# UNIVERSAL BEHAVIOR IN DIRAC SPECTRA

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## INTRODUCTION

In the chiral limit the QCD Lagrangian has an important global chiral symmetry: Specifically, for QCD with three colors and $N_f$ massless flavors, the invariance group acting on the quark fields is $SU(N_f) \times SU(N_f) \times U_A(1)$. In spite of the lightness of the up and down quark, no signatures of this symmetry have been observed in nature. On the contrary, the fact that the pion mass is much lighter than the other hadron masses indicates that $SU(N_f) \times SU(N_f)$ is spontaneous broken with the pions as Goldstone bosons. The breaking of $U_A(1)$ follows from the absence of parity doublets. For example, the mass of the $\delta$ is very different from the mass of the pion. The description of the broken phase requires the introduction of an order parameter which is zero in the restored phase. For the chiral phase transition the order parameter is the chiral condensate $\langle \bar{\psi}\psi \rangle$. Large scale lattice QCD simulations performed during the past decade show that for two light flavors the value of the chiral condensate is approximately $-(220\, MeV)^3$ at zero temperature and vanishes above a critical temperature of about 140 $MeV$ (see reviews by DeTar[1], Ukawa[2] and Smilga[3] for recent results on this topic). As was in particular pointed out by Shuryak[4], not necessarily both symmetries are restored at the same temperature[5]. This may lead to interesting observable consequences.

According the Banks-Casher formula[6], the chiral condensate is directly related to the spectrum of the Euclidean Dirac operator near zero virtuality (here and below, we use virtuality for the value of the Euclidean Dirac eigenvalues). However, the eigenvalues *fluctuate* about their average position as the gauge fields vary over the ensemble of gauge field configurations. The main question that will be addressed in these two lectures is to what extent such fluctuations are universal. Inspired by the study of spectra of complex systems[7], we will conjecture that spectral fluctuations on the scale of a typical eigenvalue spacing are universal. If this conjecture is true, such spectral fluctuations can be obtained from a broad class of theories with global symmetries of QCD as common ingredient. We will derive them from the simplest models in this class: chiral Random Matrix Theory (chRMT).

A first argument in favor of universality in Dirac spectra came from the analysis of the finite volume QCD partition function[8]. As has been shown by Gasser and Leutwyler[9], for box size $L$ in the range

$$1/\Lambda \ll L \ll 1/m_\pi, \tag{1}$$

($\Lambda$ is a typical hadronic scale and $m_\pi$ is the pion mass) the mass dependence of the QCD partition function is completely determined by its global symmetries. As a consequence, fluctuations of Dirac eigenvalues near zero virtuality are constrained by, but not determined by, an infinite family of sum rules[8] (also called Leutwyler-Smilga sum rules). For example, the simplest Leutwyler-Smilga sum rule can be obtained from the microscopic spectral density[10] (the spectral density near zero virtuality on a scale of a typical eigenvalue spacing). On the other hand, the infinite family of Leutwyler-Smilga sum rules is not sufficient to determine the microscopic spectral density. The additional ingredient is universality. A priori there is no reason that fluctuations of Dirac eigenvalues are in the same universality class as chRMT. Whether or not QCD is inside this class is a dynamical question that can only be answered by full scale lattice QCD simulations. However, the confidence in an affirmative answer to this question has been greatly enhanced by universality studies within chiral Random Matrix Theory. The aim of such studies is to show that spectral fluctuations do not depend on the details of the probability distribution. Recently, it has been shown that the microscopic spectral density is universal for a wide class of probability distributions[11, 12, 13, 14, 15, 16]. We will give an extensive review of these important new results.

Because of the $U_A(1)$ symmetry of the Dirac operator two types of spectral fluctuations can be distinguished. Spectral fluctuations near zero virtuality and spectral fluctuations in the bulk of the spectrum. The fluctuations of Dirac eigenvalues near zero virtuality are directly related to the approach to the thermodynamic limit of the chiral condensate. In particular, knowledge of the microscopic spectral density provides us with a quantitative explanation[17] of finite size corrections to the valence quark mass of dependence of the chiral condensate[18].

Recently, it has become possible to obtain *all* eigenvalues of the lattice QCD Dirac operator on reasonably large lattices[19, 20], making a direct verification of the above conjecture possible. This is one of the main objectives of these lectures. This is easiest for correlations in the bulk of the spectrum. Under the assumption of spectral ergodicity[21], eigenvalue correlations can be studied by spectral averaging instead of ensemble averaging[22, 23]. On the other hand, in order to study the microscopic spectral density, a very large number of independent gauge field configurations is required. First lattice results confirming the universality of the microscopic spectral density have been obtained recently[20].

At this point I wish to stress that there are two different types of applications of Random Matrix Theory. In the first type, fluctuations of an observable are related to its average. Because of universality it is possible to obtain exact results. In general, the average of an observable is not given by Random Matrix Theory. There are many examples of this type of universal fluctuations ranging from atomic physics to quantum field theory (a recent comprehensive review was written by Guhr, Müller-Groeling and Weidenmüller[24]). Most examples are related to fluctuations of eigenvalues. Typical examples are nuclear spectra[25], acoustic spectra[26], resonances in resonance cavities[27], $S$-matrix fluctuations[28, 29] and universal conductance fluctuations[30]. In these lectures we will discuss correlations in the bulk of Dirac spectra and the microscopic spectral density. The second type of application of Random Matrix Theory is as a schematic model of disorder. In this way one obtains *qualitative* results which may be helpful in understanding some physical phenomena. There are numerous examples in this category. We only mention the Anderson model of localization[31], neural networks[32], the Gross-Witten model of QCD[33] and quantum gravity[34] . In these lectures we will discuss chiral random matrix models at nonzero temperature and chemical potential. In particular, we will review recent work by Stephanov[35] on the quenched approximation at nonzero chemical potential.

In the first lecture we will review some general properties of Dirac spectra including the Banks-Casher formula. From the zeros of the partition function we will show that there is an intimate relation between chiral symmetry breaking and correlations of Dirac eigenvalues. Starting from Leutwyler-Smilga sum-rules the microscopic spectral density will be introduced. We will discuss the statistical analysis of quantum spectra. It will be argued that spectral correlations of 'complex' systems are given by Random Matrix Theory. We will end the first lecture with the introduction of chiral Random Matrix Theory.

In the second lecture we will compare the chiral random matrix model with QCD and discuss some of its properties. We will review recent results showing that the microscopic spectral density and eigenvalue correlations near zero virtuality are strongly universal. Lattice QCD results for the microscopic spectral density and correlations in the bulk of the spectrum will be discussed in detail. We will end the second lecture with a review of chiral Random Matrix Theory at nonzero chemical potential. Novel features of spectral universality in nonhermitean matrices will be discussed.

## THE DIRAC SPECTRUM

### Introduction

The Euclidean QCD partition function is given by

$$Z_{\rm QCD}(m,\theta) = \int dA \det(\gamma D + m) e^{-S_{\rm YM}/\hbar + i\theta\nu}, \tag{2}$$

where $\gamma D = \gamma_\mu(\partial_\mu + iA_\mu)$ is the anti-Hermitean Dirac operator and $S_{\rm YM}$ is the Yang-Mills action. The integral over field configurations includes a sum over all topological sectors with topological charge $\nu$. Each sector is weighted by $\exp(i\theta\nu)$. Phenomenologically the value of the vacuum $\theta$-angle is consistent with zero. We use the convention that the Euclidean gamma matrices are Hermitean with $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. The integral is over all gauge field configurations, and for definiteness, we assume a lattice regularization of the partition function.

Our main object of interest is the spectrum of the Dirac operator. The eigenvalues $\lambda_k$ are defined by

$$\gamma D \phi_k = i\lambda_k \phi_k. \tag{3}$$

The spectral density is given by

$$\rho(\lambda) = \sum_k \delta(\lambda - \lambda_k). \tag{4}$$

Correlations of the eigenvalues can be expressed in terms of the two-point correlation function

$$\rho_2(\lambda, \lambda') = \langle \rho(\lambda)\rho(\lambda')\rangle, \tag{5}$$

where $\langle \cdots \rangle$ denotes averaging with respect to the QCD partition function (2). The connected two-point correlation function is obtained by subtraction of the product of the average spectral densities

$$\rho_c(\lambda, \lambda') = \rho_2(\lambda, \lambda') - \langle \rho(\lambda) \rangle \langle \rho(\lambda') \rangle. \tag{6}$$

Because of the $U_A(1)$ symmetry

$$\{\gamma_5, \gamma D\} = 0, \tag{7}$$

the eigenvalues occur in pairs $\pm\lambda$ or are zero. The eigenfunctions are given by $\phi_k$ and $\gamma_5 \phi_k$, respectively. If $\gamma_5 \phi_k = \pm \phi_k$, then necessarily $\lambda_k = 0$. This happens for a solution of the Dirac operator in the field of an instanton. In a sector with topological charge $\nu$ the Dirac operator has $\nu$ exact zero modes with positive chirality. In order to represent the low energy sector of the Dirac operator for field configurations with topological charge $\nu$, it is natural to choose a chiral basis with $n$ right-handed states and $m \equiv n + \nu$ left-handed states. Then the Dirac matrix has the block structure

$$\left( \begin{array}{cc} 0 & W \\ W^\dagger & 0 \end{array} \right), \tag{8}$$

where $W$ is an $n \times m$ matrix. For $m = 2$ and $n = 1$, one can easily convince oneself that the Dirac matrix has exactly one zero eigenvalue. We leave it as an exercise to the reader to show that in general the Dirac matrix has $|m - n|$ zero eigenvalues.

In terms of the eigenvalues of the Dirac operator, the QCD partition function can be rewritten as

$$Z_{\rm QCD}(m, \theta) = \sum_\nu e^{i\nu\theta} \prod_f m_f^{|\nu|} \int_\nu dA \prod_f \prod_k (\lambda_k^2 + m_f^2) e^{-S_{YM}/\hbar} \tag{9}$$

where $\int_\nu dA$ denotes the integral over field configurations with topological charge $\nu$, and $\prod_f$ is the product over $N_f$ flavors with mass $m_f$. The partition function in the sector of topological charge $\nu$ is obtained by Fourier inversion

$$Z_\nu(m) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-i\nu\theta} Z_{\rm QCD}(m, \theta). \tag{10}$$

The fluctuations of the eigenvalues of the QCD Dirac operator are induced by the fluctuations of the gauge fields. Formally, one can think of integrating out all gauge field configurations for fixed values of the Dirac eigenvalues. The transformation of integration variables from the fields, $A$, to the eigenvalues, $\lambda_k$, leads to a nontrivial "Jacobian". Universality in Dirac spectra has its origin in this "Jacobian".

The free Dirac spectrum can be obtained immediately from the square of the Dirac operator. For a box of volume $L_1 \times L_2 \times L_3 \times L_4$ one finds

$$\lambda_{\vec n} = 2\pi \left( (\frac{n_1}{L_1})^2 + (\frac{n_2}{L_2})^2 + (\frac{n_3}{L_3})^2 + (\frac{n_4 + \frac{1}{2}}{L_4})^2 \right)^{1/2}, \tag{11}$$

where we have used periodic boundary conditions in the spatial directions and anti-periodic boundary conditions in the time direction. The spectral density is obtained by counting the total number of eigenvalues in a shell of radius $\lambda L/2\pi$. The result is

$$\rho^{\rm free}(\lambda) \sim V \lambda^3. \tag{12}$$

For future reference, we note that in the generic case, when the sides of the hypercube are related by an irrational number, asymptotically, the eigenvalues are uncorrelated, i.e.

$$\rho_2(\lambda, \lambda') = \langle \rho(\lambda) \rangle \langle \rho(\lambda') \rangle. \tag{13}$$

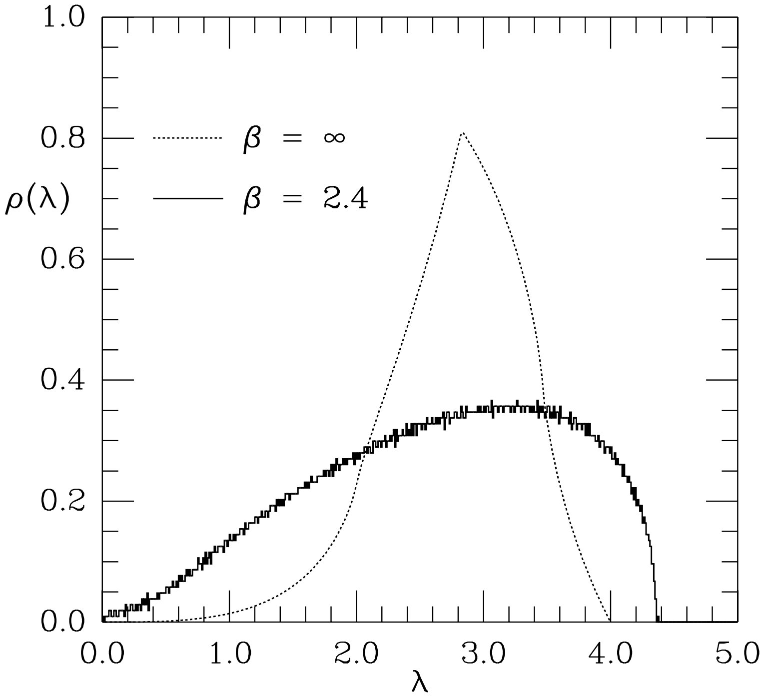


**Figure 1.** The free Dirac spectral density (dotted curve) and the spectral density of the Kogut-Susskind Dirac operator for a gauge field configuration generated with $\beta = 2.4$ (histogram). Both spectral densities are on a $12^4$ lattice and are normalized to unit area.

Two examples of Dirac spectra are shown in Fig. 1. The dotted curve represents the free Kogut-Susskind Dirac spectrum on a $12^4$ lattice with periodic boundary conditions in the spatial directions anti-periodic boundary conditions in the time direction. For an $N_1 \times N_2 \times N_3 \times N_4$ lattice this spectrum is given by

$$\lambda_{\vec{n}} = 2\left(\sin^2\left(\frac{\pi n_1}{N_1}\right) + \sin^2\left(\frac{\pi n_2}{N_2}\right) + \sin^2\left(\frac{\pi n_3}{N_3}\right) + \sin^2\left(\frac{\pi(n_4+0.5)}{N_4}\right)\right)^{1/2}. \quad (14)$$

Here, $n_i = 0, 1, \cdots, [N_i/2]$ $(i = 1, 2, 3)$ and $n_4 = 0, 1, \cdots, [(N_4+1)/2]$. The Kogut-Susskind Dirac spectrum for an $SU(2)$ gauge field configuration with $\beta = 2.4$ on the same size lattice is shown by the histogram in the same figure (full curve). We clearly observe an accumulation of small eigenvalues.

## The Banks-Casher Relation

The order parameter of the chiral phase transition, $\langle\bar\psi\psi\rangle$, is nonzero only below the critical temperature. As was shown by Banks and Casher[6], $\langle\bar\psi\psi\rangle$ is directly related to the eigenvalue density of the QCD Dirac operator per unit four-volume

$$\Sigma \equiv |\langle\bar\psi\psi\rangle| = \lim \frac{\pi\langle\rho(0)\rangle}{V}. \quad (15)$$

It is elementary to derive this relation. The chiral condensate follows from the partition function (9) (all quark masses are chosen equal),

$$\langle\bar\psi\psi\rangle = -\lim \frac{1}{VN_f}\partial_m \log Z(m) = -\lim \frac{1}{V}\langle\sum_k \frac{2m}{\lambda_k^2 + m^2}\rangle. \quad (16)$$

If we express the sum as an integral over the average spectral density, and take the thermodynamic limit before the chiral limit, so that many eigenvalues are less than $m$, we recover (15). The order of the limits in (15) is important. First we take the thermodynamic limit, next the chiral limit and, finally, the field theory limit. As can be observed from (16) the sign of $\langle\bar{\psi}\psi\rangle$ changes if $m$ crosses the imaginary axis.

An important consequence of the Bank-Casher formula (15) is that the eigenvalues near zero virtuality are spaced as

$$\Delta\lambda = 1/\rho(0) = \pi/\Sigma V. \tag{17}$$

This should be contrasted with the eigenvalue spectrum of the non-interacting Dirac operator. Then eq. (12) results in an eigenvalue spacing equal to $\Delta\lambda \sim 1/V^{1/4}$. Clearly, the presence of gauge fields leads to a strong modification of the spectrum near zero virtuality. Strong interactions result in the coupling of many degrees of freedom leading to extended states and correlated eigenvalues. On the other hand, for uncorrelated eigenvalues, the eigenvalue distribution factorizes, and for $\lambda \neq 0$, we have $\rho(\lambda) \sim \lambda^{2N_f+1}$ in the chiral limit, i.e. no breaking of chiral symmetry. One consequence of the interactions is level repulsion of neighboring eigenvalues. Therefore, the two smallest eigenvalues of the Dirac operator, $\pm\lambda_{\min}$ repel each other, and the Dirac spectrum will have a gap at $\lambda = 0$ with a width of the order of $1/\Sigma V$.

## Spectral Correlations and Zeros of the Partition Function

The study of zeros of the partition function has been a fruitful tool in statistical mechanics[37, 38]. In QCD, both zeros in the complex fugacity plane and the complex mass plane have been studied[39, 40]. Since the QCD partition function is a polynomial in $m$ it can be factorized as (all quark masses are taken to be equal to $m$)

$$Z_{\rm QCD}(m,\theta) = \prod_k (m - m_k). \tag{18}$$

Because configurations of opposite topological charge occur with the same probability, the coefficients of this polynomial are real, and the zeros occur in complex conjugate pairs. For an even number of flavors the zeros occur in pairs $\pm m_k$. In a sector with topological charge $\nu$, this is also the case for even $N_f \times \nu$. The chiral condensate is given by

$$\Sigma(m) = -\lim \frac{1}{VN_f}\partial_m \log Z_{\rm QCD}(m,\theta) = -\lim \frac{1}{VN_f}\sum_k \frac{1}{m-m_k}. \tag{19}$$

For an even number of flavors, $\Sigma(m)$ is an odd function of $m$. In order to have a discontinuity at $m = 0$, the zeros in this region have to coalesce into a cut along the imaginary axis in the thermodynamic limit.

In the hypothetical case that the eigenvalues of the Dirac operator do not fluctuate the zeros are located at $m_k = \pm i\lambda_k$. In the opposite case, of uncorrelated eigenvalues, the eigenvalue distribution factorizes and all zeros are located at $\pm i\sigma$, where $\sigma^2 = \langle\lambda_k^2\rangle$. As a result, the chiral condensate does not show a discontinuity across the imaginary axis and is equal to zero.

We hope to convince the reader that the presence of a discontinuity is intimately related to correlations of eigenvalues[41]. Let us study the effect of pair correlations for one flavor in the sector of zero topological charge. The fermion determinant can be written as

$$\langle\prod_k(m^2+\lambda_k^2)\rangle=\sum_k\left(\begin{array}{c}N\\k\end{array}\right)m^{2(N-k)}\langle\lambda_1^2\cdots\lambda_k^2\rangle. \tag{20}$$

There are

$$\left(\begin{array}{c}k\\2l\end{array}\right)(2l-1)!!$$

ways of selecting $l$ pairs from $\lambda_1^2\cdots\lambda_k^2$. The average of each pair of different eigenvalues is given by

$$\langle\lambda_m^2\lambda_n^2\rangle=\sigma^4+C_2, \tag{21}$$

where $\sigma^2$ is the expectation value of $\lambda_k^2$ and $C_2$ is the connected correlator

$$C_2=\langle\lambda_m^2\lambda_n^2\rangle-\langle\lambda_m^2\rangle\langle\lambda_n^2\rangle, \qquad m\neq n. \tag{22}$$

This results in the partition function

$$Z(m)=\sum_{k=0}^{N}\sum_{l=0}^{[\frac{k}{2}]}m^{2(N-k)}\left(\begin{array}{c}N\\k\end{array}\right)\left(\begin{array}{c}k\\2l\end{array}\right)(2l-1)!!C_2^l\sigma^{2(k-2l)}. \tag{23}$$

After interchanging the two sums, one can easily show that $Z(m)$ can be expressed as a multiple of a Hermite polynomial

$$Z(m)=(-\frac{C_2}{2})^{N/2}\mathrm{H}_N((\sigma^2+m^2)/\sqrt{-2C_2}). \tag{24}$$

In terms of the zeros of the Hermite polynomials, $z_k$, the zeros of the partition function are located at

$$m_k^2=z_k\sqrt{-2C_2}-\sigma^2. \tag{25}$$

Asymptotically, the zeros of the Hermite polynomials are given by $z_k\approx\pi k/2\sqrt{N}$ (with integer $\pm k$). In order for the zeros to join into a cut in the thermodynamic limit, they have to be spaced as $\sim 1/N$. This requires that

$$C_2\sim-\frac{1}{N}. \tag{26}$$

The density of zeros is then given by

$$\frac{dk}{dm}\sim Nm. \tag{27}$$

We conclude that pair correlations are sufficient to generate a cut of $Z(m)$ in the complex $m$-plane, but the chiral symmetry remains unbroken. Pair correlations alone cannot suppress the effect of the fermion determinant.

**Leutwyler-Smilga Sum Rules**

We have shown that pair-correlations are not sufficient to generate a discontinuity in the chiral condensate. In this subsection we start from the assumption that chiral symmetry is broken spontaneously, and look for consistency conditions that are imposed on the Dirac spectrum. As has been argued by Gasser and Leutwyler[9] and Leutwyler and Smilga[8], in the mesoscopic range (1), the mass dependence of the QCD partition function is given by (for simplicity, all quark masses have been taken equal)

$$Z^{\rm eff}(m,\theta) \sim \int_{U\in G/H} dU e^{mV\Sigma {\rm Re\,Tr}\, U e^{i\theta/N_f}}. \tag{28}$$

The integral is over the Goldstone manifold associated with chiral symmetry breaking from $G$ to $H$. For three or more colors with fundamental fermions $G/H = SU(N_f) \times SU(N_f)/SU(N_f)$. The finite volume partition function in the sector of topological charge $\nu$ follows by Fourier inversion according to (10). The partition function for $\nu = 0$ is thus given by (28) with the integration over $SU(N_f)$ replaced by an integral over $U(N_f)$. The case of $N_f = 1$ is particularly simple. Then only a $U(1)$ integration remains, and the partition function is given by[8] $Z^{\rm eff}_{\nu=0}(m) = I_0(mV\Sigma)$. Its zeros are regularly spaced along the imaginary axis in the complex $m$-plane, and, in the thermodynamic limit, they coalesce into a cut.

The Leutwyler-Smilga sum-rules are obtained by expanding the partition function $Z_\nu(m)$ in powers of $m$ before and after averaging over the gauge field configurations and equating the coefficients. This corresponds to an expansion in powers of $m$ of both the QCD partition function (2) and the finite volume partition function (28) in the sector of topological charge $\nu$. As an example, we consider the coefficients of $m^2$ in the sector with $\nu = 0$. This results in the sum-rule

$$\langle {\sum}' \frac{1}{\lambda_k^2} \rangle = \frac{\Sigma^2 V^2}{4N_f}, \tag{29}$$

where the prime indicates that the sum is restricted to nonzero positive eigenvalues.

The next order sum rules are obtained by equating the coefficients of order $m^4$. They can be combined into

$$\langle {\sum_{k,l}}' \frac{1}{\lambda_k^2 \lambda_l^2} \rangle - \langle {\sum_k}' \frac{1}{\lambda_k^2} \rangle \langle {\sum_l}' \frac{1}{\lambda_l^2} \rangle = \frac{\Sigma^4 V^4}{16 N_f^2 (N_f^2 - 1)}. \tag{30}$$

We conclude that chiral symmetry breaking leads to correlations of the inverse eigenvalues. However, if one performs an analysis similar to the one in previous section, it can be shown easily that pair correlations given by (30) do not result in a cut in the complex $m$-plane. Apparently, chiral symmetry breaking requires a subtle interplay of all types of correlations.

For two colors with fundamental fermions or for adjoint fermions the pattern of chiral symmetry breaking is different. Sum rules for the inverse eigenvalues can be derived along the same lines. The general expression for the simplest sum-rule can be summarized as[42, 43]

$$\langle {\sum}' \frac{1}{\lambda_k^2} \rangle = \frac{\Sigma^2 V^2}{4(|\nu| + (\dim(G/H) + 1)/N_f)}, \tag{31}$$

where $\dim(G/H)$ is the number of generators of the coset manifold in (28).

The Leutwyler-Smilga sum-rules can be expressed as an integral over the average spectral density and spectral correlation functions. For the sum rule (29) this results in

$$\frac{1}{V^2\Sigma^2} \int \frac{\langle \rho(\lambda) \rangle d\lambda}{\lambda^2} = \frac{1}{4N_f}. \tag{32}$$

If we introduce the microscopic variable

$$u = \lambda V \Sigma, \tag{33}$$

this integral can be rewritten as

$$\int \frac{1}{V\Sigma}\langle\rho(\frac{u}{V\Sigma})\rangle\frac{du}{u^2} = \frac{1}{4N_f}. \tag{34}$$

The thermodynamic limit of the combination that enters in the integrand,

$$\rho_S(u) = \lim_{V\to\infty}\frac{1}{V\Sigma}\langle\rho(\frac{u}{V\Sigma})\rangle, \tag{35}$$

will be called the *microscopic spectral density*[10]. This limit exists if chiral symmetry is broken. Our conjecture is that $\rho_S(u)$ is a universal function that only depends on the global symmetries of the QCD partition function. Because of universality it can be derived from the simplest theory with the global symmetries of the QCD partition function. Such theory is a chiral Random Matrix Theory which will be introduced later in these lectures.

We emphasize again that the $U_A(1)$ symmetry of the QCD Dirac spectrum leads to two different types of eigenvalue correlations: spectral correlations in the bulk of the spectrum and spectral correlations near zero virtuality. The simplest example of correlations of the latter type is the microscopic spectral density defined in (35).

We close this subsection with two unrelated side remarks. First, the QCD Dirac operator is only determined up to a constant matrix. We can exploit this freedom to obtain a Dirac operator that is maximally symmetric. For example, the Wilson lattice QCD Dirac operator, $D^W$, is neither Hermitean nor anti-Hermitean, but $\gamma_5 D^W$ is Hermitean.

Second, the QCD partition function can be expanded in powers of $m^2$ before or after averaging over the gauge field configurations. In the latter case one obtains sum rules for the inverse zeros of the partition function. As an example we quote,

$$\sum \frac{1}{m_k^2}\bigg|_{\nu=0} = \frac{\Sigma^2 V^2}{4}, \tag{36}$$

where we have averaged over field configurations with zero topological charge.

## SPECTRAL CORRELATIONS IN COMPLEX SYSTEMS

### Statistical Analysis of Spectra

Spectra for a wide range of complex quantum systems have been studied both experimentally and numerically (a excellent recent review has been given by Guhr, Müller-Groeling and Weidenmüller[24]). One basic observation is that the scale of variations of the average spectral density and the scale of the spectral fluctuations separate. This allows us to unfold the spectrum, i.e. we rescale the spectrum in units of the local average level spacing. Specifically, the unfolded spectrum is given by

$$\lambda_k^{\rm unf} = \int_{-\infty}^{\lambda_k}\langle\rho(\lambda')\rangle d\lambda', \tag{37}$$

with unfolded spectral density

$$\rho_{\rm unf}(\lambda) = \sum_k \delta(\lambda - \lambda_k^{\rm unf}). \tag{38}$$

The fluctuations of the unfolded spectrum can be measured by suitable statistics. We will consider the nearest neighbor spacing distribution, $P(S)$, and moments of the number of levels in an interval containing $n$ levels on average. In particular, we will consider the number variance, $\Sigma_2(n)$, and the first two cumulants, $\gamma_1(n)$ and $\gamma_2(n)$. Another useful statistic is the $\Delta_3(n)$-statistic introduced by Dyson and Mehta[44]. It is related to $\Sigma_2(n)$ via a smoothening kernel. The advantage of this statistic is that its fluctuations as a function of $n$ are greatly reduced. Both $\Sigma_2(n)$ and $\Delta_3(n)$ can be obtained from the pair correlation function defined as

$$Y_2(\lambda, \lambda') = -\langle \rho_{\rm unf}(\lambda) \rho_{\rm unf}(\lambda') \rangle + \langle \rho_{\rm unf}(\lambda) \rangle \langle \rho_{\rm unf}(\lambda') \rangle. \tag{39}$$

Analytical expressions for the above statistics can be obtained for the eigenvalues of the invariant random matrix ensembles. They are defined as ensembles of Hermitean matrices with Gaussian independently distributed matrix elements, i.e. with probability distribution given by

$$P(H) \sim e^{-\frac{N\beta}{2} {\rm Tr} H^\dagger H}. \tag{40}$$

Depending on the anti-unitary symmetry, the matrix elements are real, complex or quaternion real. They are called the Gaussian Orthogonal Ensemble (GOE), the Gaussian Unitary Ensemble (GUE) and the Gaussian Symplectic Ensemble (GSE), respectively. Each ensemble is characterized by its Dyson index $\beta$ which is defined as the number of independent variables per matrix element. For the GOE, GUE and the GSE we thus have $\beta = 1$, 2 and 4, respectively.

Independent of the value of $\beta$, the average spectral density is the semicircle,

$$\langle \rho(\lambda) \rangle = \frac{N}{\pi} \sqrt{2 - \lambda^2}. \tag{41}$$

Analytical results for all spectral correlation functions have been derived for each of the three ensembles[45] via the orthogonal polynomial method. We only quote the most important results. The nearest neighbor spacing distribution, which is known exactly in terms of a power series, is well approximated by

$$P(S) \sim S^\beta \exp(-a_\beta S^2), \tag{42}$$

where $a_\beta$ is a constant of order one. The asymptotic behaviour of the pair correlation function is given by[45]

$$Y_2(\lambda, \lambda') \sim \frac{1}{\pi^2 (\lambda - \lambda')^2} \quad \text{for} \quad \beta = 1, \tag{43}$$

$$Y_2(\lambda, \lambda') \sim \frac{\sin^2 \pi(\lambda - \lambda')}{\pi^2 (\lambda - \lambda')^2} \quad \text{for} \quad \beta = 2, \tag{44}$$

$$Y_2(\lambda, \lambda') \sim -\frac{\cos 2\pi(\lambda - \lambda')}{4(\lambda - \lambda')} + \frac{1 + (\pi/2) \sin 2\pi(\lambda - \lambda')}{4\pi^2 (\lambda - \lambda')^2} \quad \text{for} \quad \beta = 4. \tag{45}$$

The $1/(\lambda - \lambda')^2$ tail of the pair correlation function results in a logarithmic dependence of the asymptotic behavior of $\Sigma_2(n)$ and $\Delta_3(n)$,

$$\Sigma_2(n) \sim (2/\pi^2 \beta) \log n \quad \text{and} \quad \Delta_3(n) \sim \beta \Sigma_2(n)/2. \tag{46}$$

Characteristic features of random matrix correlations are level repulsion at short distances and a strong suppression of fluctuations at large distances.

For uncorrelated eigenvalues the level repulsion is absent and one finds

$$P(S) = \exp(-S), \tag{47}$$

and

$$\Sigma_2(n) = n \quad \text{and} \quad \Delta_3(n) = n/15. \tag{48}$$

**Spectral Universality**

The main conclusion of numerous studies of eigenvalue spectra of complex systems is that spectral correlations of classically chaotic systems are given by RMT[24]. As illustration of this so called Bohigas conjecture, we mention three examples from completely different fields. The first example is the nuclear data ensemble in which the above statistics are evaluated by superimposing level spectra of many different nuclei[25]. The second example concerns correlations of acoustic resonances in irregular quartz blocks[26]. In both cases the statistics that were considered are, within experimental accuracy, in complete agreement with the GOE statistics. The third example pertains to the zeros of Riemann's zeta function. Extensive numerical calculations[46] have shown that asymptotically, for large imaginary part, the correlations between the zeros are given by the GUE.

The Gaussian random matrix ensembles introduced above can be obtained[45] from two assumptions: i) The probability distribution is invariant under unitary transformations; ii) The matrix elements are statistically independent. If the invariance assumption is dropped it can be shown with the theory of free random variables[47] that the average spectral density is still given by a semicircle if the variance of the probability distribution is finite. For example, if the matrix elements are distributed according to a rectangular distribution, the average spectral density is a semicircle. On the other hand, if the independence assumption is released the average spectral density is typically not a semicircle. For example, this is the case if the quadratic potential in the probability distribution is replaced by a more complicated polynomial potential $V(H)$. Using the supersymmetric method for Random Matrix Theory, it was shown by Hackenbroich and Weidenmüller[48] that the same supersymmetric nonlinear $\sigma$-model is obtained for a wide range of potentials $V(H)$. This implies that spectral correlations of the unfolded eigenvalues are independent of the potential. Remarkably, this result could be proved for all three values of the Dyson index.

Several examples have been considered where both the invariance assumption and the independence assumption are relaxed. We mention $H \to H + A$, where $A$ is an arbitrary fixed matrix, and the probability distribution of $H$ is given by a polynomial $V(H)$. It was shown by P. Zinn-Justin[49] that also in this case the spectral correlations are given by the invariant random matrix ensembles. For a Gaussian probability distribution the proof was given by Brézin and Hikami[50].

The domain of universality has been extended in the direction of real physical systems by means of the Gaussian embedded ensembles[51, 52]. The simplest example is the ensemble of matrix elements of $n$-particle Slater determinants of a two-body operator with random two-particle matrix elements. It can be shown analytically that the average spectral density is a Gaussian. However, according to substantial numerical evidence, the spectral correlations are in complete agreement with the invariant random matrix ensembles[51].

A large number of examples have been found that fall into one of universality classes of the invariant random matrix ensembles. This calls out for a more general approach. Naturally, one thinks in terms of the renormalization group. This approach was pioneered by Brézin and Zinn-Justin[56]. The idea is to integrate out rows and columns of a random matrix and to show that the Gaussian ensembles are a stable fixed point. This was made more explicit in a paper by Higuchi *et al.*[57]. However, much more work is required to arrive at a natural proof of spectral universality.

Although the above mentioned universality studies provide support for the validity of the Bohigas conjecture, the ultimate goal is to derive it directly from the underlying classical dynamics. An important first step in this direction was made by Berry[55]. He showed that the asymptotics of the two-point correlation function is related to sum-rules for isolated classical trajectories. Another interesting approach was introduced by Andreev *et al.*[58] who were able to obtain a supersymmetric nonlinear sigma model from spectral averaging. In this context we also mention the work of Altland and Zirnbauer[59] who showed that the kicked rotor can be mapped onto a supersymmetric sigma model.

## CHIRAL RANDOM MATRIX THEORY

### Introduction of the Model

In this section we will introduce an instanton liquid[60, 61] inspired chiral RMT for the QCD partition function. In the spirit of the invariant random matrix ensembles we construct a model for the Dirac operator with the global symmetries of the QCD partition function as input, but otherwise Gaussian random matrix elements. The chRMT that obeys these conditions is defined by[10, 62, 63, 64]

$$Z^{\beta}_{N_f,\nu}(m_1,\cdots,m_{N_f}) = \int DW \prod_{f=1}^{N_f} \det(\mathcal{D}+m_f) e^{-\frac{N\beta}{4}{\rm Tr}V(W^\dagger W)}, \tag{49}$$

where

$$\mathcal{D} = \left( \begin{array}{cc} 0 & iW \\ iW^\dagger & 0 \end{array} \right), \tag{50}$$

and $W$ is a $n\times m$ matrix with $\nu = |n-m|$ and $N = n+m$. As is the case in QCD, we assume that $\nu$ does not exceed $\sqrt{N}$, so that, to a good approximation, $n = N/2$. The parameter $N$ is identified as the dimensionless volume of space time. The potential $V$ is defined by

$$V(\phi) = \sum_{k\ge 1} \frac{a_k}{k}\phi^k. \tag{51}$$

The simplest case is the Gaussian case when $V(\phi) = \Sigma^2\phi$. Below it will be shown that the microscopic spectral density is independent of the higher order terms in this potential provided that the average spectral density near zero remains nonzero. The matrix elements of $W$ are either real ($\beta = 1$, chiral Gaussian Orthogonal Ensemble (chGOE)), complex ($\beta = 2$, chiral Gaussian Unitary Ensemble (chGUE)), or quaternion real ($\beta = 4$, chiral Gaussian Symplectic Ensemble (chGSE)). This partition function is invariant under

$$W \to U^\dagger W V \tag{52}$$

where the $n\times n$ matrix $U$ and the $m\times m$ matrix $V$ are orthogonal matrices for $\beta = 1$, unitary matrices for $\beta = 2$, and symplectic matrices for $\beta = 4$. This invariance makes it possible to express the partition function in terms of eigenvalues of $W$ defined by

$$W = U^\dagger \Lambda V. \tag{53}$$

Here, $\Lambda$ is a diagonal matrix with diagonal matrix elements $\lambda_k \ge 0$. In terms of the eigenvalues the partition function (49) is given by

$$Z^{\beta}_{N_f,\nu}(m_1,\cdots,m_{N_f}) = \int d\lambda |\Delta(\lambda_k^2)|^\beta \prod_k \lambda_k^{2N_f+\beta\nu+\beta-1} \prod_f m_f^\nu \prod_{f,k}(\lambda_k^2+m_f^2) e^{-\frac{N\beta}{4}\sum_k V(\lambda_k^2)}, \tag{54}$$

where the Vandermonde determinant is defined by

$$\Delta(\lambda_k^2) = \prod_{k<l}(\lambda_k^2 - \lambda_l^2). \tag{55}$$

This model reproduces the following symmetries of the QCD partition function:

- The $U_A(1)$ symmetry. All nonzero eigenvalues of the random matrix Dirac operator occur in pairs $\pm\lambda$ or are zero.

- The topological structure of the QCD partition function. The Dirac matrix has exactly $|\nu| \equiv |n-m|$ zero eigenvalues. This identifies $\nu$ as the topological sector of the model.

- The flavor symmetry is the same as in QCD. For $\beta = 2$ it is $SU(N_f) \times SU(N_f)$, for $\beta = 1$ it is $SU(2N_f)$ and for $\beta = 4$ it is $SU(N_f)$.

- The chiral symmetry is broken spontaneously with chiral condensate given by

  $$\Sigma = \lim_{N\to\infty} \pi\rho(0)/N. \tag{56}$$

  ($N$ is interpreted as the (dimensionless) volume of space time.) The symmetry breaking pattern is[42] $SU(N_f) \times SU(N_f)/SU(N_f)$, $SU(2N_f)/Sp(N_f)$ and $SU(N_f)/O(N_f)$ for $\beta = 2$, 1 and 4, respectively, the same as in QCD[65].

- The anti-unitary symmetries. For three or more colors with fundamental fermions the Dirac operator has no anti-unitary symmetries, and generically, the matrix elements of the Dirac operator are complex. The matrix elements $W_{kl}$ of the corresponding random matrix ensemble are chosen arbitrary complex as well ($\beta = 2$). For $N_c = 2$, the Dirac operator in the fundamental representation satisfies

  $$[C\tau_2 K, i\gamma D] = 0, \tag{57}$$

  where $C$ is the charge conjugation matrix and $K$ is the complex conjugation operator. Because, $(C\tau_2 K)^2 = 1$, the matrix elements of the Dirac operator can always be chosen real, and the corresponding random matrix ensemble is defined with real matrix elements ($\beta = 1$). For two or more colors with gauge fields in the adjoint representation the anti-unitary symmetry of the Dirac operator is given by

  $$[CK, i\gamma D] = 0. \tag{58}$$

  Because $(CK)^2 = -1$, it is possible to rearrange the matrix elements of the Dirac operator into real quaternions. The matrix elements $W_{kl}$ of the corresponding random matrix ensemble are chosen quaternion real ($\beta = 4$).

Together with the invariant random matrix ensembles, the chiral ensembles are part of a larger classification scheme. Apart from the random matrix ensembles discussed in this review, this classification also includes random matrix models for disordered super-conductors[66]. As pointed out by Zirnbauer[67], all known universality classes of Hermitean random matrices are tangent to the large classes of symmetric spaces in the classification given by Cartan. There is a one-to-one correspondence between this classification and the classification of the large families of Riemannian symmetric superspaces[67].

## Calculation of the Microscopic Spectral Density

The joint eigenvalue distribution of the nonzero eigenvalues for zero masses follows immediately from the partition function (54). For $N_f$ flavors and topological charge $\nu$ the result for arbitrary potential is given by[62]

$$\rho_\beta(\lambda_1, \cdots, \lambda_n) = C_{\beta,n} \prod_{k<l} |\lambda_k^2 - \lambda_l^2|^\beta \prod_k \lambda_k^{2N_f+\beta\nu+\beta-1} \exp\left(-\frac{n\beta}{2}\sum_k V(\lambda_k^2)\right), \quad (59)$$

where $C_{\beta,n}$ is a normalization constant. For $\beta = 2$ the average spectral density and the spectral correlation functions can be derived from (59) with the help of the orthogonal polynomial method[45]. The orthogonal polynomials in the variable $x = \lambda^2$ are defined by

$$\int_0^\infty dx x^a e^{-nV(x)} P_n^a(x) P_m^a(x) = h_n^a \delta_{mn}. \quad (60)$$

where

$$a = N_f + |\nu| \quad \text{for} \quad \beta = 2. \quad (61)$$

In the Gaussian case, $V(\lambda) = \Sigma^2\lambda$, the associated polynomials are the generalized Laguerre polynomials. That is why this ensemble is also known as the Laguerre ensemble[68, 69]. The Vandermonde determinant can be rewritten as $\Delta(x_i) = \det_{kl}[x_k^{l-l}]$. By the addition of linear combinations of rows this determinant can be expressed in terms of the orthogonal polynomials (60), $\Delta(x_i) \sim \det_{kl}[P_{l-1}^a(x_k)]$. By multiplying the two determinants one obtains

$$\rho_\beta(x_1, \cdots, x_n) = \frac{1}{n!} \det K^a(x_k, x_l), \quad (62)$$

where the kernel is defined by

$$K^a(x,y) = \sum_{k=0}^{n-1} \frac{1}{h_k^a} P_k^a(x) P_k^a(y) (xy)^{a/2} e^{-n(V(x)+V(y))/2}. \quad (63)$$

The average spectral density is obtained by integrating over $x_2, \cdots, x_n$,

$$\rho(x) = n \int \prod_{k=2}^n dx_k \rho(x, x_2, \cdots, x_n) = K^a(x,x). \quad (64)$$

In terms of the original variables, $\lambda = \sqrt{x}$, the spectral density is given by

$$\rho(\lambda) = 2\lambda K^a(\lambda^2, \lambda^2). \quad (65)$$

In the Gaussian case the spectral density is thus given by

$$\rho(\lambda) = 2\lambda^{2a+1}\Sigma^{2a+2} n^{a+1} e^{-n\Sigma^2\lambda^2} \sum_{k=0}^{n-1} \frac{1}{h_k^a} L_k^a(\lambda^2\Sigma^2 n) L_k^a(\lambda^2\Sigma^2 n), \quad (66)$$

where the $L_k^a$ are the generalized Laguerre polynomials. With the help of the Christoffel-Darboux formula the sum can be expressed into the $n^{\rm th}$ order Laguerre polynomial and its derivative (with $L_n^{a\prime}(x) = -L_{n-1}^{a+1}(x)$),

$$\rho(\lambda) = \frac{2\Sigma n^{-a} n!}{\Gamma(a+n)} (\frac{z}{2})^{2a+1} e^{-z^2/4n} \left(L_{n-1}^a(\frac{z^2}{4n}) L_{n-1}^{a+1}(\frac{z^2}{4n}) - L_n^a(\frac{z^2}{4n}) L_{n-2}^{a+1}(\frac{z^2}{4n})\right). \quad (67)$$

Here, we introduced the microscopic variable

$$z = N\Sigma\lambda = 2n\Sigma\lambda, \tag{68}$$

and used the explicit expression for the normalization of the Laguerre polynomials, $h_k^a = \Gamma(a+k+1)/k!$. The microscopic limit is defined by

$$\lim_{N\to\infty} \frac{1}{N\Sigma}\rho(\frac{\lambda}{N\Sigma}) \tag{69}$$

and can be evaluated with the help of the asymptotic relation

$$\lim_{n\to\infty} \frac{1}{n^\alpha} L_n^\alpha(\frac{x}{n}) = x^{-\frac{\alpha}{2}} J_\alpha(2\sqrt{x}), \tag{70}$$

where $J_\alpha$ is the ordinary Bessel function of degree $\alpha$. In order to take the this limit most conveniently, we substitute the recursion relations

$$\begin{aligned} L_{n-1}^{a+1}(z) &= L_{n-1}^a(z) + L_{n-2}^{a+1}(z), \\ L_n^a(z) &= L_{n-1}^a(z) + L_n^{a-1}(z). \end{aligned} \tag{71}$$

This results in the microscopic spectral density

$$\rho_S(z) = \frac{z}{2}(J_a^2(z) - J_{a+1}(z)J_{a-1}(z)). \tag{72}$$

From the asymptotic relation for the Bessel function

$$J_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} \cos(z - \frac{\pi}{2}\nu - \frac{\pi}{4}) \quad \text{for} \quad z \to \infty \tag{73}$$

we find that

$$\lim_{x\to\infty} \rho_S(x) = \frac{1}{\pi}. \tag{74}$$

The result for the average spectral density follows from the asymptotic properties of the Laguerre polynomials. It given by the semicircular distribution

$$\rho(\lambda) = (n\Sigma^2/\pi)\sqrt{4/\Sigma^2 - \lambda^2}. \tag{75}$$

Notice that the microscopic limit of the average spectral density coincides with the asymptotic limit (74) of the microscopic spectral density.

The two-point correlation function is obtained by integrating the joint spectral density over all eigenvalues except two. The microscopic limit can again be expressed in terms of Bessel functions[63].

The spectral density and the two-point correlation function were also derived within the framework of the supersymmetric method of Random Matrix Theory[70].

The calculation of the average spectral density and the spectral correlations functions for $\beta = 1$ and $\beta = 4$ is much more complicated. However, with the help of skew-orthogonal polynomials[71, 72, 73] exact analytical results for finite $N$ can be obtained as well.

The microscopic spectral density for $SU(2)$ with fundamental fermions ($\beta = 1$) is given by[74]

$$\begin{aligned} \rho_S(z) = \frac{1}{4}J_{2a+1}(z) + \frac{1}{2}\int_0^\infty dw (zw)^{2a+1}\epsilon(z-w)\left(\frac{1}{w}\frac{d}{dw} - \frac{1}{z}\frac{d}{dz}\right) \\ \times \frac{wJ_{2a}(z)J_{2a-1}(w) - zJ_{2a-1}(z)J_{2a}(w)}{(zw)^{2a}(z^2-w^2)}, \end{aligned} \tag{76}$$

where $a$ is the combination

$$a = N_f - \frac{1}{2} + \frac{|\nu|}{2}. \tag{77}$$

The microscopic spectral density in the symmetry class with $\beta = 4$ was first calculated by Nagao and Forrester[75]. It is given by

$$\rho_S(z) = 2z^2 \int_0^1 du u^2 \int_0^1 dv \quad [J_{4a-1}(2uvz)J_{4a}(2uz) - vJ_{4a-1}(2uz)J_{4a}(2uvz)] \tag{78}$$

with $4a = N_f + 2|\nu| + 1$.

The spectral correlations in the bulk of the spectrum are given by the invariant random matrix ensemble with the same value of $\beta$. For $\beta = 2$ this was already shown three decades ago by Fox and Kahn[68]. For $\beta = 1$ and $\beta = 4$ this was only proved recently[73].

**Duality between Flavor and Topology**

As one can observe from the joint eigenvalue distribution, for $\beta = 2$ the dependence on $N_f$ and $\nu$ enters only through the combination $N_f + |\nu|$. This allows for the possibility of trading topology for flavors. In this section we will work out this duality for the finite volume partition function. This relation completes the proof of the conjectured expression[76] for the finite volume partition function for different quark masses and topological charge $\nu$.

For $\beta = 2$ the partition function (54) obeys the relation

$$\frac{Z_{N_f,\nu}(m_1, \cdots, m_{N_f})}{\prod_f m_f^\nu} \sim Z_{N_f+\nu,0}(m_1, \cdots, m_{N_f}, 0, \cdots, 0), \tag{79}$$

where the argument of the last factor has $\nu$ zeros.

As an example, the simplest nontrivial identity of this type is given by

$$Z_{1,1}(m) \sim m Z_{2,0}(m, 0). \tag{80}$$

Let us prove this identity without relying on Random Matrix Theory. According to the definition (28) we have

$$Z_{1,1}(m) \sim \int d\theta e^{i\theta} e^{mV\Sigma\cos\theta}, \tag{81}$$

and

$$Z_{2,0}(m, 0) \sim \int_{U \in U(2)} dU e^{V\Sigma \mathrm{Re\,Tr}(MU)}, \tag{82}$$

where $M$ is a diagonal matrix with diagonal elements $m$ and 0. The integral over $U$ can be performed by diagonalizing $U$ according to $U = U_1 e^{i\phi_k} U_1^{-1}$, and choosing $U_1$ and $\phi_k$ as new integration variables. The Jacobian of this transformation is

$$J \sim \Delta^2(e^{i\phi_k}). \tag{83}$$

The integral over $U_1$ can be performed using the Itzykson-Zuber formula. This results in

$$Z_{2,0} \sim \int d\phi_1 d\phi_2 \frac{|e^{-i\phi_1} - e^{-i\phi_2}|^2}{m(\cos\phi_1 - \cos\phi_2)} (e^{mV\Sigma\cos\phi_1} - e^{mV\Sigma\cos\phi_2}). \tag{84}$$

Both terms in the last factor result in the same contribution to the integral. Let us consider only the first term $\sim \exp(mV\Sigma\cos\phi_1)$. Then the integral over $\phi_2$ has to be defined as a principal value integral. If we use the identity

$$\frac{|e^{-i\phi_1} - e^{-i\phi_2}|^2}{(\cos\phi_1 - \cos\phi_2)} = 2\left(\cos\phi_1 - \frac{\sin\phi_1}{\tan((\phi_1+\phi_2)/2)}\right), \tag{85}$$

the $\phi_2$-integral of the term proportional to $\sin\phi_1$ gives zero because of the principal value prescription. The term proportional to $\cos\phi_1$ trivially results in $Z_{1,1}$. We leave it as an exercise to the reader to generalize this proof to arbitrary $N_f$ and $\nu$.

The group integrals in finite volume partition function (28) were evaluated by Leutwyler and Smilga[8] for *equal* quark masses. An expression for *different* quark masses was obtained by Jackson *et al.*[76]. The expression could only be proved for $\nu = 0$. The above duality can be used to relate a partition function at arbitrary $\nu$ to a partition function at $\nu = 0$. This completes the proof of the conjectured expression for arbitrary topological charge.

## UNIVERSALITY IN CHIRAL RANDOM MATRIX THEORY

In the chiral ensembles, two types of universality studies can be performed. First, the universality of correlations in the bulk of the spectrum. As discussed above, they are given by the invariant random matrix ensembles. Second, the universality of the microscopic spectral density and the eigenvalue correlations near zero virtuality. The aim of such studies is to show that microscopic correlations are stable against deformations of the chiral ensemble away from the Gaussian probability distribution. Recently, a number of universality studies on microscopic correlations have appeared. They will be reviewed in this section.

We wish to emphasize that all universality studies for the chiral ensembles have been performed for $\beta = 2$. The reason is that $\beta = 1$ and $\beta = 4$ are mathematically much more involved. It certainly would be of interest to extend such studies to these cases as well.

In addition to the analytical studies to be discussed below the universality of the microscopic spectral density also follows from numerical studies of models with the symmetries of the QCD partition function. In particular, we mention strong support in favor of universality from a different branch of physics, namely from the theory of universal conductance fluctuations. In that context, the microscopic spectral density of the eigenvalues of the transmission matrix was calculated for the Hofstadter[81] model, and, to a high degree of accuracy, it agrees with the random matrix prediction[82]. Other studies deal with a class random matrix models with matrix elements with a diverging variance. Also in this case the microscopic spectral density is given by the universal expressions[83] (76) and (72).

The conclusion that emerges from all numerical and analytical work on modified chiral random matrix models is that the microscopic spectral density and the correlations near zero virtuality exhibit a strong universality that is comparable to the stability of microscopic correlations in the bulk of the spectrum.

Of course, QCD is much richer than chiral Random Matrix Theory. One question that should be asked is at what scale (in virtuality) QCD spectral correlations deviate from RMT. This question has been studied by means of instanton liquid simulations. Indeed, at macroscopic scales, it was found that the number variance shows a linear dependence instead of the logarithmic dependence observed at microscopic scales[84].

More work is needed to determine the point where the crossover between these two regimes takes place.

### Invariant Deformations of the Gaussian Random Matrix Ensembles

In a first class of universality studies one considers probability distributions that maintain unitary invariance. In this case the joint probability distribution is given by (59). In this section we consider the simplest nontrivial case with only $a_1$ and $a_2$ different from zero and present the proof of Akemann, Damgaard, Magnea and Nishigaki[11, 12] for this case. We wish to point out that the general case only leads to minor complications.

This case was first studied by Brézin, Hikami and Zee[13]. They showed that the microscopic spectral density is independent of $a_2$. A general proof valid for arbitrary potential was given by Akemann, Damgaard, Magnea and Nishigaki[11, 12]. The essence of the proof is a remarkable generalization of the identity for the Laguerre polynomials,

$$\lim_{n\to\infty} L_n(\frac{x}{n}) = J_0(2\sqrt{x}) \ , \tag{86}$$

to orthogonal polynomials determined by an arbitrary potential $V$. This relation was proved by deriving a differential equation from the continuum limit of the recursion relation for orthogonal polynomials. This proof has been extended to the microscopic correlation functions of all chiral ensembles in a recent work by Akemann, Damgaard, Magnea and Nishigaki[12].

In the normalization $P_k(0) = 1$, the orthogonal polynomials (60) satisfy the recursion relation

$$xP_k(x) = -r_k\left[P_{k+1}(x) - P_k(x) - Q_k(P_k(x) - P_{k-1}(x))\right], \tag{87}$$

where

$$r_k = -\frac{p_k}{p_{k+1}} \quad \text{and} \quad Q_k = \frac{h_k r_{k-1}}{r_k h_{k-1}}. \tag{88}$$

Here, the coefficient of $x^k$ in $P_k(x)$ is denoted by $p_k$. The fractions $r_k$ and the normalizations $h_k$ can be determined from the relations

$$\int_0^\infty dx\ nV'(x)e^{-nV(x)}P_k(x)P_k(x) = -\int_0^\infty dx\frac{d}{dx}\left[e^{-nV(x)}P_k(x)P_k(x)\right] = 1, \tag{89}$$

$$\int_0^\infty dx\ nV'(x)e^{-nV(x)}xP_k(x)P_k(x) - h_k - 2kh_k = -\int_0^\infty dx\frac{d}{dx}\left[e^{-nV(x)}xP_k(x)P_k(x)\right] = 0. \tag{90}$$

For the potential

$$V(x) = a_1 x + \frac{a_2}{2}x^2, \tag{91}$$

these relations reduce to

$$na_1h_k + na_2r_kh_k(1+Q_k) = 1, \tag{92}$$

$$na_1r_kh_k(1+Q_k) + na_2r_k^2\left(h_{k+1} + h_k(1+Q_k)^2 + h_{k-1}Q_k^2\right) - h_k - 2kh_k = 0. \tag{93}$$

In order to proceed we take the continuum limit of these relations, i.e. $k \to \infty$ and $n \to \infty$ at fixed $k/n = t$. With

$$\begin{aligned} r(t) &= r(k/n) + \mathcal{O}(1/n), \\ h(t) &= \frac{1}{n} h(k/n) + \mathcal{O}(1/n^2), \end{aligned} \tag{94}$$

to leading order in $1/n$, these recursion relations can be rewritten as

$$a_1 + 2a_2 r(t) = \frac{1}{h(t)}, \tag{95}$$

$$a_1 r(t) + 3a_2 r^2(t) = t. \tag{96}$$

The functions $r(t)$ and $h(t)$ are given by the solution of these equations. However, we do not need the explicit solution. A more useful property is that they satisfy a differential equation that does not depend on $a_1$ and $a_2$:

$$r'(t)h(t) - 2h'(t)r(t) = h^2(t). \tag{97}$$

Remarkably, as was shown by Akemann, Damgaard, Magnea and Nishigaki[11, 12], this relation is valid for any polynomial potential. This relation is the essential ingredient in reducing the continuum limit of the recursion relation (87) to a Bessel equation.

To leading order in $1/n$ the r.h.s. of the recursion relation (87) is zero. We have to collect the terms of subleading order. In the continuum limit we may write

$$\begin{aligned} r_{k-1} &= r(t) - \frac{1}{n} r'(t), \\ h_{k-1} &= \frac{1}{n} h(t) - \frac{1}{n^2} h'(t), \\ P_{k-1}(x) &= P(t,x) - \frac{1}{n}\frac{d}{dt} P(t,x), \quad \text{etc..} \end{aligned} \tag{98}$$

This results in the differential equation

$$n^2 x P(t,x) = -h(t) \frac{d}{dt} \frac{r(t)}{h(t)} \frac{d}{dt} P(t,x). \tag{99}$$

We observe that the continuum limit of the recursion relation exists if we take at the same time the microscopic limit with fixed $n\sqrt{x}$. If we introduce the new variables

$$\begin{aligned} v &= n\lambda = n\sqrt{x}, \\ u(t) &= \frac{2\sqrt{r(t)}}{h(t)} = \int_0^t \frac{dt'}{\sqrt{r(t')}} \end{aligned} \tag{100}$$

the differential equation reduces to the Bessel equation

$$\frac{1}{u}\frac{d}{du} u \frac{d}{du} P(u, v^2) + v^2 P(u, v^2) = 0. \tag{101}$$

Here, and below we omit the argument $t$ of $u(t)$. The general solution of this Bessel equation is given by

$$P(u, v^2) = A J_0(uv) + B N_0(uv) \tag{102}$$

The constants are determined by the boundary condition that the orthogonal polynomials are normalized as $P_n(0) = 1$. This results in $A = 1$ and $B = 0$.

The average spectral density follows from (63) and (64). For $a = 0$ (we omit the superscript $a$) we obtain

$$\rho(\lambda) = 2\lambda e^{-nV(\lambda^2)} \sum_{k=0}^{n-1} \frac{1}{h_k} P_k(\lambda^2) P_k(\lambda^2). \tag{103}$$

With the help of the Christoffel-Darboux formula the sum over the polynomials can be expressed in $P_{n-1}(\lambda^2)$ and its derivative. This results in

$$\rho(\lambda) = 2\lambda e^{-nV(\lambda^2)} \frac{r_{n-1}}{h_{n-1}} \left( P'_{n-1}(\lambda^2) P_n(\lambda^2) - P'_n(\lambda^2) P_{n-1}(\lambda^2) \right), \tag{104}$$

where the prime denotes differentiation with respect to $\lambda^2$. The leading order terms contributing to the continuum limit of this equation cancel. The subleading terms follow from the third equation in (98). In terms of the new variables we have,

$$P_{n-1}(\lambda^2) \sim P_n(\lambda^2) - \frac{1}{n}\frac{d}{dt} P(u, \lambda^2) = P_n(\lambda^2) - \frac{1}{n}\frac{d}{dt} J_0(uv), \tag{105}$$

$$P'_n(\lambda^2) = \frac{d}{d\lambda^2} P_n(\lambda^2) \sim n^2 \frac{d}{dv^2} J_0(uv). \tag{106}$$

Taking both the continuum limit and the microscopic limit, eq. (104) reduces to

$$\begin{aligned} \rho(\lambda) &= 2\lambda n^2 \frac{r(t)}{h(t)} \left( -\frac{1}{2v} J_0(uv) \frac{d}{dv}\frac{d}{dt} J_0(uv) + \frac{1}{2v}\frac{d}{dv} J_0(uv) \frac{d}{dt} J_0(uv) \right) \\ &= \lambda n^2 \frac{r(t) u u'}{h(t)} \left( -\frac{1}{vu} J'_0(uv) J_0(uv) - J''_0(uv) J_0(uv) + J'_0(uv) J'_0(uv) \right). \end{aligned} \tag{107}$$

All polynomials that enter in the precursor to this equation are of the $n$'th order. Therefore, we can put $t = 1$. If we also use the relations $J'_0(z) = -J_1(z)$ and $J'_1 = -J_1(z)/z + J_0(z)$, and the definition of $u$ (see eq. (100)), we obtain

$$\rho(\lambda) = \frac{1}{2} \lambda n^2 u^2(1) \left( J_0(u(1)v) J_0(u(1)v) + J_1(u(1)v) J_1(u(1)v) \right). \tag{108}$$

For $\lambda \to \infty$ the microscopic spectral density should coincide with $\rho(0)$. This allows us to identify $u(1)$ as

$$u(1) = \frac{\pi \rho(0)}{n}. \tag{109}$$

For the microscopic spectral density defined by (with $z = N\Sigma\lambda = \pi\rho(0)\lambda$, notice that $N = 2n$)

$$\rho_S(z) = \lim_{N\to\infty} \frac{1}{N\Sigma} \rho(\frac{z}{N\Sigma}) = \lim_{n\to\infty} \frac{1}{\pi\rho(0)} \rho(\frac{z}{\pi\rho(0)}), \tag{110}$$

we obtain

$$\rho_S(z) = \frac{z}{2}(J_0^2(z) + J_1^2(z)), \tag{111}$$

in agreement with the result (72) derived for the Gaussian chiral ensemble. This result is independent of the specific shape of the random matrix potential. This proves its universality.

## Noninvariant Deformations of the Chiral Gaussian Random Matrix Ensembles

In a second class of universality studies one considers deformations of the Gaussian random matrix ensemble that violate unitary invariance. In particular, one has considered the case where the matrix $W$ in (50) is replaced by

$$W \to W + A, \tag{112}$$

whereas the probability distribution of $W$ is Gaussian. Because of the unitary invariance, the matrix $A$ can always be chosen diagonal. The simplest case with $A = \pi T$ times the identity was considered by Jackson *et al.*[15]. This model provides a schematic model of the chiral phase transition. In this section we will give a detailed derivation of the average spectral density. The aim is to shown that the spectrum changes dramatically with variations of the temperature parameter. Nevertheless, it could be shown[15] that the microscopic spectral density is temperature independent in the broken phase.

We will show that for large matrices, the average resolvent defined by

$$G(z) = \frac{1}{2n}\mathrm{Tr}\left\langle \frac{1}{z + i\epsilon - \mathcal{D}} \right\rangle \tag{113}$$

obeys the cubic equation[77, 78, 15] (the parameter $\Sigma = 1$ in (49))

$$G^3 - 2zG^2 + G(z^2 - \pi^2 T^2 + 1) - z = 0. \tag{114}$$

Here, $\mathcal{D}$ is the ensemble of random matrices

$$\mathcal{D} = \begin{pmatrix} 0 & W + \pi T \\ W^\dagger + \pi T & 0 \end{pmatrix}, \tag{115}$$

with probability distribution given by

$$P(W) = \exp(-n\mathrm{TrWW}^\dagger). \tag{116}$$

The average over $P(W)$ is denoted by the brackets $\langle \cdots \rangle$. Because the operator (115) has only a finite support, it is possible to expand the resolvent in a geometric series in $1/(z - K)$ for $z$ sufficiently large. Here, $K$ is the matrix

$$K = \begin{pmatrix} 0 & \pi T \\ \pi T & 0 \end{pmatrix}. \tag{117}$$

One finds by inspection that $G(z)$ satisfies

$$G(z) = \mathrm{Tr}\frac{1}{z - K} + \mathrm{Tr}\frac{1}{z - K}\left\langle \begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix} \mathcal{G} \begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix} \mathcal{G} \right\rangle \tag{118}$$

where $\mathcal{G}$ is the matrix

$$\mathcal{G} = \left\langle \frac{1}{z - H} \right\rangle. \tag{119}$$

It should be clear that $\mathcal{G}$ is block diagonal with the block structure

$$\mathcal{G} = \begin{pmatrix} g\mathbf{1}_n & h\mathbf{1}_n \\ h\mathbf{1}_n & g\mathbf{1}_n \end{pmatrix}, \tag{120}$$

where $\mathbf{1_n}$ is the $n \times n$ identity matrix. Therefore, we find that $G(z) = g$. The average over $W$ can be carried out immediately to give

$$\left\langle \begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix} \mathcal{G} \begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix} \right\rangle = \frac{1}{n} \begin{pmatrix} g\mathbf{1}_n & 0 \\ 0 & g\mathbf{1}_n \end{pmatrix} . \tag{121}$$

This yields the following matrix equation for $g$ and $h$:

$$\begin{pmatrix} z & -\pi T \\ -\pi T & z \end{pmatrix} \begin{pmatrix} g & h \\ h & g \end{pmatrix} = \mathbf{1} + \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} g & h \\ h & g \end{pmatrix} , \tag{122}$$

which leads to the two independent equations

$$\begin{aligned} zg - \pi T h &= 1 + g^2 , \\ zh - \pi T g &= gh . \end{aligned} \tag{123}$$

Elimination of $h$ yields the equation

$$zg - \frac{\pi^2 T^2 g}{z - g} = 1 + g^2 . \tag{124}$$

Evidently, it can be rewritten as a cubic equation for $g$. By taking the trace of $\mathcal{G}$ one obtains the announced cubic equation (114) for $G(z)$

The average spectral density given by

$$\rho(\lambda) = -\frac{1}{\pi} \mathrm{Im} G(z = \lambda) \tag{125}$$

is a semicircle at $\pi T = 0$ and splits into two arcs at $\pi T = 1$. For the spectral density at zero one obtains $\rho(0) = \sqrt{1 - \pi^2 T^2}$, and therefore chiral symmetry is broken for $\pi T < 1$, and is restored above this temperature. In spite of this drastic change in average spectral density, it could be shown[15] with the help of a supersymmetric formulation of Random Matrix Theory that the microscopic spectral density does not depend on $T$.

The super-symmetric method in the first paper by Jackson *et al.*[15] is not easily generalizable to higher order correlation functions. A natural way to proceed is to employ the super-symmetric method introduced by Guhr[79]. In the case of $\beta = 2$ this method results in an analytical expression for the kernel determining all correlation functions. This approach was followed in two papers, one by Guhr and Wettig[14] and one by Jackson *et al.*[16]. The latter authors studied microscopic correlation functions for $A$ in (112) proportional to the identity, whereas Guhr and Wettig considered an arbitrary diagonal matrix $A$. It was shown that independent of the matrix $A$, the correlations are given by the Bessel kernel[80]. Of course, a necessary condition on the matrix $A$ is that chiral symmetry is broken. Guhr and Wettig also showed that correlations in the bulk of the spectrum are insensitive to $A$.

The deformation $W \to W + A$ with the probability distribution for $W$ given by an arbitrary invariant potential has not yet been considered. We have no doubt that universality proofs along the lines of methods developed by P. Zinn-Justin[49] can be given.

## LATTICE QCD RESULTS

In this section we will focus ourselves on the spectral correlations of the lattice QCD Dirac operator. Both correlations in the bulk of the spectrum and the microscopic spectral density will be studied. Consistent with universality arguments presented above, we find that spectral correlations are in complete agreement with chiral Random Matrix Theory.

### Correlations in the Bulk of the Spectrum

Recently, Kalkreuter[19] calculated *all* eigenvalues of the $N_c = 2$ lattice Dirac operator both for Kogut-Susskind (KS) fermions and Wilson fermions for lattices as large as $12^4$. For the Kogut-Susskind Dirac operator, $D^{KS}$, we use the convention that it is anti-Hermitean. Because of the Wilson-term, the Wilson Dirac operator, $D^W$, is neither Hermitean nor anti-Hermitean. Its Hermiticity relation is given by $D^{W\dagger} = \gamma_5 D^W \gamma_5$. Therefore, the operator $\gamma_5 D^W$ is Hermitean. However, it does not anti-commute with $\gamma_5$, and its eigenvalues do not occur in pairs $\pm\lambda_k$.

In the case of $SU(2)$, the anti-unitary symmetry of the Kogut-Susskind and the Wilson Dirac operator is given by[85, 22],

$$[D^{KS}, \tau_2 K] = 0, \quad \text{and} \quad [\gamma_5 D^W, \gamma_5 C K \tau_2] = 0. \tag{126}$$

Because

$$(\tau_2 K)^2 = -1, \quad \text{and} \quad (\gamma_5 C K \tau_2)^2 = 1, \tag{127}$$

the matrix elements of the KS Dirac operator can be arranged into real quaternions, whereas the Wilson Dirac operator can be expressed into real matrix elements. Therefore, we expect that eigenvalue correlations in the bulk of the spectrum are described by the GSE and the GOE, respectively[22]. The microscopic correlations for KS fermions are described by the chGSE. However, the microscopic correlations for Wilson fermions are not described by the chGOE but rather by the GOE. Because of the anti-unitary symmetry, the eigenvalues of the KS Dirac operator are subject to the Kramers degeneracy, i.e. they are double degenerate.

In both cases, the Dirac matrix is tri-diagonalized by Cullum's and Willoughby's Lanczos procedure[86] and diagonalized with a standard QL algorithm. This improved algorithm makes it possible to obtain *all* eigenvalues. This allows us to test the accuracy of the eigenvalues by means of sum-rules for the sum of the squares of the eigenvalues of the lattice Dirac operator. Typically, the numerical error in the sum rule is of order $10^{-8}$.

As an example, in Fig. 1 we show a histogram of the overall Dirac spectrum for KS fermions at $\beta = 2.4$. Results for the spectral correlations are shown in Figs. 2, 3 and 4. The results for KS fermions are for 4 dynamical flavors with $ma = 0.05$ on a $12^4$ lattice. The results for Wilson fermions were obtained for two dynamical flavors on a $8^3 \times 12$ lattice. For the values of $\beta$ and $\kappa$ we refer to the labels of the figure. For $\beta > 2.4$, with our lattice parameters for KS fermions, the Dirac spectrum near zero virtuality develops a gap. Of course, this is an expected feature of the weak coupling domain. For small enough values of $\kappa$ the Wilson Dirac spectrum shows a gap at $\lambda = 0$ as well. In the scaling domain the value of $\kappa$ is just above the critical value of $\kappa$.

The eigenvalue spectrum is unfolded by fitting a second order polynomial to the integrated spectral density of a stretch of 500-1000 eigenvalues. The results for $\Sigma_2(n)$, $\Delta_3(n)$ and $P(S)$ in Fig. 2 show an impressive agreement with RMT predictions. The fluctuations in $\Sigma_2(n)$ are as expected from RMT. The advantage of $\Delta_3$-statistic is well illustrated by this figure. We also investigated[23] the $n$ dependence of the first two cumulants of the number of levels in a stretch of length $n$. Results presented in Fig. 4 show a perfect agreement with RMT. Spectra for different values of $\beta$ have been analyzed as well. It is probably no surprise that random matrix correlations are found at stronger couplings. What is surprising, however, is that even in the weak-coupling domain ($\beta = 2.8$) the eigenvalue correlations are in complete agreement with Random

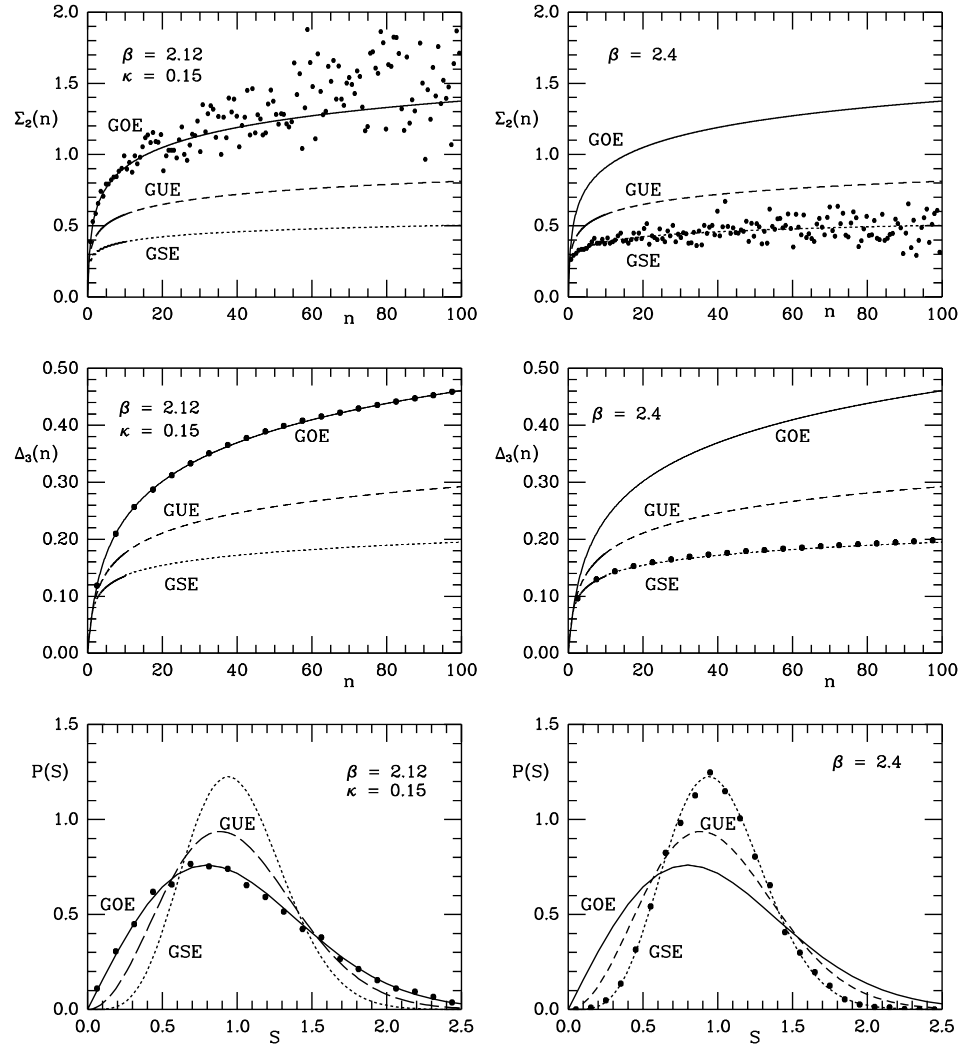


**Figure 2.** Spectral correlations of Dirac eigenvalues for Wilson fermions (left) and KS-fermions (right). Results are shown for the number variance, $\Sigma_2(n)$, the $\Delta_3-$statistic and the nearest neighbor spacing distribution, $P(S)$. The solid, dashed and dotted curves represent the analytical result for the GOE, GUE and GSE, respectively.

Matrix Theory. Finally, we have studied the stationarity of the ensemble by analyzing level sequences of about 200 eigenvalues (with relatively low statistics). No deviations from random matrix correlations were observed all over the spectrum, including the region near $\lambda = 0$. This justifies the spectral averaging which results in the good statistics in Figs. 2 and 3.

In the case of three or more colors with fundamental fermions, both the Wilson and Kogut-Susskind Dirac operator do not possess any anti-unitary symmetries. Therefore, our conjecture is that in this case the spectral correlations in the bulk of the spectrum of both types of fermions can be described by the GUE. In the case of two fundamental colors the continuum theory and Wilson fermions are in the same universality class. It is an interesting question of how spectral correlations of KS fermions evolve in the approach to the continuum limit. Certainly, the Kramers degeneracy of the eigenvalues remains. However, since Kogut-Susskind fermions represent 4 degenerate flavors in the continuum limit, the Dirac eigenvalues should obtain an additional two-fold degeneracy. We are looking forward to more work in this direction.

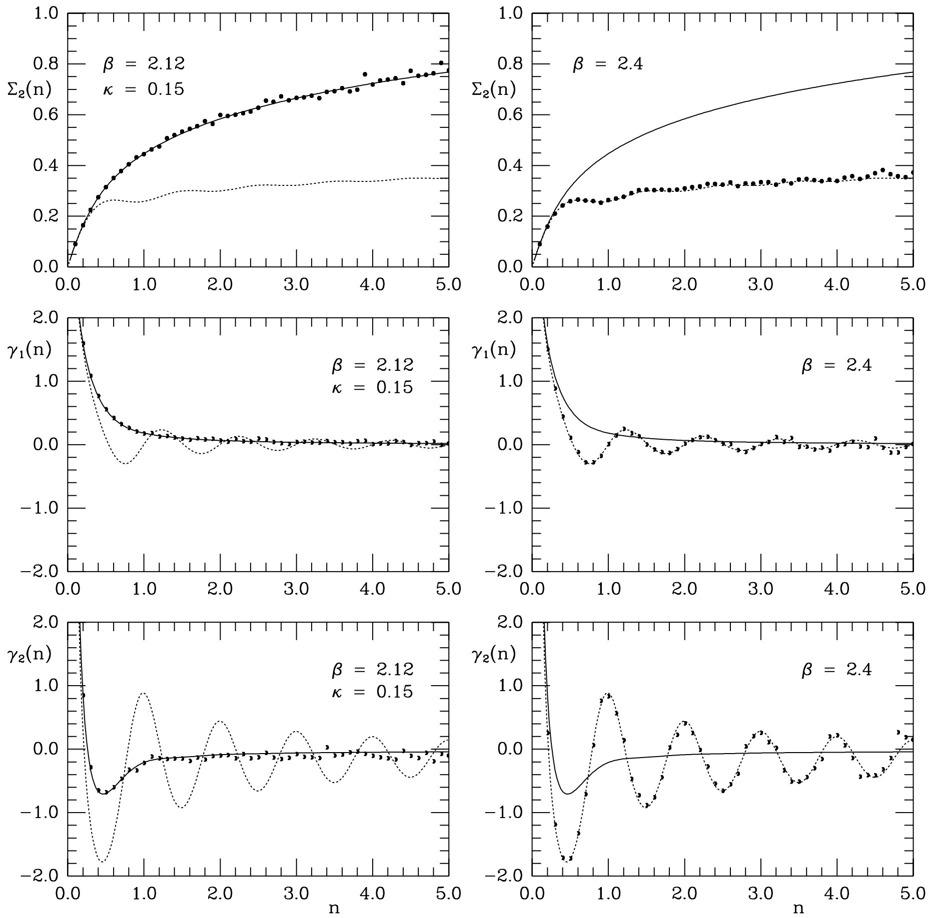


**Figure 3.** The number variance, $\Sigma_2(n)$ and the first two cumulants, $\gamma_1(n)$ and $\gamma_2(n)$ as a function of $n$ for eigenvalues of the Wilson Dirac operator (left) and the Kogut-Susskind Dirac operator (right). The full and dotted curves represent the analytical result for the GOE and the GSE, respectively.

## The Microscopic Spectral Density

The advantage of studying spectral correlations in the bulk of the spectrum is that one can perform spectral averages instead of ensemble averages requiring only a relatively small number of equilibrated configurations. This so called spectral ergodicity cannot be exploited in the study of the microscopic spectral density. In order to gather sufficient statistics for the microscopic spectral density of the lattice Dirac operator a large number of independent configurations is needed. One way to proceed is to generate instanton-liquid configurations which can be obtained much more cheaply than lattice QCD configurations. Results of such analysis[87] show that for $N_c = 2$ with fundamental fermions the microscopic spectral density is given by the chGOE. For $N_c = 3$ it is given by the chGUE. One could argue that instanton-liquid configurations can be viewed as smoothened lattice QCD configurations. Roughening such configurations will only improve the agreement with Random Matrix Theory.

Of course, the ultimate goal is to test the conjecture of microscopic universality for realistic lattice QCD configurations. In order to obtain a very large number of independent gauge field configurations one is necessarily restricted to relatively small lattices. The first study in this direction was reported recently[20, 88]. In this work, the quenched $SU(2)$ Kogut-Susskind Dirac operator was diagonalized for lattices with linear dimension of 4, 6, 8 and 10, and a total number of configurations of 9978, 9953, 3896 and 1416, respectively. The results were compared with predictions from the chGSE.

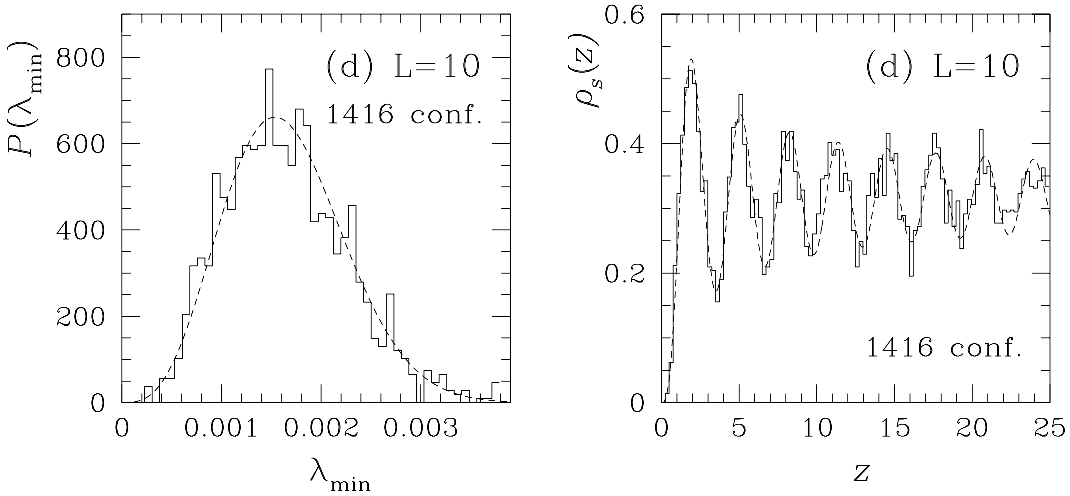


**Figure 4.** The distribution of the smallest eigenvalue (left) and the microscopic spectral density (right) of the Kogut-Susskind Dirac operator for two colors and $\beta = 2.0$. Lattice results are represented by the histogram, and the analytical results for the chGSE are given by the dashed curves.

We only show results for the largest lattice. For more detailed results, including results for the two-point correlation function, we refer to the original work. In Fig. 4 we show the distribution of the smallest eigenvalue (left) and the microscopic spectral density (right). The lattice results are given by the full line. The dashed curve represents the random matrix results. The distribution of the smallest eigenvalue was derived by Forrester[89] and is given by

$$P(\lambda_{\rm min}) = \alpha\sqrt{\frac{\pi}{2}}(\alpha\lambda_{\rm min})^{3/2}I_{3/2}(\alpha\lambda_{\rm min})e^{-\frac{1}{2}(\alpha\lambda_{\rm min})^2}, \tag{128}$$

where $\alpha = V\Sigma$. The random matrix result for the microscopic spectral density is given in eq. (78). We emphasize that the theoretical curves have been obtained without any fitting of parameters. The input parameter, the chiral condensate, is derived from the same lattice calculations. The above simulations were performed at a relatively strong coupling of $\beta = 2$. Recently, the same analysis[90] was performed for $\beta = 2.2$ and for $\beta = 2.5$ on a $16^4$ lattice. In both cases agreement with the random matrix predictions was found[90].

An alternative way to probe the Dirac spectrum is via the valence quark mass dependence of the chiral condensate[18] defined as

$$\Sigma(m) = \frac{1}{N}\int d\lambda \langle\rho(\lambda)\rangle \frac{2m}{\lambda^2+m^2}. \tag{129}$$

The average spectral density is obtained for a fixed sea quark mass. For masses well beyond the smallest eigenvalue, $\Sigma(m)$ shows a plateau approaching the value of the chiral condensate $\Sigma$. In the mesoscopic range (1), we can introduce $u = \lambda m N$ and $x = mN\Sigma$ as new variables. Then the microscopic spectral density enters in $\Sigma(m)$. For three fundamental colors the microscopic spectral density for $\beta = 2$ (eq. (72)) applies and the integral over $\lambda$ in (129) can be performed analytically. The result is given by[17],

$$\frac{\Sigma(x)}{\Sigma} = x(I_a(x)K_a(x) + I_{a+1}(x)K_{a-1}(x)), \tag{130}$$

where $a = N_f + |\nu|$ and $I_a$ and $K_a$ are modified Bessel functions. In Fig. 2 we plot this ratio as a function of $x$ (the 'volume' $V$ is equal to the total number of Dirac eigenvalues) for lattice data of two dynamical flavors with mass $ma = 0.01$ and $N_c = 3$ on a $16^3 \times 4$ lattice. We observe that the lattice data for different values of $\beta$ fall on a single curve. Moreover, in the mesoscopic range this curve coincides with the random matrix prediction for $N_f = \nu = 0$.

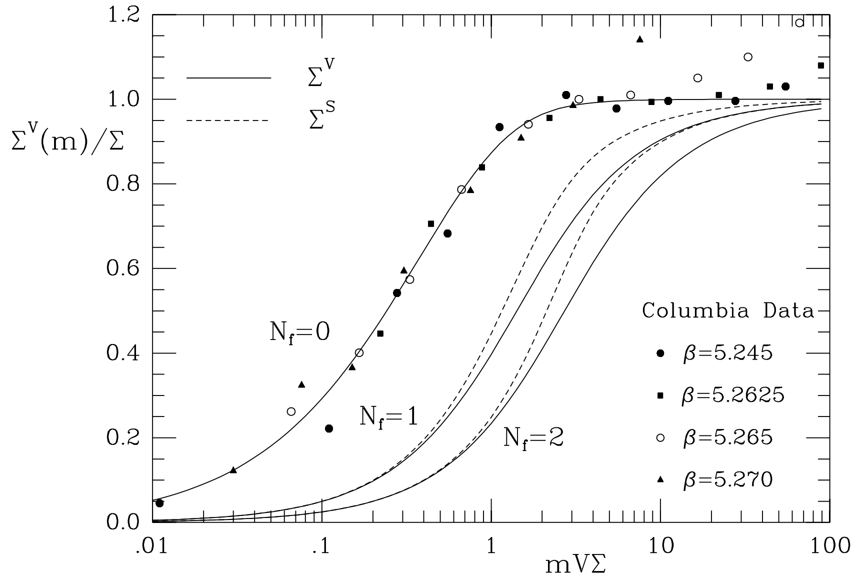


**Figure 5.** The valence quark mass dependence of the chiral condensate $\Sigma^V(m)$ plotted as $\Sigma^V(m)/\Sigma$ versus $mV\Sigma$. The dots and squares represent lattice results by the Columbia group[18] for values of $\beta$ as indicated in the label of the figure.

Apparently, the zero modes are completely mixed with the much larger number of nonzero modes. For eigenvalues much smaller than the sea quark mass, one expects quenched ($N_f = 0$) eigenvalue correlations. In the same figure the dashed curves represent results for the quark mass dependence of the chiral condensate (i.e. the mass dependence for equal valence and sea quark masses). In the sector of zero topological charge one finds[91, 9, 8]

$$\frac{\Sigma^S(u)}{\Sigma} = \frac{I_1(u)}{I_0(u)} \quad \text{for} \quad N_f = 1, \tag{131}$$

and

$$\frac{\Sigma^S(u)}{\Sigma} = \frac{I_1^2(u)}{u(I_0^2(u) - I_1^2(u))} \quad \text{for} \quad N_f = 2. \tag{132}$$

We observe that both expressions do not fit the data. Also notice that, according to Göckeler *et al.*[92], eq. (131) describes the valence mass dependence of the chiral condensate for non-compact QED with quenched Kogut-Susskind fermions. However, we were not able to derive their result (no derivation is given in the paper).

## CHIRAL RANDOM MATRIX THEORY AT $\mu \neq 0$

### Generalities

At nonzero temperature $T$ and chemical potential $\mu$ a *schematic* random matrix model of the QCD partition function is obtained by replacing the Dirac operator in

(49) by[77, 93, 78, 35]

$$\mathcal{D} = \left( \begin{array}{cc} 0 & iW + i\Omega_T + \mu \\ iW^\dagger + i\Omega_T + \mu & 0 \end{array} \right). \tag{133}$$

Here, $\Omega_T = T \otimes_n (2n+1)\pi\mathbf{1}$ are the matrix elements of $i\gamma_0\partial_0$ in a plane wave basis with anti-periodic boundary conditions in the time direction. Below, we will discuss a model with $\Omega_T$ absorbed in the random matrix and $\mu \neq 0$. The aim of this model is to explore the effects of the non-Hermiticity of the Dirac operator. For example, the random matrix partition function (49) with the Dirac matrix (133) is well suited for the study of zeros of this partition function in the complex mass plane and in the complex chemical potential plane. Numerical results for the location of zeros could be explained analytically[94].

The term $\mu\gamma_0$ does not affect the anti-unitary symmetries of the Dirac operator. This is also the case in lattice QCD where the color matrices in the forward time direction are replaced by $U \to e^\mu U$ and in the backward time direction by $U^\dagger \to e^{-\mu} U^\dagger$. For this reason the universality classes are the same as at zero chemical potential.

The Dirac operator that will be discussed in this section is thus given by

$$\mathcal{D}(\mu) = \left( \begin{array}{cc} 0 & iW + \mu \\ iW^\dagger + \mu & 0 \end{array} \right), \tag{134}$$

where the matrix elements of the $n \times n$ matrix $W$ are either real ($\beta = 1$), complex ($\beta = 2$) or quaternion real ($\beta = 4$). For all three values of $\beta$ the eigenvalues of $\mathcal{D}(\mu)$ are scattered in the complex plane.

Since many standard random matrix methods rely on convergence properties based on the Hermiticity of the random matrix, direct application of most methods is not possible. The simplest way out is the Hermitization[95] of the problem, i.e we consider the Hermitean operator

$$D^H(z, z^*) = \left( \begin{array}{cc} \kappa & z - \mathcal{D}(\mu) \\ z^* - \mathcal{D}^\dagger(\mu) & \kappa \end{array} \right). \tag{135}$$

For example, the generating function in the supersymmetric method of Random Matrix Theory[96, 97] is then given by[98, 99, 100, 101]

$$Z(J, J^*) = \left\langle \frac{\det(D^H(z+J, z^*+J^*)}{\det(D^H(z, z^*))} \right\rangle. \tag{136}$$

The determinants can be rewritten as fermionic and bosonic integrals. Convergence is assured by the Hermiticity and by the infinitesimal increment $\kappa$. The resolvent follows from the generating function by differentiation with respect to the source terms

$$G(z, z^*) = \mathrm{Tr} \left\langle \frac{1}{z - \mathcal{D}(\mu)} \right\rangle = \frac{\partial}{\partial J} Z(J, J^*) \bigg|_{J=J^*=0}. \tag{137}$$

Notice that, after averaging over the random matrix, the partition function depends in a non-trivial way on both $z$ and $z^*$. The spectral density is then given by

$$\rho(z) = \frac{1}{\pi} \frac{\partial}{\partial z^*} G(z, z^*). \tag{138}$$

Therefore, outside the domain of the eigenvalues $G(z, z^*)$ is a function of $z$ only, whereas for $z$ inside the domain of eigenvalues, $G(z, z^*)$ should depend on $z^*$ as well.

Alternatively, one can use the replica trick[102, 35] with generating function given by

$$Z(z, z^*, \mu) = \langle \det{}^{N_f} D_H(z, z^*) \rangle. \tag{139}$$

The resolvent is then given by

$$G(z, z^*) = \lim_{N_f \to 0} \frac{1}{2nN_f} \frac{\partial}{\partial z} \log Z(z, z^*, \mu). \tag{140}$$

The idea is to perform the calculation for integer values of $N_f$ and perform the limit $N_f \to 0$ at the end of the calculation. Although the replica limit, $N_f \to 0$, fails in general[104], it is expected to work for $\det D_H(z, z^*)$ because it is positive definite (or zero). Then the partition function is a smooth function of $N_f$. For other techniques addressing nonhermitean matrices we refer to the recent papers by Feinberg and Zee[95] and Nowak and co-workers[103]. One recent method that does not rely on the Hermiticity of the random matrices is the method of complex orthogonal polynomials[36]. This method was used by Fyodorov *et al.*[36] to calculate the number variance and the nearest neighbor spacing distribution in the regime of weakly nonhermitean matrices. As surprising new result, they found an $S^{5/2}$ repulsion law.

In the physically relevant case of QCD with three colors, the fermion determinant is complex for nonzero chemical potential. Its phase prevents the convergence of fully unquenched Monte-Carlo simulations (see Kogut *et al.*[105] for the latest progress in this direction). However, it is possible to perform quenched simulations. In such calculations it was found that the critical chemical potential $\mu_c \sim \sqrt{m}$, instead of a third of the nucleon mass[106]. This phenomenon was explained analytically by Stephanov[35] with the help of the above random matrix model. He could show that for small $\mu$ the eigenvalues are distributed along the imaginary axis in a band of width $\sim \mu^2$ leading to a critical chemical potential of $\mu_c \sim \sqrt{m}$. A detailed derivation of his solution will be given in the next section. As has been argued above, the quenched limit is necessarily obtained from a partition function in which the fermion determinant appears as

$$\lim_{N_f \to 0} |\det \mathcal{D}(\mu)|^{N_f}, \tag{141}$$

instead of the same expression without the absolute value signs. The partition function with the absolute value of the determinant can be interpreted as a partition function of an equal number of fermions and conjugate fermions. The critical value of the chemical potential, equal to half the pion mass, is due to Goldstone bosons with a net baryon number consisting of a quark and conjugate anti-quark. The reason that the quenched limit does not correspond to the standard QCD partition function is closely related to the failure of the replica trick in the case of a determinant with a nontrivial phase.

**The Stephanov Solution**

In this section we study the quenched limit of the partition function (139). We give a detailed derivation of the results originally obtained by Stephanov[35]. The determinants in (139) can be written as Grassmann integrals. Since Grassmann integrals are always convergent the infinitesimal increment $\kappa$ can be put equal to zero. This results in the partition function

$$\begin{aligned} Z(z, z^*, \mu) &= \int \mathcal{D}W \mathcal{D}\psi^* \mathcal{D}\psi \mathcal{D}\phi^* \mathcal{D}\phi \exp[-n\Sigma^2 \operatorname{Tr} WW^\dagger] \\ \times \exp&\left[ -\sum_{k=1}^{N_f} \psi^{k*} \begin{pmatrix} z & -iW + \mu \\ -iW^\dagger + \mu & z \end{pmatrix} \psi^k - \sum_{k=1}^{N_f} \phi^{k*} \begin{pmatrix} z^* & iW + \mu \\ iW^\dagger + \mu & z^* \end{pmatrix} \phi^k \right], \end{aligned} \tag{142}$$

where $W$ is an arbitrary complex $n \times n$ matrix. The Gaussian integrals over $W$ can be performed trivially,

$$
\begin{aligned}
Z(z, z^*, \mu) = \int \mathcal{D}\psi^* \mathcal{D}\psi \mathcal{D}\phi^* \mathcal{D}\phi \exp \Big[ & z(\psi^{f\,*}_{R\,i}\psi^f_{R\,i} + \psi^{f\,*}_{L\,k}\psi^f_{L\,k}) + z^*(\phi^{f\,*}_{R\,i}\phi^f_{R\,i} + \phi^{f\,*}_{L\,k}\phi^f_{L\,k}) \\
& + \mu(\psi^{f\,*}_{R\,i}\psi^f_{L\,i} + \psi^{f\,*}_{L\,k}\psi^f_{R\,k} + \phi^{f\,*}_{R\,i}\phi^f_{L\,i} + \phi^{f\,*}_{L\,k}\phi^f_{R\,k}) \\
- \frac{1}{n\Sigma^2} \Big( & \psi^{f\,*}_{L\,k}\psi^f_{R\,i}\psi^{g\,*}_{R\,i}\psi^g_{L\,k} - \psi^{f\,*}_{L\,k}\psi^f_{R\,i}\phi^{g\,*}_{R\,i}\phi^g_{L\,k} - \phi^{f\,*}_{L\,k}\phi^f_{R\,i}\psi^{g\,*}_{R\,i}\psi^g_{L\,k} + \phi^{f\,*}_{L\,k}\phi^f_{R\,i}\phi^{g\,*}_{R\,i}\phi^g_{L\,k} \Big) \Big] .
\end{aligned}
\tag{143}
$$

The four-fermion terms can be written as the difference of two squares. Each square can be linearized by the Hubbard-Stratonovitch transformation according to

$$
\exp(-AQ^2) \sim \int d\sigma \exp(-\frac{\sigma^2}{4A} - iQ\sigma) \quad . \tag{144}
$$

Using this, the fermionic integrals can be performed, and the partition function can be written as an integral over the complex $2N_f \times 2N_f$ matrices, $\sigma$ and $\tilde{\sigma}$,

$$
Z(z, z^*, \mu) = \int \mathcal{D}\tilde{\sigma}\mathcal{D}\sigma \exp[-n\Sigma^2 \mathrm{Tr}(\sigma - \zeta)k(\tilde{\sigma} - \zeta)k] \det{}^n \left( \sigma\tilde{\sigma} - \mu^2 \right) . \tag{145}
$$

In a $N_f \times N_f$ block structure notation, the matrices $\sigma$, $\tilde{\sigma}$, $k$ and $\zeta$ are given by

$$
\begin{aligned}
\sigma &= \begin{pmatrix} a & ib \\ ic & d \end{pmatrix}, & \tilde{\sigma} &= \begin{pmatrix} a^\dagger & ic^\dagger \\ ib^\dagger & d^\dagger \end{pmatrix}, \\
k &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \zeta &= \begin{pmatrix} z & 0 \\ 0 & z^* \end{pmatrix},
\end{aligned}
\tag{146}
$$

where $a$, $b$, $c$ and $d$ are arbitrary complex $N_f \times N_f$ matrices. The resolvent is obtained by differentiation with respect to $z$ according to (140) with the averaged partition function given by (145). This results in

$$
G = \frac{1}{2N_f} \mathrm{Tr}(2z + a + a^\dagger). \tag{147}
$$

In the thermodynamic limit the integrals in (145) can be evaluated by a saddle-point method. Notice that a variable and its complex conjugate have to be considered as independent integration variables. The saddle point equations are given by

$$
\frac{1}{\tilde{\sigma}\sigma - \mu^2}\tilde{\sigma} = k(\tilde{\sigma} + \zeta)k, \tag{148}
$$

$$
\frac{1}{\sigma\tilde{\sigma} - \mu^2}\sigma = k(\sigma + \zeta)k. \tag{149}
$$

In general the solution of the saddle point equations is not unique. However, a unique solution is obtained from the requirement that $\partial_z^* G(z)$ is positive definite. The saddle point equations have two obvious solutions

$$
\sigma = \tilde{\sigma} \quad \text{or} \quad \sigma = k\tilde{\sigma}k, \tag{150}
$$

for which the matrix equations (148) and (149) coincide.

It can be shown that the first possibility results in an unphysical solution. As usual in applications of the replica trick, we assume that the replica (or flavor) symmetry remains unbroken. This implies that, at the saddle point, each block in (146) is diagonal. With (150) the solution of our saddle point equations is reduced to a $2 \times 2$ matrix problem

$$\sigma = (\sigma k \sigma k - \mu^2)(k \sigma k + \zeta). \tag{151}$$

Consistent with the second solution in (150) we use the parametrization

$$\sigma = \begin{pmatrix} a & ib \\ -ib^* & d \end{pmatrix}. \tag{152}$$

One solution of (151) can be written down immediately, namely $b = b^* = 0$. Then the equations for $a$ and $d$ reduce to the same cubic equation. This solution results in a partition function which factorizes in a product of a $z$ dependent part and in a $z^*$ dependent part. This resolvent is therefore an analytic function of $z$ valid outside the domain of the eigenvalues. The cubic equation for the resolvent in this domain can be obtained from the cubic equation (114) by the replacement $T \to i\mu$.

Let us now focus on the solution inside the domain of eigenvalues with $b \neq 0$. From the off-diagonal elements of (151) we obtain the equations

$$\begin{aligned} (a+d)(d-z^*) - (a^2 - bb^* - \mu^2) &= 1, \\ (a+d)(a-z) - (d^2 - bb^* - \mu^2) &= 1. \end{aligned} \tag{153}$$

The difference of these equation results in

$$2(a+d)(d-a) + (a+d)(z-z^*) = 0, \tag{154}$$

with solution given by

$$a - d = \frac{1}{2}(z - z^*). \tag{155}$$

From the sum of the two equations one obtains

$$bb^* = 1 + \frac{1}{2}(a+d)(z+z^*) - ad - \mu^2. \tag{156}$$

The boundary of the domain of the eigenvalues is given by the set of points where this solution merges with the factorized solution, i.e. where $b = 0$. With (155) the condition $b = 0$ can be expressed as an equation for $a$, or by (147) ($G = a - z$ for diagonal $a$) an equation for the resolvent. On the boundary of the domain the equation for the resolvent is given by

$$(2G + 2x + iy)x - (G + x + iy)(G + x) + 1 - \mu^2 = 0. \tag{157}$$

For $b = 0$ the Stephanov solution and the solution of the cubic equation merge into each other. We thus have a second order phase transition, and on the boundary, the second derivative of the free energy corresponding to (145) vanishes.

From the diagonal elements of (151) one obtains

$$\begin{aligned} a &= (a-z)(a^2 - bb^* - \mu^2) + bb^*(a+d), \\ d &= (d-z^*)(d^2 - bb^* - \mu^2) + bb^*(a+d). \end{aligned} \tag{158}$$

In the first equation we substitute the first equation of (153) in the first term of the r.h.s. and the expression (156) for $bb^*$ in the second term of the r.h.s.. This results in

$$d - a + z + (a+d)(\frac{1}{2}(z-z^*)(a-d) - \mu^2 + zz^*) = 0. \tag{159}$$

Together with (155) this results in an equation for $a$ with solution

$$a = \frac{x}{\mu^2 - x^2} + \frac{i}{2}y. \tag{160}$$

For the resolvent one obtains

$$G = a - z = \frac{x}{\mu^2 - x^2} - x - \frac{i}{2}y. \tag{161}$$

This results in the spectral density

$$\rho(x, y) = \frac{1}{4\pi}\left(\frac{x^2 + \mu^2}{(\mu^2 - x^2)^2} - 1\right) \tag{162}$$

inside the boundary given by

$$y^2 = 4 - 4\mu^2 + \frac{4x^2}{\mu^2 - x^2} - \frac{x^2}{(\mu^2 - x^2)^2}. \tag{163}$$

The latter equation has obtained by the substitution of (161) in (157). Both (161) and (163) were first derived by Stephanov[35, 108]. The closed curves in Fig. 6 show the boundary of the domain of eigenvalues given by (163). Numerical results for the eigenvalues are represented by the dots in the same figure.

For small $\mu$, the width of the domain of eigenvalues is $\sim \mu^2$. This explains that the critical value of the chemical potential is given $\mu_c \sim \sqrt{m}$. This result explains the quenched results found in lattice QCD at nonzero chemical potential.

**Random Matrix Triality at Nonzero Chemical Potential**

In this section, we study the Dirac operator (134) for all three values of $\beta$. Both for $\beta = 1$ and $\beta = 4$ the fermion determinant, $\det(\mathcal{D}(\mu) + m)$, is real. This is obvious for $\beta = 1$. For $\beta = 4$ the reality follows from the identity $q^* = \sigma_2 q \sigma_2$ for a quaternion real element $q$, and the invariance of a determinant under transposition. We thus conclude that quenching works for an even number of flavors. Consequently, chiral symmetry will be restored for arbitrarily small nonzero $\mu$, whereas a condensate of a quark and a conjugate anti-quark develops. Indeed, this phenomenon has been observed in the strong coupling limit of lattice QCD with two colors[109].

In the quenched approximation, the spectral properties of the random matrix ensemble (134) can be easily studied numerically by simply diagonalizing a set of matrices with probability distribution (49). In Fig. 6 we show numerical results[110] for the eigenvalues of a few $100 \times 100$ matrices for $\mu = 0.15$ and $\mu = 0.5$. The dots represent the eigenvalues in the complex plane. The solid line is the analytical result[35] for the boundary of the eigenvalues which is given by the algebraic curve (163). This result was first derived by Stephanov for $\beta = 2$ (see previous section). However, the method that was used can be extended[110] to $\beta = 1$ and $\beta = 4$. Although the effective partition function is much more complicated, it can be shown without too much effort that the solutions of the saddle point equations are the same if the variance of the probability distribution is scaled as $1/\beta$. In particular, the boundary of the domain of eigenvalues is the same in each of the three cases. However, as one observes from Fig. 6, for $\beta = 1$ and $\beta = 4$ the spectral density deviates significantly from the saddle-point result. For $\beta = 1$ we find an accumulation of eigenvalues on the imaginary axis, whereas for $\beta = 4$ we find a depletion of eigenvalues in this domain. This depletion can be understood as

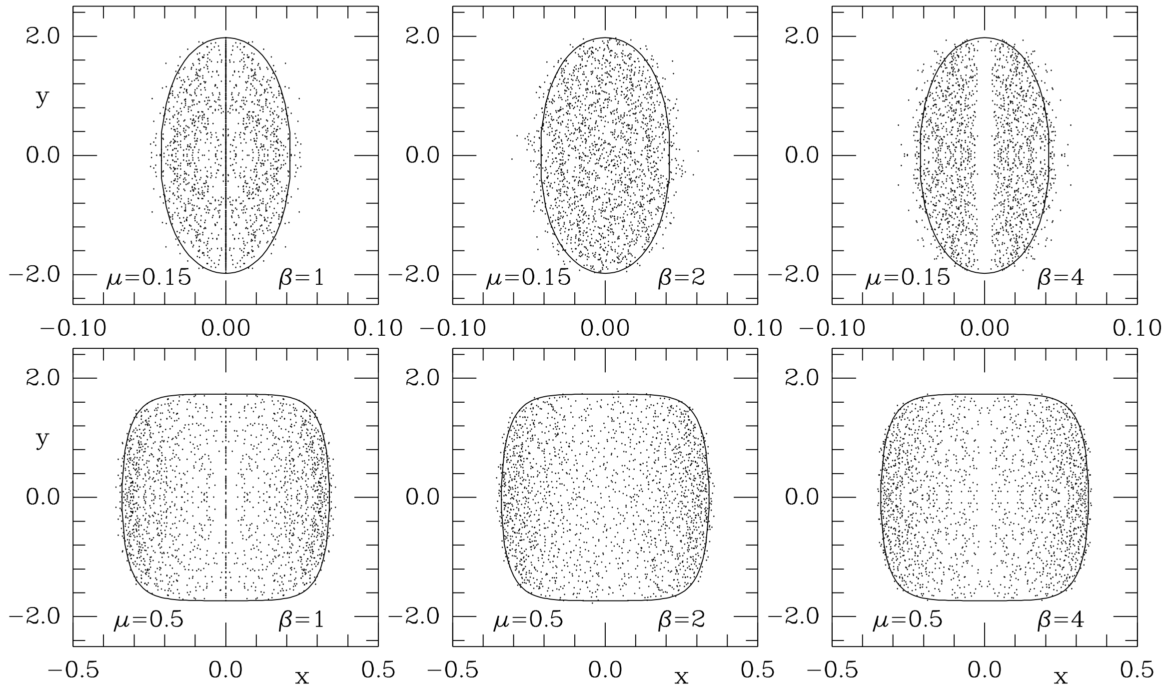


**Figure 6.** Scatter plot of the real ($x$), and the imaginary parts ($y$) of the eigenvalues of the random matrix Dirac operator. The values of $\beta$ and $\mu$ are given in the labels of the figure. The full curve shows the analytical result for the boundary.

follows. For $\mu = 0$ all eigenvalues are doubly degenerate. This degeneracy is broken at $\mu \neq 0$ which produces the observed repulsion of the eigenvalues.

The number of purely imaginary eigenvalues for $\beta = 1$ appears to scale as $\sqrt{N}$. This explains that this effect is not visible in a leading order saddle point analysis. From a perturbative analysis of (139) one obtains a power series in $1/N$. Clearly, the $\sqrt{N}$ dependence requires a truly nonperturbative analysis of the partition function (49) with the Dirac operator (134). Such a $\sqrt{N}$ scaling behavior is typical for the regime of weak non-hermiticity first identified by Fyodorov *et al.*[100]. Using the supersymmetric method for the generating function (136) the $\sqrt{N}$ dependence was obtained analytically by Efetov[101].

A similar cut below a cloud of eigenvalues was found in instanton liquid simulations[111] for $N_c = 2$ at $\mu \neq 0$ and in a random matrix model of arbitrary real matrices[99]. The depletion of the eigenvalues along the imaginary axis was observed earlier in lattice QCD simulations with staggered fermions[112]. Obviously, more work has to be done in order to arrive at a complete characterization of universal features[36] in the spectrum of nonhermitean matrices.

## CONCLUSIONS

We have argued that there is an intimate relation between correlations of Dirac eigenvalues and the breaking of chiral symmetry. In the chiral limit, the fermion determinant suppresses gauge field configurations with small Dirac eigenvalues. Correlations counteract this suppression, and are a necessary ingredient of chiral symmetry breaking. From the study of eigenvalue correlations in strongly interacting systems, we have concluded that they are described naturally by a Random Matrix Theory with the global symmetries of the physical system. In QCD, this led to the introduction of chiral Random Matrix Theories. They provided us with an analytical understanding of the statistical properties of the eigenvalues on the scale of a typical level spacing. It could be shown analytically that the microscopic spectral density is strongly universal. These results constitute the foundation of the impressive agreement between lattice QCD and chiral Random Matrix Theory for the microscopic spectral density and for spectral correlations in the bulk of the spectrum.

An extension of this model to nonzero chemical potential provided us with a complete analytical understanding of the failure of the quenched approximation observed in lattice QCD simulations at finite density. Some intriguing properties of previously obtained lattice QCD Dirac spectra and instanton liquid Dirac spectra at finite density could be explained as well.

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