INTRODUCTION TO RIGID SUPERSYMMETRIC THEORIES

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PREFACE

In these lectures we discuss the supersymmetry algebra and its irreducible representations. We construct the theories of extended rigid supersymmetry and give their superspace formulations. The perturbative quantum properties of the extended supersymmetric theories are derived, including the superconformal invariance of a large class of these theories. The superconformal transformations in four dimensional superspace are derived and encoded into one superconformal Killing superfield. It is shown that the anomalous dimensions of chiral operators in a superconformal quantum field are related to their R weight.

Some of this material follows the book[1] by the author. Certain chapters of this book are reproduced here, however, in other sections the reader is referred to the relevant parts of Ref.[1].

In this review the section on superconformal theories and two subsections on flat directions and non-holomorphicity are new material.

The aim of the lectures is to provide the reader with the material required to understand more recent developments in the non-perturbative properties of quantum extended supersymmetric theories.

THE SUPERSYMMETRY ALGEBRA

This section is identical to chapter 2 of Ref.[1]. The equation numbers are kept the same as in this book. I thank World Scientific Publishing for their permission to reproduce this material.

In the 1960's, with the growing awareness of the significance of internal symmetries such as $SU(2)$ and larger groups, physicists attempted to find a symmetry which would combine in a non-trivial way the space-time Poincaré group with an internal symmetry group. After much effort it was shown that such an attempt was impossible within the context of a Lie group. Coleman and Mandula[2] showed on very general assumptions that any Lie group which contained the Poincaré group P, whose generators P_a and

J_{ab} satisfy the relations

$$\begin{aligned}[P_a, P_b] &= 0 \\ [P_a, J_{bc}] &= (\eta_{ab}P_c - \eta_{ac}P_b) \\ [J_{ab}, J_{cd}] &= -(\eta_{ac}J_{bd} + \eta_{bd}J_{ac} - \eta_{ad}J_{bc} - \eta_{bc}J_{ad})\end{aligned} \tag{2.1}$$

and an internal symmetry group G with generators T_s such that

$$[T_r, T_s] = f_{rst}T_t \tag{2.2}$$

must be a direct product of P and G; or in other words

$$[P_a, T_s] = 0 = [J_{ab}, T_s] \tag{2.3}$$

They also showed that G must be of the form of a semisimple group with additional $U(1)$ groups.

It is worthwhile to make some remarks concerning the status of this no-go theorem. Clearly there are Lie groups that contain the Poincaré group and internal symmetry groups in a non-trivial manner; however, the theorem states that these groups lead to trivial physics. Consider, for example, two-body scattering; once we have imposed conservation of angular momentum and momentum the scattering angle is the only unknown quantity. If there were a Lie group that had a non-trivial mixing with the Poincaré group then there would be further gnerators associated with space-time. The resulting conservation laws will further constrain, for example, two-body scattering, and so the scattering angle can only take on discrete values. However, the scattering process is expected to be analytic in the scattering angle, θ, and hence we must conclude that the process does not depend on θ at all.

Essentially the theorem shows that if one used a Lie group that contained an internal group which mixed in a non-trivial manner with the Poincaré group then the S-matrix for all processes would be zero. The theorem assumes among other things, that the S-matrix exists and is non-trivial, the vacuum is non-degenerate and that there are no massless particles. It is important to realise that the theorem only applies to symmetries that act on S-matrix elements and not on all the other many symmetries that occur in quantum field theory. Indeed it is not uncommon to find examples of the latter symmetries. Of course, no-go theorems are only as strong as the assumptions required to prove them.

In a remarkable paper Gelfand and Likhtman[3] showed that provided one generalised the concept of a Lie group one could indeed find a symmetry that included the Poincaré group and an internal symmetry group in a non-trivial way. In this section we will discuss this approach to the supersymmetry group; having adopted a more general notion of a group, we will show that one is led, with the aid of the Coleman-Mandula theorem, and a few assumptions, to the known supersymmetry group. Since the structure of a Lie group, at least in some local region of the identity, is determined entirely by its Lie algebra it is necessary to adopt a more general notion than a Lie algebra. The vital step in discovering the supersymmetry algebra is to introduce generators Q^i_α, which satisfy anti-commutation relations, i.e.

$$\begin{aligned}\{Q^i_\alpha, Q^j_\beta\} &= Q^i_\alpha Q^j_\beta + Q^j_\beta Q^i_\alpha \\ &= \text{some other generator}\end{aligned} \tag{2.4}$$

The significance of the i and α indices will become apparent shortly. Let us therefore assume that the supersymmetry group involves generators P_a, J_{ab}, T_s and possibly some other generators which satisfy commutation relations, as well as the generators Q^i_α $(i = 1, 2, \dots, N)$. We will call the former generators which satisfy Eqs. (2.1), (2.2) and (2.3) to be even and those satisfying Eq. (2.4) to be odd generators.

Having let the genie out of the bottle we promptly replace the stopper and demand that the supersymmetry algebras have a Z_2 graded structure. This simply means that the even and odd generators must satisfy the rules:

$$\begin{aligned}
[\text{even}, \text{even}] &= \text{even} \\
\{\text{odd}, \text{odd}\} &= \text{even} \\
[\text{even}, \text{odd}] &= \text{odd}
\end{aligned} \tag{2.5}$$

We must still have the relations

$$[P_a, T_s] = 0 = [J_{ab}, T_s] \tag{2.6}$$

since the even (bosonic) subgroup must obey the Coleman-Mandula theorem.

Let us now investigate the commutator between J_{ab} and Q^i_α. As a result of Eq.(2.5) it must be of the form

$$[Q^i_\alpha, J_{ab}] = (b_{ab})_\alpha^{\ \beta} Q^i_\beta \tag{2.7}$$

since by definition the Q^i_α are the only odd generators. We take the α indices to be those rotated by J_{ab}. As in a Lie algebra we have some generalised Jacobi identities. If we denote an even generator by B and an odd generator by F we find that

$$\begin{aligned}
[[B_1, B_2], B_3] + [[B_3, B_1], B_2] + [[B_2, B_3], B_1] &= 0 \\
[[B_1, B_2], F_3] + [[F_3, B_1], B_2] + [[B_2, F_3], B_1] &= 0 \\
\{[B_1, F_2], F_3\} + \{[B_1, F_3], F_2\} + [\{F_2, F_3\}, B_1] &= 0 \\
[\{F_1, F_2\}, F_3] + [\{F_1, F_3\}, F_2] + [\{F_2, F_3\}, F_1] &= 0
\end{aligned} \tag{2.8}$$

The reader may verify, by expanding each bracket, that these relations are indeed identically true.

The identity

$$[[J_{ab}, J_{cd}], Q^i_\alpha] + [[Q^i_\alpha, J_{ab}], J_{cd}] + [[J_{cd}, Q^i_\alpha], J_{ab}] = 0 \tag{2.9}$$

upon use of Eq. (2.7) implies that

$$[b_{ab}, b_{cd}]_\alpha^{\ \beta} = -\eta_{ac}(b_{bd})_\alpha^{\ \beta} - \eta_{bd}(b_{ac})_\alpha^{\ \beta} + \eta_{ad}(b_{bc})_\alpha^{\ \beta} + \eta_{bc}(b_{ad})_\alpha^{\ \beta} \tag{2.10}$$

This means that the $(b_{cd})_\alpha^{\ \beta}$ form a representation of the Lorentz algebra or in other words the Q^i_α carry a representation of the Lorentz group. We will select Q^i_α to be in the $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$ representation of the Lorentz group, i.e.

$$[Q^i_\alpha, J_{ab}] = \frac{1}{2}(\sigma_{ab})_\alpha^{\ \beta} Q^i_\beta \tag{2.11}$$

We can choose Q^i_α to be a Majorana spinor, i.e.

$$Q^i_\alpha = C_{\alpha\beta}\bar{Q}^{\beta i} \tag{2.12}$$

where $C_{\alpha\beta} = -C_{\beta\alpha}$ is the charge conjugation matrix (see Appendix A of Ref.[1]). This does not represent a loss of generality since, if the algebra admits complex conjugation as an involution we can always redefine the supercharges so as to satisfy (2.12) (see Note 1 at the end of this section).

The above calculation reflects the more general result that the Q^i_α must belong to a realization of the even (bosonic) subalgebras of the supersymmetry group. This is a simple consequence of demanding that the algebra be Z_2 graded. The commutator of any even generator B_1, with Q^i_α is of the form

$$[Q^i_\alpha, B_1] = (h_1)^{i\beta}_{\alpha j} Q^j_\beta \tag{2.13}$$

The generalised Jacobi identity

$$[[Q^i_\alpha, B_1], B_2] + [[B_1, B_2], Q^i_\alpha] + [[B_2, Q^i_\alpha], B_1] = 0 \tag{2.14}$$

implies that

$$[h_1, h_2]^{i\beta}_{\alpha j} Q^j_\beta = [Q^i_\alpha [B_1, B_2]] \tag{2.15}$$

or in other words the matrices h represent the Lie algebra of the even generators.

The above remarks imply that

$$[Q^i_\alpha, T_r] = (l_r)^i_j Q^j_\alpha + (t_r)^i_j (i\gamma_5)^\beta_\alpha Q^j_\beta \tag{2.16}$$

where $(l_r)^i_j + i\gamma_5 (t_r)^i_j$ represent the Lie algebra of the internal symmetry group. This results from the fact that δ^α_β and $(\gamma_5)^\beta_\alpha$ are the only invariant tensors which are scalar and pseudoscalar.

The remaining odd-even commutator is $[Q^i_\alpha, P_a]$. A possibility that is allowed by the generalised Jacobi identities that involve the internal symmetry group and the Lorentz group is

$$[Q^i_\alpha, P_a] = c(\gamma_a)^\beta_\alpha Q^i_\beta \tag{2.17}$$

However, the $[[Q^i_\alpha, P_a], P_b] + \ldots$ identity implies that the constant $c = 0$, i.e.

$$[Q^i_\alpha, P_a] = 0 \tag{2.18}$$

More generally we could have considered $(c\gamma_a + d\gamma_a\gamma_5)Q$, on the right-hand side of (2.17), however, then the above Jacobi identity and the Majorana condition imply that $c = d = 0$. (See Note 2 at the end of this section). Let us finally consider the $\{Q^i_\alpha, Q^j_\beta\}$ anticommutator. This object must be composed of even generators and must be symmetric under interchange of $\alpha \leftrightarrow \beta$ and $i \leftrightarrow j$. The even generators are those of the Poincaré group, the internal symmetry group and other even generators which, from the Coleman-Mandula theorem, commute with the Poincaré group, i.e. they are scalar and pseudoscalar. Hence the most general possibility is of the form

$$\{Q^i_\alpha, Q^j_\beta\} = r(\gamma^a C)_{\alpha\beta} P_a \delta^{ij} + s(\sigma^{ab} C)_{\alpha\beta} J_{ab} \delta^{ij} + C_{\alpha\beta} U^{ij} + (\gamma_5 C)_{\alpha\beta} V^{ij} \tag{2.19}$$

We have not included a $(\gamma^b\gamma_5 C)_{\alpha\beta} L^{ij}_b$ term as the (Q, Q, J_{ab}) Jacobi identity implies that L^{ij}_b mixes nontrivially with the Poincaré group and so is excluded by the no-go theorem.

The fact that we have only used numerically invariant tensors under the Poincaré group is a consequence of the generalised Jacobi identities between two odd and one even generators.

To illustrate the argument more clearly, let us temporarily specialise to the case $N = 1$ where there is only one supercharge Q_α. Equation (2.19) then reads

$$\{Q_\alpha, Q_\beta\} = r(\gamma^a C)_{\alpha\beta} P_a + s(\sigma^{ab} C)_{\alpha\beta} J_{ab}.$$

Using the Jacobi identity

$$\{[P_a, Q_\alpha], Q_\beta\} + \{[P_a, Q_\beta], Q_\alpha\} + [\{Q_\alpha, Q_\beta\}, P_a] = 0,$$

we find that

$$0 = s(\sigma^{cd}C)_{\alpha\beta}[J_{cd}, P_a] = s(\sigma^{cd}C)_{\alpha\beta}(-\eta_{ac}P_d + \eta_{ad}P_c),$$

and, consequently, $s = 0$. We are free to scale the generator P_a in order to bring $r = 2$.

Let us now consider the commutator of the generator of the internal group and the supercharge. For only one supercharge, Eq. (2.16) reduces to

$$[Q_\alpha, T_r] = l_r Q_\alpha + i(\gamma_5)_\alpha^{\ \beta} t_r Q_\beta.$$

Taking the adjoint of this equation, multiplying by $(i\gamma^0)$ and using the definition of the Dirac conjugate given in Appendix A of Ref.[1], we find that

$$[\bar{Q}^\alpha, T_r] = l_r^* \bar{Q}^\alpha + \bar{Q}^\beta (it_r^*)(\gamma_5)_\beta^{\ \alpha}.$$

Multiplying by $C_{\gamma\alpha}$ and using Eq. (2.12), we arrive at the equation

$$[Q_\alpha, T_r] = l_r^* Q_\alpha + it_r^*(\gamma_5)_\alpha^{\ \beta} Q_\beta.$$

Comparing this equation with the one we started from, we therefore conclude that

$$l_r^* = l_r, \qquad t_r^* = t_r.$$

The Jacobi identity

$$[\{Q_\alpha, Q_\beta\}, T_r] + [[T_r, Q_\alpha], Q_\beta] + [[T_r, Q_\beta], Q_\alpha] = 0$$

results in the equation

$$\begin{aligned}[0 + (l_r\delta_\alpha^\gamma + it_r(\gamma_5)\alpha^\gamma)2(\gamma_a C)_{\gamma\beta}P_a] + (\alpha \ \leftrightarrow \ \beta) &= \\ 2P_a\{l_r(\gamma_a C)_{\alpha\beta} + it_r(\gamma_5\gamma_a C)_{\alpha\beta}\} + (\alpha \ \leftrightarrow \ \beta) &= 0.\end{aligned}$$

Since $(\gamma_a C)_{\alpha\beta}$ and $(\gamma_5\gamma_a C)_{\alpha\beta}$ are symmetric and antisymmetric in $\alpha\beta$ respectively, we conclude that $l_r = 0$ but t_r has no constant placed on it. Consequently, we find that we have only one internal generator R and we may scale it such that

$$[Q_\alpha, R] = i(\gamma_5)_\alpha^{\ \beta} Q_\beta.$$

The $N = 1$ supersymmetry algebra is summarised in Eq. (2.27).

Let us now return to the extended supersymmetry algebra. The even generators $U^{ij} = -U^{ji}$ and $V^{ij} = -V^{ji}$ are called central charges[4] and are often also denoted by Z. It is a consequence of the generalised Jacobi identities ((Q, Q, Q) and (Q, Q, Z)) that they commute with all other generators including themselves, i.e.

$$[U^{ij}, \text{anything}] = 0 = [V^{ij}, \text{anything}] \tag{2.20}$$

We note that the Coleman-Mandula theorem allowed a semi-simple group plus $U(1)$ factors. The details of the calculation are given in note 5 at the end of the section. Their role in supersymmetric theories will emerge in later sections.

In general, we should write, on the right-hand side of (2.19), $(\gamma^a C)_{\alpha\beta}\omega^{ij}P_a + \ldots$, where ω^{ij} is an arbitrary real symmetric matrix. However, one can show that it is possible to redefine (rotate and rescale) the supercharges, whilst preserving the Majorana condition, in such a way as to bring ω^{ij} to the form $\omega^{ij} = r\delta^{ij}$ (see Note 3 at the end of this section). The $[P_a, \{Q_\alpha^i, Q_\beta^j\}] + \ldots = 0$ identity implies that $s = 0$ and we can normalise P_a by setting $r = 2$ yielding the final result

$$\{Q_\alpha^i, Q_\beta^j\} = 2(\gamma_a C)_{\alpha\beta}\delta^{ij}P^a + C_{\alpha\beta}U^{ij} + (\gamma_5 C)_{\alpha\beta}V^{ij} \tag{2.21}$$

In any case r and s have different dimensions and so it would require the introduction of a dimensional parameter in order that they were both non-zero.

Had we chosen another irreducible Lorentz representation for Q^i_α, one other then $(j+\frac{1}{2}, j) \oplus (j, j+\frac{1}{2})$, we would not have been able to put P_a, i.e. a $(\frac{1}{2}, \frac{1}{2})$ representation, on the right-hand side of Eq. (2.21). The simplest choice is $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$. In fact this is the only possible choice (see Note 4).

Finally, we must discuss the constraints placed on the internal symmetry group by the generalised Jacobi identity. This discussion is complicated by the particular way the Majorana constraint of Eq. (2.12) is written. A two-component version of this constraint is

$$\bar{Q}_{\dot{A}i} = (Q^i_A)^*; \quad A, \dot{A} = 1, 2 \tag{2.22}$$

(see Appendix A of Ref.[1] for two-component notation). Equation (2.19) and (2.16) then become

$$\begin{aligned} \{Q^i_A, \bar{Q}_{\dot{B}j}\} &= -2i(\sigma^a)_{A\dot{B}}\delta^i_j P_a \\ \{Q^i_A, Q^j_B\} &= \varepsilon_{AB}(U^{ij} + iV^{ij}) \\ [Q^i_A, J_{ab}] &= +\frac{1}{2}(\sigma_{ab})_A^{\ B} Q^i_B \end{aligned} \tag{2.23}$$

and

$$[Q^i_A, T_r] = (l_r + it_r)^i_j Q^j_A \tag{2.24}$$

Taking the complex conjugate of the last equation and using the Majorana condition we find that

$$[Q_{\dot{A}l}, T_r] = Q_{\dot{A}k}(U^\dagger_r)^k_l \tag{2.25}$$

where $(U_r)^i_j = (l_r + it_r)^i_j$. The $(Q, \bar{Q}, T)$ Jacobi identity then implies that δ^i_j be an invariant tensor of G, i.e.

$$U_r + U^\dagger_r = 0 \tag{2.26}$$

Hence U_r is an antihermitian matrix and so represents the generators of the unitary group $U(N)$. However, taking account of the central charge terms in the (Q, Q, T) Jacobi identity one finds that there is for every central charge an invariant antisymmetric tensor of the internal group and so the possible internal symmetry group is further reduced. If there is only one central charge, the internal group is $Sp(N)$ while if there are no central charges it is $U(N)$.

To summarise, once we have adopted the rule that the algebra be Z_2 graded and contain the Poincaré group and an internal symmetry group then the generalised Jacobi identities place very strong constraints on any possible algebra. In fact, once one makes the further assumption that Q^i_α are spinors under the Lorentz group then the algebra is determined to be of the form of equations (2.1), (2.6), (2.11), (2.16), (2.18) and (2.21).

The simplest algebra is for $N = 1$ and takes the form

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= 2(\gamma_a C)_{\alpha\beta} P^a \\ [Q_\alpha, P_a] &= 0 \\ [Q_\alpha, J_{cd}] &= \frac{1}{2}(\sigma_{cd})_\alpha^{\ \beta} Q_\beta \\ [Q_\alpha, R] &= i(\gamma_5)_\alpha^{\ \beta} Q_\beta \end{aligned} \tag{2.27}$$

as well as the commutation relations of the Poincaré group. We note that there are no central charges (i.e. $U^{11} = V^{11} = 0$), and the internal symmetry group becomes just a chiral rotation with generator R.

We now wish to prove three of the statements above. This is done here rather than in the above text, in order that the main line of argument should not become obscured by technical points. These points are best clarified in two-component notation.

Note 1:

Suppose we have an algebra that admits a complex conjugation as an involution; for the supercharges this means that

$$(Q^i_A)^* = b_i{}^j Q_{\dot{A}j}; \quad (Q_{\dot{A}j})^* = d^j{}_k Q^k_A$$

There is no mixing of the Lorentz indices since $(Q^i_A)^*$ transforms like $Q_{\dot{A}i}$, namely in the $(0, \frac{1}{2})$ representation of the Lorentz group, and not like Q^i_A which is in the $(\frac{1}{2}, 0)$ representation. The lowering of the i index under $*$ is at this point purely a notational device. Two successive $*$ operations yield the unit operation and this implies that

$$(b_i{}^j)^* d^j{}_k = \delta^i_k \tag{2.28}$$

and in particular that $b_i{}^j$ is an invertible matrix. We now make the redefinitions

$$\begin{aligned} Q'^i_A &= Q^i_A \\ Q'_{\dot{A}i} &= b_i{}^j Q_{\dot{A}j} \end{aligned} \tag{2.29}$$

Taking the complex conjugate of Q'_{Ai}, we find

$$(Q'^i_A)^* = (Q^i_A)^* = b_i{}^j Q_{\dot{A}j} = Q'_{\dot{A}j}$$

while

$$(Q'_{\dot{A}i})^* = (b_i{}^j)^* (Q_{\dot{A}j})^* = (b_i{}^j)^* d^j{}_k Q^k_A = Q^i_A \tag{2.30}$$

using Eq. (2.28).

Thus the Q'^i_A satisfy the Majorana condition, as required. If the Q's do not initially satisfy the Majorana condition, we may simply redefine them so that they do.

Note 2:

Suppose the $[Q_A, P_a]$ commutator were of the form

$$[Q_A, P_a] = e(\sigma_a)_{A\dot{B}} Q^{\dot{B}} \tag{2.31}$$

where e is a complex number and for simplicity we have suppressed the i index. Taking the complex conjugate (see Appendix A of Ref.[1]), we find that

$$[Q_{\dot{A}}, P_a] = -e^*(\sigma_a)_{B\dot{A}} Q^B \tag{2.32}$$

Consideration of the $[[Q_A, P_a], P_b] + \ldots = 0$ Jacobi identity yields the result

$$-|e|^2 (\sigma_a)_{A\dot{B}} (\sigma^b)^{C\dot{B}} - (a \leftrightarrow b) = 0 \tag{2.33}$$

Consequently $e = 0$ and we recover the result

$$[Q_A, P_a] = 0 \tag{2.34}$$

Note 3:

The most general form of the $Q^{Ai}, Q^{\dot{B}}_j$ anticommutator is

$$\{Q^{Ai}, Q^{\dot{B}}_j\} = -2iU^i_j(\sigma^m)^{A\dot{B}} P_m + \text{terms involving other Dirac matrices} \tag{2.35}$$

Taking the complex conjugate of this equation and comparing it with itself, we find that U is a Hermitian matrix

$$(U^i_j)^* = U^j_i \tag{2.36}$$

We now make a field redefinition of the supercharge

$$Q'^{Ai} = B^i_j Q^{Aj} \tag{2.37}$$

and its complex conjugate

$$Q'^{\dot{A}}_i = (B^i_j)^* Q^{\dot{A}}_j \tag{2.38}$$

Upon making this redefinition in Eq. (2.35), the U matrix becomes replaced by

$$U'^i_j = B^i_k U^k_l (B^j_l)^* \quad \text{or} \quad U' = BUB^\dagger \tag{2.39}$$

Since U is a Hermitian matrix, we may diagonalise it in the form $c_i \delta^i_j$ using a unitarity matrix B. We note that this preserves the Majorana condition on Q^{Ai}. Finally, we may scale $Q^i \to (1/\sqrt{c}^i)Q^i$ to bring U to the form $U = d_i \delta^i_j$, where $d_i = \pm 1$. In fact, taking $A = B = 1$ and $i = j = k$, we realise that the right-hand side of Eq. (2.35) is a positive definite operator and since the energy $-iP_0$ is assumed positive definite, we can only find $d_i = +1$. The final result is

$$\{Q^{Ai}, Q^{\dot{B}}_j\} = -2i\delta^i_j (\sigma^m)^{A\dot{B}} P_m \tag{2.40}$$

Note 4:

Let us suppose that the supercharge Q contains an irreducible representation of the Lorentz group other than $(0,\frac{1}{2}) \oplus (\frac{1}{2},0)$, say, the representation $Q_{A_1 \ldots A_n, \dot{B}_1 \ldots \dot{B}_m}$ where the A and B indices are understood to be separately symmetrised and $n+m$ is odd in order that Q is odd and $n+m > 1$. By projecting the $\{Q, Q^\dagger\}$ anticommutator we may find the anti-commutator involving $Q_{A_1 \ldots A_n, \dot{B}_1 \ldots \dot{B}_m}$ and its hermitian conjugate. Let us consider in particular the anticommutator involving $Q = Q_{11\ldots 1, \dot{1}\dot{1}\ldots\dot{1}}$, this must result in an object of spin $n+m > 1$. However, by the Coleman-Mandula no-go theorem no such generator can occur in the algebra and so the anticommutator must vanish, i.e. $QQ^\dagger + Q^\dagger Q = 0$.

Assuming the space on which Q acts has a positive definite norm, one such example being the space of on-shell states, we must conclude that Q vanishes. However if $Q_{11\ldots 1, \dot{1}\dot{1}\ldots\dot{1}}$ vanishes, so must $Q_{A_1 \ldots A_n, \dot{B}_1 \ldots \dot{B}_m}$ by its Lorentz properties, and we are left only with the $(0,\frac{1}{2}) \oplus (\frac{1}{2},0)$ representation.

Note 5:

We now return to the proof of equation (2.20). Using the (Q,Q,Z) Jacobi identity it is straightforward to show that the supercharges Q commutes with the central charges Z. The (Q,Q,U) Jacobi identity then implies that the central charges commute with themselves. Finally, one considers the (Q, Q, T_r) Jacobi identity. this relation shows that the commutator of T_r and Z takes the generic form $[T_r,\ Z] = \ldots Z$. However, the generators T_r and Z form the internal symmetry group of the supersymmetry algebra and from the no-go theorem we know that this group must be a semisimple Lie group times $U(1)$ factors. We recall that a semisimple Lie group is one that has no normal Abelian subgoups other that the group itself and the identity element. As such, we must conclude that T_r and Z commute, and hence our final result that the central charges commute with all generator, that is they really are central.

Although the above discussion started with the Poincaré group, one could equally well have started with the conformal or (anti-)de Sitter groups and obtained the superconformal and super (anti-)de Sitter algebras. For completeness, we now list these algebras. The superconformal algebra which has the generators $P_n, J_{mn}, D, K_n, A, Q^{\alpha i}$, $S^{\alpha i}$ and the internal symmetry generators T_r and A is given by the Lorentz group plus:

$$
\begin{aligned}
[J_{mn}, P_k] &= \eta_{nk}P_m - \eta_{mk}P_n \\
[J_{mn}, K_k] &= \eta_{nk}K_m - \eta_{mk}K_n \\
[D, P_K] &= -P_K \quad [D, K_K] = +K_K \\
[P_m, K_n] &= -2J_{mn} + 2\eta_{mn}D \quad [K_n, K_m] = 0 \quad [P_n, P_m] = 0 \\
[Q^i_\alpha, J_{mn}] &= \frac{1}{2}(\gamma_{mn})_\alpha^\beta Q^i_\beta \quad [S^i_\alpha, J_{mn}] = \frac{1}{2}(\gamma_{mn})_\alpha^\beta S^{\beta i} \\
\{Q^i_\alpha, Q^j_\beta\} &= -2(\gamma^n C^{-1})_{\alpha\beta}P_n\delta^{ij} \\
\{S^i_\alpha, S^j_\beta\} &= +2(\gamma^n C^{-1})_{\alpha\beta}K_n\delta^{ij} \\
[Q^i_\alpha, D] &= \frac{1}{2}Q^i_\alpha \quad [S^i_\alpha, D] = -\frac{1}{2}S^i_\alpha \\
[Q^i_\alpha, K_n] &= -(\gamma_n)_\alpha^\beta S^i_\beta \quad [S^i_\alpha, P_n] = (\gamma_n)_\alpha^\beta Q^i_\beta \\
[Q^i_\alpha, T_r] &= (\delta_\alpha^\beta(\tau_{r_1})^i_j + (\gamma_5)_\alpha^\beta(\tau_{r_2})^i_j)Q^j_\beta \\
[S^i_\alpha, T_r] &= (\delta_\alpha^\beta(\tau_{r_1})^i_j - (\gamma_5)_\alpha^\beta(\tau_{r_2})^i_j)Q^j_\beta \\
[Q^i_\alpha, A] &= -i(\gamma_5)_\alpha^\beta Q^i_\beta\left(\frac{4-N}{4N}\right) \\
[S^i_\alpha, A] &= \frac{4-N}{4N}i(\gamma_5)_\alpha^\beta S^i_\beta \\
\{Q^i_\alpha, S^j_\beta\} &= -2(C^{-1})_{\alpha\beta}D\delta^{ij} + (\gamma^{mn}C^{-1})_{\alpha\beta}J_{mn}\delta^{ij} + 4i(\gamma_5 C^{-1})_{\alpha\beta}A\delta^{ij} \\
&\quad -2((\tau_{r_1})^{ij}(C^{-1})_{\alpha\beta} + (\tau_{r_2})^{ij}(\gamma_5 C^{-1})_{\alpha\beta})T_r
\end{aligned}
\tag{2.41}
$$

The T_r and A generate $U(N)$ and $\tau_1 + \gamma_5\tau_2$ are in the fundamental representation of $SU(N)$.

The case of $N = 4$ is singular and one can have either

$$[Q^i_\alpha, A] = 0 \quad \text{or} \quad [Q^i_\alpha, A] = -i(\gamma_5)_\alpha^\beta Q^i_\beta$$

and similarly for S^i_α and A. One may verify that both possibilities are allowed by the $N = 4$ Jacobi identities and so form acceptable superalgebras.

The anti-de Sitter superalgebra has generators M_{mn}, $T_{ij} = -T_{ji}$ and $Q^{\alpha i}$, and is given by

$$
\begin{aligned}
[M_{mn}, M_{pq}] &= \eta_{np}M_{mq} + 3 \text{ terms} \\
[M_{mn}, T_{ij}] &= 0 \quad [Q^i_\alpha, M_{mn}] = \frac{1}{2}(\gamma_{mn})_\alpha{}^\beta Q^i_\beta \\
[Q^i_\alpha, T^{jk}] &= -2i(\delta^{ij}Q^k_\alpha - \delta^{ik}Q^j_\alpha) \\
\{Q^i_\alpha, Q^j_\beta\} &= \delta^{ij}(\gamma_{mn}C^{-1})_{\alpha\beta}iM_{mn} + (C^{-1})_{\alpha\beta}T^{ij} \\
[T^{ij}, T^{kl}] &= -2i(\delta^{jk}T^{il} + 3 \text{ terms})
\end{aligned}
\tag{2.42}
$$

MODELS OF RIGID SUPERSYMMETTY

The Wess-Zumino Model

This section is identical to chapter 5 of Ref.[1]. The equation numbers are kept the same as in this book. I thank World Scientific Publishing for their permission to reproduce this material.

The first four-dimensional model in which supersymmetry was linearly realised was found by Wess and Zumino[5] by studying two-dimensional dual models[6]. In this section we discuss the Wess-Zumino model which is the simplest model of $N = 1$ supersymmetry.

Let us assume that the simplest model possesses one fermion χ_α which is a Majorana spinor, i.e.

$$\chi_\alpha = C_{\alpha\beta}\bar{\chi}^\beta \tag{5.1}$$

On shell, that is, when

$$\not{\partial}\chi = \text{ interaction} \tag{5.2}$$

χ_α has two degrees of freedom or two helicity states. Applying the rule concerning equal numbers of fermionic and bosonic degrees of freedom to the on-shell states we find that we must add two bosonic degrees of freedom to χ_α in order to form a realization of supersymmetry. These could either be two spin-zero particles or one massless vector particle which also has two helicity states on-shell. We will consider the former possibility in this section and the latter possibility, which is the $N = 1$ Yang-Mills theory, further on.

An irreducible representation of $N = 1$ supersymmetry can be carried either by one parity even spin-zero state, one parity odd spin-zero state and one Majorana spin-$\frac{1}{2}$, or by one massless spin-one and one Majorana spin-$\frac{1}{2}$. Taking the former possibility we have a Majorana spinor χ_α and two spin-zero states which we will assume to be represented by a scalar field A and pseudoscalar field B. For simplicity we will begin by constructing the free theory; the fields A, B, χ_α are then subject to

$$\partial^2 A = \partial^2 B = \not{\partial}\chi = 0 \tag{5.3}$$

We now wish to construct the supersymmetry transformations that are carried by this irreducible realization of supersymmetry. Since $\bar{\varepsilon}^\alpha Q_\alpha$ is dimensionless and Q_α has mass dimension $+\frac{1}{2}$, the parameter $\bar{\varepsilon}^\alpha$ must have dimension $-\frac{1}{2}$. On grounds of linearity, dimension, Lorentz invariance and parity we may write down the following set of transformations:

$$\begin{aligned} \delta A &= \bar{\varepsilon} Q A = \bar{\varepsilon}\chi, \qquad \delta B = i\bar{\varepsilon}\gamma_5\chi \\ \delta\chi &= \not{\partial}(\alpha A + \beta i\gamma_5 B)\varepsilon \end{aligned} \tag{5.4}$$

where α and β are undetermined parameters.

The variation of A is straightforward; however, the appearance of a derivative in $\delta\chi$ is the only way to match dimensions once the transformations are assumed to be linear. The reader will find no trouble verifying that these transformations do leave the set of field equations of Eq.(5.3) intact.

We can now test whether the $N = 1$ supersymmetry algebra of the previous section is represented by these transformations. The commutator of two supersymmetries on A is given by

$$[\delta_1, \delta_2]A = [\bar{\varepsilon}_1 Q, \bar{\varepsilon}_2 Q]A \tag{5.5}$$

which, using Eq.(2.27), becomes

$$\begin{aligned} [\delta_1, \delta_2]A &= 2\bar{\varepsilon}_2\gamma^a\varepsilon_1 P_a A \\ &= 2\bar{\varepsilon}_2\gamma^a\varepsilon_1 \partial_a A \\ \text{since} \qquad P_a &= \partial_a \end{aligned} \tag{5.6}$$

On the other hand the transformation laws of Eq.(5.4) imply that

$$\begin{aligned} [\delta_1, \delta_2]A &= \bar{\varepsilon}_2\not{\partial}(\alpha A + i\gamma_5\beta B)\varepsilon_1 - (1 \leftrightarrow 2) \\ &= 2\alpha\bar{\varepsilon}_2\not{\partial}\varepsilon_1 A \end{aligned} \tag{5.7}$$

The term involving B drops out because of the properties of Majorana spinors (see Appendix A of Ref.[1]). Provided $\alpha = +1$ this is indeed the 4-translation required by the algebra. We therefore set $\alpha = +1$. The calculation for B is similar and yields $\beta = +1$. For the field χ_α the commutator of two supersymmetries gives the result

$$
\begin{aligned}
[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\chi &= \partial\!\!\!/[\bar{\varepsilon}_1\chi + i\gamma_5\bar{\varepsilon}_1 i\gamma_5\chi]\varepsilon_2 - (1 \leftrightarrow 2) \\
&= -\frac{1}{4}\bar{\varepsilon}_1\gamma^R\varepsilon_2\partial\!\!\!/[\gamma_R + i\gamma_5\gamma_R i\gamma_5]\chi - (1 \leftrightarrow 2) \\
&= +\frac{1}{2}\bar{\varepsilon}_2\gamma^a\varepsilon_1 2\partial\!\!\!/\gamma_a\chi \\
&= 2\bar{\varepsilon}_2\partial\!\!\!/\varepsilon_1\chi - \bar{\varepsilon}_2\gamma^a\varepsilon_1\gamma_a\partial\!\!\!/\chi
\end{aligned}
\tag{5.8}
$$

The above calculation makes use of a Fierz rearrangement (see Appendix A of Ref.[1]) as well as the properties of Majorana spinors. However, χ_α is subject to its equation of motion, i.e. $\partial\!\!\!/\chi = 0$, implying the final result

$$
[\delta_1, \delta_2]\chi = 2\bar{\varepsilon}_2\partial\!\!\!/\varepsilon_1\chi \tag{5.9}
$$

which is the consequence dictated by the supersymmetry algebra. The reader will have no difficulty verifying that the fields A, B and χ_α and the transformations

$$
\begin{aligned}
&\delta A = \bar{\varepsilon}\chi, \quad \delta B = i\bar{\varepsilon}\gamma_5\chi \\
&\delta\chi = \partial\!\!\!/(A + i\gamma_5 B)\varepsilon
\end{aligned}
\tag{5.10}
$$

form a representation of the whole of the supersymmetry algebra *provided* A, B and χ_α are on-shell (i.e. $\partial^2 A = \partial^2 B = \partial\!\!\!/\chi = 0$).

We now wish to consider the fields A, B and χ_α when they are no longer subject to their field equations. The Lagrangian from which the above field equations follow is

$$
L = -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}\bar{\chi}\partial\!\!\!/\chi \tag{5.11}
$$

It is easy to prove that the action $\int d^4x L$ is indeed invariant under the transformation of Eq.(5.10). This invariance is achieved *without the use of the field equations.* The trouble with this formulation is that the fields A, B and χ_α do not form a realization of the supersymmetry algebra when they are no longer subject to their field equations, as the last term in Eq.(5.8) demonstrates. It will prove useful to introduce the following terminology. We shall refer to an irreducible representation of supersymmetry carried by fields which are subject to their equations of motion as an *on-shell representation.* We shall also refer to a Lagrangian as being *algebraically on-shell* when it is formed from fields which carry an on-shell representation, that is, do not carry a representation of supersymmetry off-shell, and the Lagrangian is invariant under these on-shell transformations. The Lagrangian of Eq.(5.11) is then an algebraically on-shell Lagrangian.

That A, B and χ_α cannot carry a representation of supersymmetry off-shell can be seen without any calculation, since these fields do not satisfy the rule of equal numbers of fermions and bosons. Off-shell, A and B have two degrees of freedom, but χ_α has four degrees of freedom. Clearly, the representations of supersymmetry must change radically when enlarged from on-shell to off-shell.

A possible way out of this dilemma would be to add two bosonic fields F and G which would restore the fermion-boson balance. However, these additional fields would have to occur in the Lagrangian so as to give rise to no on-shell states. As such, they must occur in the Lagrangian in the form $+\frac{1}{2}F^2 + \frac{1}{2}G^2$ assuming the free action to be only bilinear in the fields and consequently be of mass dimension two. On dimensional grounds their supersymmetry transformations must be of the form

$$
\delta F = \bar{\varepsilon}\partial\!\!\!/\chi \quad \delta G = i\bar{\varepsilon}\gamma_5\partial\!\!\!/\chi \tag{5.12}
$$

where we have tacitly assumed that F and G are scalar and pseudoscalar respectively. The fields F and G cannot occur in δA on dimensional grounds, but can occur in $\delta\chi_\alpha$ in the form

$$\delta\chi = [(\mu F + i\tau\gamma_5 G) + \not{\partial}(A + i\gamma_5 B)]\varepsilon \tag{5.13}$$

where μ and τ are undetermined parameters.

We note that we can only modify transformation laws in such a way that on-shell (i.e., when $F = G = \not{\partial}\chi = \partial^2 A = \partial^2 B = 0$) we regain the on-shell transformation laws of Eq.(5.10).

We must now test if these new transformations do form a realization of the supersymmetry algebra. In fact, straightforward calculation shows they do, provided $\mu = \tau = +1$. This representation of supersymmetry involving the fields A, B, χ_α, F and G was found by Wess and Zumino[5] and we now summarize their result:

$$\begin{aligned} &\delta A = \bar{\varepsilon}\chi \qquad \delta B = i\bar{\varepsilon}\gamma_5\chi \\ &\delta\chi = [F + i\gamma_5 G + \not{\partial}(A + i\gamma_5 B)]\varepsilon \\ &\delta F = \bar{\varepsilon}\not{\partial}\chi \qquad \delta G = i\bar{\varepsilon}\gamma_5\not{\partial}\chi \end{aligned} \tag{5.14}$$

The action which is invariant under these transformations, is given by the Lagrangian

$$A = \int d^4x \left\{ -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}\bar{\chi}\not{\partial}\chi + \frac{1}{2}F^2 + \frac{1}{2}G^2 \right\} \tag{5.15}$$

As expected the F and G fields occur as squares without derivatives and so lead to no on-shell states.

The above construction of the Wess-Zumino model is typical of that for a general free supersymmetric theory. We begin with the on-shell states, and construct the on-shell transformation laws. We can then find the Lagrangian which is invariant without use of the equations of motion, but contains no auxiliary fields. One then tries to find a set of auxiliary fields that give an off-shell algebra. Once this is done one can find a corresponding off-shell action. How one finds the nonlinear theory from the free theory is discussed in the later chapters of Ref.[1].

The first of these two steps is always possible; however, there is no sure way of finding auxiliary fields that are required in all models, except with a few rare exceptions. This fact is easily seen to be a consequence of our rule for equal numbers of fermi and bose degrees of freedom in any representation of supersymmetry. It is only spin 0's, when represented by scalars, that have the same number of field components off-shell as they have on-shell states. For example, a Majorana spin-$\frac{1}{2}$ when represented by a spinor χ_α has a jump of 2 degrees of freedom between on and off-shell and a massless spin-1 boson when represented by a vector A_μ has a jump of 1 degree of freedom. In the latter case it is important to subtract the one gauge degree of freedom from A_μ thus leaving 3 field components off-shell (see chapter 6 of Ref.[1]). Since the increase in the number of degrees of freedom from an on-shell state to the off-shell field representing it changes by different amounts for fermions and bosons, the fermionic-boson balance which holds on-shell will not hold off-shell if we only introduce the fields that describe the on-shell states. The discrepancy must be made up by fields, like F and G, that lead to no on-shell states. These latter type of fields are called *auxiliary fields.* The whole problem of finding representations of supersymmetry amounts to finding the auxiliary fields.

Unfortunately, it is not at all easy to find the auxiliary fields. Although the fermi-bose counting rule gives a guide to the number of auxiliary fields it does not actually tell you what they are, or how they transform. In fact, the auxiliary fields are only known for almost all $N = 1$ and 2 supersymmetry theories and for a very few $N = 4$ theories and not for the higher N theories. In particular, they are not known for the $N = 8$ supergravity theory.

Theories for which the auxiliary fields are not known can still be described by a Lagrangian in the same way as the Wess-Zumino theory can be described without the use of F and G, namely, by the so called algebraically on-shell Lagrangian formulation, which for the Wess-Zumino theory was given in Eq. (5.11). Such 'algebraically on-shell Lagrangians' are not too difficult to find at least at the linearized level. As explained in chapter 8 of Ref.[1] we can easily find the relevant on-shell states of the theory. The algebraically on-shell Lagrangian then consists of writing down the known kinetic terms for each spin.

Of course, we are really interested in the interacting theories. The form of the interactions is however often governed by symmetry principles such as gauge invariance in the above example or general coordinate invariance in the case of gravity theories. When the form of the interactions is dictated by a local symmetry there is a straightforward, although maybe very lengthy way of finding the nonlinear theory from the linear theory. This method, called Noether coupling, is described in chapter 7 of Ref.[1]. In one guise or another this technique has been used to construct nonlinear 'algebraically on-shell Lagrangians' for all supersymmetric theories.

The reader will now ask himself whether algebraically on-shell Lagrangians may be good enough. Do we really need the auxiliary fields? The following example is a warning against over-estimating the importance of a Lagrangian that is invariant under a set of transformations that mix fermi-bose fields, but do not obey any particular algebra.

Consider the Lagrangian

$$L = -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}\bar{\chi}\not{\partial}\chi \tag{5.16}$$

whose corresponding action is invariant under the transformations

$$\partial A = \bar{\varepsilon}\chi \quad \partial\chi = \not{\partial}A\varepsilon \tag{5.17}$$

However, this theory has nothing to do with supersymmetry. The algebra of transformations of Eq. (5.17) does not close on or off-shell without generating transformations which, although invariances of the free theory, can never be generalized to be invariances of an interacting theory. In fact, the on-shell states do not even have the correct fermi-bose balance required to form an irreducible representation of supersymmetry. This example illustrates the fact that the 'algebraically on-shell Lagrangians' rely for their validity, as supersymmetric theories, on their on-shell algebra.

As a final remark in this section it is worth pointing out that the problem of finding the representations of any group is a mathematical question not dependent on any dynamical considerations for its resolution. Thus the questions of which are physical fields and which are auxiliary fields is a model-dependent statement.

The $N = 1$ Yang-Mills Theory

The construction of the $N = 1$ Yang-Mills theory in x-space, presented during the lectures, follows closely chapter 6 of Ref.[1].

The Extended Theories

The $N = 2$ Yang-Mills theory and $N = 2$ matter, as well as their most general renormalizable coupling, were constructed along the lines of chapter 12 of Ref.[1].

THE IRREDUCIBLE REPRESENTATIONS OF SUPERSYMMETRY

Irreducible representations of supersymmetry were constructed using the method of induced representations. This provides a complete list of the possible supersymmetric theories in four dimensions. The material can be found in chapter 8 of Ref.[1].

SUPERSPACE

Construction of Superspace

Superspace was constructed as the coset space of the super-Poincare group divided by the Lorentz group. Details can be found in chapter 14 of Ref.[1].

Superspace Formulations of Rigid Supersymmetric Theories

The formulation of the Wess-Zumino model and $N = 1, 2$ Yang-Mills theories in superspace can be found in chapter 15 of Ref.[1].

QUANTUM PROPERTIES OF SUPERSYMMETRIC MODELS

Super-Feynman Rules and the Non-renormalisation Theorem

The super-Feynman rules of the Wess-Zumino model and $N = 1$ Yang-Mills theory were derived and the non-renormalisation theorem was proved. For details see chapter 17 of Ref.[1].

Flat Directions

The potential in a supersymmetric theory is given by the squares of the auxiliary fields. In this section we consider an $N = 1$ supersymmetric model which contains Wess-Zumino multiplets coupled to the $N = 1$ Yang-Mills multiplet with gauge group G. Let us denote the auxiliary fields of the Wess-Zumino multiplets by the complex field $\mathcal{F}^i$ where the index i labels the Wess-Zumino multiplets and those of the $N = 1$ Yang-Mills multiplet by D^a where $a = 1, \ldots,$ dimension of G. Then the classical potential is given by

$$V = \left|\mathcal{F}^i\right|^2 + \frac{1}{2}\sum_a (D^a)^2 \tag{5.2.1}$$

For a general $N = 1$ renormalizable theory the auxiliary fields are given by

$$\mathcal{F}^i = \frac{\partial W(z^j)}{\partial z^i} \tag{5.2.2}$$

and

$$D^a = -g\bar{z}_i (T^a)^i{}_j z^j + \zeta^a \tag{5.2.3}$$

In equation (5.2.2) W is the superpotential which we recall occurs in the superspace formulation of the theory as $(\int d^4x d^2\theta W + c.c)$ and z^i are the scalars of the Wess-Zumino multiplet. For a renormlizable theory, the superpotential has the form $W(z) = \frac{1}{3!}d_{ijk}z^i z^j z^k + \frac{1}{2!}m_{ij}z^i z^j + e_i z^i$. In equation (5.2.3) g is the gauge coupling constant and

$(T^a)^i_j$ are the generators of the group G to which these scalars z^i belong. The terms in the auxiliary fields which are independent of z^i can only occur when we have $U(1)$ factors for D^a and auxiliary fields $\mathcal{F}^i$ that transform trivial under G. The resulting ζ^a and e_i are constants.

Clearly, the potential is positive definite. Another remarkable feature of the potential is that it generically has flat directions. This means that minimizing the potential does not specify a unique field configuration. In other words there exists a vacuum degeneracy. The simplest example is for a Wess-Zumino model in the adjoint representation coupled to a $N = 1$ Yang-Mills multiplet. Taking the superpotential for this theory to vanish the potential is given by

$$V = \frac{1}{2}\sum_a \left(-gf_{abc}\bar{z}^b z^c\right)^2 \tag{5.2.4}$$

Clearly, the minimum is given by field configurations whose only non-zero vacuum expectation values are $< z^a > H^a$ where H^a are the Cartan generators of the algebra. This theory is precisely the $N = 2$ supersymmetric Yang-Mills theory when written in terms of $N = 1$ supermuliplets.

In a general quantum field theory such a vacuum degeneracy would be removed by quantum corrections to the potential. However, things are different in supersymmetric theories. In fact, if supersymmetry is not broken the potential does not receive any perturbative quantum corrections[7]. It obviously follows that if supersymmetry is not broken then the vacuum degeneracy is not removed by perturbative quantum corrections[7]. This result was first proved before the advent of the non-renormalisation theorem as formulated in Ref.[8], but it is particularly obvious given this theorem. For the effective potential we are interested in field configurations where the spinors vanish and the space-time derivatives of all fields are set to zero. For such configurations, the gauge invariant superfields do not contain any θ dependence as only their first component is non-zero. Quantum corrections, however, contain an integral over all of superspace and to be non-zero requires a $\theta^2\bar{\theta}^2$ factor in the integrand. For the field configurations of interest to us such an integral over the full superspace must vanish and as a result we find that there are no quantum corrections to the effective potential if supersymmetry is not broken.

Finally, we recall why the expectation values of the auxiliary fields vanish if supersymmetry is preserved. In this case the expectation value of the supersymmetry transformations of the spinors must vanish. The transformation of the spinors contain auxiliary fields which occur without space-time derivatives and the bosonic fields which correspond to the dynamical degrees of freedom of the theory. The latter occur with space-time derivative, as they have mass dimension one and ϵ has dimension $-\frac{1}{2}$. Consequently, if the expectation values of supersymmetry transformations of the spinors vanish so do the expectation values of all the auxiliary fields. By examining the supersymmetry transformations of the spinors given earlier the reader may verify that there are no loop holes in this argument.

Clearly, the rigid $N = 2$ and $N = 4$ theories can be written in terms of $N = 1$ supermultiplets and so the flat directions that occur in these theories are also not removed by quantum corrections. Although this might be viewed as a problem in these theories it has been turned to advantage in the work of Seiberg and Witten. These authors realized that the dependence of these theories on the expectation values of the scalar fields, or the moduli, obeyed interesting properties that can be exploited to solve for part of the effective action of these theories.

Non-holomorphicity

The non-renormalisation theorem states that perturbative quantum corrections to the effective action are of the form

$$\int d^4x_1 \ldots \int d^4x_n \int d^4\theta G(x_1,\ldots x_n) f(\varphi(x_1,\theta),\ldots,V(x_1,\theta),\ldots,D^A\varphi(x_1,\theta),\ldots) \tag{5.3.1}$$

where φ and V are the superfields that contain the Wess-Zumino and $N=1$ Yang-Mills fields respectively.

The most significant aspect of this result is that the corrections arise from a single superspace integral over **all** of superspace, that is they contain a $d^4\theta = d^2\theta d^2\bar{\theta}$ integral and not a sub-integral of the form $d^2\theta$ or $d^2\bar{\theta}$. Such sub-integrals play an important role in supersymmetric theories. For example, the superpotential in the superspace formulation of the Wess-Zumino model has the form $\int d^4x d^2\theta W + c.c.$

While their is no question that this formulation of the non-renormalisation theorem is correct, with the passing of time, it was taken by many workers to mean that their could never be any quantum corrections which were sub-integrals i.e. that is of the form

$$\int d^4x d^2\theta \hat{W}(\varphi) \tag{5.3.2}$$

In particular, it was often said that there could be no quantum corrections to the superpotential.

Consider, however, the expression

$$\int d^4x d^4\theta [(-\frac{D^2}{4\partial^2})\varphi^n] = \int d^4x d^2\theta (-\frac{\bar{D}^2}{4})(-\frac{D^2}{4\partial^2})\varphi^n = \int d^4x d^2\theta \varphi^n \tag{5.3.3}$$

where we have used the relation $\bar{D}^2 D^2 \psi = 16\partial^2\psi$ where ψ is any chiral superfield. This maneuver illustrates the important point that although an expression can be written as a full superspace integral, it can also be expressible as a local integral over only a subspace of superspace. The above expression when written in terms of the full superspace integral is non-local, however, any effective action contains many non-local contributions. The occurrence of the $\frac{1}{\partial^2}$ is the signal of a massless particle. For a massive particle one would instead find a factor of $\frac{1}{(\partial^2+m^2)}$ which can not be rewritten as a sub-integral. Hence, only when massless particles circulate in the quantum loops can we find a contribution to the effective action which can be written as a sub-superspace integral.

The first example of such a correction to the superpotential was found in Ref.[9]. In Ref.[10] it was shown that all the proofs of the non-renormalisation theorem allowed contributions to the effective action which were integrals over a subspace of superspace if massless particles were present. It was also shown[10] that such corrections were not some pathological exception, but that they generically occurred whenever massless particles were present. This lecture follows the first part of Ref.[10] and the reader is referred there for a much more complete discussion and several examples. In the Wess-Zumino model such corrections first occur at two loops and were calculated in[11], while in the Wess-Zumino model coupled to $N=1$ Yang-Mills theory the corrections occur even at one loop[12]. An alternative way of looking at such corrections was given in Refs.[13, 14].

The reader may wonder what such corrections have to do with non-holomorphicity. The answer is that the corrections we have been considering are non-holomorphic in the coupling constants. The situation is most easily illustrated in the context of the massless Wess-Zumino model where the superpotential is of the form $\int d^4x d^2\theta \lambda\varphi^3 + c.c.$ Since the propagator connects φ to $\bar{\varphi}$ we get no corrections at all if we do not include terms that contain both λ and $\bar{\lambda}$. Consequently, the corrections we find to the superpotential must contain λ and $\bar{\lambda}$ and so is non-holomorphic in λ.

We can of course prevent the occurrence of such terms if we give masses to all the particles or we do not integrate over the infra-red region of the loop momentum integration for the massless particles. Such is the case if we calculate the Wilsonian effective action. However, if the terms considered here affect the physics in an important way one will necessarily miss such effects and they will only become apparent when one carries out the integrations that one had previously excluded.

Perturbative Quantum Properties of Extended Theories of Supersymmetry

Many of the perturbative properties of the extended theories of supersymmetry were derived. These include the finiteness, or superconformal invariance, of the $N = 4$ Yang-Mills theory, the demonstration that $N = 2$ Yang-Mills theory coupled to $N = 2$ matter has a perturbative beta-function that only has one-loop contributions and the existence of a large class of superconformally invariant quantum $N = 2$ theories. Details can be found in chapter 18 of Ref.[1].

SUPERCONFORMAL THEORIES

The Geometry of Superconformal Transformations

The superspace that we used was defined as the coset space of the super-Poincare group divided by the Lorentz group and internal symmetry group, see chapter 14 of Ref.[1]. This superspace is called Minkowski superspace. For the case of the $N = 1$ super-Poincare group, the superspace is parameterised by the coordinates $Z^M = (x^\mu, \theta^A, \theta^{\dot{A}})$ corresponding to the generators P^μ and $Q^A, Q^{\dot{A}}$ which generate transformations that are not contained in the isotropy subgroup. We can construct on superspace a set of preferred frames with supervierbeins $E_M{}^\pi$. The covariant derivatives are given by $D_M = E_M^\pi \frac{\partial}{\partial Z^\pi}$. Their precise form being

$$D_A = \frac{\partial}{\partial \theta^A} - i(\sigma^m)_{A\dot{B}}\theta^{\dot{B}}\partial_m, \; D_{\dot{A}} = \frac{\partial}{\partial \theta^{\dot{A}}} - i(\sigma^m)_{B\dot{A}}\theta^B\partial_m, \tag{6.1}$$

and

$$D_m = \frac{\partial}{\partial x^m} \equiv \partial_m \tag{6.2}$$

We can read off the components of the inverse supervierbien from these equations.

For superconformal theories it is more natural to consider a superspace which is constructed from the coset space found by dividing the superconformal group by the subgroup which is generated by Lorentz transformations $J_{\mu\nu}$, dilations D, special translations K_μ and special supersymmetry transformations $S_{Ai}, S_{\dot{A}i}, i = 1, \ldots, N$ and the internal symmetry generators. The internal group for the superconformal algebra contains the group $U(N) = SU(N) \times U(1)$ although in the case of $N = 4$ the $U(1)$ factor does not act on the supercharges.

This coset construction leads to the same Minkowski superspace with the same transformations for the super-Poincare group, but it has the advantage that it automatically encodes the action of the superconformal transformations on the superspace. These transformations were first calculated by Martin Sohnius in Ref.[15].

The purpose of this section is to give an alternative method of calculating the superconformal transformations in four dimensions which will enable us to give a compact superspace form for the superconformal transformations. In particular, all the parameters of the transformations will be encode in one superfield which we can think of as the superspace equivalent of a conformal Killing vector. This formulation was first given by B. Conlong and P. West and can be found in Ref.[16]. One reason for reviewing this work here is that there is still not a readable account readily available in the literature. This section was written in collaboration with B. Conlong. Some reviews on this subject can be found in[17].

Conformal transformations in Minkowski space are defined to be those transformations which preserve the Minkowski metric up to scale (see chapter 25 of Ref.[1] for a review). However, superspace does not have a natural metric since the tangent space group contains the Lorentz group which does not relate the bosonic to the fermionic sectors of the tangent space. The treatment we now give follows that given in chapter 25 of Ref.[1] for the case of two dimensional superconformal transformations.

There are two methods to define a superconformal transformation.

1. We can demand that it is a superdiffeomorphism which preserves part of the bosonic part of the supersymmetric line element
$$dL^m dL^n \eta_{mn} \tag{6.3}$$
where
$$dL^m \equiv dZ^M E_M^{\ m} = dx^m + i\bar{\theta}\gamma^m d\theta \tag{6.4}$$
up to an arbitrary local scale factor.

2. We can alternatively demand that it is a superdiffeomorphism which preserves the spinor components of the superspace covariant derivatives up to an arbitrary local scale factor. More precisely, a superconformal transformation is one such that
$$D_A \mapsto D'_A = f_A{}^B(z) D_B, \quad \bar{D}_{\dot{A}} \mapsto \bar{D}'_{\dot{A}} = \bar{f}_{\dot{A}}{}^{\dot{B}}(z)\bar{D}_{\dot{B}} \tag{6.5}$$
We note that the transformation must preserve each chirality spinor derivative separately. In fact, these two definitions are equivalent and we will work with only the second definition.

Carrying out a super-reparameterisation $z^M = (x^\mu, \theta^A, \bar{\theta}^{\dot{A}}) \mapsto z^{M\prime} = (x^{\mu\prime}, \theta^{A\prime}, \bar{\theta}^{\dot{A}\prime})$ upon the spinorial covariant derivatives we find that a finite superconformal transformation obeys the constraints

$$D_A \bar{\theta}'^{\dot{B}} = 0, \quad D_A x'^\mu + i\sigma^\mu_{B\dot{B}} \bar{\theta}'^{\dot{B}} (D_A \theta'^B) = 0 \tag{6.6}$$

and
$$\bar{D}_{\dot{A}} \theta'^B = 0, \quad \bar{D}_{\dot{A}} x'^\mu + i\sigma^\mu_{B\dot{B}} \theta'^B (\bar{D}_{\dot{A}} \bar{\theta}'^{\dot{B}}) = 0 \tag{6.7}$$
The corresponding transformation of the covariant derivatives being

$$D_A = (D_A \theta'^B) D'_B, \quad \bar{D}_{\dot{A}} = (\bar{D}_{\dot{A}} \bar{\theta}'^{\dot{B}}) \bar{D}'_{\dot{B}} \tag{6.8}$$

We now consider an infinitesimal transformation

$$z^\pi \mapsto z'^\pi = z^\pi + G^\pi \tag{6.9}$$

where $G^\pi = (G^\mu, G^A, G^{\dot{A}})$ is a set of infinitesimal superfields. Equations (6.6) and (6.7) then become
$$D_A G^{\dot{A}} = 0, \quad D_A G^\mu + i\sigma^\mu_{A\dot{B}} G^{\dot{B}} + i\sigma^\mu_{B\dot{B}} \bar{\theta}^{\dot{B}} D_A G^B = 0 \tag{6.10}$$

and

$$\bar{D}_{\dot{A}}\bar{G}^{B} = 0, \quad \bar{D}_{\dot{A}}G^{\mu} + i\sigma^{\mu}_{A\dot{A}}G^{A} + i\sigma^{\mu}_{B\dot{B}}\theta^{B}\bar{D}_{\dot{A}}\bar{G}^{\dot{B}} = 0 \tag{6.11}$$

The vector field corresponding to such an infinitesimal transformation is given by $V = G^{\pi}\partial_{\pi}$. However, this can also be written as $V = F^{M}D_{M}$ where the change of basis corresponds to the relation $F^{M} = G^{\pi}E_{\pi}^{M}$. In terms of components this change is given by

$$F^{\mu} = G^{\mu} - i\sigma^{\mu}_{A\dot{A}}\bar{\theta}^{\dot{A}}G^{A} - i\sigma^{\mu}_{A\dot{A}}\theta^{A}G^{\dot{A}} \tag{6.12}$$

as well as

$$F^{A} = G^{A}, \ F^{\dot{A}} = G^{\dot{A}} \tag{6.13}$$

We shall denote the vector component by $F^{\mu}, \ldots$ or $F^{n}, \ldots$ even though it should strictly speaking carry the latter $m, n, \ldots$ indices. It is straightforward to verify that equations (6.10) and (6.11) now take the neater form

$$D_{A}F^{\mu} = -2i\sigma^{\mu}_{A\dot{A}}\bar{F}^{\dot{A}}, \ \bar{D}_{\dot{A}}F^{\mu} = -2i\sigma^{\mu}_{A\dot{A}}F^{A} \tag{6.14}$$

and

$$D_{\dot{A}}F^{B} = 0 = D_{A}F^{\dot{B}}. \tag{6.15}$$

A somewhat quicker derivation of this result can be given by first writing the infinitesimal change in the covariant derivatives under an infinitesimal superdiffeomorphism in the form $D_{M} \to D_{M} + [V, D_{M}]$. Using the form for V given above which contains the covariant derivatives and then using the fact that the only non-zero commutator or anti-commutator, where appropriate, of the covariant derivatives is

$$\{D_{A}, D_{\dot{B}}\} = -2i(\sigma^{n})_{A\dot{B}}\partial_{n} \tag{6.16}$$

we recover equations (6.14) and (6.15).

Equation (6.14) can be rewritten as

$$F^{\dot{A}} = -\frac{1}{8i}\sigma_{\mu}^{A\dot{A}}D_{A}F^{\mu}, \ F^{A} = -\frac{1}{8i}\sigma_{\mu}^{A\dot{A}}\bar{D}_{\dot{A}}F^{\mu} \tag{6.17}$$

from which it is apparent that all transformations may be expressed in terms of F^{μ} alone. Using equations (6.12) and (6.13) we find that the explicit transformations of the coordinates are given by

$$\begin{aligned}
x'^{\mu} &= x^{\mu} + (F^{\mu} - \frac{1}{8}\theta^{A}D_{A}F^{\mu} + \frac{1}{8}\theta^{A}(\sigma^{\mu}{}_{\nu})_{AC}D^{C}F^{\nu} - \frac{1}{8}\bar{\theta}^{\dot{A}}\bar{D}_{\dot{A}}F^{\mu} + \frac{1}{8}\bar{\theta}^{\dot{A}}(\bar{\sigma}^{\mu}{}_{\nu})_{\dot{A}\dot{C}}\bar{D}^{\dot{C}}F^{\nu}) \\
\theta'^{A} &= \theta^{A} - \frac{1}{8i}\sigma_{\mu}^{A\dot{A}}\bar{D}_{\dot{A}}F^{\mu} \\
\bar{\theta}'^{\dot{A}} &= \bar{\theta}^{\dot{A}} - \frac{1}{8i}\sigma_{\mu}^{A\dot{A}}D_{A}F^{\mu}
\end{aligned} \tag{6.18}$$

Let us define $F_{B\dot{B}} \equiv \sigma^{\mu}_{B\dot{B}}F_{\mu}$, whereupon equation (6.14) becomes

$$D_{A}F_{B\dot{B}} = -4i\varepsilon_{AB}F_{\dot{B}}, \ \ \bar{D}_{\dot{A}}F_{B\dot{B}} = -4i\varepsilon_{\dot{A}\dot{B}}F_{B} \tag{6.19}$$

from which we may deduce the constraint

$$D_{(A}F_{B)\dot{B}} = 0, \quad \bar{D}_{(\dot{A}}F_{|B|\dot{B})} = 0 \tag{6.20}$$

Acting with D_{C} on $D_{A}F_{B\dot{B}}$ and using equation (6.19) we conclude that

$$D_{C}D_{A}F_{B\dot{B}} = -4i\varepsilon_{AB}D_{C}F_{\dot{B}} = -D_{A}D_{C}F_{B\dot{B}} = -4i\varepsilon_{CB}D_{A}F_{\dot{B}} = 0. \tag{6.21}$$

The last step follows by tracing with ε^{AB}. Consequently, we find that equation (6.15) follows from equation (6.14) or equivalently equation (6.20) and so the superconformal transformations are encoded in $F_{B\dot{B}}$ subject to equation (6.20). We shall refer to equation (6.20) as the superconformal Killing equation, and the field $F_{B\dot{B}}$ as the superconformal Killing vector, these being the natural analogues of the conformal Killing equation and the usual Killing vector in Minkowski space.

To find the consequences for the x-space component fields within $F_{B\dot{B}}$ we expand the superfield $F_{B\dot{B}}$ as a Taylor series in θ and solve the superconformal Killing equation order by order in θ. Writing $F_{B\dot{B}}$ as

$$\begin{aligned}F_{B\dot{B}}(x,\theta,\bar{\theta}) = {} & \zeta_{B\dot{B}}(x) + \theta^A\chi_{AB\dot{B}}(x) + \bar{\theta}^{\dot{A}}\bar{\chi}_{\dot{A}B\dot{B}}(x) + \theta^A\bar{\theta}^{\dot{A}}A_{AB\dot{B}\dot{A}}(x)\\ & +\frac{1}{2}\theta^2 f_{B\dot{B}}(x) + \frac{1}{2}\bar{\theta}^2 g_{B\dot{B}}(x)\\ & +\frac{1}{2}\theta^2\bar{\theta}^{\dot{A}}\lambda_{B\dot{B}\dot{A}}(x) + \frac{1}{2}\bar{\theta}^2\theta^A\mu_{BA\dot{B}}(x) + \frac{1}{4}\theta^2\bar{\theta}^2\kappa_{B\dot{B}}(x)\end{aligned} \tag{6.22}$$

and substituting this expression into the superconformal Killing equation we find that the resulting constraints are solved by the solution

$$\begin{aligned}F_{B\dot{B}} = {} & \zeta_{B\dot{B}} + \theta_B\bar{\chi}_{\dot{B}} - \bar{\theta}_{\dot{B}}\chi_B + i\theta^A\bar{\theta}^{\dot{A}}\sigma^\mu_{B\dot{A}}\partial_\mu\zeta_{A\dot{B}}\\ & +i\theta_B\bar{\theta}_{\dot{B}}\alpha - \frac{i}{2}\theta^2\bar{\theta}^{\dot{A}}\sigma^\mu_{B\dot{B}}\partial_\mu\bar{\chi}_{\dot{B}}\\ & +\frac{i}{2}\bar{\theta}^2\theta^A\sigma^\mu_{A\dot{B}}\partial_\mu\chi_B\end{aligned} \tag{6.23}$$

In this equation α is constant, $\zeta_{B\dot{B}}$ is a conformal Killing vector which satisfies

$$\partial_{(A}{}^{(\dot{A}}\zeta_{B)}{}^{\dot{B})} = 0 \tag{6.24}$$

and χ_B and $\bar{\chi}_{\dot{B}}$ are conformal spinors which obey the relation

$$\partial_{(A}{}^{\dot{A}}\chi_{B)} = 0, \quad \partial_{(\dot{A}}{}^{A}\bar{\chi}_{\dot{B})} = 0 \tag{6.25}$$

The solutions to equations (6.24) and (6.25) are given by

$$\begin{aligned}\zeta_\mu &= a_\mu + \lambda x_\mu + \Lambda_\mu{}^\nu x_\nu + k_\mu x^2 - 2(k\cdot x)x_\mu\\ \chi_B &= 4(i\varepsilon_B + \bar{\eta}^{\dot{B}}\sigma^\mu_{B\dot{B}}x_\mu)\\ \bar{\chi}_{\dot{B}} &= -4(i\bar{\varepsilon}_{\dot{B}} + \eta^B\sigma^\mu_{B\dot{B}}x_\mu)\end{aligned} \tag{6.26}$$

where a_μ, λ, $\Lambda_{\mu\nu}$, ε_B and η_B are constant parameters.

Combining equations (6.26) and (6.23) it is clear that the parameters a_μ, λ, $\Lambda_{\mu\nu}$, k_μ, α, ε_B and η_B are translations, dilations, Lorentz rotations, special conformal transformations, chiral transformations, chiral rotations, supersymmetry transformations and special supersymmetry transformations respectively.

Having found the superconformal transformations on superspace we now turn our attention to the transformations of superfields under a superconformal transformation. If φ is a general superfield, which may carry Lorentz indices, then, its transformation is of the form

$$\delta\varphi(z) = \delta z^\pi\partial_\pi\varphi(z) + J\varphi(z) \tag{6.27}$$

where J is a superfield which arises from the non-trivial action of generators from the isotropy group acting on φ at the origin of the superspace. This factor is most pedagogically worked out by considering the superfields as induced representations.

However, here we content ourselves with the final result which for a general superfield φ is given by

$$\begin{aligned}\delta\varphi = (F\cdot\partial)\varphi - \frac{1}{8i}\sigma_\mu^{A\dot{A}}(\bar{D}_{\dot{A}}F^\mu D_A\varphi + D_A F^\mu \bar{D}_{\dot{A}}\varphi)\\ +\Big\{\frac{1}{4}(\partial\cdot F)\Delta + \frac{1}{96i}\sigma^{\mu A\dot{A}}([D_A,\bar{D}_{\dot{A}}]F_\mu)\mathcal{A} - 2(\partial^{[\mu}F^{\nu]})\sigma_{\mu\nu}\\ +\frac{1}{8}\partial^\mu(D^A F_\mu)\mathcal{S}_A + \frac{1}{8}\partial^\mu(\bar{D}^{\dot{A}}F_\mu)\bar{\mathcal{S}}_{\dot{A}} - \frac{1}{8}\partial^\mu(\partial\cdot F)\kappa_\mu\Big\}\varphi\end{aligned} \quad (6.28)$$

In this equation the symbols $\{\Delta, \Sigma_{\mu\nu}, \mathcal{A}, \mathcal{S}_A, \bar{\mathcal{S}}_{\dot{A}}, \kappa_\mu\}$ are constants that are the values of the corresponding generators of the isotropy group acting on the superfield when it is taken to be at the origin of superspace. For almost all known situations, only the parameters Δ, $\Sigma_{\mu\nu}$ and $\mathcal{A}$, which correspond to the dilation, Lorentz and $U(1)$ transformations respectively, are non-zero. The first part of the result is just the shift in the coordinates which is given by

$$\delta z^\pi \partial_\pi\varphi = (F\cdot\partial)\varphi - \frac{1}{8i}\sigma_\mu^{A\dot{A}}(\bar{D}_{\dot{A}}F^\mu D_A\varphi + D_A F^\mu\bar{D}_{\dot{A}}\varphi) \quad (6.29)$$

while J is given by

$$\begin{aligned}J = \ &\frac{1}{4}(\partial\cdot F)\Delta + \frac{1}{96i}\sigma^{\mu A\dot{A}}[D_A,\bar{D}_{\dot{A}}]F_\mu\mathcal{A} - 2\partial^{[\mu}F^{\nu]}\sigma_{\mu\nu}\\ &+\frac{1}{8}\partial^\mu(D^A F_\mu)\mathcal{S}_A + \frac{1}{8}\partial^\mu(\bar{D}^{\dot{A}}F_\mu)\bar{\mathcal{S}}_{\dot{A}} - \frac{1}{8}\partial^\mu(\partial\cdot F)\kappa_\mu\end{aligned} \quad (6.30)$$

We can verify that equation (6.28) reproduces some of the known results. Let us consider dilations which are generated by taking $F_\mu = \lambda x_\mu$. For this case, equation (6.28) becomes

$$\delta\varphi = \lambda(x^\mu\partial_\mu + \frac{1}{2}\theta^A\partial_A + \frac{1}{2}\bar{\theta}^{\dot{A}}\bar{\partial}_{\dot{A}})\varphi + \lambda\Delta\varphi \quad (6.31)$$

which we recognise as the well known result. In fact, by writing J as the most general form possible which is linear in $F_{B\dot{B}}$, contains covariant derivatives and is consistent with dimensional analysis, evaluating the result for particular transformations we can also arrive at the correct J.

We can apply equation (6.28) to the case of a chiral and anti-chiral superfield. For simplicity, let us consider a lorentz scalar chiral superfield whose $\mathcal{S}_A, \bar{\mathcal{S}}_{\dot{A}}, \kappa_\mu$ values also vanish. The result is

$$\delta\varphi = (F\cdot\partial)\varphi - \frac{1}{8i}\sigma_\mu^{A\dot{A}}\bar{D}_{\dot{A}}F^\mu D_A\varphi + \Delta(\frac{1}{4}(\partial\cdot F)\varphi + \frac{1}{48i}\sigma^{\mu A\dot{A}}([D_A,\bar{D}_{\dot{A}}]F_\mu)\varphi) \quad (6.32)$$

and

$$\delta\bar{\varphi} = (F\cdot\partial)\bar{\varphi} - \frac{1}{8i}\sigma_\mu^{A\dot{A}}D_A F^\mu\bar{D}_{\dot{A}}\bar{\varphi} + \Delta(\frac{1}{4}(\partial\cdot F)\bar{\varphi} - \frac{1}{48i}\sigma^{\mu A\dot{A}}([D_A,\bar{D}_{\dot{A}}]F_\mu)\bar{\varphi}) \quad (6.33)$$

The reader will observe that the dilation and A weights of the chiral superfield are tied together, a fact that can be established by taking the straightforward reduction of equation (6.28) and making sure the transformed superfield is still chiral or anti-chiral as appropriate. We will discuss this result from a more general perspective in the next section.

Anomalous Dimensions of Chiral Operators at a Fixed Point

Let us consider a supersymmetric theory at a fixed point of the renormalisation group, i.e. $\beta = 0$. Such a theory should be invariant under superconformal transformations. As in all supersymmetric theories some of the observables are given by chiral operators which by definition obey the equation

$$D_{\dot{A}}\varphi = 0 \tag{6.34}$$

where φ denotes the chiral operator involved. It follows that this equation must itself be invariant under any superconformal transformation i.e. $D_{\dot{A}}\delta\varphi = 0$. Choosing a special supersymmetry transformation we conclude that

$$\{D_{\dot{A}},\ S_{\dot{B}}\}\varphi = 0 \tag{6.35}$$

In this equation we can swop the covariant derivative for the generator of supersymmetry transformations using the equation

$$D_{\dot{A}} = \frac{\partial}{\partial\theta^{\dot{A}}} - i(\sigma^m)_{A\dot{A}}\theta^A\partial_m = Q_{\dot{A}} - 2i(\sigma^m)_{A\dot{A}}\theta^A\partial_m \tag{6.36}$$

We then conclude that

$$\{Q_{\dot{A}},\ S_{\dot{B}}\}\varphi = 0 \tag{6.37}$$

plus terms that contain space-time derivatives. However, in this equation the condition must hold separately on the parts of the equation containing space-time derivatives and those that do not. The advantage of writing the equation in this form is that the anti-commutator is one of the defining relations of the superconformal algebra, namely

$$\{Q_{\dot{A}},\ S_{\dot{B}}\} = \epsilon_{\dot{A}\dot{B}}(2D - 4iA) + (\bar{\sigma}^{\mu\nu})_{\dot{A}\dot{B}}J_{\mu\nu} \tag{6.38}$$

where D and A are the generators of the dilations and $U(1)$ transformations in the superconformal algebra which we gave in the first part of these lectures. If we restrict the superconformal algebra to just its super-Poincare subgroup then the A generator is identified with the generator of R transformations. The latter satisfies the relation $[Q_A,\ R] = iQ_A$ comparing this with the equivalent commutator in the superconformal group (i.e. $[Q_A,\ A] = -i\frac{3}{4}Q_A$) we thus find that the generators are related by $A = -\frac{3}{4}R$.

Consequently for a Lorentz invariant chiral operator we conclude that $D = -i\frac{3}{2}R$. One can also find this result by substituting the explicit expressions for $D_{\dot{A}}$ and $S_{\dot{A}}$ in equation (6.35) and setting $\theta = 0$. We summarise the result in the theorem

Theorem[18]

Any Lorentz invariant operator in a four dimensional supersymmetric theory at a fixed point has its anomalous dimension Δ and chiral R weight, related by the equation

$$\Delta = -i\frac{3}{2}R. \tag{6.39}$$

In any conformal theory we can determine the two and three point Green's functions using conformal invariance alone. However, one can not normally use this symmetry alone to fix the anomalous weights of any operators. Since non-trivial fixed points are outside the range of usual perturbation theory, these must be calculated using techniques such as the ϵ-expansion. The result so obtained are approximations and in some case one can not reliably calculate the anomalous dimensions at all. However, in supersymmetric theories at a fixed point one can determine the anomalous dimensions of chiral operators in superconformal theories exactly in terms of their R weight. However, in many situations one does know the R weight of the chiral operators of interest and we so can indeed exploit the above theorem to find their anomalous dimensions exactly[18]. We shall shortly demonstrate this procedure with some examples.

We must first fix the normalisation of the dilation and R weights that is implied by the superconformal algebra. The relation $[P_\mu,\ D] = P_\mu$ implies that P_μ has dilation weight one. On the other hand, the relationship $[Q_A,\ R] = iQ_A$ implies that Q_A has R weight 1. Consequently, θ^A has R weight -1 meaning that it transforms as $\theta^A \to e^{-i\alpha}\theta^A$ where α is the parameter of R transformations.

As our first example, let us consider the Wess-Zumino model in four dimensions and suppose that it had a non-trivial fixed point at which the interaction was of the usual form;

$$\int d^4x d^2\theta \varphi^3 \tag{6.40}$$

Using the above scaling of θ we find that φ transforms as $\varphi \to e^{i\frac{2}{3}\alpha}\varphi$ and as a result φ has R weight $\frac{2}{3}$. Using our theorem we find that φ had dilation weight one. This is the canonical dilation weight of φ, that is, the weight it would have in the free theory. It can be argued that if φ has its canonical weight then the theory must be free and so such a non-trivial fixed point can not exist[19]. It can also be argued that this result implies the the Wess-Zumino model is a trivial field theory meaning that the only consistent value of the coupling constant as we remove the cutoff is zero[20].

Now let us consider the Wess-Zumino model in three dimensions and suppose it has a non-trivial fixed point at which the interaction is given by

$$\int d^3x d^2\theta \varphi^3. \tag{6.41}$$

This is the supersymmetric generalisation of the Ising model. Running through the same argument as above, but taking into account the modified form of the three dimensional superconformal algebra, we find that φ has anomalous dimension $\frac{1}{6}$. However, in this case the canonical weight of φ is $1/2$. Clearly, the theory can not be free with such a dilation weight. Such a non-trivial fixed point is known to exist by using the epsilon expansion which also gives an anomalous dimension in agreement with this result[9].

The theorem in this section can also be used to fix the anomalous dimensions for the chiral operators in the two dimensional $N = 2$ supersymmetric Landau-Ginsburg models whose superpotential at the fixed point take the form

$$\int d^2x d^2\theta \varphi^n + c.c \tag{6.42}$$

The anomalous dimensions agree with the correspondence between these models at their fixed points and the $N = 2$ minimal series of superconformal models. This result was first conjectured in[21] and shown by using the epsilon expansion in Ref.[22].

The theorem can also be applied to four dimensional gauge invariant operators composed form the $N = 1$ Yang-Mills field strength W_A. Such a connection was used to argue that $N = 1$ super QED is trivial[18] and has been used extensively by Seiberg in recent work on dualities between certain $N = 1$ supersymmetric theories.

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