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Abelian projected monopoles - to be or not to be

Pierre van Baal[a]

[a]Instituut-Lorentz for Theoretical Physics, University of Leiden,
PO Box 9506, NL-2300 RA Leiden, The Netherlands.

We analyse what happens with two merging constituent monopoles for the $SU(3)$ caloron. Identified through degenerate eigenvalues (the singularities or defects of the abelian projection) of the Polyakov loop, it follows that there are defects that are not directly related to the actual constituent monopoles.

1. Introduction

Finite temperature instantons (calorons) have a rich structure if one allows the Polyakov loop, $P(\vec{x}) = P\exp(\int_0^\beta A_0(\vec{x},t)dt)$ in the periodic gauge $A_\mu(t,\vec{x}) = A_\mu(t+\beta,\vec{x})$, to be non-trivial at spatial infinity (specifying the holonomy). It implies the spontaneous breakdown of gauge symmetry. For a charge one $SU(n)$ caloron, the location of the n constituent monopoles can be identified through:
i. Points where two eigenvalues of the Polyakov loop coincide, which is where the $U^{n-1}(1)$ symmetry is partially restored to $SU(2)\times U^{n-2}(1)$.
ii. The centers of mass of the (spherical) lumps.
iii. The Dirac monopoles (or rather dyons, due to self-duality) as the sources of the abelian field lines, extrapolated back to the cores. If well separated and localised, all these coincide [1,2]. Here we study the case of two constituents coming close together for $n\geq 3$, with an example for $SU(3)$.

The eigenvalues of $\mathcal{P}_\infty \equiv \lim_{|\vec{x}|\to\infty} P(\vec{x})$ can be ordered by a constant gauge transformation W_∞

$$W_\infty^\dagger \mathcal{P}_\infty W_\infty = \mathcal{P}_\infty^0 = \exp[2\pi i\,\mathrm{diag}(\mu_1,\ldots,\mu_n)],$$
$$\mu_1 \leq \ldots \leq \mu_n \leq \mu_{n+1} \equiv 1+\mu_1, \qquad (1)$$

with $\sum_{m=1}^n \mu_m = 0$. The constituent monopoles have masses $8\pi^2\nu_i$, where $\nu_i \equiv \mu_{i+1}-\mu_i$ (using the classical scale invariance to put the extent of the euclidean time direction to one, $\beta=1$). In the same way we can bring $P(\vec{x})$ to this form by a *local* gauge function, $P(\vec{x}) = W(\vec{x})P_0(\vec{x})W^\dagger(\vec{x})$. We note that $W(\vec{x})$ (unique up to a residual abelian gauge rotation) and $P_0(\vec{x})$ will be smooth, except where two (or more) eigenvalues coincide.

The ordering shows there are n different types of singularities (called defects [3]), for each of the *neighbouring* eigenvalues to coincide. The first $n-1$ are associated with the basic monopoles (as part of the inequivalent $SU(2)$ subgroups related to the generators of the Cartan subgroup). The n^{th} defect arises when the first and the last eigenvalue (still neighbours on the circle) coincide. Its magnetic charge ensures charge neutrality of the caloron. The special status [4,5] of this defect also follows from the so-called Taubes winding [6], supporting the non-zero topological charge [1].

2. Puzzle

To analyse the lump structure when two constituents coincide, we recall the simple formula for the $SU(n)$ action density [4].

$$\mathrm{Tr}F_{\mu\nu}^2(x) = \partial_\mu^2\partial_\nu^2 \log\left[\tfrac{1}{2}\mathrm{tr}(\mathcal{A}_n\cdots\mathcal{A}_1) - \cos(2\pi t)\right],$$
$$\mathcal{A}_m \equiv \frac{1}{r_m}\begin{pmatrix} r_m & |\vec{y}_m - \vec{y}_{m+1}| \\ 0 & r_{m+1}\end{pmatrix}\begin{pmatrix} c_m & s_m \\ s_m & c_m\end{pmatrix}, \qquad (2)$$

with $\vec{y}_m$ the center of mass location of the m^{th} constituent monopole. We defined $r_m \equiv |\vec{x}-\vec{y}_m|$, $c_m \equiv \cosh(2\pi\nu_m r_m)$, $s_m \equiv \sinh(2\pi\nu_m r_m)$, as well as $\vec{y}_{n+1}\equiv\vec{y}_1$, $r_{n+1}\equiv r_1$. We are interested in the case where the problem of two coinciding constituents in $SU(n)$ is mapped to the $SU(n-1)$ caloron. For this we restrict to the case where $\vec{y}_m = \vec{y}_{m+1}$ for some m, which for $SU(3)$ is *always* the case when two constituents coincide. Since now $r_m = r_{m+1}$, one easily verifies that $\mathcal{A}_{m+1}\mathcal{A}_m = \mathcal{A}_{m+1}[\nu_{m+1}\to\nu_m+\nu_{m+1}]$, describing a *single* constituent monopole (with properly combined mass), reducing eq. (2) to the action density for the $SU(n-1)$ caloron, with $n-1$ constituents.

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The topological charge can be reduced to surface integrals near the singularities with the use of $\mathrm{tr}(P^\dagger dP)^3 = d\,3\mathrm{tr}((P_0^\dagger A_W P_0 + 2P_0^\dagger dP_0)\wedge A_W) = d\,3\mathrm{tr}(A_W\wedge(2A_W \log P_0 + P_0 A_W P_0^\dagger))$, where $A_W \equiv W^\dagger dW$. If one assumes *all* defects are pointlike, this can be used to show that for each of the n types the (net) number of defects has to equal the topological charge, the type being selected by the branch of the logarithm (associated with the n elements in the center) [3]. One might expect the defects to merge when the constituent monopoles do. A triple degeneracy of eigenvalues for $SU(3)$ implies the Polyakov loop takes a value in the center. Yet this can be shown *not* to occur for the $SU(3)$ caloron with *unequal* masses. We therefore seem to have (at least) one more defect than the number of constituents, when $\vec{y}_m \to \vec{y}_{m+1}$.

3. Example

We will study in detail a generic example in $SU(3)$, with $(\mu_1,\mu_2,\mu_3) = (-17,-2,19)/60$. We denote by $\vec{z}_m$ the position associated with the m^{th} constituent where two eigenvalues of the Polyakov loop coincide. In the gauge where $\mathcal{P}_\infty = \mathcal{P}_\infty^0$ (see eq. (1)), we established numerically [2] that

$$\begin{aligned} P_1&=P(\vec{z}_1)=\mathrm{diag}(\ e^{-\pi i\mu_3},\ e^{-\pi i\mu_3},\ e^{2\pi i\mu_3}),\\ P_2&=P(\vec{z}_2)=\mathrm{diag}(\ e^{2\pi i\mu_1},\ e^{-\pi i\mu_1},\ e^{-\pi i\mu_1}),\\ P_3&=P(\vec{z}_3)=\mathrm{diag}(-e^{-\pi i\mu_2},\ e^{2\pi i\mu_2},-e^{-\pi i\mu_2}).\end{aligned} \quad (3)$$

This is for *any* choice of holonomy and constituent locations (with the proviso they are well separated, i.e. their cores do not overlap, in which case to a good approximation $\vec{z}_m = \vec{y}_m$). Here we take $\vec{y}_1 = (0,0,10+d)$, $\vec{y}_2 = (0,0,10-d)$ and $\vec{y}_3 = (0,0,-10)$. The limit of coinciding constituents is achieved by $d \to 0$. With this geometry it is simplest to follow for changing d the location where two eigenvalues coincide. In very good approximation, as long as the first two constituents remain well separated from the third constituent (carrying the Taubes winding), P_3 will be constant in d and the $SU(3)$ gauge field [2] of the first two constituents will be constant in time (in the periodic gauge). Thus $P(\vec{z}_m) = \exp(A_0(\vec{z}_m))$ for $m = 1,2$, greatly simplifying the calculations.

When the cores of the two approaching constituents start to overlap, P_1 and P_2 are no longer diagonal (but still block diagonal, mixing the lower 2×2 components). At $d = 0$ they are diagonal again, but P_2 will be no longer in the fundamental Weyl chamber. A Weyl reflection maps it back, while for $d \neq 0$ a more general gauge rotation back to the Cartan subgroup is required to do so, see fig. 1. At $d = 0$, *each* P_m (and $\mathcal{P}_\infty$) lies on the dashed line, which is a direct consequence of the reduction to an $SU(2)$ caloron.

To illustrate this more clearly, we give the expressions for P_m (which we believe to hold for any non-degenerate choice of the μ_i) when $d \to 0$:

$$\begin{aligned} \hat{P}_1&=P(\vec{z}_1)=\mathrm{diag}(\ e^{2\pi i\mu_2},\ e^{2\pi i\mu_2},\ e^{-4\pi i\mu_2}),\\ \hat{P}_2&=P(\vec{z}_2)=\mathrm{diag}(\ e^{-\pi i\mu_2},\ e^{2\pi i\mu_2},\ e^{-\pi i\mu_2}),\\ \hat{P}_3&=P(\vec{z}_3)=\mathrm{diag}(-e^{-\pi i\mu_2},\ e^{2\pi i\mu_2},-e^{-\pi i\mu_2}).\end{aligned} \quad (4)$$

These can be factorised as $\hat{P}_m = \hat{P}_2 Q_m$, where $\hat{P}_2$ describes an overall $U(1)$ factor. In terms of $Q_1 = \mathrm{diag}(e^{3\pi i\mu_2}, 1, e^{-3\pi i\mu_2})$, $Q_2 = \mathrm{diag}(1,1,1) = \mathbf{1}$ and $Q_3 = \mathrm{diag}(-1,1,-1)$ the $SU(2)$ embedding in $SU(3)$ becomes obvious. It leads for Q_2 to the trivial and for Q_3 to the non-trivial element of the center of $SU(2)$ (appropriate for the latter, carrying the Taubes winding). On the other hand, Q_1 corresponds to $\mathrm{diag}(e^{3\pi i\mu_2}, e^{-3\pi i\mu_2})$, which for the $SU(2)$ caloron is not related to coinciding eigenvalues. For $d \to 0$, fig. 2 shows that $\vec{z}_1$ gets "stuck" at a *finite* distance (0.131419) from $\vec{z}_2$.

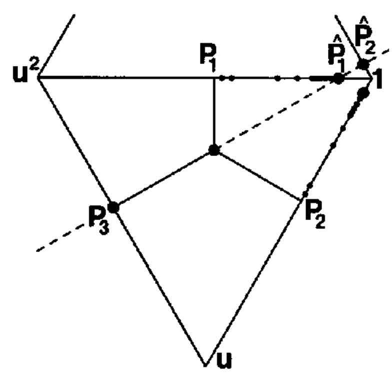

Figure 1. The fundamental Weyl chamber with the positions of P_m indicated at $d = 2$, 1, .2, .1, .05, .04, .03, .02, .01, .005, .001, .0005, and (large dots) 0. The perpendiculars point to $\mathcal{P}_\infty$ (center), and emanate from the values of P_m for well separated constituents. The dashed line shows the $SU(2)$ embedding for $d = 0$. ($u \equiv \exp(2\pi i/3)\,\mathbf{1}$)

4. Resolution

The $SU(2)$ embedding determines the caloron solution for $d = 0$, with constituent locations $\vec{y}_1' = \vec{y}_2$ and $\vec{y}_2' = \vec{y}_3$, and masses $\nu_1' = \nu_1 + \nu_2 = \mu_3 - \mu_1$ and $\nu_2' = \nu_3$. The best proof for the spurious nature of the defect is to calculate its location purely in terms of this $SU(2)$ caloron, by demanding the $SU(2)$ Polyakov loop to equal $\mathrm{diag}(e^{3\pi i\mu_2}, e^{-3\pi i\mu_2})$. For this we can use the analytic expression [7] of the $SU(2)$ Polyakov loop along the z-axis. The location of the spurious defect, $\vec{z}_1 = (0,0,z)$, is found by solving $3\pi\mu_2 = \pi\nu_2' - \frac{1}{2}\partial_z \mathrm{acosh}[\frac{1}{2}\mathrm{tr}(\mathcal{A}_2'\mathcal{A}_1')]$. For our example, $z = 10.131419$ indeed verifies this equation.

With the $SU(2)$ embedded result at hand, we find that only for $\mu_2 = 0$ the defects merge to form a triple degeneracy. Using $3\mu_2 = \nu_1 - \nu_2$, this is so for coinciding constituent monopoles of *equal* mass. For *unequal* masses the defect is always spurious, but it tends to stay within reach of the non-abelian core of the coinciding constituent monopoles, except when the mass difference approaches its extremal values $\pm(1-\nu_3)$, see fig. 2 (bottom). At these extremal values one of the $SU(3)$ constituents becomes massless and *delocalised*, which we excluded for $d \neq 0$.

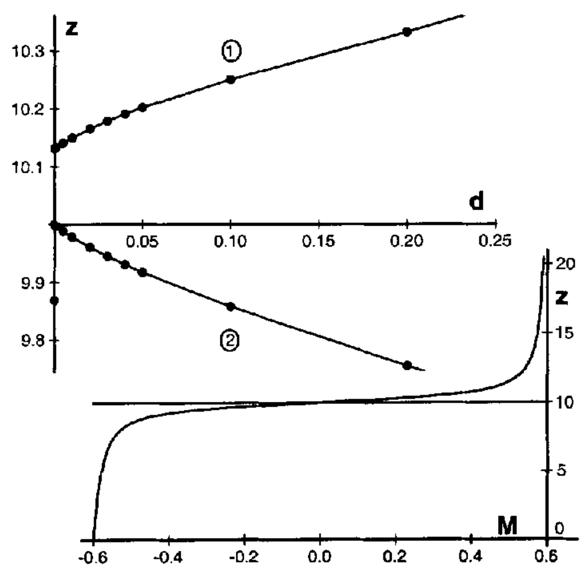

Figure 2. The defect locations $\vec{z}_1$ and $\vec{z}_2$, along the z-axis, for $M \equiv \nu_2 - \nu_1 = 0.1$ as a function of d (top), and for $d \to 0$ as a function of M (bottom).

However, the limit $d \to 0$ is singular due to the *global* decomposition into $SU(2) \times U(1)$ at $d = 0$. Gauge rotations U in the global $SU(2)$ subgroup do not affect $\hat{P}_2$, and therefore any $UQ_1U^\dagger$ gives rise to the *same* accidental degeneracy. In particular solving $-3\pi\mu_2 = \pi\nu_2' - \frac{1}{2}\partial_z \mathrm{acosh}[\frac{1}{2}\mathrm{tr}(\mathcal{A}_2'\mathcal{A}_1')]$ (corresponding to the Weyl reflection $Q_1 \to Q_1^\dagger$) yields $z = 9.868757$ for $\mu_2 = -1/30$ (isolated point in fig. 2 (top)). Indeed, $U \in SU(2)/U(1)$ traces out a (nearly spherical) *shell* where two eigenvalues of P coincide (note that for $\mu_2 = 0$ this shell collapse to a single point, $z = 10$). A perturbation tends to remove this accidental degeneracy.

5. Lesson

Abelian projected monopoles are not always what they seem to be, even though required by topology. *Topology* cannot be localised, no matter how tempting this may seem for smooth fields.

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