DUALITY AND OBLIQUE CONFINEMENT

G. 't Hooft*

Institute for Theoretical Physics
University of Utrecht, P.O.Box 80 006
3508 TA Utrecht, the Netherlands

ABSTRACT

By construction, renormalized non-Abelian gauge theories do not allow for point-like magnetic charges. However, a procedure exists called 'the Abelian projection', that transforms these theories into apparently non-renormalizable Abelian theories with magnetic point charges. The small-distance treatment of magnetic charges in gauge theories differs completely from that of electric charges. Yet the effects of these charges, in particular in the infrared region, are very similar. This is expressed in the notion called 'duality', and allows one to predict the possibility of exotic confinement modes, 'oblique confinement'. An exact duality relation is shown to exist which applies to electric and magnetic fluxes in a box.

1. THE ABELIAN PROJECTION

Consider an $SU(N)$ Yang-Mills gauge theory. Fermions and/or scalar fields may be there, but will not be looked at. In the early '70s, an important issue in these theories was the understanding of their infrared behaviour, in particular quark confinement, which cannot be understood in terms of perturbation theory. For the renormalization of the theory it was considered mandatory to fix the gauge freedom, by adding a gauge fixing term to the Lagrangian[1]. Gauge-fixing gives rise to ghost fields, and although we have learned how to handle these ghost in perturbation theory, and how to renormalize the theory in the presence of these ghosts, we now understand that these same ghosts obscure our view on the physical degrees of freedom, in particular at large distances[2].

Giving a gauge fixing condition at one point in space-time, may affect the field values at some other point, and since this change is a pure gauge transformation, this effect produces an unphysical action at a distance. This is why ghosts arise. Therefore, ghosts can be avoided if we choose a gauge fixing procedure that does not require gauge transformations elsewhere: a 'local' gauge fixing. Let us try to do this[3].

*e-mail: g.thooft@fys.ruu.nl

• *Step 1.*
Pick a field $X(\mathbf{x},t)$ (elementary or composite) in the adjoint representation. We choose the adjoint representation because, in the absence of fermions, this is the simplest non-trivial representation available. X must be a single field (a scalar or one component of a vector). Examples:

$$X^{ij} = G^{ij}_{12} \qquad \text{or} \qquad X^{ij} = (D_\mu G_{\alpha\beta} D_\mu G_{\alpha\beta})^{ij} \ . \tag{1}$$

X is a self-adjoint $N \times N$ matrix: $X = X^\dagger$. It transforms as

$$X'(\mathbf{x},t) = \Omega(\mathbf{x},t) X(\mathbf{x},t) \Omega^{-1}(\mathbf{x},t)\,, \tag{2}$$

which is completely local (*i.e.*, no $\partial_\mu \Omega$ is involved.)

• *Step 2.*
Pick the gauge Ω in which this field X is diagonal:

$$X = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \emptyset \\ \emptyset & & \ddots & \\ & & & \lambda_N \end{pmatrix} , \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \,. \tag{3}$$

Note that this can always be done, but it does not fix Ω entirely.

One may, for instance, choose as a gauge-fixing term

$$\mathcal{L}^{\text{gauge}} = \sum_{i>j} \alpha_{ij} X_{ij} \ , \tag{4}$$

where α_{ij} are Lagrange multiplier fields.

• *Step 3.*
Observe that this leaves a local gauge group,

$$\frac{U(1)^N}{U(1)_{\text{diag}}} \ = \ U(1)^{N-1} \tag{5}$$

as an invariance group:

$$\begin{gathered} X = \Omega X \Omega^{-1} \quad \text{iff} \quad [X,\Omega] = 0 \quad \to \\ \Omega = \begin{pmatrix} \omega_1 & & \emptyset \\ & \ddots & \\ \emptyset & & \omega_N \end{pmatrix} \ = \ \exp i \begin{pmatrix} \Lambda_1 & & \emptyset \\ & \ddots & \\ \emptyset & & \Lambda_N \end{pmatrix} , \\ \text{with} \quad \sum_i \Lambda_i = 0 \ . \end{gathered} \tag{6}$$

• *Step 4.*
Fix the residual (Abelian) gauge just as one is accustomed to do in QED. For instance:

$$\mathcal{L}^{\text{gauge, Abelian}} \ = \ \sum_i \beta_i \partial_\mu A^\mu_{ii} \ , \qquad (\sum_i \beta_i \equiv 0\,) \ . \tag{7}$$

This is a *non-local* gauge condition, but there is not much that can be done about that; it is not very harmful either, since this refers to an Abelian local symmetry, which is as well understood as the standard theory of quantum electrodynamics.

Adding (4) with (7) we find the total gauge fixing term:

$$\mathcal{L}^{\text{gauge}} = \sum_{i>j} \alpha_{ij} X_{ij} + \sum_i \beta_i \partial_\mu A_\mu^{ii} \,. \tag{8}$$

• *Step 5.*
Formally, we still need the corresponding ghost Lagrangian:

$$\begin{aligned}\mathcal{L}^{\text{ghost}}(\eta,\bar\eta) &= \sum_{i>j} \bar\eta_{ij}[\eta, X]_{ij} + \sum_i \bar\eta_i \partial_\mu D_\mu \eta^{ii} \;= \\ \sum_{i>j} \bar\eta_{ij}\eta_{ij}(\lambda_j-\lambda_i) &+ \sum_i \bar\eta_i \partial^2 \eta_{ii} + ig \sum_i \bar\eta_i \partial_\mu \left\{ A_\mu^{ik}\eta_{ki} - A_\mu^{ki}\eta_{ik} \right\} .\end{aligned} \tag{9}$$

It is not so difficult to convince oneself however that this ghost does not contribute to the physical amplitudes. The interaction term in Eq. (9) turns the diagonal components of the η field into off-diagonal components, but there is no way back. So, even though the diagonal gauge fixing required a propagating ghost (the propagator is the second term in (9)), this ghost is as harmless as in ordinary Abelian theories. Thus, we see that the ghost can be ignored.

• *Step 6.*
Observe that, in this gauge, we have all characteristics of a $U(1)^{N-1}$ Abelian gauge theory. All fields carry $U(1)^{N-1}$ charges q_e^i (with $\sum_i q_e^i = 0$). For instance, the off-diagonal parts of the gauge field A_μ^{ij} has $q_e^i = 1$ and $q_e^j = -1$. The diagonal components, A_μ^{ii}, behave exactly as $N-1$ massless photons, but the off-diagonal components are no longer obviously protected from obtaining a mass. Once the gauge has been fixed as in (8), all three polarizations of these vector fields are physical, as in massive vector particles.

• *Step 7.*
But this gauge fixing procedure may lead to singularities, whenever two eigenvalues λ_i of the field X coincide. Since we ordered the eigenvalues (see Eq. (3)), this may happen only for two consecutive eigenvalues: $\lambda_i = \lambda_{i+1}$. Prior to gauge fixing, the field X near such a point may take values of the form

$$X(\mathbf{x},t) = \begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda & 0 & \\ & & 0 & \lambda & \\ & & & & \ddots \end{pmatrix} + \sum_{k=1}^{3} a_k(\mathbf{x},t) \begin{pmatrix} \ddots & & & \\ & \ddots & & \\ & & \left(\sigma_k \right) & \\ & & & \ddots \end{pmatrix} . \tag{10}$$

Here, a_k are three space-time dependent functions that vanish at the point of the singularity. If X were to be taken to be a C^∞ function of space-time, the loci of points where all three a_k vanish are isolated points in 3-space, forming trajectories in time, and therefore, they are particle-like. Indeed, they have all characteristics of magnetic monopoles. They are magnetically charged with respect to the Abelian subgroup $U(1)_i \otimes U(1)_{i+1}^{-1}$, so we lable this charge as

$$g_i = (0, \cdots, 0, 1, -1, 0, \cdots, 0) \,. \tag{11}$$

In conclusion: the original non-Abelian $SU(N)$ theory is formally equivalent to an Abelian $U(1)^{N-1}$ theory containing electric charges as described under Step 6 and magnetic charges as described by (11). All these charges are bare, pointlike objects.

2. OBLIQUE CONFINEMENT

In a Θ-vacuum, the magnetically charged objects receive an electric charge, in addition to their magnetic charges. This is seen as follows[4]. Let us write the Abelian parts of the Lagrangian, after fixing the non-Abelian part of the gauge (Abelian projection) as

$$\begin{aligned} \mathcal{L} &= -\tfrac{1}{2}\mathrm{Tr}\, F_{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}i\Theta\mathrm{Tr}\, g^2 F_{\mu\nu}\tilde{F}_{\mu\nu} + \cdots \\ &= \mathrm{Tr}\,(\dot{\mathbf{A}}^2 - \mathbf{B}^2) - g^2\Theta \cdot 2\mathrm{Tr}\,\dot{\mathbf{A}}\mathbf{B} + \cdots \\ &= \mathrm{Tr}\,(\dot{\mathbf{A}} - g\Theta\mathbf{B})^2 - (1 + g^2\Theta^2)\mathrm{Tr}\,\mathbf{B}^2 + \cdots\,. \end{aligned} \tag{12}$$

Here, $\mathbf{B}$ describes the magnetic fields of the monopoles, whose values are fixed by topological constraints. This is exactly the Lagrangian of a theory without Θ term but with a background $\mathbf{E}$ field of strength $-g\Theta\mathbf{B}$.

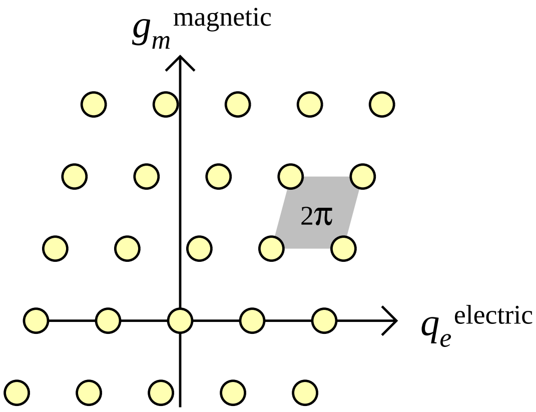

Figure 1. The oblique lattice of electric and magnetic charges.

The fact that these monopoles carry fractional electric charge is not in contradiction with the Dirac quantization condition[5] for electric and magnetic charges, because, given two particles 1 and 2 in a $U(1)$ theory, with electric charges q_e^1 and q_e^2 and magnetic charges g_m^1 and g_m^2, this condition reads

$$q_e^1 g_m^2 - q_e^2 g_m^1 = 2\pi n\,, \tag{13}$$

where n is an integer. If we plot the possible electric and magnetic charges as in Fig. 1, this condition amounts to the requirement that the area of the shaded box in Fig. 1 be a multiple of 2π. In our theory the Abelian gauge group is $U(1)^{N-1}$, in which case the Dirac condition reads

$$\sum_i (q_e^{1i} g_m^{2i} - q_e^{2i} g_m^{1i}) = 2\pi n\,. \tag{14}$$

If the *Higgs mechanism* is activated (for instance if a scalar charged field develops a vacuum expectation value), then all electrically charged particles only carry short range electric fields (since the A_μ^{ii} bosons obtain masses); they can roam about freely.

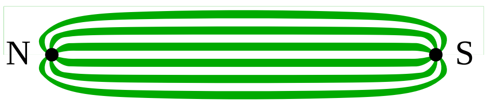

Figure 2. The Abrikosov vortex connecting magnetic monopoles in a super conductor.

Magnetic monopoles in this case will be *confined* by Nielsen-Olesen vortex tubes[6] (the

Abrikosov vortex in a super conductor, see Fig. 2). In a plane that dissects the vortex, the Higgs field makes a full rotation; the vortex stays located near the zero of the Higgs field.

At large distance scales, this vortex behaves just as a string. It displays string degrees of freedom, whereas all other effects due to the exchanges of quantum fields will only be of short range (there are no massless particle types).

It is important to note that the *Higgs phase* is an aggregation phase in which the color-electrically charged particles have undergone Bose condensation (super conductivity). *Confinement* is a different aggregation phase of the system. Instead of the electrically charged particles, we expect the magnetically charged ones to undergo Bose condensation. In terms of the long distance degrees of freedom, this phenomenon could occur equally well. Because there is a complete dual symmetry between the electric fields and the magnetic fields, and since both electrically charged and magnetically charged particles were present, magnetic superconductivity is the dual counterpart of the Higgs mechanism. The particles that are bose condensed may roam about freely in the vacuum. They screen all other particles that carry the same combination of electric and magnetic charges. All other particles, however, undergo the same fate as what happens with the magnetic monopoles in the Higgs phase: they are confined. This is how we can understand the confinement phenomenon in theories such as QCD. Confinement is the dual analogue of the Higgs mechanism.

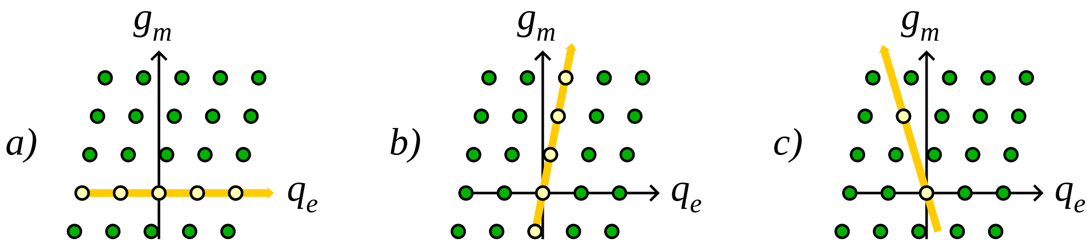

Figure 3. The light circles are liberated; the dark ones are confined.
a) The Higgs phase, b) The confinement phase, c) Oblique confinement.

It is also possible that *no* Bose condensation takes place at all. In this case, al particles carry long-range Abelian force fields. We call this the Coulomb phase. The Coulomb phase is self-dual, apart from the fact that the electric charge units q_e^i need not have the same values as the magnetic charge units g_m^i. They must obey Eq. (14).

As it was argued at the beginning of this section, the tilt angle in Fig. 1 and Fig. 3 is determined by the Θ angle. If we could vary Θ continuously from 0 to 2π, we would see the pattern of Fig. 1 shifting continuously, such that the configuration at $\Theta = 2\pi$ coincides with the one for $\Theta = 0$. Since the physics of the system must be periodic in Θ, we conclude that there is no fundamental distinction between magnetic monopoles and dyons (particles with both magnetic and electric charge).

If confinement takes place (Fig. 3b), we see however that one series of charge combination Bose condenses, and particles with the same charge ratios are liberated, the others confined. But, symmetry arguments tell us that, if Θ comes close to 2π, the series of charges that Bose condenses must be a different one. There must be intermediate values for Θ, at which the system jumps from one confinement mode to the other. Alternatively, it could be that at some Θ values, the system jumps back into the Higgs phase or the Coulomb phase. Thus, we see that QCD must exhibit phase transitions as Θ is varied from 0 to 2π. There must be at least one phase transition, which would then occur at $\Theta = \pi$.

A more exotic aggregation phase is conceivable. It could be that the objects that Bose condense carry more exotic charge combinations, for instance a dyon with $g_m = 2q_e$. (that is, a combination that would be impossible for monopoles with just one unit of magnetic charge. See Fig. 3c. In particular when Θ is close to π, such confinement modes are conceivable. It is this phase that we call oblique confinement.

3. FLUX IN A BOX

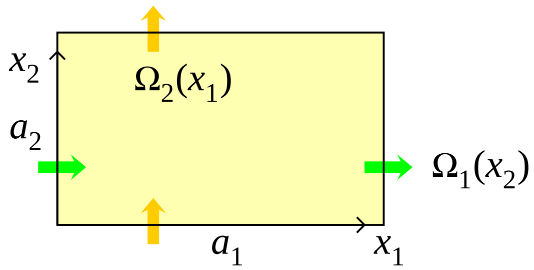

Figure 4. Boundary conditions in two-dimensional plane.

Again consider a pure $SU(N)$ gauge theory inside a 4 dimensional box with periodic boundary conditions. The lengths of the sides of the box are $a_1,\ \dots,\ a_4$. We choose the gauge fields to be periodic *modulo* gauge transformations. Introducing the shorthand notation

$$\begin{aligned} A'_\mu = \Omega A_\mu\ , \qquad &\text{or} \qquad A'_\mu(x') = \Omega(x_\perp)A_\mu(x) \\ &\text{to mean} \qquad A'_\mu(x') = \Omega(x_\perp)\Big(A_\mu(x) + \frac{1}{gi}\partial_\mu\Big)\Omega^{-1}(x_\perp)\ , \end{aligned} \tag{15}$$

we first formulate our boundary conditions in a two-dimensional plane: (see Fig. 4)

$$A_\mu(a_1,\ x_2) = \Omega_1(x_2)A_\mu(0,\ x_2)\,; \tag{16}$$
$$A_\mu(x_1,\ a_2) = \Omega_2(x_1)A_\mu(x_1, 0)\,. \tag{17}$$

This gives *two* conditions relating $A_\mu(a_1, a_2)$ to $A_\mu(0,0)$, which clearly should not conflict with each other:

$$A_\mu(a_1,\ a_2) = \Omega_1(a_2)\Omega_2(0)A_\mu(0,\ 0) = \Omega_2(a_1)\Omega_1(0)A_\mu(0,\ 0)\,. \tag{18}$$

From this we deduce the requirement that

$$\Omega_1(a_2)\Omega_2(0) = \Omega_2(a_1)\Omega_1(0)\, Z_{12}\,, \tag{19}$$

where Z_{12} is an element of the center on the gauge group (in this case $SU(N)$):

$$Z_{12} = e^{\frac{2\pi i}{N}}\, n_{12}\,, \tag{20}$$

where N is the number of 'colors' in the gauge group $SU(N)$, and n_{12} is an integer defined *modulo* N. Because of continuity these integers cannot depend on any of the (other) coordinates, but they do depend on the topological orientations of the plane. There are 6 orientations (in 4 dimensions), so we have 6 numbers $n_{\mu\nu}$ with $\mu\nu = 12,\ \dots,\ 34$. For future reference we introduce the relabeling

$$\begin{matrix} n_{12} = m_3 & & n_{14} = n_1 \\ n_{23} = m_1 & \quad\text{and}\quad & n_{24} = n_2 \\ n_{31} = m_2 & & n_{34} = n_3 \end{matrix}\ . \tag{21}$$

Besides these six numbers, there is one more topological index that can take any (positive or negative) integral value, the instanton winding number ν. This number is defined by multiplying[†] the gauge transformation matrices $\Omega(x_\perp)$ mentioned above by another matrix $\Omega(x_\perp)$ with a winding number ν from the mapping of the boundary $S(3)$ of the box onto $SU(N)$.

One can show that

$$\int \tfrac{1}{2}\mathrm{Tr}\, F_{\mu\nu}\tilde{F}_{\mu\nu} = \frac{8\pi}{g^2}\Big(\nu + \frac{\mathbf{n}\cdot\mathbf{m}}{N}\Big)\,. \tag{22}$$

This is proven, for instance, by constructing some simple field configurations, and then arguing that all other cases are obtained by continuous deformations. One may write (see Eqs. 21)

$$\frac{\mathbf{n}\cdot\mathbf{m}}{N} = \frac{n_{\mu\nu}\,\tilde{n}_{\mu\nu}}{8N}\,. \tag{23}$$

Now we continue by demanding that the box is in Euclidean space, and we define the amplitudes

$$W(\mathbf{n},\,\mathbf{m},\,\nu,\;a_1,\,a_2,\,a_3,\,a_4)\;\equiv\;\int \mathcal{D}A\;e^{iS}\Big|_{\mathbf{n},\mathbf{m},\nu,a_\mu}\,. \tag{24}$$

The interpretation of these amplitudes is elaborated in Ref[7]. The integers m_1, m_2, m_3 are the *magnetic* fluxes in the three spatial directions of a three-dimensional box. The length a_4 in the Euclidean time direction is the inverse temperature β. If we wish to know the free energy F of a gauge field configuration in the box with *electric* fluxes e_1, e_2, e_3 in the same box, then one can show that

$$e^{-\beta F(\mathbf{e},\mathbf{m},\Theta)} \;=\; C\sum_{n_i=0}^{N-1}\sum_{\nu=-\infty}^{\infty} e^{i\Theta(\nu+\frac{\mathbf{n}\cdot\mathbf{m}}{N})\,-\,\frac{2\pi i}{N}(\mathbf{n}\cdot\mathbf{e})}\;W(\mathbf{n},\,\mathbf{m},\,\nu)\,. \tag{25}$$

We see that this expression depends on $\mathbf{e}$ and $\mathbf{m}$ only via the combination

$$\mathbf{e}\;-\;\frac{\Theta}{2\pi}\,\mathbf{m}\,, \tag{26}$$

which again shows that any object emitting magnetic flux behaves as if carrying (fractional) electric flux as well.

4. DUALITY

A striking feature of this expression is that is obeys an exact duality relation[7]. One can relate the free energy of magnetic fluxes with that of electric ones. The relation is obtained by means of a rotation over 90° in Euclidean space. Let us interchange the axes 1 and 2, and also the axes 3 and 4, using the $SO(4)$ rotation matrix

$$\mathcal{O}\;=\;\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & \multicolumn{2}{c}{\emptyset} \\ & \emptyset & 0 & 1 \\ & & -1 & 0 \end{pmatrix}\,. \tag{27}$$

Defining the following notation for the transverse components of a vector,

$$\tilde{x} \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\,,\qquad \hat{x} \equiv \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}\,, \tag{28}$$

[†]To some extent, the result of this operation depends on how the equations (18) are implemented in the construction of the matrices Ω_i, but Eq. (22) defines the index ν uniquely.

we find that this rotation produces the substitutions:

$$\begin{aligned} &\tilde{n} \leftrightarrow \tilde{m}\ , \qquad a_1 \leftrightarrow a_2\ , \qquad a_3 \leftrightarrow \beta\ ; \\ &n_3 \leftrightarrow n_3\ , \qquad m_3 \leftrightarrow m_3\ , \end{aligned} \tag{29}$$

and the final result is

$$\begin{aligned} &e^{-\beta F(\tilde{e}, e_3,\, \tilde{m}, m_3,\, \tilde{a}, a_3,\, \beta)} \;= \\ &\frac{1}{N^2}\sum_{\tilde{n},\tilde{\ell}} e^{-\frac{i}{N}\,\tilde{n}(2\pi\tilde{e} - \Theta\tilde{m})\; +\; \frac{i}{N}\,\tilde{m}(2\pi\tilde{\ell} - \Theta\tilde{n})}\ .\ e^{-\,a_3 F(\tilde{\ell}, e_3,\, \tilde{n}, m_3,\, \hat{a}, \beta,\, a_3)} \\ &=\ \frac{1}{N^2}\sum_{\tilde{n},\tilde{\ell}} e^{\frac{2\pi i}{N}(\tilde{m}\cdot\tilde{\ell} - \tilde{n}\cdot\tilde{e}) - a_3 F(\tilde{\ell}, e_3,\, \tilde{n}, m_3,\, \hat{a}, \beta,\, a_3)}\ . \end{aligned} \tag{30}$$

Thus, the Θ dependence drops out.

From this duality relation, one can derive several interesting features concerning the behaviour of electric and magnetic fluxes. If, for instance, there is quark confinement, then the electric flux lines will carry energy proportional to their lengths, which can be written as

$$E_{e1} \approx \varrho\, a_1\ , \tag{31}$$

(for the flux in the 1-direction), where ϱ is the string constant for the electric flux. One may derive[7] that then the energy E_{m1} of a magnetic flux in the 1-direction drops as

$$E_{m1} \approx C\, a_1\, e^{-\varrho a_2 a_3}\ . \tag{32}$$

If the Coulomb phase is realized, the duality relation agrees with the usual expressions for the energy of electric and magnetic Abelian fields.

REFERENCES

1. G. 't Hooft and M. Veltman, Nucl.Phys. **B50** (1972) 318.
2. V. Gribov, Nucl. Phys. **B139** (1978) 1.
3. G. 't Hooft, Nucl. Phys. **B190** [FS3] (1981) 455; Phys. Scripta **25** (1982) 133.
4. E. Witten, Phys. Lett, **B86** (1979) 283.
5. P.A.M. Dirac, Proc. Roy. Soc. **A133** (1934) 60; Phys. Rev. **74** (1948) 817.
6. H.B. Nielsen and P. Olesen, Phys. Lett. **32B** (1970) 203.
7. G. 't Hooft, Acta Phys. Austr., Suppl.**22** (1980) 531.