c}$ then come out automatically antisymmetric, and the square brackets in Eq. (2.8) remind us of this antisymmetrization. *A priori*, the number of the variables B and $\tilde{B}$ is $C_{N_f}^{N_c}$ each, where $C_{N_f}^{N_c}$ are the combinatorial coefficients,

$$C_{N_f}^{N_c} = \frac{N_f!}{N_c!(N_f - N_c)!} \,,$$

since one can pick up N_c flavors out of the total set of N_f in various ways. If we try to calculate now the number of moduli, assuming that all those indicated in Eq. (2.8) are independent, we will see that this number does not match the number of the massless degrees of freedom. Let us consider two examples, $N_f = N_c$ and $N_f = N_c + 1$.

$\mathbf{N_f = N_c}$

The original number of the chiral supefields is $2N_f N_c$; since the gauge symmetry is completely broken the number of the "eaten" superfields is $N_c^2 - 1$; the number of the massless degrees of freedom is thus $N_f^2 + 1 = N_c^2 + 1$. The number of moduli in Eq. (2.8) is $N_f^2 + 2$. One chiral variable is therefore redundant.

$\mathbf{N_f = N_c + 1}$

The number of the chiral supefields is $2N_f N_c$; since the gauge symmetry is completely broken the number of the "eaten" superfields is $N_c^2 - 1$; the number of massless degrees of freedom is N_f^2. The number of moduli in Eq. (2.8) is $N_f^2 + 2N_f$, i.e. $2N_f$ chiral variables are therefore redundant.

In the first case, $N_f = N_c$, the constraint eliminating the redundant chiral variable is

$$\det\{M\} = B\tilde{B} \,, \tag{2.9}$$

while in the second example, $N_f = N_c + 1$, by using merely the definitions of the moduli in Eq. (2.8), it is not difficult to obtain

$$B_i \{M\}_j^i = \{M\}_j^i \tilde{B}^j = 0 \,, \tag{2.10}$$

Table 1. The quantum numbers of the matter fields with respect to the global symmetries (2.13) in SQCD with N_f flavors. The R charges indicated refer to the lowest components of the superfields.

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
Q	N_f	1	1	$(N_f - N_c)/N_f$
$\tilde{Q}$	1	$\bar{N}_f$	-1	$(N_f - N_c)/N_f$

and

$$\text{minor}\{M\}_i^j = B_i \tilde{B}^j \,, \tag{2.11}$$

where the left-hand side of the last equation is the minor of the matrix $\{M\}$ (i.e. $(-1)^{i+j}\times$ determinant of the matrix obtained from M by omitting the i-th row and the j-th column. Note that $\det\{M\}$ vanishes in this case. For brevity we will sometimes write Eq. (2.11) in a somewhat sloppy form

$$\det M \left(M^{-1}\right)_i^j = B_i \tilde{B}^j \,. \tag{2.12}$$

At the classical level one could, in principle, eliminate the redundant chiral variables using Eqs. (2.9) or (2.12). One should not hurry with this elimination, however, since at the quantum level the classical moduli fields are replaced by the vacuum expectation values of M_i^j and B, $\tilde{B}$, and although generically the total number of massless degrees of freedom does not change, the quantum version of constraints (2.9) and (2.12) may (and will) be different. Moreover, at some specific points of the valley the number of massless degrees of freedom may increase, as we will see shortly.

SQCD with N_f flavors and no tree-level superpotential has the following global symmetries free from internal anomalies:

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R \tag{2.13}$$

where the conserved R current was introduced in Eq. (1.65), and the quantum numbers of the matter multiplets with respect to these symmetries are collected in Table 1. (For discussion of the subtleties in the R current definition see Ref.[37]. These subtleties, being conceptually important, are irrelevant for our considerations).

The $SU(N_f)_L \times SU(N_f)_R \times U(1)_B$ transformations act only on the matter fields in an obvious way, and do not affect the superspace coordinate θ. As for the extra global symmetry $U(1)_R$ it is defined in such a way that it acts nontrivially on the supercoordinate θ and, therefore, acts differently on the spinor and the scalar or vector components of superfields. The R charges in Table 1 are given for the lowest component of the chiral superfields. If the R charge of the boson component of the given superfield is r then the R charge of the fermion component is, obviously, $r-1$. A part of the above global symmetries is spontaneously broken by the vacuum expectation values of M_j^i and/or $B, \tilde{B}$.

Unlike the $N_c = 2$, $N_f = 1$ model discussed before, instantons do not lift the classical degeneracy, and the bottom of the valley remains flat. The easiest way to see this is to consider a generic point of the bottom of the valley, far away from the origin, where the theory is in the weak coupling regime, and try to write the most general superpotential, compatible with all exact symmetries[18, 19] (it must be symmetric even under those symmetries which may turn out to be spontaneously broken). The symmetry under $SU(N_f)_L \times SU(N_f)_R \times U(1)_B$ is guaranteed if we assume that the superpotential $\mathcal{W}$ depends on $\det M$. What about the R symmetry?

For $N_f = N_c$ the R charge of the matter superfield vanishes, as is clear from Table 1. Since the superpotential must have the R charge 2, it is obvious that it cannot be generated. For $N_f > N_c$ the R charge of the matter fields does not vanish, and, in principle, one could have written

$$\mathcal{W} \propto \left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{\frac{1}{N_c-N_f}} ,$$

an expression which has the right dimension (three) and the correct R charge (two). However, the dimension of Λ does not match the instanton expression which can produce only $\Lambda^{3N_c-N_f}$ (and in the weak coupling regime the instanton is the only relevant nonperturbative contribution). What is even more important, for $N_f > N_c$ the determinant of M vanishes identically. This fact alone shows that no superpotential can be generated, and the flat direction remains flat[61, 18, 19].

The argument above demonstrates again the power of holomorphy. In non-supersymmetric theories one could built a large number of invariants involving $Q, \tilde{Q}, Q^\dagger$ and $\tilde{Q}^\dagger$. In SUSY theories, as far as the F terms are concerned, one is allowed to use only Q and $\tilde{Q}$ which constraints the possibilities to the extent that nothing is left.

In summary, for $N_f \geq N_c$ the vacuum degeneracy is not lifted. At the origin of the space of moduli, where $B = \tilde{B} = 0$ and M has fewer than $N_c - 1$ non-zero eigenvalues, the gauge symmetry is not fully broken. At this point, the classical moduli space is singular. Far away from the origin, when the expectation values of the squark fields are large, the distinction between the classical and quantum moduli space should be unimportant. In the vicinity of the origin, however, this distinction may be crucial. Our next task is to investigate this distinction. Needless to say that the vicinity of the origin is just the domain of most interesting dynamics. Since the Higgs fields are in the fundamental representation, we are always in the Higgs/confining phase. Far away from the origin the theory is in the weak coupling regime and is fully controllable by well understood methods of weak coupling. In the vicinity of the origin the theory is in the strong coupling regime. The issues to be investigated are the patterns of the spontaneous breaking of the global symmetries and the occurrence of the composite massless degrees of freedom at large distances. Here each non-trivial theoretical result or assertion is a precious asset, a miraculous achievement.

The quantum moduli space

Relations (2.9) and (2.12) are constraints on the classical composite fields. Since in the quantum theory the vacuum valley is parametrized by the *expectation values* of the fields, which may get a contribution from quantum fluctuations, these relations may alter. In other words, the quantum moduli space need not exactly coincide with the classical one. Only in the limit when the vacuum expectation values of the fields parametrizing the vacuum valley become large, much larger than the scale parameter of the underlying theory, we must be able to return to the classical description.

To see that the quantum moduli space does indeed differ from the classical one we will consider here, following Ref.[25], the same two examples, $N_f = N_c$ and $N_f = N_c + 1$. The general strategy used in these explorations is the same as was discussed in detail in connection with $SU(2)$ SQCD[10]: in order to analyze the theory along the classically flat directions one adds the appropriately chosen mass terms (sometimes, other superpotential terms as well), solves the theory in the weak coupling regime, and then analytically continues to the limit where the classical superpotential vanishes.

$N_f = N_c$

Introduce a mass term for all quark flavors, or more generically, the quark mass matrix

$$m_i^j Q^i \tilde{Q}_j \,, \tag{2.14}$$

(it can always be diagonalized, of course). If we additionally assume that the mass terms for $N_f - 1$ flavors are small and the mass term for one flavor is large then we find ourselves in a situation where an effective low-energy theory is that of $N_c - 1$ flavors. From the first part of this lecture course we know already that in this case the $SU(N_c)$ symmetry is totally broken spontaneously, the theory is in the weak coupling phase, instantons generate a superpotential, and this superpotential, being combined with Eq. (2.14), leads to[18, 19, 43, 33]

$$M_j^i = \langle Q^i \tilde{Q}_j \rangle = \Lambda^2 (\det m)^{\frac{1}{N_c}} \left(\frac{1}{m}\right)_j^i \,,$$

$$B = \langle Q^{i_1} ... Q^{i_{N_c}} \rangle \epsilon_{i_1 ... i_{N_c}} = 0 \,, \;\; \tilde{B} = \langle \tilde{Q}_{j_1} ... \tilde{Q}_{j_{N_c}} \rangle \epsilon^{j_1 ... j_{N_c}} = 0 \,. \tag{2.15}$$

Although this result was obtained under a very specific assumption on the values of the mass terms, holomorphy tells us that it is exact. In particular, one can let $m_i^j \to 0$, thus returning to the original massless theory. Equation (2.15) obviously implies that

$$\det M - B\tilde{B} = \Lambda^{2N_c} \,. \tag{2.16}$$

It is instructive to check that this relation stays valid even if $B \neq 0$, $\tilde{B} \neq 0$. To this end one must introduce, additionally, a superpotential $\beta B + \tilde{\beta}\tilde{B}$ where β and $\tilde{\beta}$ are some constants, and redo the instanton calculations. If $\beta \neq 0$ and $\tilde{\beta} \neq 0$, the instanton-induced superpotential changes, non-vanishing values of B and $\tilde{B}$ are generated, the vacuum expectation values M_j^i change as well, but the relation (2.16) stays intact.

Far from the origin, where the semiclassical analysis is applicable, the quantum moduli space (2.16) is close to the classical one. A remarkable phenomenon happens near the origin[25]. In the classical theory where the gluons were massless near the origin, the classical moduli space was singular. Quantum effects eliminated the massless modes by creating a mass gap†. Correspondingly, the singular points with $B = \tilde{B} = 0$ and vanishing eigenvalues of M are eliminated from the moduli space.

In the weak coupling regime dynamics is rather trivial and boring. Let us consider the most interesting domain of the vacuum valley, near the origin, in more detail, "under a microscope". There are several points that are special, they are characterized by an enhanced global symmetry. For instance, if

$$B = \tilde{B} = 0 \quad \text{and} \quad M_j^i = \Lambda^2 \delta_j^i \,, \tag{2.17}$$

the original global $SU(N_f) \times SU(N_f)$ symmetry is spontaneously broken down to the diagonal $SU(N_f)$, while the $U(1)_R$ remains unbroken (the R charges of Q and $\tilde{Q}$ vanish, see Table 1). We are in the vicinity of the origin, where all moduli are either of order of Λ^2 or vanish. Hence, the fundamental gauge dynamics of the quark (squark) matter is strongly coupled. We are in the strong coupling regime.

The spontaneous breaking of the global symmetry implies the existence of the massless Goldstone mesons which, through supersymmetry, entails, in turn, the occurrence of the massless (composite) fermions. These fermions reside in the superfields M, B, and $\tilde{B}$. Their quantum numbers with respect to the unbroken symmetries are indicated in Table 2.

†The latter statement is not quite correct. Massless moduli fields still persist. What is important, however, is that the gluons acquire a dynamical "mass".

Table 2. The quantum numbers of the composite massless fermions with respect to the unbroken global symmetries in SQCD with $N_f = N_c$

	$SU(N_f)$	$U(1)_B$	$U(1)_R$
ψ_M	$N_f^2 - 1$	0	-1
ψ_B	1	N_f	-1
$\psi_{\tilde{B}}$	1	$-N_f$	-1
ψ_Q	N_f	1	-1
$\psi_{\tilde{Q}}$	$\bar{N}_f$	-1	-1
λ	1	0	1

For convenience Table 2 summarizes also the quantum numbers of the fundamental fermions – quarks and gluino. A remark is in order concerning the multiplet of the massless fermions ψ_M. Since M is an $N_f \times N_f$ matrix, naively one might think that the number of these fermions is N_f^2. Actually we must not forget that we are interested in small fluctuations of the moduli fields M, B and $\tilde{B}$ around the expectation values (2.17) subject to the constraint (2.16). It is easy to see that this constraint implies that the matrix of fluctuations $M^i_j - \Lambda^2 \delta^i_j$ is traceless, i.e. the fluctuations form the adjoint ($(N_f^2 - 1)$-dimensional) representation of the diagonal $SU(N_f)$.

Massless composite fermions in gauge theories are subject to a very powerful constraint known as the *'t Hooft consistency condition*[62]. As was first noted in[63], the triangle anomalies of the AVV type in the gauge theories with the fermion matter imply the existence of infrared singularities in the matrix elements of the axial currents. (Here A and V stand for the axial and vector currents, respectively). These singularities are unambiguously fixed by the short-distance (fundamental) structure of the theory even if the theory at hand is in the strong coupling regime and cannot be solved in the infrared. The massless composite fermions in the theory, if present, must arrange themselves in such a way as to match these singularities. If they cannot, the corresponding symmetry is spontaneously broken, and the missing infrared singularity is provided by the Goldstone-boson poles coupled to the corresponding broken generators. This device – the 't Hooft consistency condition, or anomaly matching – is widely used in strongly coupled gauge theories: from QCD to technicolor, to supersymmetric models; it allows one to check various conjectures about the massless composite states. (For a pedagogical review see e.g.[64].)

In our case we infer the existence of the massless fermions from the fact that a set of moduli exists, plus supersymmetry. Why do we need to check the matching of the AVV triangles? If we know for sure the pattern of the symmetry breaking – which symmetry is spontaneously broken and which is realized linearly – the matching of the AVV triangles for the unbroken currents must be automatic. The condensates indicated in Eq. (2.17) suggest that the axial $SU(N_f)$ is spontaneously broken while the R current and the baryon current are unbroken. Suggest, but do not prove! For in the strong coupling regime other (non-chiral) condensates might develop too. For instance, on general grounds one cannot exclude the condensate of the type $\langle M^i_j B^\dagger \rangle$ which will spontaneously break the baryon charge conservation. Since this superfield is non-chiral the holomorphy consideration is inapplicable. If the anomalous triangles with the baryon current do match, it will be a strong argument showing that no additional condensates develop, and the pattern of the spontaneous symmetry breaking can be read off from Eq. (2.17). Certainly, this is not a completely rigorous proof, but, rather, a very strong indication.

What is extremely unusual in the pattern implied by Eq. (2.17) is the survival of an unbroken axial current (the axial component of the R current). We must verify that this scheme of symmetry breaking is compatible with the spectrum of the massless composite fermions residing in the superfields M, B, and $\tilde{B}$.

The 't Hooft consistency conditions, to be analyzed in the general case, refer to the so called *external* anomalies of the AVV type. More exactly, one considers those axial currents, corresponding to global symmetries of the theory at hand, which are non-anomalous inside the theory *per se*, but acquire anomalies in weak external backgrounds. For instance, in QCD with several flavors the singlet axial current is internally anomalous – its divergence is proportional to $G\tilde{G}$ where G is the gluon field strength tensor. Thus, it should not be included in the set of the 't Hooft consistency conditions to be checked. The non-singlet currents are non-anomalous in QCD itself, but become anomalous if one includes the photon field, external with respect to QCD. These currents must be checked. The anomaly in the singlet current does not lead to the statement of the infrared singularities in the current while the anomaly in the non-singlet currents does. Those symmetries that are internally anomalous, are non-symmetries.

In our case we first list all those symmetries which are supposedly realized linearly, i.e. unbroken. After listing all relevant currents we then saturate the corresponding triangles. The diagonal $SU(N_f)$ symmetry which remains unbroken is induced by the vector current, not axial. The same is true with regards to $U(1)_B$. The conserved (unbroken) R current has the axial component. Therefore, the list we must consider includes the following triangles

$$U(1)_R^3\,,\;\; U(1)_R SU(N_f)^2\,,\;\; U(1)_R U(1)_B^2\,.$$

One more triangle is of a special nature. One can consider the gravitational field as external, and study the divergence of the R current in this background. This divergence is also anomalous,

$$\partial^\mu(\sqrt{-g}R_\mu) \;=\; \frac{1}{192\pi^2}\,[\text{const.}]\,\epsilon_{\mu\nu\lambda\delta}R^{\mu\nu\sigma\rho}R^{\lambda\delta}_{\sigma\rho}\,, \tag{2.18}$$

where g is the metric and $R^{\mu\nu\sigma\rho}$ is the curvature of the gravitational background. The constant in the square brackets depends on the particle content of the theory, and must be matched at the fundamental and composite fermion level. This gravitational anomaly in the R current is routinely referred to as $U(1)_R$. Thus, altogether we have to analyze four triangles. Let us start, for instance, from $U(1)_R SU(N_f)^2$. The relevant quantum numbers of the fundamental-level fermions (ψ_Q, $\psi_{\tilde{Q}}$ and λ), and the composite fermions (ψ_M, ψ_B and $\psi_{\tilde{B}}$) are collected in Table 2. At the fundamental level we have to take into account only ψ_Q and $\psi_{\tilde{Q}}$ since only these fields have both $U(1)_R$ charge and transform non-trivially with respect to $SU(N_f)$. The corresponding triangle is proportional to $-N_f T_{\rm fund} - N_f T_{\rm anti-fund} \equiv -N_f$. The factor N_f appears since we have N_f fundamentals and N_f anti-fundamentals. Here T is (one half of) the Dynkin index defined as follows. Assume we have the matrices of the generators of the group G in the representation R. Then

$$\text{Tr}\,(T^aT^b)_R = T_R\,\delta^{ab}\,.$$

For the fundamental representation $T = 1/2$ while for the adjoint representation of $SU(N)$ the index $T = N$. Now, let us calculate the same triangle at the level of the composite fermions. From Table 2 it is obvious that we have to consider only ψ_M, and the corresponding contribution is $-T_{\rm adjoint} = -N_f$. The match is perfect.

The balance in the $U(1)_R^3$ triangle looks as follows. At the fundamental level we include ψ_Q, $\psi_{\tilde{Q}}$ and λ and get $2N_f^2(-1)^3 + (N_f^2 - 1) = -N_f^2 - 1$. At the composite fermion level we include ψ_M, ψ_B and $\psi_{\tilde{B}}$ and get $(N_f^2 - 1)(-1)^3 - 2 = -N_f^2 - 1$.

By the same token one can check that $U(1)_B^2 U(1)_R$ triangle gives $-2N_f^2$ both at the fundamental and composite levels. The $U(1)_R$ case requires a special comment. The coupling of all fermions to gravity is universal. Therefore, the coefficient in Eq. (2.18) merely counts the number of the fermion degrees of freedom weighed with their R charges. At the fundamental level we, obviously, have $-N_f^2 - N_f^2 + (N_f^2 - 1) = -N_f^2 - 1$, while at the composite level the coefficient is $-(N_f^2 - 1) - 1 - 1 = -N_f^2 - 1$. Again, the match is perfect.

Thus, the massless fermion content of the theory is consistent with the regime implied by Eq. (2.17) – spontaneous breaking of the chiral $SU(N_f)_R \times SU(N_f)_L$ down to vector $SU(N_f)$. The baryon and the $U(1)_R$ currents remain unbroken. This regime is rather similar to what we have in ordinary QCD. The unconventional aspect, as was stressed above, is the presence of the conserved unbroken R current which has the axial component.

This does not mean, however, that all points from the vacuum valley are so reminiscent of QCD. Other points are characterized by different dynamical regimes, with drastic distinctions in the most salient features of the emerging picture. To illustrate this statement let us consider, instead of Eq. (2.17), another point

$$B = -\tilde{B} = \Lambda^{N_f}\,, \quad M_j^i = 0\,. \tag{2.19}$$

This point is characterized by a fully unbroken chiral $SU(N_f) \times SU(N_f)$ symmetry, in addition to the unbroken R symmetry. The only broken generator is that of $U(1)_B$.

This regime is exceptionally unusual from the point of view of the QCD practitioner. As a matter of fact, the emerging picture is directly opposite to what we got used to in QCD: the axial $SU(N_f)$ generators remain unbroken while the vector baryon charge generator is spontaneously broken.

As is well-known, spontaneous breaking of vector symmetries is forbidden in QCD[65]. The no-go theorem of Ref.[65] is based only on very general features of QCD – namely on the vector nature of the quark-gluon vertex. Where does the no-go theorem fail in SQCD?

The answer is quite obvious. The spontaneous breaking of the baryon charge generator in SQCD, apparently defying the no-go theorem of Ref.[65], is due to the fact that in SQCD we have scalar quarks (and the quark-squark-gluino interaction) which invalidates the starting assumptions of the theorem.

Moreover, in QCD general arguments, based on the 't Hooft consistency condition and N_c counting, strongly disfavor[66] the possibility of the linearly realized axial $SU(N_f)$. Although I do not say here that the consideration of Ref.[66] proves the axial $SU(N_f)$ to be spontaneously broken in QCD, there is hardly any space left over for a linear realization. The linear realization is not ruled out only because the argument of Ref.[66] is based on an assumption regarding the N_c dependence (discussed below) which is absolutely natural but still was not derived from first principles. Certain subtleties which I cannot explain now due to time limitations might, in principle, invalidate this assumption. Leaving aside these – quite unnatural – subtleties one can say that the linear realization of the axial $SU(N_f)$ is impossible in QCD.

At the same time, this is exactly what happens in SQCD in the regime specified by Eq. (2.19). Again, the scalar quarks are to blame for the failure of the argument presented in Ref.[66]. In QCD it is difficult to imagine how massless baryons could saturate anomalous triangles since the baryons are composed of N_c quarks; the corresponding contribution naturally tends to be suppressed as $\exp(-N_c)$ at large N_c. In SQCD there exist fermion states built from one quark and one (anti)squark whose contribution to the triangle is not exponentially suppressed.

After this introductory remark it is time to check that the 't Hooft consistency conditions are indeed saturated. The triangles to be analyzed are

$$SU(N_f)^3\,,\;\; SU(N_f)^2U(1)_R\,,\;\; U(1)_R^3\,,\;\; \text{and}\;\; U(1)_R\,.$$

The $SU(N_f)$ symmetry is either $SU(N_f)_R$ or $SU(N_f)_L$, but the triangles are the same for both. It is necessary to take into account the fact that the fluctuations around the expectation values (2.19) subject to the constraint (2.16) are slightly different than those indicated in Table 2. Namely, the matrix of fluctuations M^i_j need not be traceless any longer; correspondingly, there are N_f^2 fermions in this matrix transforming as the $(N_f,\,\bar{N}_f)$ representation of $SU(N_f)_R\times SU(N_f)_L$. At the same time the fluctuations of B and $\tilde{B}$ are not independent now, so that $B-\Lambda^{N_f}=\tilde{B}+\Lambda^{N_f}$. One should count only one of them. The $U(1)_R$ quantum numbers remain intact, of course.

With this information in hands, matching of the triangles becomes a straightforward exercise. For instance, the $SU(N_f)^3$ triangle obviously yields $N_f D_{\rm fund}$ both at the quark and composite levels. Here $D_{\rm fund}$ is the cubic Casimir operator for the fundamental representation defined as follows

$$\mathrm{Tr}\,(T^a\{T^b,T^c\})=d^{abc}D\,;$$

the matrices of the generators T^a are taken in the given representation, and the braces denote the anticommutator; d^{abc} stand for the d symbols. The $SU(N_f)^2U(1)_R$ triangle yields $-N_f T_{\rm fund}$ both at the quark and composite levels. Both triangles, $SU(N_f)^3$ and $SU(N_f)^2U(1)_R$, are saturated by ψ_M. Passing to $U(1)_R^3$ we must add the gluino contribution at the fundamental level and that of ψ_B at the composite level. At both levels the coefficient of the $U(1)_R^3$ triangle is $-N_f^2-1$. Finally, $U(1)_R$ counts the number of degrees of freedom weighed with the corresponding R charges. The corresponding coefficient again turns out to be the same, $-N_f^2-1$.

Summarizing, the massless composite fermions residing in the moduli superfields $M, B, \tilde{B}$ saturate all anomalies induced by the symmetries that are supposed to be realized linearly. The conjecture of the unbroken $SU(N_f)_R\times SU(N_f)_L$ and spontaneously broken $U(1)_B$ at the point (2.19) goes through. As a matter of fact, some of these anomaly matching conditions were observed long ago, in Refs.[18, 19].

Sometimes it is convenient to mimic the constraint (2.16) by introducing a Lagrange multiplier superfield X with the superpotential

$$\mathcal{W}=X\left(\det M-B\tilde{B}-\Lambda^{2N_f}\right)\,. \qquad (2.20)$$

We could treat, in a similar fashion, any point belonging to the quantum moduli space (2.16). For instance, we could travel from (2.17) to (2.19) observing how the regime continuously changes from the broken axial $SU(N_f)$ to the broken baryon number.

Concluding this part we remind that the case of the gauge group $SU(2)$ is exceptional. Indeed, in this case, the matter sector consisting of 2 fundamentals and 2 anti-fundamentals has $SU(4)$ global flavor symmetry, rather than $SU(2)\times SU(2)$. This is because all representations of $SU(2)$ are (pseudo)real, and fundamentals can be transformed into anti-fundamentals and *vice versa* by applying the $\epsilon^{\alpha\beta}$ symbol. This peculiarity was already discussed in detail in the first part of these lectures. Under these circumstances the pattern of the global symmetry breaking is somewhat different and the saturation of the anomaly triangles must be checked anew. Although this is a relatively simple exercise, we will not do it here. The interested reader is referred to[25].

$\mathbf{N_f = N_c + 1}$

The general strategy is the same as in the previous case. We introduce the mass term (2.14) assuming that two eigenvalues of the mass matrix are large while others are small. Then two heavy flavors can be integrated over, leaving us with the theory with $N_f = N_c - 1$, which can be analyzed in the weak coupling regime. A superpotential is generated on the vacuum valley. Using this superpotential it is not difficult to get the vacuum expectation values of the moduli fields $M, B_i, \tilde{B}^j$. They turn out to be constrained by the following relation[25]:

$$\det M \left(\frac{1}{M}\right)_i^j - B_i \tilde{B}^j = \Lambda^{2N_c - 1} m_i^j \,. \tag{2.21}$$

Note that the vanishing of the determinant, $\det M = 0$, which at the classical level automatically follows from the definition of M_j^i, is gone for the quantum VEV's $\langle Q^i \tilde{Q}_j \rangle$. This is most readily seen if the mass matrix $m_i^j \to m\delta_i^j$. In this case $\langle Q^i \tilde{Q}_j \rangle \sim (\Lambda^{2N_c-1} m)^{1/N_c} \delta_j^i$, $i, j = 1, ..., N_f$.

In the massless limit $m_i^j \to 0$ the quantum constraint (2.21) coincides with the classical one (2.11), or (2.12). Thus, the quantum and classical moduli spaces are identical. Every point from the vacuum valley can be reached by adding appropriate perturbations to the Lagrangian (i.e. mass terms and/or $\beta B + \tilde{\beta}\tilde{B}$).

The only point which deserves special investigation is the origin, which, unlike the situation $N_f = N_c$, remains singular. This is a signature of massless fields. Classically we have massless gluons and massless moduli fields. In the strong coupling regime we expect the gluons to acquire a dynamical mass gap. The classical moduli subject to the constraint (2.12) need not be the only composite massless states, however. Other composite massless states may form too. We will see shortly that they actually appear. At the origin, when $M = B = \tilde{B} = 0$, *all* global symmetries of the Lagrangian are presumably unbroken. In particular, the axial $SU(N_f)$ is realized linearly. Although we have already learned, from the previous example, that such a regime seems to be attainable in SQCD (in sharp contradistinction with QCD), the case $N_f = N_c + 1$ is even more remarkable – we want *all* global symmetries to be realized linearly. (For $N_f = N_c$, in the vacuum where the axial $SU(N_f)$ symmetry was unbroken, the baryon charge generators were spontaneously broken.)

At the origin (and near the origin) the theory is in the strong coupling regime. Let us examine the behavior of the theory in this domain more carefully. When the expectation values of all moduli fields vanish, the global symmetry $SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_R$ is unbroken provided no other (non-chiral) condensates develop. Is this solution self-consistent?

To answer this question we will try to match all corresponding anomalous AVV triangles; in this case we have seven triangles,

$$SU(N_f)^3\,,\;\; SU(N_f)^2 U(1)_B\,,\;\; SU(N_f)^2 U(1)_R\,,$$

$$U(1)_R U(1)_B^2\,,\;\; U(1)_R^3\,,\;\; U(1)_R^2 U(1)_B\,,\;\; U(1)_R\,. \tag{2.22}$$

They must be matched by the composite massless baryons residing in M, B_i and $\tilde{B}^j$. As we will see shortly, to achieve the matching we will need to consider all components of M, B_i and $\tilde{B}^j$ as independent, ignoring the constraint

$$\det M \left(\frac{1}{M}\right)_i^j - B_i \tilde{B}^j = 0 \tag{2.23}$$

Table 3. The quantum numbers of the massless fermions in SQCD with $N_f = N_c + 1$

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
ψ_M	N_f	$\bar{N}_f$	0	$\frac{2}{N_f} - 1$
ψ_B	$\bar{N}_f$	1	$N_f - 1$	$-\frac{1}{N_f}$
$\psi_{\tilde{B}}$	1	N_f	$-N_f + 1$	$-\frac{1}{N_f}$
ψ_Q	N_f	1	1	$\frac{1}{N_f} - 1$
$\psi_{\tilde{Q}}$	1	$\bar{N}_f$	-1	$\frac{1}{N_f} - 1$
λ	1	1	0	1

defining the vacuum valley both at the classical and the quantum levels. In other words, we will have to deal with a larger number of massless fields than one could infer from the parametrization (2.23) of the vacuum valley. The constraint (2.23) on the vacuum valley will reappear due to the fact that the expanded set of massless fields gets a superpotential $\mathcal{W}(M, B, \tilde{B})$. The requirement of the vanishing of the F term will give us Eq. (2.23).

Thus, our first task is to verify the matching. The quantum numbers of the fundamental quarks and the composite massless fermions can be inferred from Table 1. For convenience we collect them in Table 3.

Since we already have a considerable experience in matching the AVV triangles, I will not discuss all triangles from Eq. (2.22). As an exercise let us do just one of them, namely $U(1)_R^3$. In this case, at the fundamental level we have the $\psi_Q, \psi_{\tilde{Q}}$ and λ triangles which yield

$$2N_cN_f\left(\frac{1}{N_f} - 1\right)^3 + N_c^2 - 1\,.$$

At the composite level the $U(1)_R^3$ anomalous triangle is contributed by ψ_M (N_f^2 degrees of freedom), ψ_B and $\psi_{\tilde{B}}$ (each has N_f degrees of freedom). Thus, we get

$$N_f^2\left(\frac{2}{N_f} - 1\right)^3 - 2N_f\left(\frac{1}{N_f}\right)^3\,.$$

Both expressions reduce to

$$-N_f^2 + 6N_f - 12 + \frac{8}{N_f} - \frac{2}{N_f^2}\,.$$

Other triangles match too, in a miraculous way. Namely,

$$\begin{gathered} SU(N_f)^3 \to (N_f - 1)D_{\rm fund}\,, \\ SU(N_f)^2U(1)_R \to -\frac{(N_f-1)^2}{N_f}T_{\rm fund}\,, \\ U(1)_B^2U(1)_R \to -2(N_f-1)^2\,, \\ SU(N_f)^2U(1)_B \to (N_f - 1)T_{\rm fund}\,, \\ U(1)_R^2U(1)_B \to 0\,, \\ U(1)_R \to -N_f^2 + 2N_f - 2\,. \end{gathered}$$

The matching discussed above, was observed many years ago in Ref.[33] where the spectrum of the composite massless particles corresponding to the unconstrained M, B and $\tilde{B}$ was conjectured.

Thus, the above spectrum of the composite massless particles appearing at the origin of the vacuum valley in the $N_f = N_c + 1$ theory is self-consistent. We know, however, that the vacuum valley in the model at hand is characterized by Eq. (2.23). The situation seems rather puzzling. How the constraint (2.23) might appear?

The answer to this question was given by Seiberg[25]. If the massless fields, residing in the unconstrained M, B and $\tilde{B}$, acquire a superpotential, then the vacuum values of the moduli fields are obtained through the condition of vanishing F terms. The "right" superpotential will lead to Eq. (2.23) automatically.

So, what is the right superpotential? If it is generated, several requirements are to be met. First, it must be invariant under all global symmetries of the model, including the R symmetry. Second, the vacuum valleys obtained from this superpotential must correspond to Eq. (2.23). Third, away from the origin the only massless fields must be those compatible with the constraint (2.23).

All these requirements are satisfied by the following superpotential[25]:

$$\mathcal{W} = \frac{1}{\Lambda^{2N_f-3}}\left(B_i M^i_j \tilde{B}^j - \det M\right) . \tag{2.24}$$

It is obvious that the condition of vanishing of the F terms corresponding to M^i_j identically coincides with Eq. (2.23); moreover, vanishing of the F terms corresponding to B and $\tilde{B}$ yields two remaining constraints, $B_i M^i_j = \tilde{B}^j M^i_j = 0$. Once we move away from the origin, the moduli M^i_j grow, the fields $B_i, \tilde{B}^j$ acquire masses and can be integrated out. This eliminates $2N_f$ degrees of freedom. This is exactly the amount of the redundant degrees of freedom, see the section on $SU(N_c)$ QCD with N_f flavors - preliminaries. The emerging low-energy theory for the remaining degrees of freedom M^i_j has no superpotential. When $M^i_j \gg \Lambda^2$, the fields $B_i, \tilde{B}^j$ are very heavy, and the low-energy description based on Eq. (2.24) is no longer legitimate. It is interesting to trace the fate of the "baryons" $B_i, \tilde{B}^j$ in the process of this evolution from small to large values of M^i_j. This question has not been addressed in the literature so far.

Let us pause here to summarize the features of the dynamical regime taking place in the $N_f = N_c + 1$ model. The space of vacua is the same at the classical and quantum levels, the origin being singular due to the existence of the massless degrees of freedom. Since we have Higgs fields in the fundamental representation the theory is in the Higgs/confining phase; at the origin and near the origin the theory is strongly coupled and "confines" in the sense that physics is adequately described in terms of gauge invariant composites and their interactions. We think that at the origin all global symmetries of the Lagrangian are unbroken. The number of the massless degrees of freedom here is larger than the dimensionality of the space of vacua. To get the right description of the space of vacua one needs a superpotential, and such a superpotential is generated dynamically. It is a holomorphic function of the massless composites. The vacuum valley for this superpotential coincides with the quantum moduli space of the original theory. As we move along the vacuum valley away from the origin there is no phase transition – the theory smoothly goes into the weak coupling Higgs phase. The "extra" massless fields become massive, and irrelevant for the description of the vacuum valley.

The dynamical regime with the above properties got a special name – now it is referred to as *s-confinement*.

Seiberg's example of the s-confining theory was the first, but not the last. Other theories with similar behavior were found, see e.g.[67-73]. The set of s-confining models includes even such exotic one as the gauge group G_2 (this is an exceptional group), with five fundamentals[72, 73]. As a matter of fact, it is not difficult to work out a general strategy allowing one to carry out a systematic search of all s-confining theories. This was done in Ref.[74]. Without submerging into excessive technical detail let me outline just one basic point of the procedure suggested in[74].

A necessary condition of the s-confinement is generation of a superpotential at the origin of the moduli space, a holomorphic function of relevant moduli fields. Generically, the form of this superpotential, dictated by the R symmetry plus dimensional arguments is

$$\mathcal{W} \propto \left[\prod_{\ell} S_{\ell}^{2T(R_\ell)} \Lambda^{3T(G)-\sum_\ell T(R_\ell)}\right]^{1/(\sum_\ell T(R_\ell)-T(G))} . \tag{2.25}$$

The product (sum) runs over all matter fields present in the theory. For instance, in the case of SQCD for each flavor we have to include two subflavors. I remind that $T(R)$ is (one half of) the Dynkin index. Particular combinations of the superfields in the product are not specified; they depend on the particular representations of the matter fields with respect to the gauge group. What is important is only the fact that they all are homogeneous functions of S_ℓ's, of order $2T(R_\ell)$. Note that the combination appearing in Eq. (2.25) is the only one which has correct properties under renormalization, i.e. compatible with the NSVZ β function.

Now, if we want the origin to be analytic (and this is a feature of the s-confinement, by definition), we must ensure that

$$\sum_\ell T(R_\ell) - T(G) = 1 \tag{2.26}$$

(more generically, 1/integer). This severely limits the choice of possible representations since the Dynkin indices are integers. For instance, if the matter sector is vector-like, there exist only two options: (i) Seiberg's model, $SU(N_c)$ color with N_c+1 flavors (i.e. N_c+1 fundamentals and N_c+1 anti-fundamentals; (ii) $SU(N_c)$ color with one antisymmetric tensor plus its adjoint plus three flavors.

Not to make a false impression I hasten to add that some models that satisfy Eq. (2.26), are not s-confining. A few simple requirements to be met, which comprise a sufficient condition for s-confinement, are summarized in Ref.[74], which gives also a full list of the s-confining theories.

Conformal window. Duality

Our excursion towards larger values of N_f must be temporarily interrupted here – the methods we used so far fail at $N_f = N_c+2$. One can show that the quantum moduli space coincides with the classical one, just as in the case $N_f = N_c + 1$. However, at the origin of the moduli space, description of the large-distance behavior of the theory in terms of the massless fields residing in M, B and $\tilde{B}$ does not go through. These degrees of freedom are irrelevant for this purpose; the dynamical regime of the theory in the infrared is different. To see that M, B and $\tilde{B}$ do not fit suffice to try to saturate the 't Hooft triangles corresponding to the unbroken global symmetries, in the same vein as we did previously for $N_f = N_c + 1$. There is no matching! As we will see shortly, the dynamical regime does indeed change in passing from $N_f = N_c + 1$ to $N_f = N_c + 2$. The correlation functions of the $N_f = N_c + 2$ theory at large distances are those of a free theory, like in massless electrodynamics. But the number of free degrees of freedom ("photons" and "photinos") is different from from what one might expect naively. Namely, we will have three "photons" and three "photinos" in the case at hand, in addition to $(N_c+2)\times 2$ free "fermion" fields. These photons and photinos, in a sense, may be considered as the bound states of the original gluons, gluinos, quarks and squarks.

To elucidate this, rather surprising, picture we will have to make a jump in our travel along the N_f axis, leave the domain of N_f close to N_c for a while, and turn to much larger values of N_f. The critical points on the N_f axis are $3N_c$ and $3N_c/2$. That's where a *conformal window* starts and ends. We will return to the $N_f = N_c + 2$ and $N_f = N_c + 1$ theories later on.

At first, let me recall a few well-known facts from ordinary non-supersymmetric QCD. The Gell-Mann-Low function in QCD has the form[75]

$$\beta_{\rm QCD}(\alpha_s) = -\beta_0 \frac{\alpha_s^2}{2\pi} - \beta_1 \frac{\alpha_s^3}{4\pi^2} - \ldots ,$$

$$\beta_0 = 11 - \frac{2}{3} N_f\,, \quad \beta_1 = 51 - \frac{19}{3} N_f\,. \tag{2.27}$$

At small α_s it is negative since the first term always dominates. This is the celebrated asymptotic freedom. With the scale μ decreasing the running gauge coupling constant grows, and the second term becomes important. Generically the second term takes over the first one at $\alpha_s/\pi \sim 1$, when all terms in the α_s expansion are equally important, i.e. in the strong coupling regime. Assume, however, that for some reasons the first coefficient β_0 is abnormally small, and this smallness does not propagate to higher orders. Then the second term catches up with the first one when $\alpha_s/\pi \ll 1$, we are in the weak coupling regime, and higher order terms are inessential. Inspection of Eq. (2.27) shows that this happens when N_f is close to 33/2, say 16 or 15 (N_f has to be less than 33/2 to ensure asymptotic freedom). For these values of N_f the second coefficient β_1 turns out to be negative! This means that the β function develops a zero in the weak coupling regime, at

$$\frac{\alpha_s^*}{2\pi} = \frac{\beta_0}{-\beta_1} \ll 1\,. \tag{2.28}$$

(Say, if $N_f = 15$ the critical value is at 1/44.) This zero is nothing but the infrared fixed point of the theory. At large distances $\alpha_s \to \alpha_s^*$, and $\beta(\alpha_s^*) = 0$, implying that the trace of the energy-momentum vanishes, and the theory is in the conformal regime. There are no localized particle-like states in the spectrum of the theory; rather we deal with massless unconfined interacting quarks and gluons; all correlation functions at large distances exhibit a power-like behavior. In particular, the potential between two heavy static quarks at large distances R will behave as $\sim \alpha_s^*/R$. The situation is not drastically different from conventional QED. The corresponding dynamical regime is, thus, a non-Abelian Coulomb phase. As long as α_s^* is small, the interaction of the massless quarks and gluons in the theory is weak at all distances, short and large, and is amenable to the standard perturbative treatment (renormalization group, etc.). QCD becomes a fully calculable theory.

There is nothing remarkable in the observation that, for a certain choice of N_f, quantum chromodynamics becomes conformal and weakly coupled in the infrared limit. Belavin and Migdal played with this model over 20 years ago[76]. They were quite excited explaining how great it would be if N_f in our world were close to 16, and the theory would be in the infrared conformal regime, with calculable anomalous dimensions. Later on this idea was discussed also by Banks and Zaks[77]. Alas, we do not live in a world with $N_f \approx 16$...

What is much more remarkable is the existence of the infrared conformal regime in SQCD for large couplings, $\alpha_*/\pi \sim 1$. This fact, as many others in the given range of questions, was established by Seiberg[78]. The discovery of the strong coupling conformal regime[78] is based on the so-called *electric-magnetic duality.* Although the term suggests the presence of electromagnetism and the same kind of duality under the substitution $\vec{E} \leftrightarrow \vec{B}$ one sees in the Maxwell theory, actually both elements, "electric-magnetic" and "duality" in the given context are nothing but remote analogies, as we will see shortly.

Analysis starts from consideration of SQCD with N_f slightly smaller than $3N_c$. More exactly, if

$$\varepsilon \equiv 1 - \frac{N_f}{3N_c} \tag{2.29}$$

we assume that $N_c \to \infty$, and $0 < \varepsilon \ll 1$. It is assumed also that we are at the origin of the moduli space – no fields develop VEV's. By examining the NSVZ β function, Eq. (1.61), it is easy to see that in this limit the first coefficient of the β function is abnormally small, and the second coefficient is positive and is of a normal order of magnitude,

$$\beta_0 = 3N_c\varepsilon\,, \quad \beta_1 = -3N_c^2 + \mathcal{O}(\varepsilon)\,. \tag{2.30}$$

To get β_1 I used the fact that

$$\gamma(\alpha) = -\frac{N_c^2 - 1}{2N_c}\,\frac{\alpha}{\pi} + \mathcal{O}(\alpha^2) \tag{2.31}$$

in the model considered (for a pedagogical review see e.g. the last paper in Ref.[44]). There is a complete parallel with conformal QCD, with 15 or 16 flavors, discussed above. The numerator of the NSVZ β function vanishes at

$$\gamma(\alpha_*) \equiv \gamma_* = 1 - 3\frac{N_c}{N_f} \to -\varepsilon \quad \text{or} \quad \frac{N_c\alpha_*}{2\pi} = \varepsilon\,. \tag{2.32}$$

The vanishing of the β function marks the onset of the conformal regime in the infrared domain; the fact that α_* is small means that the theory is weakly coupled in the infrared (it is weakly coupled in the ultraviolet too since it is asymptotically free).

Here comes the breakthrough observation of Seiberg. Compelling arguments can be presented indicating that the original theory with the $SU(N_c)$ gauge group (let us call this theory "electric"), and another theory, with the $SU(N_f - N_c)$ gauge group, the same number of flavors N_f as above, and a specific Yukawa interaction (let us call this theory "magnetic"), flow to one and the same limit in the infrared asymptotics. The corresponding Gell-Mann-Low functions of both theories vanish at their corresponding critical values of the coupling constants. Both theories are in the non-Abelian Coulomb (conformal) phases. By inspecting Eq. (1.61) it is easy to see that, when α_* in the electric theory approaches zero (i.e. $\varepsilon \to 0$), in the magnetic theory $\gamma(\alpha_*)$ approaches -1, i.e. the theory becomes strongly coupled. The opposite is also true. When the magnetic theory becomes weakly coupled, i.e.

$$N_f \to \frac{3}{2}N_c \text{ from above}\,, \quad (\alpha_*)_{\rm magn} \to 0\,, \tag{2.33}$$

in the electric theory $(\gamma_*)_{\rm electr} \to -1$, and the electric theory is strongly coupled in the infrared. This reciprocity relation is, probably, the reason why the correspondence between the two theories is referred to as the electric-magnetic duality. It is worth emphasizing that the correspondence takes place only in the infrared limit. By no means are the above two theories totally equivalent to each other; their ultraviolet behavior is completely different. If $N_f < (3N_c)/2$ the magnetic theory looses asymptotic freedom. Thus, the *conformal window*, where both theories are asymptotically free

Table 4. The quantum numbers of the massless fields of the "electric" theory from the dual pair

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
ψ_Q	N_f	0	1	$-N_c/N_f$
$\psi_{\tilde{Q}}$	0	$\bar{N}_f$	-1	$-N_c/N_f$

Table 5. The quantum numbers of the massless fields of the "magnetic" theory from the dual pair

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
$\psi_{\mathcal{Q}}$	$\bar{N}_f$	0	$N_c/(N_f-N_c)$	$(N_c-N_f)/N_f$
$\psi_{\tilde{\mathcal{Q}}}$	0	N_f	$-N_c/(N_f-N_c)$	$(N_c-N_f)/N_f$
$\chi=\psi_{\mathcal{M}}$	N_f	$\bar{N}_f$	0	$(N_f-2N_c)/N_f$

in the ultraviolet and conformally invariant in the infrared extends in the interval $3N_c/2 \le N_f \le 3N_c$.

The fact that the conformal window cannot extend below $3N_c/2$ is seen from consideration of the electric theory *per se*, with no reference to the magnetic theory. Indeed, the total (normal + anomalous) dimension of the matter field $Q\tilde{Q}$ in the infrared limit is equal to

$$d = 2+\gamma_* = \frac{3(N_f-N_c)}{N_f}\,.$$

No physical field can have a dimension less than unity; this is forbidden by the Källén-Lehmann spectral representation. If $d=1$ the field is free. The dimension d reaches unity exactly at $3N_c/2$. Decreasing N_f further and assuming that the conformal regime $\beta(\alpha_*)=0$ is still preserved would violate the requirement $d \ge 1$.

Let us describe the electric and magnetic theories in more detail. I will continue to denote the quark fields of the first theory as Q, while those of the magnetic theory will be denoted by $\mathcal{Q}$. Both have N_f flavors, i.e. $2N_f$ chiral superfields in the matter sector. The same number of flavors is necessary to ensure that global symmetries of the both theories are identical.

The magnetic theory, additionally, has N_f^2 colorless "meson" superfields $\mathcal{M}^i_j$ whose quantum numbers are such as if they were built from a quark and an antiquark. The meson superfields are coupled to the quark ones of the magnetic theory through a superpotential

$$\mathcal{W} = f\mathcal{M}^i_j\mathcal{Q}_i\tilde{\mathcal{Q}}^j\,. \tag{2.34}$$

The quantum numbers of the fields belonging to the matter sectors of the magnetic and electric theories are summarized in Tables 4 and 5.

The quantum numbers of the meson superfield are fixed by the superpotential (2.34). Note a very peculiar relation between the baryon charges of the quarks in the electric and magnetic theories. This relation shows that the quarks of the magnetic theory cannot be expressed, in any polynomial way, through quarks of the electric theory. The connection of one to another is presumably extremely non-local and complicated. The explicit connection between the operators in the dual pairs is known only for a handful of operators which have a symmetry nature[79].

We can now proceed to the arguments establishing the equivalence of these two theories in the infrared limit. The main tool we have at our disposal for establishing the equivalence is again the 't Hooft matching, the same line of reasoning as was used above in verifying various dynamical regimes in $N_f=N_c$ and $N_f=N_c+1$ models.

Since we are at the origin of the moduli space, all global symmetries are unbroken, and one has to check six highly non-trivial matching conditions corresponding to various triangles with the $SU(N_f)$, $U(1)_R$ and $U(1)_B$ currents at the vertices.

The presence of fermions from the meson multiplet $\mathcal{M}$ is absolutely crucial for this matching. Specifically, one finds for the one-loop anomalies in both theories[78]:

$$\begin{aligned}
SU(N_f)^3 &\to N_c D_{\text{fund}}, \\
SU(N_f)^2 U(1)_R &\to -T_{\text{fund}} N_c^2/N_f, \\
SU(N_f)^2 U(1)_B &\to N_c T_{\text{fund}}, \\
U(1)_B^2 U(1)_R &\to -2N_c^2, \\
U(1)_R^3 &\to N_c^2 - 1 - 2N_c^4/N_f^2, \\
U(1)_R &\to -N_c^2 - 1\,.
\end{aligned} \tag{2.35}$$

For example, in the $U(1)_R^3$ anomaly in the electric theory the gluino contribution is proportional to N_c^2-1 and that of quarks to $-(N_c/N_f)^3 2N_f N_c = -2N_c^4/N_f^2$; altogether $N_c^2 - 1 - 2N_c^4/N_f^2$ as in (2.35). In the dual theory one gets from gluino and quarks another contribution, $(N_f - N_c)^2 - 1 - 2(N_f - N_c)^4/N_f^2$. Then the fermions χ from the meson multiplet $\mathcal{M}$ add an extra $[(N_f - 2N_c)/N_f]^3 N_f^2$, which is precisely the difference.

The last line in Eq. (2.35) corresponds to the anomaly of the R current in the background gravitational field. In the electric $SU(N_c)$ theory the corresponding coefficient is

$$N_c^2 - 1 - 2(N_c/N_f)N_c N_f = -N_c^2 - 1$$

while in the magnetic theory one has $-(N_f - N_c)^2 - 1$ from quarks and gluinos and $[(N_f - 2N_c)/N_f]N_f^2$ from the $\mathcal{M}$ fermions, i.e. the sum is again $-N_c^2 - 1$.

It is not difficult to check the matching of other triangles from Eq. (2.35). The dependence on N_c and N_f is rather sophisticated, and it is hard to imagine that this is an accidental coincidence. The fact that the electric and magnetic theories described above have the same global symmetries is an additional argument in favor of their (infrared) equivalence. Of course, they have different gauge symmetries: $SU(N_c)$ in the first case and $SU(N_f - N_c)$ in the second. The gauge symmetry, however, is not a regular symmetry; in fact, it is not a symmetry at all. Rather, it is a redundancy in the description of the theory. One introduces first more degrees of freedom than actually exist, and then the redundant variables are killed by the gauge freedom. That's why the gauge symmetry has no reflection in the spectrum of the theory. Therefore, distinct gauge groups do not preclude the theories from being dual, generally speaking. On the contrary, the fact that such dual pairs are found is very intriguing; it allows one to look at the gauge dynamics from a new angle.

I have just said that various dual pairs of supersymmetric gauge theories are found. To avoid misunderstanding I hasten to add that although Seiberg's line of reasoning is very compelling it still falls short of *proving* the infrared equivalence. The theory in the strong coupling regime is not directly solved, and we are hardly any closer now to the solution than we were a decade ago. The infrared equivalence has the status of a good solid conjecture substantiated by a number of various indirect arguments we have at our disposal (see the next section).

If we accept this conjecture we can make a remarkable step forward compared to the conformal limit of QCD studied in the weak coupling regime in the 70's and 80's. Indeed, if N_f is close to $3N_c$ (but slightly lower), i.e. we are near the right edge of the conformal window, the weakly coupled electric theory is in the conformal regime. Since it is equivalent (in the infrared) to the magnetic theory, which is strongly coupled at these values of N_f we, thus, establish the existence of a *strongly coupled* superconformal gauge theory. Moreover, when N_f is slightly higher than $3N_c/2$, i.e. near the left edge of the conformal window, the magnetic theory is weakly coupled and in the conformal regime. Its dual, the electric theory, which is strongly coupled near the left edge of the conformal window, must then be in the conformal regime too. In the middle of the conformal window, when both theories are strongly coupled, strictly speaking we do not know whether or not they stay superconformal. In principle, it is possible that they both leave the conformal regime. This could happen, for instance, if α_*, the solution of the equation $\gamma(\alpha_*) = 1 - 3N_c N_f^{-1}$, (temporary) becomes larger than $2\pi/T(G)$, the position of the zero of the *denominator* of the NSVZ β function, as we go further away from the point $N_f = 3N_c$ in the direction of $N_f = 3N_c/2$, and then α_* becomes smaller than $2\pi/T(G)$ again, as we approach $N_f = 3N_c/2$. Such a scenario, although not ruled out, does not seem likely, however.

Traveling along the valleys

So far, the dual pair of theories was considered at the origin of the vacuum valley. Both theories, electric and magnetic, have vacuum valleys and a natural question arises as to what happens if we move away from the origin[‡]. As a matter of fact, this question is quite crucial, since if the theories are equivalent in the infrared, a certain correspondence between them should persist not only at the origin, but at any other point belonging to the vacuum valley. If a correspondence can be found, it will only strengthen the conjecture of duality.

Thus, let us start from the electric theory and move away from the origin. Consider for simplicity a particular direction in the moduli space, namely,

$$Q = \tilde{Q} = \begin{pmatrix} a_1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}, \tag{2.36}$$

where $Q, \tilde{Q}$ are the superfields comprising, say, the first flavor. Moving along this direction we break the gauge symmetry $SU(N_c)$ down to $SU(N_c - 1)$; $2N_c - 1$ chiral superfields are eaten up in the (super)-Higgs mechanism providing masses to $2N_c - 1$ W bosons. Below the mass scale of these W bosons the effective theory is SQCD with a $SU(N_c - 1)$ gauge group and $N_f - 1$ flavors. (Additionally there is one $SU(N_c - 1)$ singlet, but it plays no role in the gauge dynamics.) It is not difficult to see that decreasing both N_f and N_c by one unit in the electric theory we move to the right along the axis $N_f/3N_c$. In other words, we move towards the right edge of our conformal window, making the electric theory weaker. From what we already know, we should then expect that the magnetic theory becomes stronger.

Let us have a closer look at the magnetic theory. The vacuum expectation value (2.36) is reflected in the magnetic theory as the expectation value of the $(1,1)$ component of the meson field $\mathcal{M}$. No Higgs phenomenon takes place, but, rather, $\mathcal{M}_1^1 \neq 0$. Then, thanks to the superpotential (2.34), the magnetic quark $\mathcal{Q}_1, \tilde{\mathcal{Q}}_1$ gets a mass, and becomes irrelevant in the infrared limit. The gauge group remains the same, $SU(N_f - N_c)$, but the number of active flavors reduces by one unit (we are left with

[‡]This question was suggested to me by C. Wetterich. Note that if in the electric theory the vacuum degeneracy manifests itself in arbitrary vacuum expectations of Q and $\tilde{Q}$, in the magnetic theory the expectation values of $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$ vanish. The flat direction corresponds to an arbitrary expectation value of $\mathcal{M}$.

$N_f - 1$ active flavors). This means that the first coefficient of the Gell-Mann-Low function of the magnetic theory becomes more negative and the critical value $\alpha_{*\mathrm{magn}}$ increases. The theory becomes coupled stronger, in full accord with our expectations.

Let us now try the other way around. What happens if we introduce the mass term to one of the quarks in the electric theory, say the first flavor? The gauge group remains, of course, the same, $SU(N_c)$. However, in the infrared domain the first flavor decouples, and we are left with $N_f - 1$ active flavors. The first coefficient in the β function of the electric theory becomes more negative; hence, the critical value $\alpha_{*\mathrm{electr}}$ increases. We move leftwards, towards the left edge of the conformal window. Correspondingly, the electric theory becomes stronger coupled, and we expect that the magnetic one will be coupled weaker.

What is the effect of the mass term in the magnetic theory? It is rather obvious that the corresponding impact reduces to introducing a mass term in the superpotential (2.34),

$$\mathcal{W}' = f\mathcal{M}^i_j \mathcal{Q}_i \tilde{\mathcal{Q}}^j + m\mathcal{M}^1_1 \,. \tag{2.37}$$

Extending the superpotential is equivalent to changing the vacuum valley. Indeed, the expectation values of $\mathcal{Q}_1$ and $\mathcal{Q}_1$ do not vanish anymore. Instead, the condition of the vanishing of the F term implies

$$\frac{\partial \mathcal{W}}{\partial \mathcal{M}^1_1} = \mathcal{Q}_1 \tilde{\mathcal{Q}}^1 + m = 0 \,. \tag{2.38}$$

If m is large, Eq. (2.38) implies, in turn, that the magnetic squarks of the first flavor develop a vacuum expectation value, the magnetic theory turns out to be in the Higgs phase, the gauge group $SU(N_f - N_c)$ is spontaneously broken down to $SU(N_f - N_c - 1)$, and one magnetic flavor is eaten up in the super-Higgs mechanism. We end up with a theory with the gauge group $SU(N_f - N_c - 1)$ and $N_f - 1$ flavors. The $\mathcal{M}^i_1$ and $\mathcal{M}^1_i$ components of the meson field become sterile in the infrared limit. In this theory the first coefficient of the β function is less negative, $\alpha_{*\mathrm{magn}}$ is smaller, we are closer to the left edge of the conformal window, as was expected.

Summarizing, we see that Seiberg's conjecture of duality is fully consistent with the vacuum structure of both theories. As a matter of fact, this observation may serve as additional evidence in favor of duality. Simultaneously it makes perfectly clear the fact that, if duality does take place, it can be valid only in the infrared limit; by no means the two theories specified above are fully equivalent.

One may ask what happens if we continue adding mass terms to the electric quarks of the first, second, third, *etc.* flavors. Adding large mass terms we eliminate flavors one by one. In other words, we launch a cascade taking us back to smaller values of N_f. The electric theory becomes stronger and stronger coupled. Simultaneously the dual magnetic theory is coupled weaker and weaker. When the number of active flavors reaches $3N_c/2$, $\alpha_{*\mathrm{magn}} = 0$. Eliminating the quark flavors further we leave the conformal window – the magnetic theory looses asymptotic freedom and becomes infrared-trivial, with the interaction switching off at large distances. We find ourselves in the free magnetic phase, or the Landau phase. By duality, the correlation functions in the electric theory (which is superstrongly coupled in this domain of N_f) must have the same trivial behavior at large distances. To be perfectly happy we would need to know the relation between all operators of the electric and magnetic theories, so that given a correlation function in the electric theory we could immediately translate it in the language of essentially free magnetic theory. Alas ... As was already mentioned, this relation is basically unknown. It can be explicitly found only for some operators of a geometric nature. Even though the general relation was not found, the achievement is remarkable. For the first time ever the gauge bosons of the weakly coupled theory (magnetic) are shown to be "bound states" of a strongly coupled theory (electric).

The last but one step in the reduction process is when the number of active flavors is N_c+2. The gauge group of the electric theory is $SU(N_c)$ while that of the magnetic one is $SU(2)$. Under the duality conjecture the large distance behavior of the superstrongly coupled electric theory is determined by the massless modes of the essentially free magnetic theory: three "photons", three "photinos", $2(N_c+2)$ fields of the type $\mathcal{Q}$ and $2(N_c+2)$ fields of the type $\tilde{\mathcal{Q}}$. We can return to the remark made in the very beginning of this section – it is explained now. It becomes clear why all attempts to describe the infrared behavior of the theory in terms of the variables $M, B, \tilde{B}$ failed in the $N_f = N_c+2$ case: these are not proper massless degrees of freedom.

The last step in the cascade that still can be done is reducing one more flavor, by adding a large mass term in the electric theory, or the corresponding entry of the matrix $\mathcal{M}$ in the superpotential of the magnetic theory. At this last stage the super-Higgs mechanism in the magnetic theory completely breaks the remaining gauge symmetry. We end up with N_c+1 massless flavors interacting with the $(N_c+1)\times(N_c+1)$ meson superfield $\mathcal{M}^i_j$ (plus a number of sterile fields inessential for our consideration). This remaining meson superfield $\mathcal{M}^i_j$ is assumed to have no (or small) expectation values, so that we stay near the origin of the vacuum valley. Moreover, it is not difficult to see that the instantons of the broken $SU(2)$ generates the superpotential of the type $\det\mathcal{M}$. This is nothing but a supergeneralization of the 't Hooft interaction[80]. Indeed, the instanton generates fermion zero modes, one mode for each $\psi_{\mathcal{Q}}$ and $\psi_{\tilde{\mathcal{Q}}}$. In the absence of the Yukawa coupling $f\mathcal{M}\mathcal{Q}\tilde{\mathcal{Q}}|_F$, there is no way to contract these zero modes, and their proliferation results in the vanishing of the would-be instanton-induced superpotential. The Yukawa coupling $f\phi_{\mathcal{M}}\psi_{\mathcal{Q}}\psi_{\tilde{\mathcal{Q}}}$ lifts the zero modes, much in the same way as the mass term does. (Supersymmetrization of the result is achieved through the vertex $f\psi_{\mathcal{M}}\phi_{\mathcal{Q}}\psi_{\tilde{\mathcal{Q}}}$ where the boson field $\phi_{\mathcal{Q}}$ is induced by the zero mode of $\psi_{\mathcal{Q}}$ and the gluino zero mode). In this way we arrive at the instanton-induced superpotential $\det\mathcal{M}$. The full superpotential of the magnetic theory, describing the interaction of massless (or nearly massless) degrees of freedom which are still left there, is

$$\mathcal{W} = f\mathcal{M}\mathcal{Q}\tilde{\mathcal{Q}} + \det\mathcal{M}\,. \tag{2.39}$$

Compare it with Eq. (2.24) which was derived for the interaction of the massless degrees of freedom of the $N_f = N_c+1$ model by exploiting a totally different line of reasoning. Up to a renaming of the fields involved, the coincidence is absolute! Thus, the duality conjecture allows us to rederive the s-confining potential in the "electric" theory, SQCD with the gauge group $SU(N_c)$ and $N_f = N_c+1$. The fact that two different derivations lead to one and the same result further strengthens the duality conjecture, making one to think that actually it is more than a conjecture: the full infrared equivalence of Seiberg's "electric" and "magnetic" theories does indeed take place.

Patterns of Seiberg's duality in more complicated gauge theories than that discussed above, including non-chiral matter sectors, were studied in a number of publications. The dual pairs proliferate! By now a whole zoo of dual pairs is densely populated. Finding a "magnetic" counterpart to the given "electric" theory remains an art, rather than science – no general algorithm exists which would allow one to generate dual pairs automatically, although there is a collection of some helpful hints and recipes. Systematic searches for dual partners to every given supersymmetric theory is an intriguing and fascinating topic. At the present stage it is too technical, however, to be included in this lecture course. Even a brief discussion of the corresponding advances would lead us far astray. The interested reader is referred to the original literature, see e.g.[81, 82, 83].

DOMAIN WALLS, OR NEW EXACT RESULTS "AFTER SEIBERG"

After 1994 many followers worked on dynamical aspects of non-Abelian SUSY gauge theories. In many instances the development went not in depth but, rather, on the surface. Various "exotic" gauge groups and matter representations were considered, supplementing the list of models considered in the previous section by many new examples with essentially the same dynamical behavior. The corresponding discussion might be interesting to experts but is hardly appropriate here. In this part we will focus on some new findings "after Seiberg" and "Seiberg and Witten". It turns out that little miracles of supersymmetric gauge dynamics are not exhausted by fascinating phenomena considered in the previous sections: the exact β function, the electric-magnetic duality in the conformal window and so on. Here we will discuss a fresh topic: exact results for supersymmetric domain walls. The walls emerge in SUSY gluodynamics, the simplest and the least studied of all supersymmetric gauge theories. In spite of the incredible complexity of this model, and unknown intricacies of its strong dynamics, we will be able to *exactly* calculate the wall energy density (sometimes referred to as the wall tension ε). Again, as in all previous cases, our calculation will be indirect. We will heavily exploit our magic tool kit, presented in the first part of these lectures, supplemented by a couple of new devices. You will need some patience to make yourself familiar with these devices. The effort will be rewarded – eventually we will get an elegant formula expressing ε in terms of the gluino condensate $\langle\lambda\lambda\rangle$.

Domain walls built of supersymmetric glue

Let me remind that SUSY gluodynamics describes gauge interactions of gluons and gluinos. For simplicity we will limit ourselves to the gauge group $SU(N)$. As we learned in the section on supersymmetric gluodynamics several degenerate vacuum states, labeled by the value of the gluino condensate, exist in this theory. There are N vacua in which the discrete chiral Z_{2N} symmetry is spontaneously broken down to Z_2,

$$\langle\lambda\lambda\rangle = \Lambda^3 e^{2\pi i k/N}\,, \tag{3.1}$$

where $k = 0, 1, ..., N-1$ and the vacuum angle ϑ is set equal to zero; Λ is the scale parameter of the theory. Furthermore, it is plausible that an additional chirally symmetric vacuum state exists, with a vanishing value of the gluino condensate, $\langle\lambda\lambda\rangle = 0$ (the so called Kovner-Shifman state[11]). All $N+1$ vacua are supersymmetric, i.e. the vacuum energy density vanishes.

The theories with a discrete set of degenerate vacuum states admit a peculiar class of excitations. Let φ be an order parameter distinguishing between distinct vacua, say, in the first vacuum $\langle\varphi\rangle = v_1$ and in the second vacuum $\langle\varphi\rangle = v_2$ (it is assumed that $v_1 \neq v_2$). Then one can consider a (static) field configuration such that φ depends only on z, the third spatial coordinate, and $\varphi \to v_1$ at $z \to -\infty$ while $\varphi \to v_2$ at $z \to +\infty$. A rapid transition from one asymptotics to another occurs in a thin layer, extending in the xy plane near $z = 0$. Far to the left of $z = 0$ we find ourselves in the first vacuum, $\langle\varphi\rangle = v_1$, far to the right in the second, $\langle\varphi\rangle = v_2$. Thus, far away from the plane $z = 0$ there is no energy stored in the field φ. However, in the transition layer adjacent to the plane $z = 0$ the field φ has to restructure itself from v_1 to v_2, which costs some energy, both kinetic and potential. The energy density profile is characterized by a sharply peaked energy distribution centered at $z = 0$. This is nothing but a domain wall, a phenomenon familiar to everybody from the schooldays, from the theory of ferromagnets. Integrating over z the volume energy density of the interpolating field configuration, we get the wall tension ε.

The total energy of the domain wall $\mathcal{E}$ is obviously proportional to its area A, so that when the xy extensions of the wall grow to infinity, the total energy becomes infinite too. It is the ratio $\varepsilon = \mathcal{E}/A$ that stays finite. The wall tension ε is a close relative of the string tension σ. The string tension σ in QCD is expected to be $\sim \Lambda^2$. The total energy of the string grows linearly with the string dimension L. The total energy of the wall in supersymmetric gluodynamics grows quadratically with L, and ε is expected to be Λ^3.

The domain wall is not a particle-like configuration, of course; it is an extended object. Nevertheless, the domain walls become an important dynamical feature of any theory where they occur. The walls are topologically stable. After the wall is formed it cannot be removed by any local perturbation. Likewise, one cannot produce a wall by a local source. They appear only as global topological defects. The sectors of the theory with a given number of domain walls (zero, one, two and so on) are totally decoupled from each other. As a matter of fact, if we happen to live in a world with a domain wall, we should perceive this field configuration as our "vacuum state" rather than an excitation, although with respect to the full theory, which includes all sectors (as it might be viewed by God), the domain wall is certainly an excitation.

In supersymmetric gluodynamics the relevant order parameter is the gluino density, $\lambda\lambda$. This parameter is quadratic in the fermion field. This fact alone tells us that we will not be able to treat the wall in a quasiclassical approximation, routinely used in the studies of the wall-like solutions in weakly coupled theories. The literature devoted to the domain walls in field theory is quite rich, but next to nothing is said about the quantitative aspects of the wall configurations in the strongly coupled theories. In supersymmetric gluodynamics the domain wall is a genuinely strong-coupling phenomenon, the realm of non-perturbative physics. Thus, once again, we find ourselves in *terra incognita*, and it is only the power of supersymmetry that will eventually lead us to an exact solution for the wall tension.

Central extension of $N = 1$ superalgebra

The title of this section may sound like a heresy for those who are familiar with the basics of supersymmetry. Indeed, in every respectable text book it is written that in four dimensions only extended supersymmetries ($N = 2$ and higher) admit central extensions, while $N = 1$ theories cannot have central charges. Thus, we are definitely going to violate one of the most sacred theorems of supersymmetry. Some time ago I gave a talk at Imperial College in London. One of the mathematicians in the audience got very excited at this point and said that the existence of the central extensions $N = 1$ theories cannot be true because it can never be true because if "one calculates the Hochschild-Serre spectral sequence for the algebra, the second cohomology should be zero!". Although I did not understand a single word in the above statement, I nevertheless insist that supersymmetric gluodynamics, being $N = 1$ theory, still does have a non-trivial central extension[84].

Since this element is absolutely crucial in the given range of questions, I will explain in detail what a central extension in general means, and how it emerges in supersymmetric gluodynamics.

The defining relation of supersymmetry is the anticommutator of two supercharges,

$$\{Q_\alpha \, \bar{Q}_{\dot\beta}\} = 2P_{\alpha\dot\beta}\,, \tag{3.2}$$

where $P_{\alpha\dot\beta}$ is the energy-momentum operator. To close the algebra one must consider a few other (anti)commutation relations between Q's, P's and other conserved quantities.

All these relations are well known and are irrelevant for our purposes. Let us concentrate on the only relevant anticommutation relation,

$$\{Q_\alpha \, Q_\beta\} = 2T_{\alpha\beta}\,. \tag{3.3}$$

Note that Eq. (3.2) contains the supercharge Q and its Hermitean conjugate $\bar{Q}$, while Eq. (3.3) contains only Q's. Certainly, one can consider a similar anticommutator of two $\bar{Q}$'s too. Since the supercharge Q is conserved, T must be conserved too. On general grounds one can show[85] that $T_{\alpha\beta}$ must commute with all other conserved operators of the theory. That is why $T_{\alpha\beta}$ is called the central extension. In many instances T reduces to a number (the central charge). Why it was universally believed that $T \equiv 0$ in $N = 1$ theories? The general classification of superalgebras dates back to the classical paper[85]. Should the central extension appear in the anticommutator (3.3), it will clearly belong to $(0, 1)$ representation of the Lorentz group. This fact is obvious, since $T_{\alpha\beta}$, by construction, is symmetric with respect to two undotted indices, while the only Lorentz-invariant combination would be proportional to $\epsilon_{\alpha\beta}$. The existence of an extra conserved quantity that is not a Lorentz scalar, in addition to four-momentum, is forbidden by the famous Coleman–Mandula theorem[86] for all theories with non-trivial S matrix. The essence of this theorem is very simple: if we have too many conserved operators that are not Lorentz scalars, the S matrix is constrained too strongly. The energy-momentum conservation still allows two-by-two scattering amplitudes to continuously depend on the scattering angle θ. If we want this property to persist, the only other conserved charges we can introduce in the theory are "external" Lorentz-scalar charges, e.g. the electromagnetic charge, the baryon charge and so on. If we add a conserved operator transforming as $(0, 1)$ with respect to the Lorentz group, we will kill any possibility of non-trivial scattering. Note that in two-dimensional theories there is no scattering angle θ, the Coleman-Mandula theorem is not applicable, and the bookkeeping works differently: central extensions are perfectly possible in the minimal $N = 1$ superalgebras. And indeed, centrally extended two-dimensional $N = 1$ theories are very well known in the literature[87].

In order to have a central extension in four-dimensional $N = 1$ theories we must violate one or more assumptions of the Coleman–Mandula theorem. The central assumption is the Lorentz-invariance. Let us look at situations when the Lorentz symmetry is spontaneously broken. This is exactly what happens in the presence of the domain wall. The original theory where the domain wall develops (say, supersymmetric gluodynamics) is perfectly Lorentz-invariant. However, after the wall is formed, in the sector with the domain wall, the translation invariance in the direction perpendicular to the wall is spontaneously broken. The wall has infinite energy – it is impossible to boost it.

It is not difficult to see that in this case a non-vanishing central charge transforming as $(0, 1)$ with respect to the Lorentz group does not forbid a non-trivial S matrix. In this way we bypass the Coleman-Mandula theorem and open the possibility for central extensions of $N = 1$ superalgebras. Field configurations of the wall type that interpolate between distinct vacua at spatial infinities are extended objects which are not invariant under the action of the Poincaré group. If we choose such a field configuration as our "vacuum", then it may well happen that $T_{\alpha\beta} \neq 0$. The central charge, however, must vanish in the sector with the Lorentz-invariant vacuum.

The central extension in SUSY gluodynamics: a new old anomaly

The argument presented above does not necessarily mean that $T_{\alpha\beta} \neq 0$ develops in any $N = 1$ theory. Moreover, evaluating the anticommutator (3.3) in supersymmetric gluodynamics using the standard canonic commutation relations in a straightforward manner gives zero. Perhaps, this was the reason why this basic property was not discovered[84] until 1996, 22 years after the discovery of SUSY Yang-Mills theory. The central charge does not appear at the tree level. $T_{\alpha\beta} \neq 0$ is a *quantum anomaly*.

At the classical level supersymmetric gluodynamics is conformally invariant, there are no dimensional parameters in the Lagrangian of this theory. Correspondingly, the trace of the energy-momentum tensor vanishes. Supersymmetry entails then that $\partial^\mu J_\mu = 0$ and $\gamma^\mu J_{\mu\alpha} = 0$ where J_μ is the chiral current (see the section on supersymmetric gluodynamics) and $J_{\mu\alpha}$ is the supercurrent[§]. As a matter of fact, all three "geometric" conserved operators form a unified supermultiplet of currents[88]

$$J_{\alpha\dot\alpha} = R^0_{\alpha\dot\alpha} - \frac{1}{2}\left\{ i\theta^\beta (J_{\beta\alpha\dot\alpha} - \frac{2}{3}\epsilon_{\beta\alpha}\epsilon^{\gamma\delta}J_{\delta\gamma\dot\alpha}) + \text{H.c.}\right\} -$$

$$\theta^\beta\bar\theta^{\dot\beta}\left(J_{\alpha\dot\alpha\beta\dot\beta} - \frac{1}{3}\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\epsilon^{\gamma\delta}\epsilon^{\dot\gamma\dot\delta}J_{\gamma\dot\gamma\delta\dot\delta}\right) + \ldots \tag{3.4}$$

where $J_{\delta\gamma\dot\alpha}$ is the supercurrent and $J_{\alpha\dot\alpha\beta\dot\beta}$ is the energy-momentum tensor (see the Appendix, Eq. (A.28)). The statement of the classical conformal invariance can be written as $\bar D^{\dot\alpha} J_{\alpha\dot\alpha} = 0$. The good old anomalies in the trace of the energy-momentum tensor and the divergence of the chiral current imply that, at the quantum level,[15, 36]

$$\bar D^{\dot\alpha} J_{\alpha\dot\alpha} = -\frac{T(G)}{16\pi^2} D_\alpha \left\{ \text{Tr} W^2 \right\} . \tag{3.5}$$

All three conventional anomalies reside in Eq. (3.5). The anomaly in the trace of the energy-momentum tensor, in particular, is responsible for the generation of the scale parameter Λ.

Equation (3.5) presents the superanomaly relation of SUSY gluodynamics. It is written in the operator form. In this form the coefficient of the anomaly is exhausted by one loop[36]; the expression on the right-hand side is exact, there are no corrections.

Let us return now to the issue of the central extension $T_{\alpha\beta}$. As was mentioned, the non-vanishing anticommutator $\{Q_\alpha Q_\beta\}$ is not seen at the classical level. Does it mean that in the problem of $T_{\alpha\beta}$ we deal with a new anomaly? Taking into account the geometric nature of the supercharges, the occurrence of a new geometric anomaly seems unlikely. And indeed, one can show[89] that the "old" anomaly (3.5) automatically entails $T_{\alpha\beta} \neq 0$. In view of the importance of the issue, let us sketch the corresponding derivation below, even though it is a little bit more technical than I would like.

The lowest component of the current supermultiplet $J_{\alpha\dot\alpha}$ is the R_0 current, while the θ and $\bar\theta$ components of the supermultiplet are related to the supercurrent $J_{\beta\alpha\dot\alpha}$, see Eq. (3.4). Now, let us investigate the anticommutator of the supercharge $\bar Q_{\dot\alpha}$ with the supercurrent,

$$\{\bar Q_{\dot\alpha}\bar Q_{\dot\beta}\} = \frac{1}{2}(\bar\sigma^0)^{\gamma\dot\gamma}\int d^3x \{\bar Q_{\dot\alpha}\bar J_{\dot\beta\dot\gamma\gamma}\} . \tag{3.6}$$

§The precise definition of the supercurrent and many useful relations are given in the Appendix. There the reader will find, in particular, the component form of $J_{\alpha\dot\alpha}$. The matrix $(\vec\sigma)_{\dot\alpha\dot\beta}$ that will appear in our master formula (3.12) for the central extension is also defined there.

By inspecting Eq. (3.4) we readily observe that the supercurrent can be expressed in terms of the $\bar\theta$ component of the current supermultiplet,

$$\frac{-i}{2}\bar J_{\dot\beta\dot\gamma\gamma} = 2\{J_{\gamma\dot\beta}\}_{\bar\theta^{\dot\gamma}} - \{J_{\gamma\dot\gamma}\}_{\bar\theta^{\dot\beta}}\,. \tag{3.7}$$

Then

$$\frac{1}{2}\{\bar Q_{\dot\alpha}\bar J_{\dot\beta\dot\gamma\gamma}\} = 2i\{\bar Q_{\dot\alpha}J_{\gamma\dot\beta}\}_{\bar\theta^{\dot\gamma}} - i\{\bar Q_{\dot\alpha}J_{\gamma\dot\gamma}\}_{\bar\theta^{\dot\beta}}. \tag{3.8}$$

The $\bar\theta$ component of the anticommutators on the right-hand side reduces to the lowest component of the superderivative of the current,

$$2i\{\bar Q_{\dot\alpha}J_{\gamma\dot\beta}\}_{\bar\theta^{\dot\gamma}} - i\{\bar Q_{\dot\alpha}J_{\gamma\dot\gamma}\}_{\bar\theta^{\dot\beta}} = \left\{\epsilon_{\dot\gamma\dot\alpha}\bar D^2 J_{\gamma\dot\beta} - \frac{1}{2}\epsilon_{\dot\beta\dot\alpha}\bar D^2 J_{\gamma\dot\gamma}\right\}_{\theta=0}\,. \tag{3.9}$$

The last term, being antisymmetric with respect to the indices $\dot\alpha$, $\dot\beta$, does not contribute to the anticommutator of the supercharges and I drop it, while for the first term it is easy to obtain,

$$\left\{\epsilon_{\dot\gamma\dot\alpha}\bar D^2 J_{\gamma\dot\beta}\right\}_{\theta=0} = 2\left\{\epsilon_{\dot\gamma\dot\alpha}\bar D_{\dot\beta}\bar D^{\dot\delta}J_{\gamma\dot\delta}\right\}_{\theta=0}. \tag{3.10}$$

Assembling everything together we arrive at

$$\{\bar Q_{\dot\alpha}\bar Q_{\dot\beta}\} = -2i(\sigma^0)^{\gamma}_{\dot\alpha}\int d^3x\left\{\bar D_{\dot\beta}\bar D^{\dot\delta}J_{\gamma\dot\delta}\right\}_{\theta=0}\,. \tag{3.11}$$

The last step is substituting the anomaly equation (3.5) into the superderivative of the current,

$$\{\bar Q_{\dot\alpha}\bar Q_{\dot\beta}\} = -4(\vec\sigma)_{\dot\alpha\dot\beta}\int d^3x\vec\nabla\left\{\frac{T(G)}{16\pi^2}\,\mathrm{Tr}W^2\right\}_{\substack{\bar\theta=0\\ \theta=0}} = \frac{N}{4\pi^2}(\vec\sigma)_{\dot\alpha\dot\beta}\int d^3x\vec\nabla(\mathrm{Tr}\,\lambda\lambda)\,. \tag{3.12}$$

It is evident that the right-hand side generally speaking is non-vanishing. In deriving Eq. (3.12) we took advantage of the fact that the term with the time derivative ∂_0 is proportional to $\epsilon_{\dot\alpha\dot\beta}$; it cancels out after symmetrization over the indices $\dot\alpha$, $\dot\beta$, while those with the spatial derivatives are proportional to the matrix $(\vec\sigma)_{\dot\alpha\dot\beta}$

$$(\sigma^0)^{\gamma}_{\dot\alpha}(\sigma^i)_{\gamma\dot\beta}\partial^i = (\vec\sigma)_{\dot\alpha\dot\beta}\vec\nabla\,.$$

This rather long technical digression is intended for a single purpose – to demonstrate that the anomalous central charge in supersymmetric gluodynamics is an overlooked consequence of the familiar anomalies. The right-hand side of Eq. (3.12) is the spatial integral of a total derivative. This feature is welcome. The integral of the total derivative vanishes for all field configurations satisfying $\lambda\lambda \to$ const. at all spatial infinities, in full accord with the Coleman-Mandula theorem and common wisdom. If, however, the value of the gluino condensate is different at $z\to\infty$ and $z\to-\infty$, i.e. for a field configuration interpolating between distinct vacua, T_{α_β} is proportional to a jump in the value of $\langle\lambda\lambda\rangle$.

I would like to emphasize that Eq. (3.12) is exact, much in the same way as the calculation of the gluino condensate discussed before was exact. There are no perturbative or non-perturbative corrections. As we will see shortly, this result implies an exact prediction for the wall tension.

SUSY preserving walls

Now we are finally prepared to do the calculation of the wall tension. Assume that the wall lies in the xy plane, and the vacua between which it interpolates are characterized by real values of the gluino condensate,

$$\langle\lambda\lambda\rangle_{z\to-\infty} \neq \langle\lambda\lambda\rangle_{z\to\infty}\,.$$

The first assumption is a matter of choice of the reference frame, the second can be readily lifted. We will get rid of it shortly. The gluino condensate is real in the Kovner-Shifman vacuum (where it vanishes), and in one or two chirally asymmetric vacua (provided that the vacuum angle $\vartheta = 0$). For SU(N) gauge groups with even N the gluino condensate is real if k in Eq. (3.1) is chosen 0 or $N/2$, while for $SU(N)$ groups with odd N the chirally asymmetric vacuum with the real value of the gluino condensate corresponds to $k = 0$.

P_z, the generator of translations in the z direction, is spontaneously broken in the sector of the theory with the given domain wall. Since the supercharges are related to the energy-momentum operator, see Eq. (3.2), generically one might expect that the breaking of P_z implies that all SUSY generators are spontaneously broken too. In other words, building a wall in supersymmetric theory, one eliminates all standard consequences of supersymmetry, such as the Fermi-Bose degeneracy.

Is it possible to salvage at least a part of supersymmetry?

Before answering this question, let us pose another one: "can the wall tension be arbitrarily small?" Since the only dimensional parameter of SUSY gluodynamics is Λ, it is natural to expect that the wall width is proportional to Λ^{-1}. Since in the transitional layer the vacuum energy density is $\sim\Lambda^4$, the wall tension ε must be proportional to Λ^3. Can it be $\Lambda^3/10$? Or $\Lambda^3/100$?

It turns out that both questions are interrelated. The following argument answers them simultaneously. Consider a (Hermitean) linear combination of supercharges

$$K = \beta^\alpha Q_\alpha + \bar\beta^{\dot\alpha}\bar Q_{\dot\alpha} \tag{3.13}$$

where β is an arbitrary complex parameter, with two components, $\beta = \{\beta^1, \beta^2\}$ that are treated as c-numbers rather than the Grassmann numbers. Denote the state of the world with a wall in the xy plane, centered at $z = 0$, by $|w\rangle$. Since the operator K is Hermitean

$$\langle w|K^2|w\rangle \geq 0\,, \tag{3.14}$$

in accordance with the general rules of quantum mechanics. On the other hand

$$K^2 = \beta^\alpha\bar\beta^{\dot\alpha}\{Q_\alpha \bar Q_{\dot\alpha}\} + \frac{1}{2}\left[\beta^\alpha\beta^\gamma\{Q_\alpha Q_\gamma\} + \text{H.c.}\right] =$$

$$2\beta^\alpha\bar\beta^{\dot\alpha}\,P_{\alpha\dot\beta} + \left[\beta^\alpha\beta^\gamma T_{\alpha\gamma} + \text{H.c.}\right]. \tag{3.15}$$

The second line is a direct consequence of the general relations (3.2) and (3.3). We need to examine it in more detail in our specific circumstances. First, the wall is at rest; therefore, only the time component of the energy-momentum operator contributes to $P_{\alpha\dot\beta}$,

$$\beta^\alpha\bar\beta^{\dot\alpha}\,\langle w|P_{\alpha\dot\beta}|w\rangle \to \beta^\alpha\bar\beta^{\dot\alpha}\,(\sigma_0)_{\alpha\dot\beta}\,\langle w|P^0|w\rangle = \mathcal{E}(|\beta^1|^2 + |\beta^2|^2)\,, \tag{3.16}$$

where $\mathcal{E} = \varepsilon A$ is the total wall energy, A is its area (the matrix σ_0 is defined in the Appendix).

Second, since the wall lies in the xy plane, the central extension $T_{\alpha\beta}$ takes the form

$$\beta^\alpha \beta^\gamma \, T_{\alpha\gamma} = \frac{NA}{8\pi^2} \, \beta^\alpha \beta^\gamma \, (\tau^1)_{\alpha\gamma} \, [\langle\lambda\lambda\rangle_{z\to\infty} - \langle\lambda\lambda\rangle_{z\to-\infty}] =$$

$$\frac{NA}{4\pi^2} \, \beta^1 \beta^2 \, [\langle\lambda\lambda\rangle_{z\to\infty} - \langle\lambda\lambda\rangle_{z\to-\infty}] \, , \tag{3.17}$$

where the master formula (3.12) is used ($(\vec{\sigma})_{\alpha\beta}$ is again defined in the Appendix).

We are already very close to our goal. One last effort: start from the positivity condition (3.14), substitute there Eqs. (3.16) and (3.17) and arrive at

$$\mathcal{E}(|\beta^1|^2 + |\beta^2|^2) \geq -\frac{NA}{8\pi^2} \, \mathrm{Re} \left\{ \beta^1 \beta^2 \, [\langle\lambda\lambda\rangle_{z\to\infty} - \langle\lambda\lambda\rangle_{z\to-\infty}] \right\} . \tag{3.18}$$

This inequality must be valid for all values of the parameters $\beta^{1,2}$. We can choose these parameters wisely in order to make this inequality as informative as possible.

By an appropriate definition of the gluino field one can always achieve that the expression in the square bracket (the jump of the gluino condensate) is positive. Then the optimal choice of β's is

$$\beta^1 = -\beta^2 = 1 \, .$$

Thus, the wall tension turns out to be constrained from below

$$\varepsilon \geq \frac{N}{8\pi^2} \, [\langle\lambda\lambda\rangle_{\infty} - \langle\lambda\lambda\rangle_{-\infty}] \, . \tag{3.19}$$

It is clear that the equality is achieved only provided K annihilates the wall state,

$$(Q_1 - Q_2 + \bar{Q}_{\dot{1}} - \bar{Q}_{\dot{2}})|w\rangle = 0 \, . \tag{3.20}$$

In this case the wall tension is minimal. If the wall tension exceeds the minimal possible value indicated in Eq. (3.19), K acts on $|w\rangle$ non-trivially.

In the case of the minimal wall tension a linear combination of the supercharges acts on the wall trivially. In other words, if the wall is treated as a "vacuum state" (which is legitimate in the given sector of the theory), a linear combination of the supercharges annihilates it. This means that a part of supersymmetry remains unbroken. Such walls are called Bogomolny-Prasad-Sommerfield-saturated walls, or just BPS walls[90], for historical reasons that I do not have time to explain (the above gentlemen had nothing to do with the wall solutions, neither did they consider supersymmetry; nevertheless, the name became common). What part of supersymmetry is unbroken? This is easy to find out on general grounds. Since P_0, P_x and P_y remain unbroken, effectively we deal with a minimal supersymmetry in three dimensions. If in four dimensions the minimal $N = 1$ supersymmetry requires four complex supercharges altogether, two Q's and two $\bar{Q}$'s, in three dimensions the minimal supersymmetry can be built with four real supercharges, or two complex. Thus, building a BPS wall eliminates 1/2 of the original supersymmetry and preserves the other half. Non-BPS wall destroys all supersymmetry.

Let us return to the beginning of this section and ask ourselves what happens if the values of the gluino condensate in the vacua between which the wall interpolates are not real. The consideration above changes in an insignificant way. We will not go into details here, leaving this straightforward exercise to the reader. In the general case Eq. (3.19) becomes

$$\varepsilon \geq \frac{N}{8\pi^2} \, |\langle\lambda\lambda\rangle_{\infty} - \langle\lambda\lambda\rangle_{-\infty}| \, . \tag{3.21}$$

A linear combination of the supercharges annihilating the BPS wall is different, but the very fact that the BPS wall preserves 1/2 of supersymmetry remains intact, as well as the exact prediction for the wall tension,

$$\varepsilon = \frac{N}{8\pi^2} \left| \langle\lambda\lambda\rangle_\infty - \langle\lambda\lambda\rangle_{-\infty} \right| . \tag{3.22}$$

The line of reasoning outlined above brings us to the conclusion that SUSY preserving walls are possible. Whether or not they actually exist is a dynamical question which must be addressed separately in every given $N = 1$ theory. In weakly coupled theories, where quasiclassical methods for finding the wall solutions can be exploited, this question can be easily answered, see e.g.[91, 92]. A plethora of various dynamical regimes was observed in these works, with extremes being the models with all walls that are BPS, or all walls non-BPS. In strongly coupled theories, the prime subject of this lecture course, one has to resort to more subtle and sophisticated analysis, however.

In SUSY gluodynamics two arguments make us believe that the walls interpolating between distinct vacua do preserve 1/2 of supersymmetry, which would automatically imply the exact formula (3.22). First, the wall can be explicitly constructed[93] within framework of the (amended) Veneziano-Yankielowicz effective Lagrangian[12, 11]. I mentioned this Lagrangian in passing in the first part of these lectures. The advantage of this approach is that it provides us with an explicit dynamical model for the order parameter $\lambda\lambda$. A disadvantage is obvious too: the Veneziano-Yankielowicz Lagrangian is *not* a genuinely Wilsonian construction; therefore, theoretical derivations based on it are somewhat shaky.

The second argument comes from a totally different direction. Glimpses of the desired walls are seemingly seen by D-braners, from higher dimensions[94]. Both arguments go well beyond the scope of the present lectures, and I will leave this topic here, with a hope that further exciting developments will be reported at the next school.

CONCLUSIONS

This lecture course summarizes advances in theoretical understanding of nonperturbative phenomena in the strong coupling regime. If before the SUSY era, the number of exact nonperturbative results in four-dimensional field theory could be counted on one hand, with the advent of supersymmetry a wide spectrum of problems relevant to the most intimate aspects of strong gauge dynamics found exact solutions. Mysteries unravel. Our understanding of gauge theories is dramatically deeper now than it was a decade ago. When preparing these lectures, I intended to share with you, all the excitement and joys associated with the continuous advances in this field spanning over 15 years. Hopefully, the message I tried to convey will be appreciated in full.

Supersymmetric gauge dynamics is very rich, but life is richer, still. The world surrounding us is not supersymmetric. It remains to be seen whether the remarkable discoveries and elegant, powerful methods developed in supersymmetric gauge theories will prove to be helpful in solving the messy problems of real-life particle physics. So far, not much has been done in this direction. In today's climate it is rare that the question of practical applications is even posed. I hope that we reached a turning point: high-energy theory will return to its empirical roots. The command we obtained of supersymmetric gauge theories will be a key which will open to us Pandora's box of problems of Quantum Chromodynamics, *the* theory of our world. Pandora opened the jar that contained all human blessings, and they were gone. Will the achievements obtained in supersymmetric gauge theories be lost in the Planckean nebula?

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APPENDIX: Notation, Conventions, Useful Formulae

In this Appendix the key elements of the formalism used in supersymmetric gauge theories are outlined. Basic formulae are collected for convenience.

The notation we follow is close to that of the canonic text book of Bagger and Wess[21]. There are some distinctions, though. The most important of them is the choice of the metric. Unlike Bagger and Wess, we use the standard metric $g_{\mu\nu} = (+ - - -)$. There are also distinctions in normalization, see Eq. (A.19).

The left-handed spinor is denoted by undotted indices, e.g. η_β. The right-handed spinor is denoted by dotted indices, e.g. $\bar\xi^{\dot\beta}$. (This convention is standard in supersymmetry but is opposite to one accepted in the text-book[95]). The Dirac spinor Ψ then takes the form

$$\Psi = \left(\begin{array}{c} \bar\xi^{\dot\beta} \\ \eta_\beta \end{array} \right) . \tag{A.1}$$

Lowering and raising of the spinorial indices is done by applying the Levi-Civita tensor from the left,

$$\chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta\, , \quad \chi_\alpha = \epsilon_{\alpha\beta}\chi^\beta, \tag{A.2}$$

and the same for the dotted indices, where

$$\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}\, , \quad \epsilon^{12} = \, - \, \epsilon_{12} = 1\, . \tag{A.3}$$

The products of the undotted and dotted spinors are defined as follows:

$$\eta\chi = \eta^\alpha\chi_\alpha = -\eta_\alpha\chi^\alpha\, , \quad \bar\eta\bar\chi = \bar\eta_{\dot\alpha}\bar\chi^{\dot\alpha}\, . \tag{A.4}$$

Under this convention $(\eta\chi)^+ = \bar\chi\bar\eta$. Moreover,

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta^2\, , \quad \theta_\alpha\theta_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\theta^2\, , \quad \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = \frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\bar\theta^2\, , \quad \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar\theta^2\, . \tag{A.5}$$

The vector quantities (representation $(\frac{1}{2}\, , \frac{1}{2})$) are obtained in the spinorial formalism by multiplication by

$$(\, \sigma^\mu\,)_{\alpha\dot\beta} \; = \; \{\, 1\, , \vec\tau\, \}_{\alpha\dot\beta} \tag{A.6}$$

where $\vec\tau$ stands for the Pauli matrices, for instance,

$$A_{\alpha\dot\beta} = A_\mu(\sigma^\mu)_{\alpha\dot\beta}. \tag{A.7}$$

Note that

$$A_\mu B^\mu \; = \; \frac{1}{2}A_{\alpha\dot\beta}B^{\alpha\dot\beta}\, , \quad A_{\alpha\dot\beta}A^{\gamma\dot\beta} = \delta_\alpha^\gamma\, A_\mu A^\mu. \tag{A.8}$$

The square of the four-vector is understood as

$$A^2 \equiv A_\mu A^\mu = \frac{1}{2}A_{\alpha\dot\beta}A^{\alpha\dot\beta}\, . \tag{A.9}$$

If the matrix $(\sigma^\mu)_{\alpha\dot\beta}$ is "right-handed" it is convenient to introduce its "left-handed" counterpart,

$$(\bar\sigma^\mu)^{\dot\beta\alpha} = \{\, 1\, , -\vec\tau\,\}_{\dot\beta\alpha}\, . \tag{A.10}$$

The matrices that appear in dealing with representations $(1,0)$ and $(0,1)$ are

$$(\vec\sigma)_\beta^\alpha = \vec\tau_{\alpha\beta}\, , \quad (\vec\sigma)^{\alpha\beta} = \epsilon^{\beta\delta}\vec\sigma_\delta^\alpha\, , \tag{A.11}$$

and the same for the dotted indices. The matrices $(\vec\sigma)^{\alpha\beta}$ are symmetric, $(\vec\sigma)^{\alpha\beta} = (\vec\sigma)^{\beta\alpha}$. In the explicit form

$$(\vec\sigma)^{\alpha\beta} = \{\, \tau^3\, , \; -i\mathbf{1}\, , \; -\tau^1\,\}_{\alpha\beta}\, . \tag{A.12}$$

Note that with our definitions

$$(\vec\sigma)_{\alpha\beta} = \{\, -\tau^3\, , \; -i\mathbf{1}\, , \; \tau^1\,\}_{\alpha\beta}\, .$$

The left (right) coordinates $x_{L,R}$ and covariant derivatives are

$$\begin{aligned} &(x_L)_{\alpha\dot\alpha} = x_{\alpha\dot\alpha} - 2\, i\, \theta_\alpha\bar\theta_{\dot\alpha}\, , \quad (x_R)_{\alpha\dot\alpha} = x_{\alpha\dot\alpha} + 2\, i\, \theta_\alpha\bar\theta_{\dot\alpha}\, , \\ &D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\, \partial_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\, , \quad \bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\, \theta^\alpha\partial_{\alpha\dot\alpha}, \end{aligned} \tag{A.13}$$

so that

$$\begin{aligned} \{D_\alpha\, \bar D_{\dot\alpha}\} &= 2i\partial_{\alpha\dot\alpha}\, , \\ \bar D_{\dot\beta}(x_L)_{\alpha\dot\alpha} = 0\, , &\qquad D_\beta(x_R)_{\alpha\dot\alpha} = 0\, , \\ \bar D_{\dot\beta}(x_R)_{\alpha\dot\alpha} = -4\, i\, \theta_\alpha\epsilon_{\dot\beta\dot\alpha}\, , &\qquad D_\beta(x_L)_{\alpha\dot\alpha} = 4\, i\, \bar\theta_{\dot\alpha}\epsilon_{\beta\alpha}\, . \end{aligned} \tag{A.14}$$

The law of the supertranslation is

$$\begin{aligned} &\theta \to \theta + \epsilon\, , \quad \bar\theta \to \bar\theta + \bar\epsilon\, , \quad x_{\alpha\dot\beta} \to x_{\alpha\dot\beta} - 2\, i\, \theta_\alpha\bar\epsilon_{\dot\beta} + 2\, i\, \epsilon_\alpha\bar\theta_{\dot\beta}\, , \\ &(x_L)_{\alpha\dot\beta} \to (x_L)_{\alpha\dot\beta} - 4\, i\, \theta_\alpha\bar\epsilon_{\dot\beta}\, , \quad (x_R)_{\alpha\dot\beta} \to (x_R)_{\alpha\dot\beta} + 4\, i\, \epsilon_\alpha\bar\theta_{\dot\beta}\, . \end{aligned} \tag{A.15}$$

It corresponds to the infinitesimal transformation of the superfield in the form

$$\delta\Phi = i\left(\epsilon Q + \bar\epsilon\bar Q\right)\Phi \tag{A.16}$$

where

$$Q_\alpha = -i\frac{\partial}{\partial\theta^\alpha} - \partial_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\, , \qquad \bar Q_{\dot\alpha} = i\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + \theta^\alpha\partial_{\alpha\dot\alpha}\, . \tag{A.17}$$

The integrals over the Grassmann variable are normalized as follows

$$\int d^2\theta\, \theta^2 = 2\, , \quad \int d^4\theta\, \theta^2\, \bar\theta^2 = 4\, , \tag{A.18}$$

and we define

$$\{\ldots\}_F = \frac{1}{2}\int d^2\theta\{\ldots\}\, , \quad \{\ldots\}_D = \frac{1}{4}\int d^4\theta\{\ldots\}. \tag{A.19}$$

A generic non-Abelian SUSY gauge theory has the Lagrangian

$$\mathcal{L} = \left\{\frac{1}{4g_0^2}\mathrm{Tr}\int d^2\theta\, W^2 + \text{H.c.}\right\} + \frac{1}{4}\sum_i\int d^4\theta\bar Q_i e^V Q_i + \left\{\frac{1}{2}\int d^2\theta\mathcal{W}(\{Q_i\}) + \text{H.c.}\right\}, \tag{A.20}$$

where

$$\frac{1}{g_0^2} = \frac{1}{g^2} - \frac{i\vartheta}{8\pi^2}\,,$$

is the (complexified) gauge coupling constant, the sum in Eq. (A.20) runs over all matter superfields Q_i present in the theory, and $\mathcal{W}(\{Q_i\})$ is a generic superpotential. Most commonly one deals with the superpotential corresponding to the mass term of the matter fields. In many models, cubic terms are gauge invariant; then they are allowed too (and do not spoil renormalizability of the theory).

Furthermore, the superfield W_α, which includes the gluon strength tensor, is defined as follows:

$$W_\alpha = \frac{1}{8}\,\bar{D}^2\left(\mathrm{e}^{-V}\,D_\alpha\,\mathrm{e}^{V}\right) \tag{A.21}$$

where V is the vector superfield. In the Wess-Zumino gauge

$$V = -2\theta^\alpha\bar{\theta}^{\dot\alpha}A_{\alpha\dot\alpha} - 2i\bar{\theta}^2(\theta\lambda) + 2i\theta^2(\bar{\theta}\bar{\lambda}) + \theta^2\bar{\theta}^2 D\,, \tag{A.22}$$

$V = V^aT^a$ and T^a stands for the generators of the gauge group G. In the fundamental representation of $SU(N)$, a case of most practical interest,

$$\mathrm{Tr}\left(T^aT^b\right) = \frac{1}{2}\,\delta^{ab}\,.$$

The supergauge transformation has the form

$$Q_i \to e^{i\Lambda}\,Q_i\,,\quad e^V \to e^{i\bar{\Lambda}}e^V\,e^{-i\Lambda}\,,\quad W_\alpha \to e^{i\Lambda}W_\alpha e^{-i\Lambda}\,, \tag{A.23}$$

where Λ is an arbitrary chiral superfield ($\bar{\Lambda}$ is antichiral). In components

$$W_\alpha = i\left(\lambda_\alpha + i\theta_\alpha D - \theta^\beta\,G_{\alpha\beta} - i\theta^2\mathcal{D}_{\alpha\dot\alpha}\bar{\lambda}^{\dot\alpha}\right) \tag{A.24}$$

where λ_α is the gluino (Weyl) field, $\mathcal{D}_{\alpha\dot\alpha}$ is the covariant derivative, and $G_{\alpha\beta}$ is the gluon field strength tensor in the spinorial notation.

The standard gluon field strength tensor transforms as $(1,0)+(0,1)$ with respect to the Lorentz group. Projecting out pure $(1,0)$ is achieved by virtue of the $(\sigma)_{\alpha\beta}$ matrices,

$$G_{\alpha\beta} \;=\; -\frac{1}{2}G_{\mu\nu}(\sigma^\mu)_{\alpha\dot\beta}(\sigma^\nu)_{\beta\dot\delta}\epsilon^{\dot\beta\dot\delta} = (\vec{E} - i\,\vec{B})(\vec{\sigma})_{\alpha\beta}\,. \tag{A.25}$$

Then

$$G^{\alpha\beta}G_{\alpha\beta} \;=\; 2(\vec{B}^2 - \vec{E}^2 + 2\,i\,\vec{E}\vec{B}) = G_{\mu\nu}G_{\mu\nu} - i\,G_{\mu\nu}\tilde{G_{\mu\nu}}$$

where

$$\tilde{G}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G^{\alpha\beta}\,,\quad (\epsilon_{0123} = -1)\,. \tag{A.26}$$

The supercurrent supermultiplet has the following general form

$$\begin{aligned} J_{\alpha\dot\alpha} = R^0_{\alpha\dot\alpha} &- \frac{1}{2}\left\{i\theta^\beta(J_{\beta\alpha\dot\alpha} - \frac{2}{3}\epsilon_{\beta\alpha}\epsilon^{\gamma\delta}J_{\delta\gamma\dot\alpha}) + \text{H.c.}\right\} - \\ &\theta^\beta\bar{\theta}^{\dot\beta}\left(J_{\alpha\dot\alpha\beta\dot\beta} - \frac{1}{3}\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\epsilon^{\gamma\delta}\epsilon^{\dot\gamma\dot\delta}J_{\gamma\dot\gamma\delta\dot\delta}\right) + \ldots \end{aligned} \tag{A.27}$$

where $R^0_{\alpha\dot\alpha}$ is the R_0 current, $J_{\beta\alpha\dot\alpha}$ is the supercurrent, and $J_{\alpha\dot\alpha\beta\dot\beta}$ is related to the energy-momentum tensor, $\theta_{\mu\nu}$, in the following way

$$J_{\alpha\dot\alpha\beta\dot\beta} = -(\sigma^i)_{\alpha\beta}(\sigma^j)_{\dot\alpha\dot\beta}\left\{\theta^{ij} + \theta^{00}g^{ij} - \epsilon^{ijk}\theta^{0k}\right\} + \frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\theta^\mu_\mu\,; \tag{A.28}$$

here $i, j, k = 1, 2, 3$, $g_{\mu\nu}$ is the metric tensor and the matrices $(\sigma^i)_{\alpha\beta}$ are defined in Eq. (A.11).

The general anomaly relation (three "geometric" anomalies) is

$$\bar{D}^{\dot\alpha} J_{\alpha\dot\alpha} = \frac{1}{3} D_\alpha \left\{ \left[3\mathcal{W} - \sum_i Q_i \frac{\partial \mathcal{W}}{\partial Q_i} \right] - \left[\frac{3T(G) - \sum_i T(R_i)}{16\pi^2} \mathrm{Tr} W^2 + \frac{1}{8} \sum_i \gamma_i \bar{D}^2 (\bar{Q}_i e^V Q_i) \right] \right\}, \tag{A.29}$$

where γ_i are the anomalous dimensions of the matter fields Q_i. The general Konishi anomaly has the form

$$\frac{1}{8} \bar{D}^2 (\bar{Q}_i^+ e^V Q_i) = \frac{1}{2} Q_i \frac{\partial \mathcal{W}}{\partial Q_i} + \frac{\sum_i T(R_i)}{16\pi^2} \mathrm{Tr} W^2 . \tag{A.30}$$

In conclusion let us present the full component expression for the simplest $SU(2)$ model with one flavor (two subflavors), assuming that the superpotential in the case at hand reduces to the mass term of the quark (squark) fields. This model was discussed in detail in the first part of these lectures. If the index f denotes the subflavors, $f = 1, 2$

$$\begin{aligned}
\mathcal{L} = & \frac{1}{g^2} \left\{ -\frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \lambda^{\alpha,a} i \mathcal{D}_{\alpha\dot\alpha} \bar\lambda^{\dot\alpha,a} + \frac{1}{2} D^a D^a \right\} \\
& + \psi^{f\alpha} i \mathcal{D}_{\alpha\dot\alpha} \bar\psi^{f\dot\alpha} + (\mathcal{D}_\mu \phi^{+f})(\mathcal{D}_\mu \phi^f) + F^{+f} F^f \\
& + i\sqrt{2} (\phi_1^+ \lambda \psi_1 + \phi_1 \bar\lambda \bar\psi_1 + \phi_2^+ \lambda \psi_2 + \phi_2 \bar\lambda \bar\psi_2) \\
& + \frac{1}{2} D^a \, (\phi_1^+ T^a \phi_1 + \phi_2^+ T^a \phi_2) \\
& + m \, (\phi_1 F_2 + \phi_1^+ F_2^+ + \phi_2 F_1 + \phi_2^+ F_1^+ + \psi_1 \psi_2 + \bar\psi_1 \bar\psi_2) .
\end{aligned} \tag{A.31}$$

In this model the component expression for the supercurrent is

$$\begin{aligned}
J_{\alpha\beta\dot\beta} = 2 \Bigg\{ & \frac{1}{g^2} \left[i G^a_{\beta\alpha} \bar\lambda^a_{\dot\beta} - 3 \epsilon_{\beta\alpha} D^a \bar\lambda^a_{\dot\beta} \right] + \\
& \sqrt{2} \left[(\partial_{\alpha\dot\beta} \phi^+) \psi_\beta - i \, \epsilon_{\beta\alpha} F \bar\psi_{\dot\beta} \right] - \frac{\sqrt{2}}{6} \left[\partial_{\alpha\dot\beta} (\psi_\beta \phi^+) + \partial_{\beta\dot\beta} (\psi_\alpha \phi^+) - 3 \epsilon_{\beta\alpha} \partial^\gamma_{\dot\beta} (\psi_\gamma \phi^+) \right] \Bigg\} .
\end{aligned} \tag{A.32}$$

RECOMMENDED LITERATURE

It is assumed that the reader is familiar with the text-books on supersymmetry:

- J. Bagger and J. Wess, *Supersymmetry and Supergravity*, (Princeton University Press, 1983).
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A solid introduction to supersymmetric instanton calculus is given in:

- *Instantons in Gauge Theories*, ed. M. Shifman, (World Scientific, Singapore, 1994), Chapter VII.

A brief survey of those aspects of supersymmetry which are most relevant to the recent developments can be found in:

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