CONTINUUM AND LATTICE COULOMB-GAUGE HAMILTONIAN

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INTRODUCTION

In the Hamiltonian formalism one can calculate the spectrum directly[1, 2, 3, 4]. In the Coulomb gauge, the color form of Gauss's law, which would appear to be essential for confinement, is satisfied exactly. Moreover an obvious confinement mechanism suggests itself in the Coulomb gauge namely, a long-range instantaneous color-Coulomb potential.

This raises the question, can the Coulomb gauge be regularized? Certainly, on the lattice any configuration may be fixed to the Coulomb gauge. The lattice Coulomb hamiltonian which will be described in this lecture is an implementation of the Coulomb-gauge dynamics with lattice regularization. As to whether the Coulomb gauge is perturbatively renormalizable in the continuum theory, the question is under active investigation[5, 6]. The answer appears to be yes.

We review the canonical quantization of continuum Yang-Mills theory, and derive the continuum Coulomb-gauge Hamiltonian by a simplification of the Christ-Lee method. We then analogously derive, by a simple and elementary method, the lattice Coulomb-gauge Hamiltonian in the minimal Coulomb gauge (and in other Coulomb gauges) from the known Kogut-Susskind Hamiltonian.

CLASSICAL YANG-MILLS EQUATIONS

Classical Yang-Mills theory is designed to be invariant under the group of local gauge transformations $g(x)$. We shall consider the local $SU(N)$ theory, $g(x) \in SU(N)$. The quark field transforms covariantly, $\psi(x) \to {}^g\psi(x) = g(x)\psi(x)$. The gauge-covariant derivative $D = D(A)$, defined by $(D_\mu\psi)(x) = [\partial_\mu + t^a A_\mu^a(x)]\psi(x)$ also transforms covariantly, $(D_\mu\psi)(x) \to g(x)(D_\mu\psi)(x)$, provided that the connection A transforms according to

$$A_\mu(x) \to {}^g A_\mu(x) = g(x)\partial_\mu g^\dagger(x) + g(x)A_\mu(x)g^\dagger(x). \tag{1}$$

Here the anti-hermitian matrices t^a form a representation of the Lie algebra of SU(N), $[t^a, t^b] = f^{abc}t^c$, and $A_\mu \equiv t^a A_\mu^a$. The *local gauge principle* states that A and ${}^gA$ are

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physically identical $A \sim^g A$. The physical configuration space $\mathcal{Q} = \mathcal{A}/\mathcal{G}$ is the space of connections $\mathcal{A} = \{A\}$ modulo the local gauge transformations $\mathcal{G} = \{g\}$. This space is non-trivial because the local gauge transformation is non-linear.

The Yang-Mills field tensor,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = t^a F_{\mu\nu}, \tag{2}$$

transforms homogeneously,

$$F_{\mu\nu}(x) \to^g F^a_{\mu\nu}(x) = g(x) F_{\mu\nu}(x) g^\dagger(x). \tag{3}$$

The Yang-Mills action

$$S_{\rm YM} = g_0^{-2} \int d^4x (1/4)(-g^{\mu\kappa} g^{\nu\lambda}) F^a_{\mu\nu} F^a_{\kappa\lambda}, \tag{4}$$

is gauge-invariant and yields the field equations $D^\mu F_{\mu\nu} \equiv \partial^\mu F_{\mu\nu} + [A^\mu, F_{\mu\nu}] = 0$. Here the metric tensor is defined by $g_{\mu,\nu} = \mathrm{diag}(-1, 1, 1, 1)$, and g_0 is the coupling constant. (If quarks were present we would have $D^\mu F_{\mu\nu} = -j_{\mathrm{qu},\nu}$.)

The color-electric and magnetic fields are defined by $E_i \equiv F_{i0} = -\dot{A}_i + D_i A_0$ and $B_i = F_{jk}$ for i, j, k cyclic.

GAUSS's LAW AND COLOR-COULOMB POTENTIAL

The time derivative of A_0 nowhere appears in the preceding equations, and the field equation which corresponds to variation with respect to A_0 is the time-independent constraint which constitutes the color Gauss law,

$$D_i E_i \equiv \nabla_i E_i + A_i \times E_i = 0 \quad (\text{or} = \rho_{\rm qu}), \tag{5}$$

where $(A \times E)^a \equiv f^{abc} A^b E^c$. This law may be used to fix the longitudinal part of the color-electric field.

We write

$$E_i = E_i^{\rm tr} - \nabla_i \Omega. \tag{6}$$

where $\nabla_i E_i^{\rm tr} = 0$, and Ω is the *color-Coulomb potential.* We have

$$D_i E_i = -D_i \nabla_i \Omega + A_i \times E_i^{\rm tr} = \rho_{\rm qu}. \tag{7}$$

The three-dimensional Faddeev-Popov operator is defined by

$$M = M(A) \equiv -D_i(A) \nabla_i \tag{8}$$

Gauss's law fixes the color-Coulomb potential

$$\Omega(x) = (M^{-1}\rho)(x) = \int d^3y \, M^{-1}(x, y; A) \rho(y). \tag{9}$$

This equation, which appears to be neither Lorentz nor gauge covariant, in fact holds in every Lorentz frame and every gauge. Here ρ is a color-charge density

$$\rho \equiv -A \times E^{\rm tr} + \rho_{\rm qu} \tag{10}$$

which does not contain the complete color-charge density of the gluons, $\rho_{\rm gl} = -A \times E$, but only the part that comes from $E^{\rm tr}$.

The color-Coulomb potential is propagated by the Green's function

$$M^{-1} = (-D_i \nabla_i)^{-1} = (-\vec{\nabla}^2 - A_i \times \nabla_i)^{-1} \tag{11}$$

of the three-dimensional Faddeev-Popov operator. The famous anti-screening property of non-Abelian gauge theory arises from this circumstance. For, whereas $(-\vec{\nabla}^2)$ is a positive operator, the second term $(-A_i \times \nabla_i)$ may be positive or negative. In the latter case there is a cancellation in the denominator which produces an enhanced color-Coulomb potential.

CLASSICAL CANONICAL FORMALISM

The Yang-Mills lagrangian is of the form

$$L_{\rm YM} = \int d^3x \mathcal{L}_{\rm YM} \tag{12}$$

where the Lagrangian density is

$$\mathcal{L}_{\rm YM} = g_0^{-2}(1/2)(\vec{E}^2 - \vec{B}^2). \tag{13}$$

The time derivatives are contained in $E_i = -\dot{A}_i + D_i A_0$, and the canonical momentum densities are given by

$$\begin{aligned} \pi_i &= \partial\mathcal{L}/\partial\dot{A}_i = -g_0^{-2}E_i \\ \pi_0 &= \partial\mathcal{L}/\partial\dot{A}_0 = 0. \end{aligned} \tag{14}$$

The classical Hamiltonian

$$\begin{aligned} H &= \int d^3x(\pi_i\dot{A}_i - \mathcal{L}) \\ H &= \int d^3x \; [\; (1/2)(g_0^2\,\vec{\pi}^2 + g_0^{-2}\vec{B}^2) - A_0 D_i\pi_i \;] \end{aligned} \tag{15}$$

produces the equations of motion

$$\begin{aligned} \dot{A}_i = \delta H/\delta\pi_i &\Rightarrow \pi_i = -g_0^{-2}E_i \\ \dot{\pi}_i = -\delta H/\delta A_i &\Rightarrow D_0E_i + \epsilon_{ijk}D_jB_k = 0, \end{aligned} \tag{16}$$

and the Gauss law constraint

$$0 = \delta H/\delta A_0 \Rightarrow D_i\pi_i = 0 \tag{17}$$

QUANTIZATION IN THE WEYL GAUGE

Before deriving the lattice Coulomb Hamiltonian, we shall derive the Christ-Lee Coulomb Hamiltonian for continuum gauge theory which it resembles. Christ and Lee start by canonical quantization in the Weyl gauge, $A_0 = 0$. This is not a complete gauge-fixing and allows time-independent local gauge transforamtions $g = g(\vec{x})$. In this gauge we have $E_i = -\dot{A}_i$, and the previous Hamiltonian reduces to

$$H_W = \int d^3x \; (1/2)(\; g_0^2\,\vec{\pi}^2 + g_0^{-2}\vec{B}^2 \;). \tag{18}$$

This Hamiltonian generates the previous equations of motion in the Weyl gauge, but not the Gauss law constraint $D_i\pi_i = 0$. In the classical theory this is imposed as an initial condition that holds at $t = 0$, and which is preserved by the equations of motion.

In the Christ-Lee method, one quantizes in the Weyl gauge by treating A and π as Cartesian variables, with canoncal equal-time commutation relations

$$[A_i^a(\vec{x}), A_j^b(\vec{y})] = [\pi_i^a(\vec{x}), \pi_j^b(\vec{y})] = 0 \tag{19}$$

$$[\pi_i^a(\vec{x}), A_j^b(\vec{y})] = -i\delta^{ab}\ \delta_{ij}\ \delta(\vec{x} - \vec{y}). \tag{20}$$

There is then no ordering problem in the Weyl Hamiltonian, nor in the definition of the left-hand side of Gauss's law

$$G^a(x) \equiv (D_i\pi_i)^a(x) = \nabla_i\pi_i^a + f^{abc}A_i^b(x)\pi_i^c(x). \tag{21}$$

Moreover with the above canonical commuation relations one observes that $G^a(x)$ is the generator of local time-independent gauge transformations,

$$[G^a(x), \pi_i^b(y)] = i\delta(x - y)f^{abc}\pi_i^c(y) \tag{22}$$

$$[G^a(x), A_i^b(y)] = -iD_i^{ab}\delta(x - y), \tag{23}$$

under which π transforms homogeneously. The $G^a(x)$ satisfy the Lie algebra of the time-independent local gauge group,

$$[G^a(x), G^b(y)] = i\delta(x - y)f^{abc}G^c(y) \tag{24}$$

and moreover these local time-independent gauge transformations are a symmetry of the Hamiltonian $[G^a(x), H_W] = 0$. Therefore it is consistent to impose Gauss's law as a subsidiary condition which must be satisfied for a wave function $\Psi = \Psi(A)$ to represent a physical state,

$$G^a(x)\Psi = 0. \tag{25}$$

This condition is also the statement that the wave function is invariant under time-independent local gauge transformations.

The inner product is defined by the functional integral

$$(\Phi, \Psi) = \int dA\ \Phi^*(A)\Psi(A), \tag{26}$$

and the Hamiltonian by $H_W = T + V$, where T and V correspond respectively to kinetic (electrical) and potential (magnetic) energies. Here

$$V = \int d^3x\ (1/2)g_0^{-2}[B_i^a(x)]^2 \tag{27}$$

acts by multiplication, and we define T by its expression as a quadratic form

$$(\Phi, T\Psi) = \int dA \int d^3x\ (g_0^2/2)\ \delta\Phi^*(A)/\delta A_i^a(x)\ \delta\Psi(A)/\delta A_i^a(x). \tag{28}$$

MINIMAL CONTINUUM COULOMB GAUGE

In the Christ-Lee method[7], one solves the Gauss law subsidiary condition explicitly to obtain the Hamiltonian in the physical subspace. In the original Christ-Lee paper, this is done by choosing coordinates in A-space adapted to the gauge orbit. A gauge orbit is a set of gauge-equivalent configurations $A \sim {}^gA$. (Here we refer only to time-independent local gauge transformations g.) This may be done by writing $A = {}^gA^{\rm tr}$, where $A^{\rm tr}$ is transverse, $\nabla_i A_i^{\rm tr} = 0$, and g runs over the set of all possible gauge transformations. Here the transverse configuration $A^{\rm tr}$ labels the gauge orbit and g labels the point on the orbit. One way to address the problem of Gribov copies[8], is to restrict $A^{\rm tr}$ to a subset Λ of transverse configurations Γ known as the fundamental modular region. For an alternative approach, see[9].

A convenient way to choose Λ is by means of a minimizing function

$$F_A(g) \equiv \parallel {}^gA \parallel^2 = \int d^3x |g\vec{\nabla} g^\dagger + g\vec{A}g^\dagger|^2. \tag{29}$$

The fundamental modular region Λ is defined to be the set of A that are an absolute minimum of this function,

$$\Lambda \equiv \{A : \parallel A \parallel^2 \leq \parallel {}^gA \parallel^2 \text{ for all } g\}. \tag{30}$$

With suitable assumptions on the space of configurations, one may show that such a minimum exists[10]. This definition (almost) uniquely fixes a single point on each gauge orbit, to within global gauge transformations, because the absolute minimum of the functional is unique apart from "accidental" degeneracies. These are degenerate (equal) absolute minima of the minimizing functional. They constitute the boundary of Λ. The fundamental modular region is obtained from Λ by topologically identifying the boundary points that are gauge-equivalent degenerate absolute minima. (For a more extensive discussion of Λ see[11].)

We may easily derive some properties of the fundamental modular region. At a relative or absolute minimum the functional is stationary to first order and its second-order variation is positive. With $g = \exp(\omega) = 1 + \omega + (1/2)\omega^2 + \ldots$ an elementary calculation gives

$$\parallel {}^gA \parallel^2 = \parallel A \parallel^2 - 2(\nabla_i A_i, \omega) + (1/2)(\omega, M(A)\omega) + \ldots\ , \tag{31}$$

where $M(A) = -D_i(A)\nabla_i$ is the three-dimensional Faddeev-Popov operator. Since this must be positive for all ω, we obtain the Coulomb gauge condition $\nabla_i A_i = 0$. *In addition* we find that $M(A)$ is a positive operator, $(\omega, M(A)\omega) \geq 0$ for all ω and for all $A \in \Lambda$. These two conditions define the Gribov region Ω which is the set of all relative and absolute minima, of which the fundamental modular region Λ is a proper subset, $\Lambda \subset \Omega$. Note that because the Faddeev-Popov operator $M(A)$ is a positive operator for $A \in \Lambda$, M(A) is invertible in the minimal Coulomb gauge and has non-negative eigenvalues.

CONTINUUM COULOMB-GAUGE HAMILTONIAN

In the present lecture we shall not explicitly change variables from A to the adapted coordinates $A^{\rm tr}$ and g, as was originally done by Christ and Lee[7], but instead use the more efficient Faddeev-Popov "trick" to obtain $H_{\rm coul}$.

The Faddeev-Popov identity reads

$$1 = \int dg \; \delta_\Lambda(\nabla_i \, {}^g\!A) \det[M({}^g\!A)], \tag{32}$$

where $M(A) = -\nabla_i D_i(A)$ is the Faddeev-Popov operator. Here $\delta_\Lambda(\nabla_i A_i)$ is the restriction of $\delta(\nabla_i A_i)$ to the fundamental modular region Λ, namely $\delta_\Lambda(\nabla_i A_i) = \delta(\nabla_i A_i)\chi_\Lambda(A)$, where $\chi_\Lambda(A) = 1$ for (transverse) $A \in \Lambda$, and $\chi_\Lambda(A) = 0$ otherwise.

We require wave functions to be gauge invariant, $\Psi({}^g\!A) = \Psi(A)$, and we wish to express the inner-product of such wave-functions (Φ, Ψ) as an integral over the parameters $A^{\rm tr} \in \Lambda$ that label the orbit space. For this purpose we insert the Faddeev-Popov identity into the inner product and obtain

$$\begin{aligned}
(\Phi, \Psi) &= \int dA \; \Phi^*(A)\Psi(A) \\
&= \int dAdg \; \delta_\Lambda(\nabla_i \, {}^g\!A_i) \det[M({}^g\!A)] \; \Phi^*(A)\Psi(A) \\
&= \int dAdg \; \delta_\Lambda(\nabla_i \, {}^g\!A_i) \det[M({}^g\!A)] \; \Phi^*({}^g\!A)\Psi({}^g\!A) \\
&= \int dA'dg \; \delta_\Lambda(\nabla_i A'_i) \det[M(A')] \; \Phi^*(A')\Psi(A') \\
&= N \int dA \; \delta_\Lambda(\nabla_i A_i) \det[M(A)] \; \Phi^*(A)\Psi(A) \\
&= N \int_\Lambda dA^{\rm tr} \; \det[M(A^{\rm tr})] \; \Phi^*(A^{\rm tr})\Psi(A^{\rm tr}).
\end{aligned} \tag{33}$$

Here $N = \int dg$ is the (infinite) volume of the gauge orbit which is an irrelevant normalization constant, and $\int_\Lambda dA^{\rm tr}$ is the functional intregral over all transverse configurations in Λ. We have used the invariance of the measure dA under local gauge transformation, $dA = dA'$ for $A' = {}^g\!A_i$. These manipulations are somewhat formal in the continuum theory, but are well defined in lattice gauge theory.

We next express the Hamiltonian $H_{\rm W} = T + V$ on the reduced space of wave functions that depend on $A^{\rm tr}$. Because $V(A)$ is gauge invariant, it acts on the reduced space as the operator of multiplication by $V(A^{\rm tr})$.

The expression for T as a quadratic form is particularly convenient for our purpose because it only involves first derivatives, and this allows us to apply Gauss's law. We have seen (22) that $\pi_i^a(x) = -i\delta/\delta A_i^a(x)$ transforms homogeneously under gauge transformations, so for gauge-invariant wave functions the quantity $\delta\Phi^*/\delta A_i^a(x) \; \delta\Psi/\delta A_i^a(x)$ is also gauge-invariant. Consequently when we insert the Faddeev-Popov identity into the quadratic form that defines T, we obtain, as for the inner product,

$$\begin{aligned}
(\Phi, T\Psi) = {} & N \int dA \; \delta_\Lambda(\nabla_i A_i) \det[M(A)] \\
& \times \int d^3x \; (g_0^2/2) \; \delta\Phi^*(A)/\delta A_i^a(x) \; \delta\Psi(A)/\delta A_i^a(x).
\end{aligned} \tag{34}$$

To proceed, we shall use Gauss's law $D_i\delta\Psi/\delta A_i = 0$ to express $\delta\Psi/\delta A_i$ as a derivative with respect to $A^{\rm tr}$. For this purpose we write $E_i = E_i^{\rm tr} - \nabla_i\Omega$, where $E_i = i\delta/\delta A_i$ and $\nabla_i E_i^{\rm tr} = 0$. Here Ω is the color-Coulomb potential operator, and $E^{a,{\rm tr}} = i\delta/\delta A_i^{a,{\rm tr}}$. With $D_iE_i = \nabla_i E_i + A_i \times E_i$, Gauss's law, reads

$$-\,D_i\nabla_i\Omega\Psi = \rho\Psi. \tag{35}$$

where we have introduced the color-charge density operator corresponding to the dynamical degrees of freedom

$$\rho \equiv -A_i \times E_i^{\rm tr}, \tag{36}$$

As in the classical theory, we may solve for $\Omega\Psi$ by inverting the Faddeev-Popov operator $M(A) = -D_i(A)\nabla_i$. We have $\Omega\Psi = M^{-1}\rho\Psi$, which gives

$$E_i\Psi = E_i^{\rm tr}\Psi - \nabla_i M^{-1}(A)\,(-A_i \times E_i^{\rm tr})\Psi. \tag{37}$$

Only derivatives with respect to $A^{\rm tr}$ now appear in the expression for the quadratic form T,

$$\begin{aligned}(\Phi, T\Psi) = N \int_\Lambda dA^{\rm tr}\ \det[M(A^{\rm tr})] \\ \times \int d^3x\ (g_0^2/2)\ (E_i^a(x)\Phi)^*(A^{\rm tr})\ (E_i^a(x)\Psi)(A^{\rm tr}),\end{aligned} \tag{38}$$

where $E\Psi$ and $E\Phi$ are defined by the preceding equation. The transverse and longitudinal parts of the electric field contribute separately to the kinetic energy,

$$\begin{aligned}(\Phi, T\Phi) = N \int_\Lambda dA^{\rm tr}\ \det[M(A^{\rm tr})]\ \int d^3x\ (g_0^2/2) \\ \times\ [\ |E_i^{{\rm tr},a}(x)\Phi|^2 + |(\nabla_i M^{-1}\rho)^a(x)\Phi|^2\ ].\end{aligned} \tag{39}$$

We have expressed T as a quadratic form on functions of transverse configurations, $A^{\rm tr} \in \Lambda$ that parametrize the orbit space, and V as the operator of multiplication by $V(A^{\rm tr})$. These expressions define the Hamiltonian in the minimal Coulomb gauge, $H_{\rm coul} = T + V$, defined by the gauge choice $A_i = A_i^{tr} \in \Lambda$.

KOGUT-SUSSKIND LATTICE HAMILTONIAN

In a Euclidean lattice theory the Hamilton H may be derived from the partition function Z. The partition function is expressed in terms of the transfer matrix $\mathcal{T}$ by $Z = \lim_{N\to\infty} {\rm tr}\ \mathcal{T}^N$, where N is the number of Euclidean "time" slices. One chooses an asymmetric lattice with lattice unit a_0 in the Euclidean time direction and lattice unit a in spatial directions. Then $\mathcal{T} = \lim_{a_0\to 0}\exp(-a_0 H)$.

In[11] the lattice Coulomb-gauge Hamiltonian was derived directly from this formula, where the configurations were fixed to the lattice Coulomb gauge. In the present lecture we present a much briefer and elementary derivation. We shall start with the known Kogut-Susskind Hamiltonian[12], $H_{\rm KS}$, which is the lattice analog of the Weyl Hamiltonian. We then fix the gauge, and solve the lattice form of the Gauss law constraint to obtain the lattice Coulomb-gauge Hamiltonian on the reduced space, just as we did in the continuum theory.

We designate points on the three-dimensional periodic cubic lattice by three-vectors with integer components, which we denote by $x, y, \ldots$. Gauge transformations are site variables, $g_x \in SU(N)$. The basic variables of the theory are the link variables $U_{(xy)} \in SU(N)$ defined for all links (xy) of the lattice, with $U_{(yx)} = U_{(xy)}^\dagger$. Local gauge transformations are defined by $U_{(xy)} \to {}^gU_{(xy)} = g_x U_{(xy)} g_y^\dagger$.

The $U_0 = 1$ gauge is the lattice analog of the Weyl $A_0 = 0$ gauge. In this gauge, the Hamiltonian obtained from Wilson's lattice action and the formula $\mathcal{T} = \lim_{a_0\to 0}\exp(-a_0 H)$ is

$$H_{\rm KS} = T + V \tag{40}$$

where

$$\begin{aligned}T &= \sum_{(xy)} (g_0^2/2a)\ J_{(xy)}^2 \\ V &= (2N/ag_0^2)\sum_p (1 - N^{-1}{\rm Re}\,{\rm tr}U_p).\end{aligned} \tag{41}$$

Here a is the (spatial) lattice unit. Henceforth we set $a = 1$. In the last expression the sum extends over all plaquettes p of the (spatial) lattice, and U_p is the product of the link variables $U_{(xy)}$ on the links around the plaquette p. The potential energy operator is gauge invariant $V({}^gU) = V(U)$, as are physical wave functions, $\Psi({}^gU) = \Psi(U)$.

The variables $J_{(xy)}$ are electric field operators associated to each link of the lattice, defined as follows. Let the link variable $U_{(xy)}$ be parametrized by a set of variables $\theta^\alpha_{(xy)}$. Then, for each link (xy),

$$J_{(xy),a} \equiv i\ J_a^\alpha(\theta_{(xy)})\ \partial/\partial\theta^\alpha_{(xy)} \tag{42}$$

is the Lie generator of the $SU(N)$ group, that satisfies the commutation relations

$$[J_{(xy),a}, J_{(uv),b}] = i\delta_{(xy),(uv)} f^{abc} J_{(xy),c}. \tag{43}$$

For the SU(2) group, T would be the Hamiltonian for a set of non-interacting tops, located on every link of the lattice. The potential energy operator V provides an interaction between tops on the same plaquette.

Both T and V are gauge invariant, as is the inner product

$$(\Psi, \Phi) \equiv \int \prod_{(xy)} dU_{(xy)} \Psi^*(U)\Phi(U). \tag{44}$$

Here $dU_{(xy)}$ is the Haar measure on each link,

$$dU_{(xy)} \equiv \det \psi_{(xy)} \prod_\alpha d\theta^\alpha_{(xy)}, \tag{45}$$

where $\psi(\theta_{(xy)})$ is the inverse of the matrix $J(\theta_{(xy)})$,

$$J_a^\alpha(\theta_{(xy)})\psi_\alpha^b(\theta_{(xy)}) = \delta_a^b. \tag{46}$$

MINIMAL LATTICE COULOMB GAUGE

We wish to express the hamiltonian as on operator on the reduced space of gauge orbits. For this purpose we require a set of parameters that label the gauge orbits. A convenient way to identify the gauge orbits is to introduce a minimizing function, as in the continuum theory,

$$F_U(g) \equiv \sum_{x,i} \{\ 1 - N^{-1}\mathrm{Re\ tr}[({}^gU)_{x,i}]\ \}. \tag{47}$$

Here we have introduced the notation $U_{x,i} \equiv U_{(xy)}$ for $y = x + e_i$ where e_i is a unit vector in the positive i-direction. The gauge-transformed link variable is given by $({}^gU)_{x,i} = g_x U_{x,i} g^\dagger_{x+\hat{i}}$, where we have written $\hat{i} \equiv e_i$. The fundamental modular region Λ is defined to be the set of configurations U that are absolute minima of this function,

$$\Lambda \equiv \{U : F_U(1) \leq F_U(g) \text{ for all } g\}. \tag{48}$$

With this gauge choice, the link variables $U_{x,i}$ are made as close to unity as possible, equitably, over the whole lattice.

The advantage of this method is that we are assured that a minimizing configuration exists on each gauge orbit, because the minimizing function is defined on a compact space (a finite product of $SU(N)$ group manifolds). An alternative procedure that is sometimes followed is to posit a gauge condition. For example with the exponential mapping $U_{x,i} = \exp(A_{x,i})$, one may posit the condition $\sum_i (A_{x,i} - A_{x-\hat{i},i}) = 0$ which is a lattice analog of the continuum Coulomb-gauge transversality condition. An obstacle to this procedure is that one does not know whether every gauge orbit intersects this gauge-fixing surface. On the other hand for the purpose of formal expansions the alternative procedure may be used.

As in the continuum case, we observe that, at a relative or absolute minimum, the minimizing function is stationary to first order, and its second order variation is positive. In order to conveniently exploit these properties, we introduce a one-parameter subgroup of the local gauge group $g_x(s) = \exp(s\omega_x)$, where $\omega_x = t^a \omega_x^a$ is anti-hermitian, $\omega_x^\dagger = -\omega_x$. We write $F_U(s) \equiv F_U(g(s))$ and we have

$$\begin{aligned} dF_U(s)/ds = -N^{-1} \sum_{x,i} \text{ Re tr}[\omega_x ({}^gU)_{x,i} - ({}^gU)_{x,i}\omega_{x+\hat{i}}\,] \\ = N^{-1} \sum_{x,i} \text{ Re tr}\{[\omega_{x+\hat{i}} - \omega_x]({}^gU)_{x,i}\,\} \end{aligned} \tag{49}$$

$$= -(2N)^{-1} \sum_{x,i} \text{ tr}\{\omega_x[({}^gU)_{x,i} - ({}^gU)^\dagger_{x,i} - ({}^gU)_{x-\hat{i},i} + ({}^gU)^\dagger_{x-\hat{i},i}]\,\}. \tag{50}$$

At a relative or absolute minimum U we have $dF_U(s)/ds|_{s=0} = 0$. To express this condition compactly, we introduce the link variables

$$t^a A^a_{x,i} \equiv A_{x,i} \equiv (1/2)(U_{x,i} - U^\dagger_{x,i})_{\text{traceless}}. \tag{51}$$

We shall use these variables as coordinates to parametrize the $SU(N)$ group, and we write $U_{x,i} = U(A_{x,i})$. The manifold of the $SU(N)$ group requires more than one coordinate patch, but we shall not trouble to account for this explicitly. The formulas of the last section hold, with $\theta^\alpha_{x,i} = A^\alpha_{x,i}$. These coordinates agree with the exponential mapping, $U_{x,i} = \exp A_{x,i}$, to second order, for with the above definition one may show

$$U_{x,i} = 1 + A_{x,i} + (1/2)(A^2_{x,i}) + (6N)^{-1}\text{tr}A^3 + O(A^4), \tag{52}$$

so these variables approach the continuum connection $A_i(x)$ in the continuum limit.

In terms of these variables, the condition $dF_U(s)/ds|_{s=0} = 0$ reads

$$\sum_x \text{tr}[\omega_x \sum_i(-A_{x,i} + A_{x-\hat{i},i})] = 0. \tag{53}$$

Since this holds for arbitrary ω, we conclude that, at a minimum of the minimizing function, A satisfies

$$\sum_i \,(A_{x,i} - A_{x-\hat{i},i}) = 0. \tag{54}$$

This is the lattice analog of the transversality condition that characterizes the continuum Coulomb gauge, and we call our gauge choice the "minimal lattice Coulomb gauge". The lattice Coulomb-gauge condition is linear in the variables $A_{x,i}$, and it may be solved by lattice fourier transform.

To see the geometrical meaning of the gauge condition, and for future use, it is helpful to introduce some definitions. We define the lattice "gradient" by

$$(\nabla\omega)_{x,i} \equiv (\omega_{x+\hat{i}} - \omega_x), \tag{55}$$

It is a matrix that linearly maps site variables into link variables. The identity,

$$\sum_{x,i} \text{tr}[(\omega_{x+i} - \omega_x)A_{x,i}] = \sum_x \text{tr}[\omega_x \sum_i(-A_{x,i} + A_{x-\hat{i},i})], \tag{56}$$

holds for all ω and A. We write it

$$(\nabla\omega, A) = (\omega, \nabla^* A) \tag{57}$$

which defines the dual ∇^* of the lattice gradient. It is a matrix that linearly maps link variables into site variables, that is given explicitly by

$$(\nabla^* A)_x = -\sum_i (A_{x,i} - A_{x-\hat{i},i}), \tag{58}$$

and we have $\nabla^* =$ - (lattice "divergence"). These geometric definitions may be extended to arbitrary lattices that need not be cubic or periodic[11]. In the minimal Coulomb gauge, the lattice divergence of A vanishes, $(\nabla^* A)_x = 0$.

At a minimum of the minimizing function we also have the condition $d^2F_U(s)/ds^2|_{s=0} \geq 0$. From (49) we obtain

$$d^2F_U(s)/ds^2 = (2N)^{-1} \sum_{x,i} \mathrm{tr}\{[\omega_{x+\hat{i}} - \omega_x](d/ds)[(^gU)_{x,i} - (^gU)^\dagger_{x,i}\,]\}. \tag{59}$$

With

$$A_{x,i}(s) \equiv A[(^{g(s)}U)_{x,i}], \tag{60}$$

the positivity condition reads

$$d^2F_U(s)/ds^2|_{s=0} = N^{-1} \sum_{x,i} \mathrm{tr}\{[\omega_{x+\hat{i}} - \omega_x]\; dA_{x,i}(s)/ds\,\}|_{s=0} \geq 0. \tag{61}$$

To make explicit the geometrical meaning this condition, and for future use, we define the lattice gauge covariant "derivative" $D(A)$ by

$$[D(A)\omega]_{x,i} \equiv -dA_{x,i}(s)/ds|_{s=0}. \tag{62}$$

It is the matrix that maps an infinitesimal gauge transformation ω into the corresponding first-order change in the coordinates. (The minus sign is to be coherent with the continuum formula ${}^gA = A - D(A)\omega$.) Like the lattice gradient ∇, it maps site variables into link variables, and we have $D(0) = \nabla$. Its dual $D(A)^*$ maps link variables into site variables, and is defined by $(D^*(A)V, \omega) \equiv (V, D(A)\omega)$, for arbitary link variables V and site variables ω. For the coordinates A defined above, $D(A)$ is given explicitly by

$$[D(A)\omega]_{x,i} = t^b[D(A)\omega]^b_{x,i} = (1/2)[(\,U_{x,i}\omega_{x+\hat{i}} - \omega_x U_{x,i}) - (\mathrm{h.c.})]_{\mathrm{traceless}}, \tag{63}$$

where $U = U(A)$.

The positivity condition $d^2F_U(s)/ds^2|_{s=0} \geq 0$, which holds at absolute or relative minima of the minimizing function, thus reads

$$\sum_{x,i} (\nabla\omega)^b_{x,i}(D(U)\omega)^b_{x,i} \geq 0 \text{ for all } \omega. \tag{64}$$

We define the lattice Faddeev-Popov matrix

$$M(A) \equiv D^*(A)\nabla, \tag{65}$$

which maps site variables into site variables. For transverse A, one may verify that this matrix is symmetric,

$$D(A)^*\nabla = \nabla^* D(A) \quad \text{for } \nabla^* A = 0. \tag{66}$$

We conclude that at a minimum of the minimizing function, this matrix is also non-negative

$$M(A) \geq 0 \quad \text{for } A \in \Lambda. \tag{67}$$

It has a trivial null space consisting of x-independent eigenvectors, $\nabla\omega = 0$. This is a reflection of the fact that the minimal Coulomb gauge does not fix global (x-independent) gauge transformations. On the orthogonal subspace, $M(A)$ is strictly positive for configurations A that are interior points on the fundamental modular region Λ. (For additional properties of Λ, see[11].)

LATTICE COULOMB-GAUGE HAMILTONIAN

In the preceeding section we obtained a parametrization of the gauge orbit space by means of transverse configurations restricted to the fundamental modular region. We shall now calculate the lattice Coulomb-gauge Hamiltonian H_{coul} as the restriction of the Kogut-Susskind Hamiltonian H_{KS} to gauge-invariant wave functions, $\Phi({}^gU) = \Phi(U)$.

We represent states $\Phi(U)$ as functions of the coordinates, $\Phi = \Phi(A)$. The coordinates transform according to ${}^gA^b_{x,i} = A^b_{x,i}({}^gU)$ under local gauge transformation g_x. For an infinitesimal gauge transformation $g_x = \exp(\omega_x) = 1 + \omega_x$, we have

$${}^gA^b_{x,i} = A^b_{x,i} - [D(A)\omega]^b_{x,i}, \tag{68}$$

by the definition (62) of the lattice gauge-covariant derivative $D(A)$. With $\Phi({}^gA) = \Phi(A - D(A)\omega)$, we conclude that gauge-invariant wave functions, which satisfy $\Phi({}^gA) = \Phi(A)$ for all g, satisfy the first order differential equation,

$$\sum_{x,i} (D(A)\omega)^a_{x,i} \partial\Phi / \partial A^a_{x,i} = 0, \tag{69}$$

for all ω. This is equivalent to the condition

$$(D^*(A)\partial/\partial A)^b_x \Phi = 0 \tag{70}$$

for all lattice sites x, which is the lattice version of Gauss's law constraint. This equation holds for any coordinates $A_{x,i}$ on the group manifold, provided only that the lattice gauge-covariant derivative D is defined by (62).

We now proceed as in the continuum theory, and introduce the lattice Faddeev-Popov identity

$$1 = \int \prod_x dg_x {\prod_x}' \delta_\Lambda[(\nabla^* \, {}^gA)_x] \det M'({}^gA). \tag{71}$$

The δ-function δ_Λ is defined by strict analogy to the continuum theory. The product $\prod_x dg_x$ extends over all sites x of the lattice, where dg_x represents Haar measure. However the primed product, ${\prod_x}' \delta_\Lambda[(\nabla^* A)_x]$ extends over all but one site x_0 of the lattice because the lattice divergences satisfy the identity $\sum_x (\nabla^* A)_x = 0$, and are thus not all linearly independent. The primed matrix $M'(A)$ is the lattice Faddeev-Popov matrix $M(A) = D^*(A)\nabla$, with the rows and columns labelled by x_0 deleted. One may show that the integrand is independent of x_0 and that, apart from an overall normalization constant, $\det M'(A)$ for $\nabla^* A = 0$ is the determinant of $M(A)$ on the space orthogonal to its trivial null space. In the minimal Coulomb gauge $M'(A)$ is a strictly positive matrix, in the interior of the fundamental modular region Λ, and its determinant is positive.

Lattice gauge-covariant derivative

The above Faddeev-Popov identity also holds for other coordinates, such as that provided by the exponential mapping $U_{x,i} = \exp A_{x,i}$, with gauge condition $(\nabla^* A)_x = 0$. We now derive an explicit expression for the lattice gauge-covariant derivative $D(\theta)$ in any coordinate system $\theta^\alpha_{x,i}$, where $D(\theta)$ is defined by the linear change in the coordinates induced by an infinitesimal local gauge transformation. We start with the Maurer-Cartan differential

$$U_{x,i} dU^\dagger_{x,i} = t^a \psi^a_\alpha(\theta_{x,i}) d\theta^\alpha_{x,i}, \tag{72}$$

which relates infinitesimal changes in the coordinates $d\theta$ to infinitesimal changes dU in U. For an infinitesimal gauge transformation $g_x = 1 + \omega_x$ we have

$$\begin{aligned}
{}^g U_{x,i} &= U_{x,i} + \omega_x U_{x,i} - U_{x,i}\omega_{x+\hat{i}} \\
dU_{x,i} &= \omega_x U_{x,i} - U_{x,i}\omega_{x+\hat{i}} \\
dU^\dagger_{x,i} &= -U^\dagger_{x,i}\omega_x + \omega_{x+\hat{i}} U^\dagger_{x,i} \\
U_{x,i} dU^\dagger_{x,i} &= U_{x,i}\omega_{x+\hat{i}} U^\dagger_{x,i} - \omega_x,
\end{aligned} \tag{73}$$

where we have used $\omega^\dagger_x = -\omega_x$. The last expression represents the difference between the parallel-transport of ω, from $x + \hat{i}$ to x, and ω_x. By analogy with the continuum theory, it is natural to call this quantity the lattice gauge-covariant derivative in the *site* basis,

$$[\mathcal{D}(U)\omega]_{x,i} \equiv U_{x,i}\omega_{x+\hat{i}} U^\dagger_{x,i} - \omega_x. \tag{74}$$

If we choose a basis in the Lie algebra, $\omega_x = t^a \omega^a_x$, we have $U_{x,i} t^b U^\dagger_{x,i} = t^a R^{ab}(U)$, where $R^{ab}(U)$ is the adjoint representative of U. In the same Lie algebra basis we have the expansion $[\mathcal{D}(U)\omega]_{x,i} = t^a [\mathcal{D}(U)\omega]^a_{x,i}$. This gives the explicit expression for the lattice gauge-covariant derivative in the site basis

$$[\mathcal{D}(U)\omega]^a_{x,i} = R^{ab}(U)\omega^b_{x+\hat{i}} - \omega^a_x. \tag{75}$$

By analogy with the continuum formula for an infinitesimal gauge transformation ${}^g A = A - D(A)\omega$, we define the lattice gauge-covariant derivative in the *link* basis $D\omega$ as the first order change in the link coordinates induced by an infinitesimal gauge transformation

$$d\theta^\alpha_{x,i} \equiv [D(\theta)\omega]^\alpha_{x,i} \ . \tag{76}$$

The two quantities are related by use of the Maurer-Cartan differential

$$t^a \psi^a_\alpha(\theta_{x,i}) d\theta^\alpha_{x,i} = U_{x,i} dU^\dagger_{x,i} = t^a [\mathcal{D}(U)\omega]^a_{x,i} \tag{77}$$

which gives

$$\psi^a_\alpha(\theta_{x,i})[D(\theta)\omega]^\alpha_{x,i} = [\mathcal{D}(U)\omega]^a_{x,i} \ . \tag{78}$$

The last formula may be inverted using $J^\beta_a(\theta_{x,i})\psi^a_\alpha(\theta_{x,i}) = \delta^\beta_\alpha$,

$$[D(\theta)\omega]^\alpha_{x,i} = J^\alpha_a(\theta_{xi})[\mathcal{D}(\theta)\omega]^a_{x,i} \ , \tag{79}$$

and we obtain for the lattice gauge-covariant derivative in the link basis the explicit formula, valid in any coordinate system

$$[D(\theta)\omega]^\alpha_{x,i} = J^\alpha_a(\theta_{xi})[R^{ab}(\theta)\omega^b_{x+\hat{i}} - \omega^a_x] \ , \tag{80}$$

where we have written $R^{ab}(\theta) = R^{ab}(U(\theta))$.

Hamiltonian

We shall now derive the lattice Coulomb-gauge Hamiltonian $H_{\rm coul}$ in any coordinate system with gauge condition $\sum_{x,i}(A_{x,i} - A_{x-\hat{i}})$. We insert the Faddeev-Popov identity into the formula for the inner-product (Φ, Ψ). Because the Haar measure $dU_{x,i} = \det\psi(A_{x,i})dA_{x,i}$ is invariant under left and right group multiplication, it is invariant under local gauge transformation. Consequently we obtain, just as in the continuum theory,

$$(\Phi, \Psi) = N \int_\Lambda \prod_{x,i} dA_{x,i}\psi(A_{x,i}) {\prod_x}' \delta[(\nabla^* A)_x] \det M'(A)\Phi^*(A)\Psi(A), \tag{81}$$

where N is the finite volume of the local gauge group, and we have written ψ for $\det\psi$. The primed product $\prod_x{}'$ extends over all sites but one, as explained in the preceding section, and correspondingly for M'.

The gauge condition $\nabla^* A = 0$ may be solved by a fourier decomposition of $A_{x,i}$ on a finite, periodic, cubic lattice, using longitudinal and transverse polarization vectors for finite k, and including $k = 0$ or harmonic modes. This allows to write

$$A_{x,i,\alpha} = A^{\rm Tr}_{x,i,\alpha} - (\nabla\sigma_\alpha)_{x,i}, \tag{82}$$

where $(\nabla^* A^{\rm Tr}_\alpha)_x = 0$. Here $A^{\rm Tr}$ includes both transverse and harmonic modes, $A^{\rm Tr} = A^{\rm tr} + A^{\rm h}$. The integral over link variables which represents the inner product becomes an integral over fourier coefficients. Because of $\delta(\nabla^* A)$, only the transverse and harmonic modes survive. We express the result as

$$(\Phi, \Psi) = N \int_\Lambda dA^{\rm Tr} \prod_{x,i} \psi(A^{\rm Tr}_{x,i}) \det M'(A^{\rm Tr})\Phi^*(A^{\rm Tr})\Psi(A^{\rm Tr}), \tag{83}$$

where $dA^{\rm Tr}$ designates the integral over all transverse and harmonic modes.

We wish to express the Hamiltonian $H_{KS} = T + V$ as an operator on the reduced space of gauge orbits parametrized by $A^{\rm Tr} \in \Lambda$. The potential energy operator V acts in the larger space by multiplication by a gauge-invariant function $V({}^gA) = V(A)$. On the reduced space it acts simply by multiplication by $V = V(A^{\rm Tr})$.

The Lie generator J_a is symmetric with respect to Haar measure, so we may write the kinetic energy operator T in the Kogut-Susskind representation as the quadratic form

$$(\Phi, T\Psi) = \int \prod_{x,i} dU_{x,i} (g_0^2/2) \sum_{x,i,a} (\mathcal{E}_{x,i,a}\Phi)^* \ (\mathcal{E}_{x,i,a}\Psi). \tag{84}$$

Only first derivatives of the wave-function appear here, which will allow us to apply Gauss's law as in the continuum theory. Here we have written

$$\mathcal{E}_{x,i,a} \equiv J_{x,i,a} \tag{85}$$

in order to emphasize the analogy with the continuum electric field. The operator $\mathcal{E}_{x,i,a}$ represents the electric field in the site basis. We also introduce the electric field in the link basis

$$E_{x,i,\alpha} \equiv i\partial/\partial A^\alpha_{x,i}. \tag{86}$$

The two are related by

$$\mathcal{E}_{x,i,a} = J^\alpha_a(A_{x,i})E_{x,i,\alpha}. \tag{87}$$

In this notation, Gauss's law constraint reads

$$(D(A)^* E)_x \Phi = 0. \tag{88}$$

The operator $\mathcal{E}_{x,i,a} = J_{x,i,a}$ transforms homogeneously under gauge transformation, so the integrand of the quadratic form T is gauge invariant. We insert the lattice Faddeev-Popov identity into this integral, and obtain

$$\begin{aligned}(\Phi, T\Psi) = N \int_\Lambda \prod_{x,i} dA_{x,i}\, \psi(A_{x,i}) \det M'(A) \\ \times {\prod_x}' \delta[(\nabla^* A)_x](g_0^2/2) \sum_{x,i,a} (\mathcal{E}_{x,i,a}\Phi)^* \; (\mathcal{E}_{x,i,a}\Psi).\end{aligned} \tag{89}$$

We wish to use the Gauss law constraint to express $\mathcal{E}$ as a derivative operator that acts only on transverse variables. For this purpose we make a Fourier decomposition of the electric field $E_{x,i,\alpha}$ on the periodic cubic lattice. For the $A^\alpha_{x,i}$-field we write the fourier decomposition, as

$$A_X = \sum_K \phi_{X,K} \tilde{A}_K, \tag{90}$$

where X represents position and vector indices, and K represents wave vector and polarization indices. Here the $\phi_{X,K}$ form a real orthonormal basis, and the inversion reads

$$\tilde{A}_K = \sum_X \phi_{X,K} A_X. \tag{91}$$

We have

$$\frac{\partial}{\partial A_X} = \sum_K \frac{\partial \tilde{A}_K}{\partial A_X} \frac{\partial}{\partial \tilde{A}_K} = \sum_K \phi_{X,K} \frac{\partial}{\partial \tilde{A}_K}, \tag{92}$$

so the fourier decompositions of A_X and of $\frac{\partial}{\partial A_X}$ are the same. For the electric field E_X we make the decomposition onto the same basis

$$E_X = \sum_K \phi_{X,K} \tilde{E}_K. \tag{93}$$

Consequently, from

$$E_X = i \frac{\partial}{\partial A_X}, \tag{94}$$

we obtain

$$\tilde{E}_K = i \frac{\partial}{\partial \tilde{A}_K}. \tag{95}$$

We choose a polarization basis which is adapted to the decomposition into transverse and longitudinal and harmonic parts. Corresponding to the decomposition $A = A^{\mathrm{Tr}} - \nabla\sigma$, we have

$$E = E^{\mathrm{Tr}} - \nabla\Omega, \tag{96}$$

where $\nabla^* A^{\mathrm{Tr}} = \nabla^* E^{\mathrm{Tr}} = 0$, and we have introduced the lattice color-Coulomb potential operator Ω. It follows that if A^{Tr} has the fourier decomposition

$$A^{\mathrm{Tr}}_X = \sum_{K \in T} \phi_{X,K} \tilde{A}_K, \tag{97}$$

then E^{Tr} has the fourier decomposition

$$E^{\mathrm{Tr}}_X = i \sum_{K \in T} \phi_{X,K} \frac{\partial}{\partial \tilde{A}_K}, \tag{98}$$

where T is the set of transverse and harmonic modes K. This expression defines E^{Tr} as a differential operator that acts within the space of functions of A^{Tr}.

We shall now use the Gauss law constraint $(D^*E)_x\Phi = 0$ to determine the color-Coulomb potential $\Omega_x\Phi$ on the reduced subspace, parametrized by $A^{\rm Tr} \in \Lambda$. We write Gauss's law in the form

$$D^*(E^{\rm Tr} - \nabla\Omega)\Phi = 0, \tag{99}$$

which gives

$$D^*\nabla\,\Omega\,\Phi = \rho\,\Phi, \tag{100}$$

where

$$(\rho)_x \equiv (D^*E^{\rm Tr})_x = [(D-\nabla)^*E^{\rm Tr}]_x\ . \tag{101}$$

This operator represents the color-charge density carried by the dynamical degrees of freedom of the gluons. We shall only need to solve Gauss's law for $\Omega\Phi$ in the reduced space where the Faddeev-Popov matrix is symmetric in the minimal Coulomb gauge, $M(A^{\rm Tr}) = D^*(A^{\rm Tr})\nabla = \nabla^*D(A^{\rm Tr})$. Thus we have also

$$(\nabla^*D\,\Omega)_x\,\Phi = \rho_x\,\Phi. \tag{102}$$

Because $\sum_x(\nabla^*V)_x = 0$ for any link variable V, the last equation is consistent only if the total charge $Q \equiv \sum_x \rho_x$ of the state Φ vanishes. Thus for gauge-invariant states in the minimal Coulomb gauge we have the constraint on the reduced space

$$Q\Phi = 0. \tag{103}$$

In the preceding section we have seen that in the lattice Coulomb gauge $D^*(A^{\rm Tr})\nabla$ is also a strictly positive matrix for $A^{\rm Tr}$ in the interior of the fundamental modular region Λ. Consequently the equation for Ω always has a solution,

$$\Omega\,\Phi = (D^*\nabla)^{-1}\rho\,\Phi\ . \tag{104}$$

With $\rho = D^*(A^{\rm Tr})E^{\rm Tr}$, this expresses the color-Coulomb potential operator Ω on the reduced space as a derivative with respect to the components of $A^{\rm Tr}$. We conclude that in the reduced space the color-electric field $E = E^{\rm Tr} - \nabla\Omega$ acts according to

$$E\Phi = E^{\rm Tr}\,\Phi - \nabla(D^*\nabla)^{-1}D^*E^{\rm Tr}\,\Phi, \tag{105}$$

where $E^{\rm Tr}$ is defined by its fourier expansion (98).

With this result we may express the kinetic energy operator T as a quadratic form on the reduced space,

$$\begin{aligned}(\Phi, T\Psi) = N\int_\Lambda dA^{\rm Tr}\prod_{x,i}\,\psi(A^{\rm Tr}_{x,i})\det M'(A^{\rm Tr})\\ \times(g_0^2/2)\sum_{x,i,a}(\mathcal{E}_{x,i,a}\Phi)^*\,(\mathcal{E}_{x,i,a}\Psi),\end{aligned} \tag{106}$$

where $\Phi = \Phi(A^{\rm Tr})$ and similarly for Ψ. Here $\mathcal{E}\Phi$ is defined by $\mathcal{E}_{x,i,a}\Phi \equiv J_a^\alpha(A^{\rm Tr}_{x,i})E_{x,i,\alpha}\Phi$, and $E\Phi$ is defined by the preceding equation. This completes the specification of the lattice Coulomb-gauge Hamiltonian, $H_{\rm coul} = T + V$.

CONCLUSION

We have derived by elementary methods the lattice Coulomb-gauge Hamilonian $H_{\rm coul}$ from the Kogut-Susskind Hamiltonian $H_{\rm KS}$. The essential step was to write $H_{\rm KS}$

as a quadratic form, involving only first derivatives, so Gauss's law $D^* E\Phi = 0$ could be used directly to solve for the color-Coulomb potential.

We have not addressed here invariance with respect to finite gauge transformations, which involves identification of points of the boundary of the fundamental region Λ, so that Gribov copies are avoided. These questions are addressed in[11], and have an important influence on the spectrum[1, 2, 3, 4].

However we believe that Gauss's law is essential to the phenomenon of confinement in QCD[5]. We have shown how, in Hamiltonian theory with lattice regularization, the imposition of gauge invariance in the form of Gauss's law leads to an instantaneous color-Coulomb potential in the lattice Coulomb-gauge Hamiltonian H_{coul}.

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