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Zooming-in on the SU(2) fundamental domain

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Abstract

For SU(2) gauge theories on the three-sphere we analyse the Gribov horizon and the boundary of the fundamental domain in the 18-dimensional subspace that contains the tunnelling path and the sphaleron and on which the energy functional is degenerate to second order in the fields. We prove that parts of this boundary coincide with the Gribov horizon with the help of bounds on the fundamental modular domain.

Dedicated to the memory of Dick Cutkosky

1. Introduction

From a perturbative point, the hamiltonian formulation of gauge theories is cumbersome, and the covariant path integral approach of Feynman is vastly superior. This remains true for certain non-perturbative features, like instanton contributions, which vanish to all orders in perturbation theory, and are determined by expanding around euclidean (classically forbidden) solutions by means of semiclassical or steepest descent approximations. When non-perturbative effects will be important and start to affect quantities that do not vanish perturbatively, the method breaks down dramatically [1,2]. When this happens, the hamiltonian formulation becomes superior, especially as long as only for a limited set of low-lying energy modes non-perturbative effects become appreciable. This has been our strategy in dealing with gauge theories in a finite volume. Due to asymptotic freedom, keeping the volume small allows us to keep the number of modes which behave non-perturbatively low. An essential feature of the non-perturbative behaviour is that the wave functional spreads out in configuration space to become sensitive to its non-trivial geometry. If wave functionals are localized within regions much smaller than the inverse curvature of the field space, the curvature has no effect on the wave functionals. At the other extreme, if the configuration space has non-contractible circles, the wave functionals are drastically affected

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by the geometry, or topology, when the support extends over the entire circle (i.e. bites in its own tail). We know from Singer [3] that the topology of the Yang–Mills configuration space $\mathcal{A}/\mathcal{G}$ ($\mathcal{A}$ is the collection of connections, $\mathcal{G}$ the group of local gauge transformations) is highly non-trivial. It also has a riemannian geometry [4] that can be made explicit, once explicit coordinates are chosen on $\mathcal{A}/\mathcal{G}$.

The geometry of the finite volume, which will be considered in this paper, is the one of a three-sphere [5]. The general arguments are of course independent of this geometry, in which case we will denote (compactified) three-space by M. Nevertheless, the details of the way $\mathcal{A}/\mathcal{G}$ is parametrized will crucially depend on M. This is already evident from Singer's argument [3] as the topology of $\mathcal{A}/\mathcal{G}$ does depend on M. We will come back to the consequences of this for the physics of the problem at the end of this paper. The physical interpretation of a hamiltonian [6] is clearest in the Coulomb gauge, $\partial_i A_i = 0$. But it has been known since Gribov's work [7] that this does not uniquely fix the gauge. Furthermore, there are coordinate "singularities" where the Faddeev–Popov determinant vanishes. Here the mapping between $\mathcal{A}/\mathcal{G}$ and the transverse vector potentials becomes degenerate.

2. Gribov and fundamental regions

Like using stereographic coordinates for a sphere, which leads to a coordinate singularity at one of the poles, coordinate singularities can be removed at the price of having different coordinate patches with transition functions at the overlaps. In gauge theory, these different coordinate patches can simply be seen as different gauge choices [2,8]. But this is somewhat cumbersome to formulate and most, but (as we shall see) not all coordinate singularities can be avoided if one restricts the set of transverse vector potentials to a fundamental region which constitutes a one to one mapping with $\mathcal{A}/\mathcal{G}$. This is achieved by minimizing the L^2 norm of the vector potential along the gauge orbit [9,10]

$$\|{}^g A\|^2 \equiv -\int_M d^3x \, \mathrm{tr}\left(\left(g^{-1}A_i g + g^{-1}\partial_i g\right)^2\right), \tag{1}$$

where the vector potential is taken anti-hermitian. For SU(2), in terms of the Pauli matrices τ_a, one has

$$A_i(x) = iA_i^a(x)\frac{\tau_a}{2},$$
$$g(x) = \exp\left(X(x)\right), \qquad X(x) = iX^a(x)\frac{\tau_a}{2}. \tag{2}$$

Expanding around the minimum of eq. (1), one easily finds

$$\|{}^g A\|^2 = \|A\|^2 + 2\int_M \mathrm{tr}\left(X\partial_i A_i\right) + \int_M \mathrm{tr}\left(X^\dagger \mathrm{FP}(A) X\right)$$

$$+\frac{1}{3}\int_M \mathrm{tr}\,(X\,[[A_i,X],\partial_i X]) + \frac{1}{12}\int_M \mathrm{tr}\,([D_i X,X][\partial_i X,X]) + \mathrm{O}(X^5)\,, \tag{3}$$

where $\mathrm{FP}(A)$ is the Faddeev–Popov operator ($\mathrm{ad}(A)X \equiv [A,X]$)

$$\mathrm{FP}(A) = -\partial_i D_i(A) = -\partial_i^2 - \partial_i \mathrm{ad}(A_i). \tag{4}$$

At the absolute minimum the vector potential is hence transverse, $\partial_i A_i = 0$, and $\mathrm{FP}(A)$ is a positive operator. The set of all transverse vector potentials with positive Faddeev–Popov operator is by definition the Gribov region Ω. It is a convex subspace of the set of transverse connections Γ, with a boundary $\partial\Omega$ that is called the Gribov horizon. At the Gribov horizon, the *lowest* eigenvalue of the Faddeev–Popov operator vanishes, and points on $\partial\Omega$ are hence associated with coordinate singularities. Any point on $\partial\Omega$ has a finite distance to the origin of field space and in some cases even uniform bounds can be derived [11,12].

The Gribov region is the set of *local* minima of the norm functional (3) and needs to be further restricted to the absolute minima to form a fundamental domain, which will be denoted by Λ. The fundamental domain is clearly contained within the Gribov region and can easily be shown to also be convex [9,10]. Its interior is devoid of gauge copies, whereas its boundary $\partial\Lambda$ will in general contain gauge copies, which are associated to those vector potentials where the absolute minima of the norm functional are degenerate [13]. If this degeneracy is continuous one necessarily has at least one zero eigenvalue for $\mathrm{FP}(A)$ and the Gribov horizon will touch the boundary of the fundamental domain at these so-called singular boundary points. By singular we mean here a coordinate singularity. There are so-called reducible connections [14], and $A = 0$ is the most important example, which are left invariant by a subgroup of $\mathcal{G}$. As here $\mathcal{G}$ does not act transitively, $\mathcal{A}/\mathcal{G}$ has curvature singularities at these reducible connections. They can be "blown up" by not dividing by their stabilizer. For S^3 one can proof $A = 0$ is the only such reducible connection in Λ. (Note $\mathcal{G}$ is the set of all gauge transformations, *including* those that are homotopically non-trivial.) The stabilizer of $A = 0$ is the group G (= SU(2)) of constant gauge transformations. This gauge degree of freedom is *not* fixed by the Coulomb gauge condition and therefore one still needs to divide by G to get the proper identification

$$\Lambda/\mathrm{G} = \mathcal{A}/\mathcal{G}\,. \tag{5}$$

Here Λ is considered to be the set of absolute minima modulo the boundary identifications, where the absolute minimum might be degenerate. It is these boundary identifications that restore the non-trivial topology of $\mathcal{A}/\mathcal{G}$. Furthermore, the existence of non-contractible spheres allows one to prove that singular boundary points cannot be avoided [13]. However, not all singular boundary points, even those associated with continuous degeneracies, need to be associated with non-contractible spheres. Note that absolute minima of the norm functional are degenerate along the constant gauge transformations, this is a trivial degeneracy, also giving rise to trivial zero-modes for the Faddeev–Popov operator, which we ignore. The action of G is essential to remove the curvature singularities mentioned above and also greatly facilitates the standard hamiltonian formulation of the theory [6]. There is no problem in dividing out G by demanding wave functionals to be gauge singlets (colourless states) with respect to G. In practice this means effectively that one minimizes the norm functional over $\mathcal{G}/\mathrm{G}$.

3. Gauge fields on the three-sphere

We will now specialize to the case of S^3, for which we will summarize the formalism that was developed in [1]. We embed S^3 in $\mathbb{R}^4$ by considering the unit sphere parametrized by a unit vector n_μ. We introduce the unit quaternions σ_μ and their conjugates $\bar{\sigma}_\mu = \sigma_\mu^\dagger$ by

$$\sigma_\mu = (\mathrm{id}, i\boldsymbol{\tau}), \qquad \bar{\sigma}_\mu = (\mathrm{id}, -i\boldsymbol{\tau}). \tag{6}$$

They satisfy the multiplication rules

$$\sigma_\mu\bar{\sigma}_\nu = \eta^\alpha_{\mu\nu}\sigma_\alpha, \qquad \bar{\sigma}_\mu\sigma_\nu = \bar{\eta}^\alpha_{\mu\nu}\sigma_\alpha, \tag{7}$$

where we used the 't Hooft η-symbols [15], generalised slightly to include a component symmetric in μ and ν for $\alpha = 0$. We can use η and $\bar{\eta}$ to define orthonormal framings of S^3, which were motivated by the particularly simple form of the instanton vector potentials in these framings. The framing for S^3 is obtained from the framing of $\mathbb{R}^4$ by restricting in the following equation the four-index α to a three-index a (see also ref. [22]; for $\alpha = 0$ one obtains the normal on S^3):

$$e^\alpha_\mu = \eta^\alpha_{\mu\nu}n_\nu, \qquad \bar{e}^\alpha_\mu = \bar{\eta}^\alpha_{\mu\nu}n_\nu. \tag{8}$$

The orthogonal matrix V that relates these two frames is given by

$$V^i_j = \bar{e}^i_\mu e^j_\mu = \frac{1}{2}\,\mathrm{tr}\,((n\cdot\sigma)\sigma_i(n\cdot\bar{\sigma})\sigma_j). \tag{9}$$

Note that e and $\bar{e}$ have opposite orientations. Each framing defines a differential operator

$$\partial^i = e^i_\mu\frac{\partial}{\partial x^\mu}, \qquad \bar{\partial}^i = \bar{e}^i_\mu\frac{\partial}{\partial x^\mu}, \tag{10}$$

to which belong SU(2) angular momentum operators, which for historical reasons will be denoted by $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$:

$$L^i_1 = \frac{i}{2}\,\partial^i, \qquad L^i_2 = \frac{i}{2}\,\bar{\partial}^i. \tag{11}$$

They are easily seen to satisfy the condition

$$\boldsymbol{L}^2_1 = \boldsymbol{L}^2_2. \tag{12}$$

The (anti-)instantons [16] in these framings, obtained from those on $\mathbb{R}^4$ by interpreting the radius in $\mathbb{R}^4$ as the exponential of the time t in the geometry $S^3\times\mathbb{R}$, become ($\boldsymbol{\varepsilon}$ and $\boldsymbol{A}$ are defined with respect to the framing e^a_μ for instantons and with respect to the framing $\bar{e}^a_\mu$ for anti-instantons)

$$A_0 = \frac{\boldsymbol{\varepsilon}\cdot\boldsymbol{\sigma}}{2(1+\varepsilon\cdot n)}, \qquad A_a = \frac{\boldsymbol{\sigma}\wedge\boldsymbol{\varepsilon} - (u+\varepsilon\cdot n)\boldsymbol{\sigma}}{2(1+\varepsilon\cdot n)}, \tag{13}$$

where

$$u = \frac{2s^2}{1+b^2+s^2}, \qquad \varepsilon_\mu = \frac{2sb_\mu}{1+b^2+s^2}, \qquad s = \lambda \mathrm{e}^t. \tag{14}$$

The instanton describes tunnelling from $A=0$ at $t=-\infty$ to $A_a = -\sigma_a$ at $t=\infty$, over a potential barrier that is lowest when $b_\mu \equiv 0$. This configuration (with $b_\mu = 0$, $u=1$) corresponds to a sphaleron [17], i.e. the vector potential $A_a = -\sigma_a/2$ is a saddle point of the energy functional with one unstable mode, corresponding to the direction (u) of tunnelling. At $t=\infty$, $A_a = -\sigma_a$ has zero energy and is a gauge copy of $A_a = 0$ by a gauge transformation $g = n\cdot\bar{\sigma}$ with winding number one, since

$$n\cdot\sigma\partial_a n\cdot\bar{\sigma} = -\sigma_a. \tag{15}$$

We will be concentrating our attention on the modes that are degenerate in energy to lowest order with the modes that describe tunnelling through the sphaleron and "anti-sphaleron". The latter corresponds to the configuration with the minimal barrier height separating $A=0$ from its gauge copy by a gauge transformation $g = n\cdot\sigma$ with winding number -1. The anti-sphaleron is actually a copy of the sphaleron under this gauge transformation, as can be seen from eq. (13), since

$$n\cdot\bar{\sigma}e^a_\mu\sigma_a n\cdot\sigma = -\bar{e}^a_\mu\sigma_a, \tag{16}$$

(with which we correct a typo in eq. (12) of ref. [1]. This also affected the sign of eq. (83) of this reference. We stick to the present more natural conventions.) The two-dimensional space containing the tunnelling paths through the sphalerons is consequently parametrized by u and v through

$$A_\mu(u,v) = \left(-ue^a_\mu - v\bar{e}^a_\mu\right)\frac{\sigma_a}{2} = A_i(u,v)e^i_\mu,$$
$$A_i(u,v) = (-u\delta^a_i - vV^a_i)\frac{\sigma_a}{2} = -u\frac{\sigma_i}{2} + v\,n\cdot\bar{\sigma}\frac{\sigma_i}{2}n\cdot\sigma. \tag{17}$$

The gauge transformation with winding number -1 is easily seen to map $(u,v) = (w,0)$ into $(u,v) = (0,2-w)$. In particular, as discussed above, it maps the sphaleron $(1,0)$ to the anti-sphaleron $(0,1)$.

The Gribov and fundamental regions will be discussed in the next section. After that we will investigate the 18-dimensional space defined by

$$A_\mu(c,d) = \left(c_i^a e_\mu^i + d_j^a \bar{e}_\mu^j\right) \frac{\sigma_a}{2} = A_i(c,d) e_\mu^i,$$
$$A_i(c,d) = \left(c_i^a + d_j^a V_i^j\right) \frac{\sigma_a}{2}. \tag{18}$$

One easily verifies that the c and d modes are mutually orthogonal and that $A(c,d)$ satisfies the Coulomb gauge condition

$$\partial_i A_i(c,d) = 0. \tag{19}$$

This space contains the (u,v) plane through $c_i^a = -u\delta_i^a$ and $d_i^a = -v\delta_i^a$. The significance of this 18-dimensional space is that the energy functional [1]

$$\mathcal{V}(c,d) \equiv -\int_{S^3} \frac{1}{2} \operatorname{tr}(F_{ij}^2)$$
$$= \mathcal{V}(c) + \mathcal{V}(d) + \frac{2\pi^2}{3} \left\{ (c_i^a)^2 (d_j^b)^2 - (c_i^a d_j^a)^2 \right\}, \tag{20}$$
$$\mathcal{V}(c) = 2\pi^2 \left\{ 2(c_i^a)^2 + 6 \det c + \frac{1}{4} [(c_i^a c_i^a)^2 - (c_i^a c_j^a)^2] \right\} \tag{21}$$

is degenerate to second order in c and d. Indeed, the quadratic fluctuation operator $\mathcal{M}$ in the Coulomb gauge, defined by

$$-\int_{S^3} \frac{1}{2} \operatorname{tr}(F_{ij}^2) = -\int_{S^3} \operatorname{tr}(A_i \mathcal{M}_{ij} A_j) + O(A^3),$$
$$\mathcal{M}_{ij} = 2\boldsymbol{L}_1^2 \delta_{ij} + 2\left(\boldsymbol{L}_1 + \boldsymbol{S}\right)_{ij}^2, \quad S_{ij}^a = -i\varepsilon_{aij} \tag{22}$$

has $A(c,d)$ as its eigenspace for the eigenvalue 4. Contrary to what was claimed in ref. [1] this is the lowest eigenvalue. The 12-fold degenerate eigenvalue 3 turns out to exist of purely longitudinal modes, rather than transverse modes.

4. Gribov and fundamental regions for $A(u,v)$

Let us analyse the condition for $\|{}^g A\|^2$ to be minimal a little closer. We can write

$$\|{}^g A\|^2 - \|A\|^2 = \int \operatorname{tr}\left(A_i^2\right) - \int \operatorname{tr}\left(\left(g^{-1} A_i g + g^{-1} \partial_i g\right)^2\right)$$
$$= \int \operatorname{tr}\left(g^\dagger \mathrm{FP}_{1/2}(A)\, g\right) \equiv \langle g, \mathrm{FP}_{1/2}(A)\, g \rangle, \tag{23}$$

where $\mathrm{FP}_{1/2}(A)$ is the Faddeev–Popov operator generalised to the fundamental representation:

$$\mathrm{FP}_t(A) = -\partial_i^2 - \frac{i}{t} A_i^a T_t^a \partial_i. \tag{24}$$

Here $\boldsymbol{T}_t$ are the hermitian gauge generators in the spin-t representation:

$$T^a_{1/2} = \frac{\tau_a}{2}, \qquad T^a_1 = \mathrm{ad}(\frac{\tau_a}{2}). \tag{25}$$

They are angular momentum operators that satisfy $\boldsymbol{T}_t^2 = t(t+1)\mathrm{id}$. At the critical points $A \in \Gamma$ of the norm functional, (recall $\Gamma = \{A \in \mathcal{A} | \partial_i A_i = 0\}$), $\mathrm{FP}_t(A)$ is an hermitian operator. Furthermore, $\mathrm{FP}_1(A)$ in that case coincides with the Faddeev–Popov operator $\mathrm{FP}(A)$ in eq. (4).

In eq. (23) $\mathrm{FP}_{1/2}(A)$ is defined as an hermitian operator acting on the vector space $\mathcal{L}$ of functions g over S^3 with values in the space of the quaternions $\mathbb{H} = \{q_\mu \sigma_\mu | q_\mu \in \mathbb{R}\}$. To be precise, we should require $g \in W_2^1(S^3, \mathbb{H})$, with $W_2^1(M, V)$ the Sobolev space of functions on M with values in the vector space V, whose first derivative is continuous and square integrable. We use the standard isomorphism between the complex spinors ψ (on which $\boldsymbol{T}_{1/2}$ acts in the standard way) and the quaternions, by combining ψ and $\bar{\sigma}_2\psi^*$. To be specific, if $\psi_1 = q_0 + iq_3$ and $\psi_2 = iq_1 - q_2$, then $g = (\psi, \bar{\sigma}_2\psi^*) = q \cdot \sigma$ is a quaternion (on which $\boldsymbol{T}_{1/2}$ now acts by matrix multiplication). Charge conjugation symmetry, $C\psi = \bar{\sigma}_2\psi^*$, implies that $[\mathrm{FP}_{1/2}(A), C] = 0$ and guarantees that the operator preserves this isomorphism. Also note that this symmetry implies that all eigenvalues are two-fold degenerate. The gauge group $\mathcal{G}$ is contained in $\mathcal{L}$ by restricting to the unit quaternions: $\mathcal{G} = \{g \in \mathcal{L} | g = g_\mu \sigma_\mu, g_\mu \in \mathbb{R}, g_\mu g_\mu = 1\}$.

We can define Λ in terms of the absolute minima (apart from the boundary identifications) over $g \in \mathcal{G}$ of $\langle g, \mathrm{FP}_{1/2}(A)\ g\rangle$

$$\Lambda = \{A \in \Gamma | \min_{g \in \mathcal{G}} \langle g, \mathrm{FP}_{1/2}(A)\ g\rangle = 0\}. \tag{26}$$

When minimizing the same functional over the larger space $\mathcal{L}$ one obviously should find a smaller result, i.e.

$$\mathcal{G} \subset \mathcal{L} \Rightarrow \min_{g \in \mathcal{G}} \langle g, \mathrm{FP}_{1/2}(A)\ g\rangle \geqslant \min_{g \in \mathcal{L}} \langle g, \mathrm{FP}_{1/2}(A)\ g\rangle. \tag{27}$$

Writing

$$\tilde{\Lambda} = \{A \in \Gamma | \min_{g \in \mathcal{L}} \langle g, \mathrm{FP}_{1/2}(A)\ g\rangle = 0\}, \tag{28}$$

it follows directly from eq. (27) that $\tilde{\Lambda} \subset \Lambda$. Since $\tilde{\Lambda}$ is related to the minimum of a functional on a *linear* space, it will be easier to analyse $\tilde{\Lambda}$ than Λ. We were inspired by appendix A of ref. [18] for this consideration. Remarkably, we will be able to prove that the boundary $\partial\tilde{\Lambda}$ will touch the Gribov horizon $\partial\Omega$. This establishes the existence of singular points on the boundary of the fundamental domain due to the inclusion $\tilde{\Lambda} \subset \Lambda \subset \Omega$.

In the (u, v) plane one easily finds that

$$\mathrm{FP}_t(A(u,v)) = 4\boldsymbol{L}_1^2 + \frac{2}{t} u \boldsymbol{L}_1 \cdot \boldsymbol{T}_t + \frac{2}{t} v \boldsymbol{L}_2 \cdot \boldsymbol{T}_t \tag{29}$$

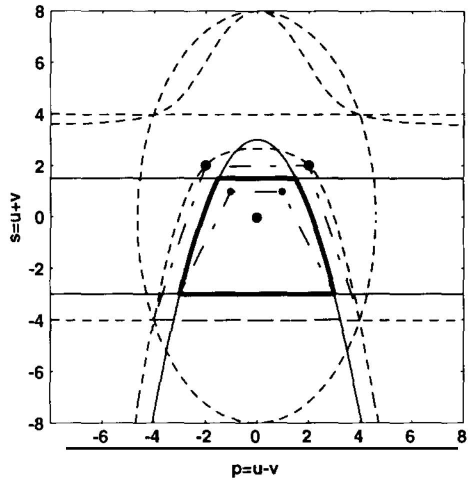

Fig. 1. Location, for the (u,v) plane of the classical vacua (large dots), sphalerons (smaller dots), bounds on the Faddeev-Popov operator for $l=\frac{1}{2}$ and $l=1$ (short-long dashed curves), zeros of the adjoint determinant (solid lines for $l=\frac{1}{2}$, dashed lines for $l=1$) and the Gribov horizon (fat sections).

For fixed angular momentum $l\neq 0$ (where $\boldsymbol{L}_1^2=\boldsymbol{L}_2^2=l(l+1)$), the eigenvalues of $\boldsymbol{L}_a\cdot\boldsymbol{T}_{1/2}$ (which is a kind of spin-orbit coupling) are $-(l+1)/2$ and $l/2$. This is easily seen to imply that for g with $\boldsymbol{L}_1^2 g=l(l+1)g$ $(l\neq 0)$

$$-\frac{l+1}{2}\|g\|^2\leqslant\langle g,\boldsymbol{L}_a\cdot\boldsymbol{T}_{1/2}\,g\rangle\leqslant\frac{l}{2}\|g\|^2, \tag{30}$$

and hence

$$\langle g,\mathrm{FP}_{1/2}(A(u,v))\,g\rangle\geqslant\|g\|^2\,[4l(l+1)-(2l+1)(|u|+|v|)-u-v], \tag{31}$$

whereas of course for $l=0$ we have $\langle g,\mathrm{FP}_{1/2}(A)\,g\rangle=0$. Now let $\tilde{\Lambda}_l$ be the region in the (u,v) plane where the right-hand side of eq. (31) is positive:

$$\tilde{\Lambda}_l=\{(u,v)|\,[4l(l+1)-(2l+1)(|u|+|v|)-u-v]\geqslant 0\}\,. \tag{32}$$

Then one easily verifies that $\tilde{\Lambda}_l\subset\tilde{\Lambda}_{l+1/2}$ for $l\neq 0$. For illustration we have drawn the boundaries of $\tilde{\Lambda}_{1/2}$ and $\tilde{\Lambda}_1$ in Fig. 1 (the two nested trapeziums). Consequently, restricted to the (u,v) plane $\tilde{\Lambda}_{1/2}$, the trapezium spanned by the four points $(1,0)$, $(0,1)$, $(-3,0)$ and $(0,-3)$, is contained in $\tilde{\Lambda}$. As one easily checks, the vector potentials belonging to the sphalerons at $(1,0)$ and $(0,1)$ have the same norm. Since they are related by a gauge transformation (as was proved earlier) and lie on the boundary of $\tilde{\Lambda}_{1/2}$, these sphalerons have to be on the boundary of the fundamental domain $\partial\Lambda$. Hence, $\tilde{\Lambda}_{1/2}$ is seen to provide already quite a strong bound.

Before constructing $\tilde{\Lambda}$ in the (u, v) plane, it is instructive to consider first the Gribov horizon, which is given by the zeros of the Faddeev–Popov determinant $\det(\mathrm{FP}_1(A))$. The operator $\mathrm{FP}_t(A(u,v))$ as given by eq. (29) not only commutes with $\boldsymbol{L}_1^2 = \boldsymbol{L}_2^2$, but also with $\boldsymbol{J}_t$, where

$$\boldsymbol{J}_t = \boldsymbol{L}_1 + \boldsymbol{L}_2 + \boldsymbol{T}_t \tag{33}$$

Using the quantum numbers (l, j_t, j_t^z) one can easily diagonalize $\mathrm{FP}_t(A(u,v))$ for low values of l. Note that the eigenvalues are independent of j_t^z. Defining the scalar and pseudoscalar helicity combinations

$$s = u + v, \qquad p = u - v, \tag{34}$$

we take from ref. [19] the results

$$\det\left(\mathrm{FP}_1(A(u,v))\,|_{l=1/2}\right) = (3-2s)\left(9-3s-2p^2\right)^3(3+s)^5, \tag{35}$$

$$\det\left(\mathrm{FP}_1(A(u,v))\,|_{l=1}\right) = 512(8-2s)((8-2s)^2 - s^2 + p^2(2s-7))^3 \times(64-s^2-3p^2)^5(8+2s)^7, \tag{36}$$

the zeros of which are also exhibited in Fig. 1 (solid lines for $l = \frac{1}{2}$, dashed lines for $l = 1$). The Gribov horizon in the (u, v) plane is indicated by the fat lines and is completely determined by the $l = \frac{1}{2}$ sector, a fact that we will now prove. Note that the set of infinitesimal gauge transformations $L_g = \{X : S^3 \to \mathrm{su}(2)\}$, where su(2) is the Lie algebra for SU(2) (i.e. the traceless quaternions), is contained in $\mathcal{L}$. It is easy to verify that for $X \in L_g$, we have for all vector potentials A

$$\langle X, \mathrm{FP}_1(A)\,X\rangle = \langle X, \mathrm{FP}_{1/2}(A)\,X\rangle. \tag{37}$$

This fact will enable us to use the same bounds for FP_1 and $\mathrm{FP}_{1/2}$ (cf. eq. (27)):

$$L_g \subset \mathcal{L} \Rightarrow \min_{X\in L_g}\langle X, \mathrm{FP}_1(A)\,X\rangle \geqslant \min_{g\in\mathcal{L}}\langle g, \mathrm{FP}_{1/2}(A)\,g\rangle. \tag{38}$$

Hence all zeros of the Faddeev–Popov determinant with $l \geqslant 1$ lie outside the trapezium $\tilde{\Lambda}_1$, spanned by the four points (2,0), (0,2), (−4,0), (0,−4).

This proves that $\mathrm{FP}_1(A) \geqslant 0$ within the region bounded by the zeros of eq. (35). We see from Fig. 1 that along the line $s = u+v = 3$, for $|p| = |u-v| \leqslant 3$, the Gribov horizon coincides with $\partial\Lambda$ and consequently these are singular boundary points. Note that therefore it is necessary that the term third order in X in eq. (3) has to vanish if $\mathrm{FP}(A)\,X = 0$. As on the Gribov horizon any non-trivial zero-mode has $l = \frac{1}{2}$, whereas $A(u,v)$ has $l = 0$ or $l = 1$, this third order term vanishes along the whole Gribov horizon in the (u, v) plane (all its points are therefore bifurcation points [13]). It can, however, also be shown that these singular boundary points are *not* associated with non-contractible spheres (see appendix A).

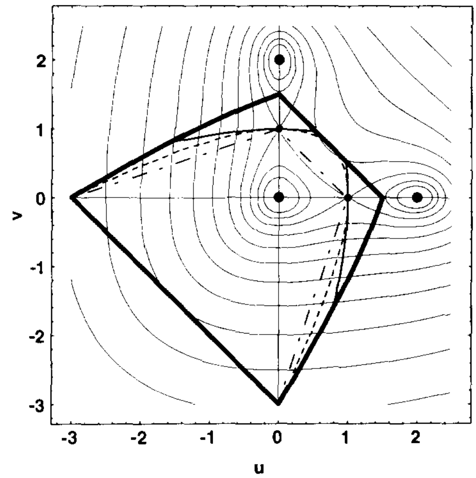

Fig. 2. Location of the classical vacua (large dots), sphalerons (smaller dots), bounds on the Faddeev–Popov operator for $l = \frac{1}{2}$ (short–long dashed lines), the Gribov horizon (fat sections), zeros of the fundamental determinant in the sector $l = \frac{1}{2}$ (dashed curves) and part of the boundary of the domain (full curves). Also indicated are the lines of equal potential in units of 2^n times the sphaleron energy.

Next we will construct $\tilde{\Lambda}$ in the (u, v) plane to get an even sharper bound on Λ. It is by now obvious that this will follow from finding $\det(\mathrm{FP}_{1/2}(A(u, v)))$ in the $l = \frac{1}{2}$ sector. A straightforward computation yields

$$\det\left(\mathrm{FP}_{1/2}(u, v)\mid_{l=1/2}\right) = 9\,(3+s)^4\left(3 - 2s - p^2\right)^2, \tag{39}$$

where the multiplicity of 4 comes from the $j = \frac{3}{2}$ state and the multiplicity 2 from the two $j = \frac{1}{2}$ states in the decomposition $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}$. In Fig. 2 the bordering parabola going through the points $(0, -3)$, $(1, 0)$, $(\frac{3}{4}, \frac{3}{4})$, $(0, 1)$ and $(-3, 0)$, cut off by the line $s = -3$, forms the boundary of $\tilde{\Lambda}$. As in the case of the Gribov horizon, $\tilde{\Lambda}$ is completely determined by the $l = \frac{1}{2}$ sector, since also the zeros of $\det\left(\mathrm{FP}_{1/2}(A(u, v))\right)$ with $l \geqslant 1$ lie outside the trapezium $\tilde{\Lambda}_1$. Notice that we have now also shown that $(u, v) = (\frac{3}{4}, \frac{3}{4})$ is a singular boundary point.

We recall that in ref. [19] part of $\partial\Lambda$ in the (u, v) plane was constructed by expanding around the sphalerons, which are known to be on $\partial\Lambda$. One solves for fixed (u, v) near $(0, 1)$ for the extremum of $\langle g, \mathrm{FP}_{1/2}(A(u, v))\, g\rangle$ with respect to $g = n \cdot \bar{\sigma} \exp(X)$, where it can be shown that $X = -\boldsymbol{n} \cdot \boldsymbol{\sigma} f(n_0)$. This leads to a second-order differential equation, solved by

$$f(x) = x \sum_{j=1} \sum_{k=0}^{j-1} a_{j,k}(v) u^j x^{2k}, \tag{40}$$

with

$$a_{1,0}(v) = \frac{2}{2+v}, \qquad a_{2,0} = \frac{-2(v^2+6v-16)}{(2+v)^3(10+v)},$$
$$a_{2,1}(v) = \frac{4(6+v)}{(v+2)^2(v+10)}, \cdots \tag{41}$$

Substituting this now back in eq. (23) and demanding equality of norms yields $v(u)$

$$v(u) = 1 - \frac{1}{9}u^2 - \frac{2}{81}u^3 - \frac{25}{2673}u^4 - \frac{1238}{264627}u^5$$
$$-\frac{172\,442}{66\,950\,631}u^6 - \frac{687\,429\,956}{457\,339\,760\,361}u^7 + \mathrm{O}(u^8), \tag{42}$$

giving the part of $\partial\Lambda$ in the (u,v) plane going through the anti-sphaleron at $u=0$. We have drawn the maximal extension to the Gribov horizon, but not all of it is expected to coincide with $\partial\Lambda$. Interchanging the two coordinates gives the part of $\partial\Lambda$ going through the sphaleron. Both parts are indicated by the curves in Fig. 2. They are consistent with the inclusion $\tilde{\Lambda} \subset \Lambda$. In this figure also lines of equal potential (eq. (20)) are drawn.

5. Gribov and fundamental regions for $A(c,d)$

We will now generalize our discussion to the 18-dimensional field space, parametrized by $A(c,d)$ in eq. (18). For this case one has

$$\mathrm{FP}_t(A(c,d)) = 4\boldsymbol{L}_1^2 - \frac{2}{t}c_i^a T_t^a L_1^i - \frac{2}{t}d_i^a T_t^a L_2^i. \tag{43}$$

This still commutes with $\boldsymbol{L}_1^2 = \boldsymbol{L}_2^2$, but for arbitrary (c,d) there are in general no other commuting operators (except for the charge conjugation operator C for $t=\frac{1}{2}$).

We first calculate the analogues of the regions $\tilde{\Lambda}_l$, as defined in eq. (32). We decompose

$$c_i^a = \sum_{j=1}^{9} c_j\,(b_j)_i^a, \qquad d_i^a = \sum_{j=1}^{9} d_j\,(b_j)_i^a, \tag{44}$$

with c_j and d_j coefficients and the set $\{b_j\}$ a basis of $\mathbb{R}^{3,3}$, consisting of orthogonal matrices ($b_j^T = b_j^{-1}$) with unit determinant. We then have

$$\mathrm{FP}_t(A(c,d)) = 4\boldsymbol{L}_1^2 - \frac{2}{t}c_j T_t^a\,(b_j)_i^a L_1^i - \frac{2}{t}d_j T_t^a\,(b_j)_i^a L_2^i$$
$$= 4\boldsymbol{L}_1^2 - \frac{2}{t}c_j \boldsymbol{T}_t\cdot\boldsymbol{L}_{1,j} - \frac{2}{t}d_j \boldsymbol{T}_t\cdot\boldsymbol{L}_{2,j}, \tag{45}$$

with $L^a_{k,j} \equiv (b_j)^a_i L^i_k$ $(k = 1,2)$ proper angular momentum operators. As in eq. (31), for g an eigenfunction of $\boldsymbol{L}_1^2$ with eigenvalue $l(l+1)$ $(l \neq 0)$, we find the bound

$$\langle g, \mathrm{FP}_{1/2}(A(c,d))\, g\rangle \geqslant \|g\|^2 \left[4l(l+1) - (2l+1)\sum_{j=1}^{9}(|c_j| + |d_j|) + \sum_{j=1}^{9}(c_j + d_j)\right]. \tag{46}$$

As before, we define $\tilde{\Lambda}_l$ as the polyhedra where the right-hand side of eq. (46) is positive. They are nested polyhedra, i.e. $\tilde{\Lambda}_l \subset \tilde{\Lambda}_{l+1/2}$. Hence we have the inclusion $\tilde{\Lambda}_{1/2} \subset \tilde{\Lambda} \subset \Lambda \subset \Omega$. If we restrict ourselves to the two-dimensional subspace where all but one of the c_j $(-u)$ and all but one of the d_j $(-v)$ are zero, we precisely recover the situation of the previous section. The bounds will, however, depend on the particular choice of the b_j matrices. The sharpest bound is obtained by forming the *union* of all $\tilde{\Lambda}_l$ obtained by these various choices.

We now turn to the computation of the Faddeev–Popov determinants. In the sector $l = \frac{1}{2}$, which is $4(2t+1)$ dimensional, the problem of computing $\det(\mathrm{FP}_t(A(c,d)))$ is still manageable. A suitable basis is given by $|s_1, s_2, s_3\rangle$, with s_i the eigenvalues of the third component of the three angular momentum operators $\boldsymbol{L}_1$, $\boldsymbol{L}_2$ and $\boldsymbol{T}_t$. For $t = 1$ it is actually more convenient to consider $|s_1, s_2\rangle_a$, where a is the vector component. Using

$$L_1^{\pm}|s_1,s_2\rangle_a = (\tfrac{1}{2} \mp s_1)|-s_1,s_2\rangle_a\,, \quad L_2^{\pm}|s_1,s_2\rangle_a = (\tfrac{1}{2} \mp s_2)|s_1,-s_2\rangle_a\,,$$
$$L_i^3|s_1,s_2\rangle_a = s_i|s_1,s_2\rangle_a\,, \quad T_1^b|s_1,s_2\rangle_a = i\varepsilon_{bac}|s_1,s_2\rangle_c\,, \tag{47}$$

where as usual $L_a^{\pm} = L_a^1 \pm iL_a^2$, one easily writes down the matrix for $M \equiv \mathrm{FP}_1(A(c,d))$ in this sector ($c_{\pm}^a \equiv c_1^a \mp ic_2^a$ and $d_{\pm}^a \equiv d_1^a \mp id_2^a$):

$$M|s_1,s_2\rangle_b = -i\sum_{\alpha=\pm}\left\{(\tfrac{1}{2} - \alpha s_1)c_\alpha^a\varepsilon_{abc}|-s_1,s_2\rangle_c + (\tfrac{1}{2} - \alpha s_2)d_\alpha^a\varepsilon_{abc}|s_1,-s_2\rangle_c\right\} + (3\delta_{bc} - 2i\varepsilon_{abc}(s_1c_3^a + s_2d_3^a))\,|s_1,s_2\rangle_c\,. \tag{48}$$

In particular for the choice

$$c_i^a = x_i\delta_i^a\,, \qquad d_i^a = y_i\delta_i^a\,, \tag{49}$$

one can, with the help of Mathematica [21], check that the following holds:

$$\det\left(\mathrm{FP}_1(A(c,d))|_{l=\frac{1}{2}}\right) = F(x_1+y_1, x_2+y_2, x_3+y_3) \times F(x_1-y_1, x_2-y_2, x_3+y_3) \times F(x_1-y_1, x_2+y_2, x_3-y_3)$$

$$\times F(x_1+y_1,x_2-y_2,x_3-y_3),$$
$$F(z)\equiv 2\prod_i z_i-3\sum_i z_i^2+27. \tag{50}$$

To obtain the result for general (c,d) we first observe that we have invariance under rotations generated by $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ and under constant gauge transformations generated by $\boldsymbol{T}_t$, implying that $\det(\mathrm{FP}_t(A(c,d))|_{l=1/2})$ is invariant under

$$c_i^a\to(ScR_1)_i^a,\qquad d_i^a\to(SdR_2)_i^a, \tag{51}$$

with R_1, R_2 and S orthogonal matrices with unit determinant (note that the $\boldsymbol{L}_{k,j}$, introduced in eq. (45), are nothing but the $\boldsymbol{L}_k$ generators, rotated by $R_1=R_2=b_j$). This also allows us to understand the large amount of symmetry in eq. (50), as a permutation of the x_i (y_i) and a simultaneous change of the sign of two of the x_i (y_i) is the remnant of this symmetry, when restricted to the diagonal configurations of eq. (49). The result for the generalization of eq. (50) to arbitrary (c,d) is presented in appendix B. Here we will treat the case $d=0$. Using eq. (51) we first diagonalize c_i^a and then express $F(\boldsymbol{x})$ in terms of the complete set of rotational and gauge invariant parameters of c_i^a

$$\det c=\prod_i x_i,\quad \mathrm{Tr}\,(cc^{\mathrm t})=\sum_i x_i^2,\quad \mathrm{Tr}\,(cc^{\mathrm t}cc^{\mathrm t})=\sum_i x_i^4, \tag{52}$$

which implies $F(\boldsymbol{x})=2\det c-3\,\mathrm{Tr}\,(cc^{\mathrm t})+27$ and

$$\det\Big(\mathrm{FP}_1(A(c,0))\,|_{l=1/2}\Big)=\big(2\det c-3\,\mathrm{Tr}\,(cc^{\dagger})+27\big)^4. \tag{53}$$

This can also be easily derived by constructing the three dimensional invariant subspace for $(c_i^a=x_i\delta_i^a,\ d=0)$, spanned by the three vectors $n_i\sigma_i$ (no sum over i), with respect to which the matrix for M takes the form

$$M(n_1\sigma_1,n_2\sigma_2,n_3\sigma_3)=\begin{pmatrix}3&x_3&x_2\\x_3&3&x_1\\x_2&x_1&3\end{pmatrix}\begin{pmatrix}n_1\sigma_1\\n_2\sigma_2\\n_3\sigma_3\end{pmatrix}, \tag{54}$$

whose determinant coincides with $F(\boldsymbol{x})$. It is not too difficult to construct the three other three-dimensional invariant subspaces with identical determinants.

Two special cases in this class were first considered by Cutkosky [20]:

$$\begin{aligned}&\mathrm{I}:\quad c_i^a=\mathrm{diag}(-u+y,-u+y,-u-2y),\\&\mathrm{II}:\quad c_i^a=\mathrm{diag}(-u+x,-u-x,-u).\end{aligned} \tag{55}$$

For F, which determines the Faddeev–Popov determinant at $l=\frac{1}{2}$, we find [20]

$$\begin{aligned}F_{\mathrm I}&=(u+2y+3)[(u+3)(3-2u)+2(2u-3)y-2y^2],\\F_{\mathrm{II}}&=(u+3)^2(3-2u)+2(u-3)x^2.\end{aligned} \tag{56}$$

The associated zeros are drawn respectively in Figs. 3 and 4. Note that the (u, y) plane admits a global gauge symmetry $(u, y) \to \frac{1}{3}(4y - u, y + 2u)$ generated by $S = \text{diag}(-1, -1, 1)$, which maps the vacuum at $(u, y) = (2, 0)$ to a vacuum at $(-\frac{2}{3}, \frac{4}{3})$. To conclude that these zeros coincide with the Gribov horizon, we have to show that the Faddeev–Popov operator for all $l \geqslant 1$ is positive within the region bounded by these zeros. Using eq. (38), it is sufficient to show that these zeros lie within $\tilde{\Lambda}_1$, the region obtained from the bound on $\text{FP}_{1/2}(A)$ in eq. (46). Clearly we should try to construct this bound by taking for b_j the diagonal orthogonal matrices diag(1, 1, 1), diag(1, −1, −1), diag(−1, −1, 1) and diag(−1, 1, −1). It turns out to be sufficient to consider the union of the bounds obtained by applying eq. (46) for the four triplets of possible choice of diagonal b_j. In terms of general x_i, this leads to the four bounds

$$\begin{aligned}
&4l(l+1) - \tfrac{1}{2}(2l+1)(|x_1 + x_2| + |x_1 + x_3| + |x_2 + x_3|) - x_1 - x_2 - x_3 \geqslant 0,\\
&4l(l+1) - \tfrac{1}{2}(2l+1)(|x_1 - x_2| + |x_1 - x_3| + |x_2 + x_3|) - x_1 + x_2 + x_3 \geqslant 0,\\
&4l(l+1) - \tfrac{1}{2}(2l+1)(|x_1 + x_2| + |x_1 - x_3| + |x_2 - x_3|) + x_1 + x_2 - x_3 \geqslant 0,\\
&4l(l+1) - \tfrac{1}{2}(2l+1)(|x_1 - x_2| + |x_1 + x_3| + |x_2 - x_3|) + x_1 - x_2 + x_3 \geqslant 0.
\end{aligned} \tag{57}$$

The union of these polyhedra respects the gauge and rotation symmetry and we take it as the definitions of $\tilde{\Lambda}_l$ for $d = 0$. They are again nested, such that it is sufficient to show that the convex regions bounded by the zeros of the Faddeev–Popov operator in the $l = \frac{1}{2}$ sector are contained within $\tilde{\Lambda}_1$. From Figs. 3 and 4 we see that this is indeed the case, allowing the identification of $\partial\Omega$ (fat curves) and $\partial\tilde{\Lambda}$ (dashed or full lines) with the zeros of respectively $\det(\text{FP}_1(A)|_{l=1/2})$ and $\det(\text{FP}_{1/2}(A)|_{l=1/2})$.

We now turn to the calculation of $\det(\text{FP}_{1/2}(A(c,d))|_{l=1/2})$ which will allow us to construct $\tilde{\Lambda}$ and to find possible further singular points on the boundary of the fundamental domain. In this case the basis $|s_1, s_2, s_3\rangle$, which was defined earlier, is a convenient one for the $l = \frac{1}{2}$ invariant subspace. Using the invariance as given by eq. (51), we can take c_i^a diagonal and d_i^a symmetric:

$$c_i^a = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}, \qquad d_i^a = \begin{pmatrix} y_1 & z_1 & z_2 \\ z_1 & y_2 & z_3 \\ z_2 & z_3 & y_3 \end{pmatrix}. \tag{58}$$

With $L_a^\pm$ as before and $T_{1/2}^\pm = T_{1/2}^1 \pm iT_{1/2}^2$, we obtain the following expression:

$$\begin{aligned}
\text{FP}_{1/2}(A(c,d)) = 3 &+ 2iz_1\left(T_{1/2}^+ L_2^+ - T_{1/2}^- L_2^-\right)\\
&-2(z_2 - iz_3)\left(T_{1/2}^3 L_2^+ + T_{1/2}^+ L_2^3\right)\\
&-2(z_2 + iz_3)\left(T_{1/2}^3 L_2^- + T_{1/2}^- L_2^3\right)
\end{aligned}$$

$$-4x_3 T^3_{1/2} L^3_1 - (x_1 + x_2)\left(T^+_{1/2} L^-_1 + T^-_{1/2} L^+_1\right)$$
$$-(x_1 - x_2)\left(T^+_{1/2} L^+_1 + T^-_{1/2} L^-_1\right)$$
$$-4y_3 T^3_{1/2} L^3_2 - (y_1 + y_2)\left(T^+_{1/2} L^-_2 + T^-_{1/2} L^+_2\right)$$
$$-(y_1 - y_2)\left(T^+_{1/2} L^+_2 + T^-_{1/2} L^-_2\right). \qquad (59)$$

In order to express the final result in invariants, we introduce the matrices X and Y via

$$X^a_b = (cc^t)^a_b, \; Y^a_b = (dd^t)^a_b. \qquad (60)$$

Using Mathematica [21] and expressing the result in terms of traces of products of X and Y, we obtain an expression which is manifestly invariant:

$$\det\left(\mathrm{FP}_{1/2}(A(c,d))\,|_{l=1/2}\right) = \mathcal{F}^2, \qquad (61)$$
$$\mathcal{F} \equiv 81 - 18\,\mathrm{Tr}\,(X + Y) + 24(\det c + \det d)$$
$$-(\mathrm{Tr}\,(X - Y))^2 + 2\,\mathrm{Tr}\,((X - Y)^2).$$

With this, one easily reproduces the result of eq. (39) ($x_i = -u, y_i = -v, s = u + v, p = u - v$). Note the overall square, which is a consequence of the two-fold degeneration of the eigenvalues due to the fact that $\mathrm{FP}_{1/2}$ commutes with the charge conjugation operator $\mathcal{C}$. Such a non-trivial commuting operator does not exist for FP_1, whose determinant does not factorise and was hence much more difficult to calculate (see appendix B).

For $d = 0$ we find

$$\det\left(\mathrm{FP}_{1/2}(A(c,0))\,|_{l=1/2}\right) = \mathcal{F}^2$$
$$= \left\{81 - 18\,\mathrm{Tr}\,X + 24\det c - (\mathrm{Tr}\,X)^2 + 2\,\mathrm{Tr}\,(X^2)\right\}^2. \qquad (62)$$

In Figs. 3 and 4 we have drawn $\partial\tilde{\Lambda}$ obtained from the zeros of eq. (62) for the two cases of eq. (55):

$$\mathcal{F}_{\mathrm{I}} = 3(3 + u + 2y)^2(u - 1)(4y - 3 - u),$$
$$\mathcal{F}_{\mathrm{II}} = 3(u + 3)(u - 1)[4x^2 - (3 + u)^2], \qquad (63)$$

which indeed provides further singular boundary points (since $\partial\tilde{\Lambda} \cap \partial\Omega$ is not empty). Also the part of $\partial\Lambda$ that contains the sphaleron is easily derived from the fact that the gauge transformation with winding number -1, $g = n \cdot \sigma$, leads to

$$^{g}A(x_i, 0) = -(2 + x_i)n \cdot \bar{\sigma}\frac{\sigma_i}{2} n \cdot \sigma, \qquad (64)$$

for arbitrary diagonal configurations. Equality of norms implies the equation $\sum x_i + 3 = 0$. This means, since $\tilde{\Lambda} \subset \Lambda$, that in Fig. 3 the edges of $\tilde{\Lambda}$ passing

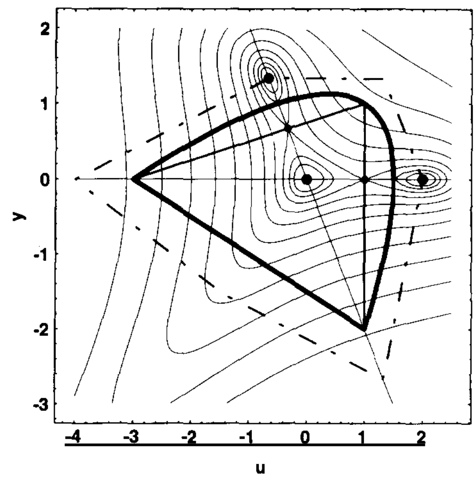

Fig. 3. Location, for the (u, y) plane of the classical vacua (large dots), sphalerons (smaller dots), bounds on the fundamental Faddeev–Popov operator for $l = 1$ (short–long dashed curves), boundary of the fundamental domain (solid lines) and the Gribov horizon (fat curves), as well as the lines of equal potential.

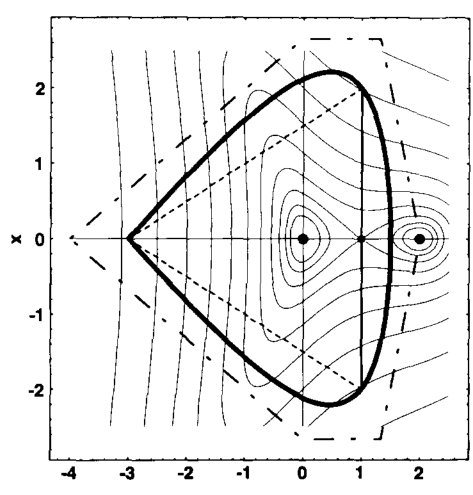

Fig. 4. Location, for the (u, x) plane of the classical vacua (large dots), sphalerons (smaller dots), bounds on the fundamental Faddeev–Popov operator for $l = 1$ (short–long dashed curves), boundary of the fundamental domain (solid line), zeros of the fundamental determinant for $l = \frac{1}{2}$ (dashed lines) and the Gribov horizon (fat curves), as well as the lines of equal potential.

through the sphalerons coincide with $\partial \Lambda$, a fact that can also be concluded from the convexity of Λ. Hence, in Fig. 3 $\tilde{\Lambda}$ coincides with Λ and the line $u + 2y = -3$ consists of singular boundary points. In Fig. 4 it is not excluded that, at the dashed lines, $\partial\tilde{\Lambda}$ does *not* coincide with $\partial\Lambda$, as was also the case for the (u,v) plane, see Fig. 2. We can settle this issue by considering the embedding of the (u,x) plane within the three dimensional space of the x_i.

All surfaces to be constructed have to respect the symmetries of the permutations and the double sign flips of the x_i coordinates. We first consider $\tilde{\Lambda}_1$, see eq. (57), which can be seen as a tetrahedron spanned by the points $(4,4,4)$, $(-4,-4,4)$, $(4,-4,-4)$ and $(-4,4,-4)$, enlarged by adding to each face a symmetric pyramid, whose tips are given by the points $(-2,-2,-2)$, $(2,2,-2)$, $(-2,2,2)$ and $(2,-2,2)$ (corresponding to the copies of the classical vacuum at $\boldsymbol{x}=0$). For general l, $\tilde{\Lambda}_l$ can be constructed from this twelve faced polygon by scaling the corners of the tetrahedron with $l+1$ and the tips of the pyramids with l, from which their nested nature is obvious. A special case arises for $l=\frac{1}{2}$, where the pyramids are of zero height, such that $\tilde{\Lambda}_{1/2}$ is a tetrahedron. It is a remarkable fact that the fundamental Faddeev–Popov determinant in the sector $l=\frac{1}{2}$ (eq. (62)) vanishes on $\partial\tilde{\Lambda}_{1/2}$. As this is enclosed by $\tilde{\Lambda}_1$, where all eigenvalues of $\mathrm{FP}_{1/2}(A)$ with $l\geqslant 1$ are strictly positive, we conclude that $\tilde{\Lambda}=\tilde{\Lambda}_{1/2}$ (the tetrahedron spanned by $(3,3,3)$, $(-3,-3,3)$, $(3,-3,-3)$ and $(-3,3,-3)$). The convex region bounded by the zeros of the adjoint determinant (eq. (53)) can be shown to form a surface contained in $\tilde{\Lambda}_1$ that can be visualized by stretching a rubber sheet around this tetrahedron, fixed at its edges and slightly inflated. This surface forms the Gribov horizon $\partial\Omega$, since also all eigenvalues of $\mathrm{FP}_1(A)$ with $l\geqslant 1$ are strictly positive inside $\tilde{\Lambda}_1$. Because of the inclusion $\tilde{\Lambda}\subset\Lambda\subset\Omega$, all points on the edges of the tetrahedron are singular boundary points. As all the faces of this tetrahedron contain a sphaleron, which we have proven earlier to be on the boundary of the fundamental domain (the edges of the tetrahedron are singular points on the same boundary), we conclude (using the convexity of Λ) that $\tilde{\Lambda}=\Lambda$. This is consistent with eq. (64), where equality of norms gives the equation that describes the face of the tetrahedron through the sphaleron at $(-1,-1,-1)$, $\sum x_i+3=0$. The other three faces follow from flipping the sign of two of the x_i, which is a symmetry. In Fig. 5 we have drawn the fundamental modular domain for $d=0$ in $\boldsymbol{x}$ space and in Fig. 6 we give the Gribov horizon and the edges of $\partial\tilde{\Lambda}_1$ (dashed lines). This completes the construction of the fundamental domain for $d=0$.

6. Discussion

In this paper we have analysed in detail the boundary of the fundamental domain for SU(2) gauge theories on the three-sphere. We have constructed it completely for the gauge fields with $\boldsymbol{L}_1^2=0$ and have provided partial results for the 18-dimensional space of modes that are degenerate with these in energy to second order in the fields. Especially, the interesting point of explicitly demonstrating the presence of singular boundary points, i.e. points where the boundary of the fundamental domain coincides with the Gribov horizon, was addressed. In ref. [13] existence of singular boundary points was proven on the basis of the

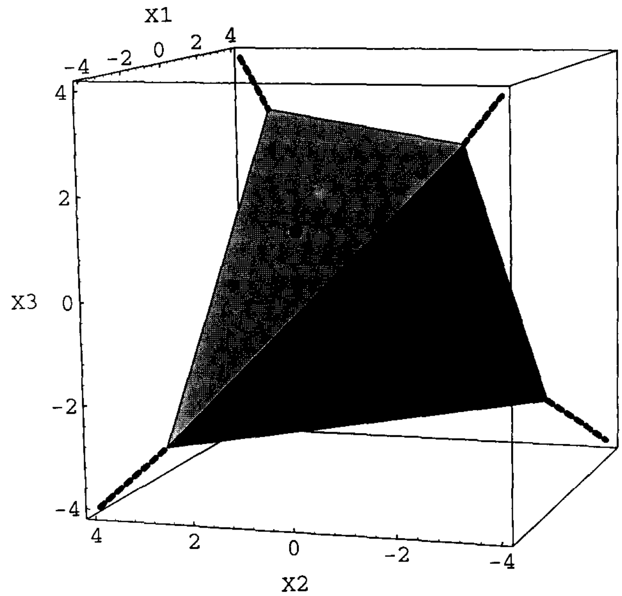

Fig. 5. The fundamental modular domain for constant gauge fields on S^3, with respect to the "instanton" framing e_μ^a, in the diagonal representation $A_a = x_a\sigma_a$ (no sum over a). By the dots on the faces we indicate the spalerons, whereas the dashed lines represent the symmetry axes of the tetrahedron.

presence of non-contractible spheres [3] in the physical configuration space $\mathcal{A}/\mathcal{G}$. This does not prove that all singular boundary points are necessarily associated with such non-contractible spheres, which we demonstrated for the case at hand (see appendix A). It is also important to note that it is necessary to divide $\mathcal{A}$ by the set of *all* gauge transformations, including those that are homotopically non-trivial, to get the physical configuration space. All the non-trivial topology is then retrieved by the identifications of points on the boundary of the fundamental domain. Zwanziger [12] (appendix E) has constructed, for the case of $M = T^3$, a gauge function parametrized by a two-sphere for which the norm functional is degenerate, but its vector potential lies outside the fundamental domain when also the anti-periodic gauge transformations are considered as part of $\mathcal{G}$ [13].

As we already mentioned in the introduction, the knowledge of the boundary identifications is important in the case that the wave functionals spread out in configuration space to such an extent that they become sensitive to these identifications. This happens at large volumes, whereas at very small volumes the wave functional is localized around $A = 0$ and one need not worry about these non-perturbative effects. That these effects can be dramatic, even at relatively small volumes (above a tenth of a fermi across), was demonstrated for the case of the torus [2,8]. However, for that case the structure of the fundamental domain (restricted to the abelian zero-energy modes) is a hypercube [13] and deviates

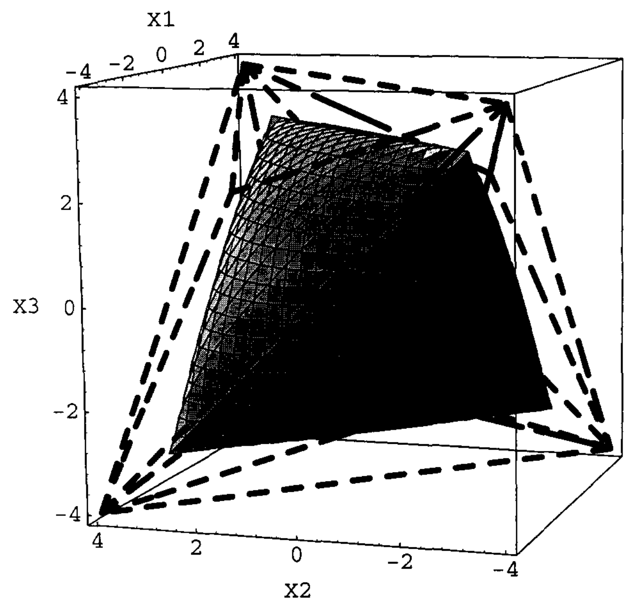

Fig. 6. The Gribov horizon for constant gauge fields on S^3, with respect to the "instanton" framing e^a_μ, in the diagonal representation $A_a = x_a \sigma_a$ (no sum over a). The dashed lines represent the edges of $\tilde{\Lambda}_1$, which encloses the Gribov horizon, whereas the latter encloses the fundamental modular domain, coinciding with it at the singular boundary points along the edges of the tetrahedron of Fig. 5.

considerably from the fundamental domain of the three-sphere. One can hence conclude that something needs to happen to the structure of the theory, to avoid that the infinite volume limit in the infrared depends on the way this limit is taken, e.g. by scaling different geometries, like T^3 or S^3. One way to avoid this undesirable effect is that the vacuum is unstable against domain formation. We have discussed this at length elsewhere and refer the reader to refs. [1,8,13] for further details.

To conclude, let us return to the issue of the singular boundary points. Many of the coordinate singularities due to the vanishing of the Faddeev–Popov determinant (which plays the role of the jacobian for the change of variables to the gauge fixed degrees of freedom [4] in the hamiltonian formulation) are screened by the boundary of the fundamental domain. Although the singular boundary points form a set of zero measure in the configuration space, they can nevertheless be important for the dynamics. Near these points we have to choose different coordinates and formulate the necessary transition functions to move from one to the other choice. It is clear that this is difficult to formulate in all rigour in the infinite dimensional field space. As the domain formation is anticipated to be due to the fact that the energies of the low-lying states flow over the sphaleron energy, we can study the dynamics of the domain formation as long as the energies of all singular boundary points are well above the sphaleron energy. From Figs. 2–4 we see that this is indeed the case in the 18-dimensional subspace we have considered. For the higher energy modes the tail of the wave functional will be so small at the singular boundary points, that we need not worry about their influence on the spectrum. In this way we have a well-defined window in which the non-perturbative treatment of a finite number of modes will allow us to calculate the low-lying spectrum of the theory (see ref. [1] for the set-up of this analysis).

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Appendix A

In this appendix we shall demonstrate that the singular part of the boundary of the fundamental domain in the (u,v) plane does not contain points associated to non-contractible spheres. Such a non-contractible sphere implies at least a one parameter gauge function $g(t)$ along which the norm functional is degenerate and minimal. We will first show that this implies that the fourth order term in eq. (3) needs to be negative. After that we show that this is not the case for the singular boundary points under consideration. We write

$$g(t) = \exp(X(t)), \quad X(t) = tX_1 + t^2X_2 + t^3X_3 + t^4X_4 + \mathrm{O}(t^5). \quad \text{(A.1)}$$

For all t one should have that $\partial_i({}^{g(t)}A_i) = 0$. Using the fact that

$$\partial_i \frac{\mathrm{d}}{\mathrm{d}t}({}^{g(t)}A_i) = \partial_i D_i({}^{g(t)}A)\ (g^\dagger(t)\frac{\mathrm{d}}{\mathrm{d}t}g(t)), \quad \text{(A.2)}$$

one easily concludes that X_1 is a zero-mode for the Faddeev–Popov operator at $t = 0$, whereas the first-order term in t gives the equation

$$\mathrm{FP}(A)X_2 = -\tfrac{1}{2}[\partial_a X_1, [X_1, A_a]]. \quad \text{(A.3)}$$

By considering the inner product of both sides of this equation with X_1, we conclude that it can only have a solution provided the third-order term (for $X = X_1$) in eq. (3) vanishes, as should be obviously true since we are considering $A \in \Lambda$, i.e. the norm functional is at its absolute minimum. We now have sufficient information to compute $\|{}^{g(t)}A\|$ to fourth order in t (the explicit forms of X_3 and X_4 drop out of the expression for this order when we use that $\mathrm{FP}(A)\ X_1 = 0$ and $\partial_a A_a = 0$, respectively)

$$\left\|{}^{g(t)}A\right\|^2 = \|A\|^2 + t^4 \left\{ 3\langle X_2, \mathrm{FP}(A)\, X_2\rangle - \frac{1}{12}\langle [D_a X_1, X_1], [\partial_a X_1, X_1]\rangle \right\} + \mathrm{O}(t^5) . \tag{A.4}$$

To obtain this result we used the Jacobi identity, partial integration, eq. (A.3) and the assumption that X_1 is an eigenfunction for $\boldsymbol{L}_1^2$. Since the first term is positive definite, the norm functional can only be degenerate if the second term is negative.

We now specialize to the case $A(u,v)$ and the singular boundary points that occur at $(u,v) = (\frac{3}{4}, \frac{3}{4})$ and $u+v = -3$ for $|u-v| \leqslant 3$. The zero-modes for the Faddeev–Popov operator at these configurations are easily seen to be given by

$$X_1 = n_a Q^{ab} \sigma_b, \qquad Q^{ab} = Q^{ba}, \tag{A.5}$$

where the trace part of the symmetric (real) tensor corresponds to the $j = 0$ (eq. (33)) zero-mode (at $u + v = \frac{3}{2}$) and the traceless part to the $j = 2$ zero-mode (at $u + v = -3$). It is now straightforward to substitute X_1 (eq. (A.5)) in eq. (A.4). After some algebra we find

$$-\langle [D_a X_1, X_1], [\partial_a X_1, X_1]\rangle / 2\pi^2 = 2\,\mathrm{Tr}\,(Q^4) - 2(\mathrm{Tr}\,(Q^2))^2 + \frac{1}{3}(u+v)\left[(\mathrm{Tr}\,Q)^2\,\mathrm{Tr}\,(Q^2) + 2\,\mathrm{Tr}\,Q\,\mathrm{Tr}\,(Q^3) - (\mathrm{Tr}\,(Q^2))^2 - 2\,\mathrm{Tr}\,(Q^4)\right] \tag{A.6}$$

With $Q^{ab} = \delta_{ab}$ and $u + v = \frac{3}{2}$ we find for the right-hand side 3, which is positive. For a traceless Q we find $4\,\mathrm{Tr}\,(Q^4) - (\mathrm{Tr}\,(Q^2))^2$ at $u+v = -3$, which is likewise strictly positive. None of the singular boundary points can therefore be associated with a continuous degeneracy.

Appendix B

In this appendix we will calculate $\det(\mathrm{FP}_1(A(c,d))|_{l=1/2})$ for general (c,d), thus extending the result in eq. (50). We will write the result in terms of the matrices X and Y defined in eq. (60). It is useful to introduce the following quantities:

$$D_1 \equiv \left(\varepsilon_{abc} c_i^a c_j^b d_k^c\right)^2 = \mathrm{Tr}\,Y\left((\mathrm{Tr}\,X)^2 - \mathrm{Tr}\,(X^2)\right) + \mathrm{Tr}\,(X^2 Y) - 2\,\mathrm{Tr}\,X\,\mathrm{Tr}\,(XY),$$

$$D_2 \equiv \left(\varepsilon_{abc} c_i^a d_j^b d_k^c\right)^2 = \mathrm{Tr}\,X\left((\mathrm{Tr}\,Y)^2 - \mathrm{Tr}\,(Y^2)\right) + \mathrm{Tr}\,(XY^2) - 2\,\mathrm{Tr}\,Y\,\mathrm{Tr}\,(XY),$$

$$D_3 \equiv \left(\mathrm{Tr}\,(X^2) - (\mathrm{Tr}\,X)^2\right)\left(\mathrm{Tr}\,(Y^2) - (\mathrm{Tr}\,Y)^2\right) + 2\,\mathrm{Tr}\,\left((XY)^2\right)$$

$$-2\,(\mathrm{Tr}\,(XY))^2,$$

$$F_0 \equiv 27 + 2\det c + 2\det d - 3\,\mathrm{Tr}\,X - 3\,\mathrm{Tr}\,Y,$$

$$F_1 \equiv 36\,\mathrm{Tr}\,(XY) + 24(\mathrm{Tr}\,X - \det c)(\mathrm{Tr}\,Y - \det d) + 2D_1 + 2D_2,$$

$$\begin{aligned}F_2 \equiv\ & 6D_3 + 8\det c\det d\,(27 - \det c - \det d + 3\,\mathrm{Tr}\,X + 3\,\mathrm{Tr}\,Y)\\ &+4\det c\left(9(\mathrm{Tr}\,(Y^2) - (\mathrm{Tr}\,Y)^2) - D_2\right)\\ &+4\det d\left(9(\mathrm{Tr}\,(X^2) - (\mathrm{Tr}\,X)^2) - D_1\right),\end{aligned}$$

$$\begin{aligned}F_3 \equiv\ & 48\,(\det c\det d)\,D_3 + 1296\,\mathrm{Tr}\left((XY)^2\right)\\ &+1728\left(\det c\det d\,\mathrm{Tr}\,(XY) - \det c\,\mathrm{Tr}\,(XY^2) - \det d\,\mathrm{Tr}\,(X^2Y)\right)\\ &+96(\det c)^2\left(9\,\mathrm{Tr}\,(Y^2) - 6(\mathrm{Tr}\,Y)^2 - 2\det d(\mathrm{Tr}\,X + \mathrm{Tr}\,Y)\right)\\ &+96(\det d)^2\left(9\,\mathrm{Tr}\,(X^2) - 6(\mathrm{Tr}\,X)^2 - 2\det c(\mathrm{Tr}\,X + \mathrm{Tr}\,Y)\right)\\ &+4\left(D_1 + 4\det c\det d - 2(\det d)^2 - 12\det c\,\mathrm{Tr}\,Y\right)^2\\ &+4\left(D_2 + 4\det c\det d - 2(\det c)^2 - 12\det d\,\mathrm{Tr}\,X\right)^2\\ &-16\,(\det c + \det d)^2\left((\det c)^2 - 6\det c\det d + (\det d)^2\right),\end{aligned}$$

$$\begin{aligned}R \equiv\ & 576(\mathrm{Tr}\,X + \mathrm{Tr}\,Y)\left(\mathrm{Tr}\,(X^2Y^2) - \mathrm{Tr}\left((XY)^2\right)\right)\\ &+576\left(\mathrm{Tr}\,(X^2YXY) - \mathrm{Tr}\,(X^3Y^2) + \mathrm{Tr}\,(Y^2XYX) - \mathrm{Tr}\,(Y^3X^2)\right).\end{aligned} \tag{B.1}$$

In terms of this list of invariants we have

$$\det\left(\mathrm{FP}_1(A(c,d))\,|_{l=1/2}\right) = 2(F_0^4 + F_3) - (F_0^2 + F_1)^2 - 8F_0F_2 + R. \tag{B.2}$$

The significance of this expansion becomes clear when we substitute the diagonal choices for c and d, eq. (49), for which

$$F_0 = E_0, \quad F_1 = \prod_{i=1}^{3} E_i, \quad F_2 = \sum_{i=1}^{3} E_i^2, \quad F_3 = \sum_{i=1}^{3} E_i^4, \tag{B.3}$$

with

$$E_0 = 27 + 2\prod x_i + 2\prod y_i - 3\sum x_i^2 - 3\sum y_i^2,$$

$$E_1 = 6x_1y_1 - 2y_1x_2x_3 - 2x_1y_2y_3,$$

$$E_2 = 6x_2y_2 - 2y_2x_3x_1 - 2x_2y_3y_1,$$

$$E_3 = 6x_3y_3 - 2y_3x_1x_2 - 2x_3y_1y_2. \quad \text{(B.4)}$$

Most important is that R vanishes identically for eq. (49). Hence the way we came to eq. (B.2) was to first extend E_μ to invariant combinations, which is unique up to invariant polynomials that vanish identically for the diagonal configuration of eq. (49). The space of invariant polynomials with this property is 11 dimensional and by fitting the determinant with numerical values substituted for c and d, one can easily (with the help of Mathematica [21]) solve for the 11 coefficients. The final result of eq. (B.2) has then been checked for a large number of random choices of c and d.

References

[1] P. van Baal and N. D. Hari Dass, Nucl. Phys. B385 (1992) 185

[2] J. Koller and P. van Baal, Nucl. Phys. B302 (1988) 1;
P. van Baal, Acta Phys. Pol. B20 (1989) 295

[3] I. Singer, Commun. Math. Phys. 60 (1978) 7

[4] O. Babelon and C. Viallet, Commun. Math. Phys. 81 (1981) 515

[5] R.E. Cutkosky, J. Math. Phys. 25 (1984) 939;
R.E. Cutkosky and K. Wang, Phys. Rev. D37 (1988) 3024;
R.E. Cutkosky, Czech. J. Phys. 40 (1990) 252

[6] N.M. Christ and T.D. Lee, Phys. Rev. D22 (1980) 939

[7] V. Gribov, Nucl. Phys. B139 (1978) 1

[8] P. van Baal, *in* Probabilistic methods in quantum field theory and quantum gravity, ed. P.H. Damgaard et al. (Plenum, New York, 1990) p. 131

[9] M.A. Semenov-Tyan-Shanskii and V.A. Franke, Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V.A. Steklov AN SSSR, 120 (1982) 159 [Translation: (Plenum, New York, 1986) p. 1999]

[10] G. Dell'Antonio and D. Zwanziger, *in* Probabilistic methods in quantum field theory and quantum gravity, ed. P.H. Damgaard et al. (Plenum Press, New York, 1990) p. 107;
G. Dell'Antonio and D. Zwanziger, Commun. Math. Phys. 138 (1991) 291

[11] G. Dell'Antonio and D. Zwanziger, Nucl. Phys. B326 (1989) 333

[12] D. Zwanziger, Nucl. Phys. B378 (1992) 525

[13] P. van Baal, Nucl. Phys. B369 (1992) 259

[14] S. Donaldson and P. Kronheimer, The geometry of four manifolds (Oxford Univ. Press, Oxford, 1990);
D. Freed and K. Uhlenbeck, Instantons and four-manifolds, M.S.R.I. publ. Vol. I (Springer, New York, 1984)

[15] G. 't Hooft, Phys. Rev. D14 (1976) 3432

[16] A. Belavin, A. Polyakov, A. Schwarz and Y. Tyupkin, Phys. Lett. B59 (1975) 85;
M. Atiyah, V. Drinfeld, N. Hitchin and Yu. Manin, Phys. Lett. A65 (1978) 185

[17] F.R. Klinkhamer and M. Manton, Phys. Rev. D30 (1984) 2212

[18] D. Zwanziger, Fundamental modular region, Boltzann factor and area law in lattice gauge theory, preprint NYU-TP-9351, August 1993

[19] P. van Baal and R.E. Cutkosky, Int. J. Mod. Phys. A (Proc. Suppl.) 3A (1993) 323

[20] R.E. Cutkosky, private communication, November 1992

[21] S. Wolfram et al., Mathematica (Addison-Wesley, New York, 1991)

[22] M. Lüscher, Phys. Lett. B70 (1977) 321