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Stability of Dirac sheet configurations

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Using cooling for SU(2) lattice configurations, purely Abelian constant magnetic-field configurations were left over after the annihilation of constituents that formed metastable $Q=0$ configurations. These so-called Dirac sheet configurations were found to be stable if emerging from the confined phase, close to the deconfinement phase transition, provided their Polyakov loop was sufficiently nontrivial. Here we show how this is related to the notion of marginal stability of the appropriate constant magnetic-field configurations. We find a perfect agreement between the analytic prediction for the dependence of stability on the value of the Polyakov loop (the holonomy) in a finite volume and the numerical results studied on a finite lattice in the context of the Dirac sheet configurations.

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I. INTRODUCTION

In lattice gauge theory cooling is used to remove the high-frequency fluctuations to be left with classical solutions [1,2]. This allows one to extract the underlying topological content of the gauge field configurations and determine to what extent instantons have a role to play. It is known that when using the ordinary Wilson action, the lattice artifacts are such that one can further lower the action by reducing the size of the instantons (whereas in the continuum the classical action does not depend on the size). Ultimately the instanton falls through the lattice and in general one relaxes to the trivial minimum with zero action. However, at finite temperature, when the Polyakov loop away from the instanton is nontrivial, the relevant instanton (called a caloron) actually consists of n constituents for SU(n) [3,4]. These can be shown to be 't Hooft–Polyakov (BPS) monopoles when identifying A_0 with the (adjoint) Higgs field. From the Euclidean four-dimensional point of view, due to the self-duality of the gauge field, these are dyons with their magnetic charge equal to their electric charge, with overall electric and magnetic neutrality.

Under cooling in the confined phase, due to the discreteness artifacts of the Wilson action, these constituents will attract and approach each other. When they are no longer visible as separate entities, the solutions behave like ordinary instantons localized in space and time. The distance between the constituents is [for SU(2)] given by $\pi\rho^2/b$, where b is the inverse temperature (the period in the Euclidean time direction). Another possibility is the annihilation of dyons and antidyons left over from different caloron and anticaloron solutions. As a result, with an action near the one-instanton action, a metastable configuration can be either a dyon-dyon pair that shrinks and falls through the lattice or a dyon-antidyon pair that finally annihilates [5]. Sometimes this annihilation process leaves behind a constant Abelian magnetic field, which subsequently turns out to be stable or unstable under further cooling [6], strongly correlated to the asymptotic value of the Polyakov loop (the holonomy) which has been acquired in this stage of cooling. In the deconfined phase no dyonic structure was observable under cooling. The Polyakov loop remains always close to its trivial value but quasiconstant magnetic-field configurations were seen to emerge as well, although they never happened to be stable. In this paper we present an explanation for these observations.

II. CONSTANT MAGNETIC FIELDS

It is well known that Abelian constant magnetic fields are embedded solutions of the (non-Abelian) equations of motion. They tend to be unstable, due to the self-coupling of the gauge fields [7], which formed the basis for the studies of the so-called Copenhagen vacuum picture [8].

In the four-dimensional context a constant field is stable if it is self-dual [9,10]. Here we will be interested in the degenerate case with magnetic, but no electric flux and periodic boundary conditions (the general case allows for center fluxes, but requires twisted boundary conditions [11]). For SU(2) these gauge fields are Abelian and there is the freedom of adding to it a constant Abelian vector potential, which does not change the field strength, $F_{\mu\nu}=\pi i\tau_3 n_{\mu\nu}/(L_\mu L_\nu)$. This field strength is unique up to a constant gauge rotation, and $n_{\mu\nu}$ is an integer (even in the case of periodic boundary conditions) antisymmetric tensor, fixed by flux quantization. In the degenerate case $n_{\mu\nu}$ has two nonzero eigenvectors, and computing the gauge-invariant Polyakov-loop observables in this subspace it is easily seen that no translation

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invariance holds. Adding a constant Abelian vector potential can consequently be absorbed by a translation and therefore cannot affect the fluctuation spectrum. But in this degenerate case there are also two zero eigenvectors, and the vector potential is invariant under translations in this subspace. Its Polyakov loops label the gauge invariant parameters on which the fluctuation spectrum does depend!

It had been found [12,13] that on a symmetric torus there was one class of constant magnetic-field solutions that for a certain range of values of the Polyakov loop were stable. This example, involving the smallest possible nonzero magnetic field, possesses nontrivial center flux and requires twisted boundary conditions. Therefore it could not explain the findings of Ref. [6]. However, at finite temperature involving a nonsymmetric box, more room exists to obtain stable constant magnetic-field solutions.

III. FLUCTUATION SPECTRUM

For SU(2) all constant curvature solutions in a finite box have been classified. Also the spectrum of fluctuations has been calculated [14]. For the ‘‘charged’’ isospin components, in the subspace of nonzero eigenvectors of $n_{\mu\nu}$, the problem is equivalent to that of Landau levels. The eigenfunctions are described by Θ functions to incorporate the boundary conditions. In the subspace of zero eigenvectors one simply has plane waves, with properly discretized momenta. These momenta are, however, shifted due to the constant vector potential which determines the Polyakov loops in this subspace, thereby obviously modifying the fluctuation spectrum. The ‘‘neutral’’ isospin component is described by ordinary plane waves.

The following gauge field for SU(2) gives the most general solution with constant field strength on a torus [13]:

$$A_\nu(x)=\frac{1}{2}i(-\pi n_{\nu\mu}x_\mu/L_\mu+C_\nu)\tau_3/L_\nu. \tag{1}$$

It is periodic up to the gauge transformation

$$A_\nu(x+\hat{\mu}L_\mu)=\Omega_\mu(x)[A_\nu(x)+\partial_\nu]\Omega_\mu^{-1}(x), \tag{2}$$

where $\hat{\mu}$ is the unit vector in the μ direction and

$$\Omega_\mu(x)=\exp\left(\frac{1}{2}i\pi x_\nu n_{\nu\mu}\tau_3/L_\nu\right). \tag{3}$$

With $n_{\mu\nu}$ even, these Abelian boundary conditions are, however, gauge equivalent (in general by a non-Abelian gauge transformation) to periodic boundary conditions (as long as $Q=0$). Following Ref. [6] we assume $L_0=L_t=b$, $L_1=L_2=L_3=L_s$. The data in all cases can be interpreted in terms of a (nearly) constant magnetic field with $n_{0\nu}=-n_{\nu 0}=0$ and $\vec{m}=(0,0,2)$, where $m_i=\frac{1}{2}\varepsilon_{ijk}n_{jk}$. Therefore we compute the fluctuation eigenvalues for this case (compare Refs. [12–14])

$$\lambda_\pm=4\pi(2n+1\pm 2)/L_s^2+(2\pi p+C_3)^2/L_s^2+(2\pi q+C_0)^2/L_t^2, \tag{4}$$

$$\lambda_0=(2\pi k_\mu/L_\mu)^2.$$

The multiplicities are 4 for $\lambda_\pm$ and 2 for λ_0, with all quantum numbers (n,p,q,k_μ) integer (but $n\geqslant 0$).

As argued above the spectrum depends on the constant Abelian gauge field described by the constants C_0 and C_3. These are only defined modulo 2π, as a shift over 2π is related to a gauge transformation that shifts the relevant momenta by one unit. The Polyakov-loop observables are given by

$$P_\mu=\frac{1}{2}\mathrm{Tr}\exp(iC_\mu\tau_3/2)=\cos(C_\mu/2),\quad \mu=0,3. \tag{5}$$

Note that these are antiperiodic under a shift over 2π, whereas the fluctuation spectrum is periodic. This is simply because the fluctuations involve fields in the adjoint representation, whereas the Polyakov loop is in the fundamental representation. Indeed P_0^2 and P_3^2, relevant for the adjoint representation, are periodic under a shift of C_0 and C_3 over 2π.

From the lattice data it is clear that $P_3=1$, and we can put $C_3=0$, as well as $n=p=q=0$ (we may restrict $|C_0|\leqslant\pi$) to find the lowest eigenvalue $\lambda_-=-4\pi/L_s^2+C_0^2/L_t^2$ to be negative unless the Polyakov loop is sufficiently nontrivial ($P_0=\pm 1$ being associated to a trivial Polyakov loop). The lowest eigenvalue is positive when

$$L_t/L_s<\sqrt{\pi}/2,$$

$$|P_0|=\cos(C_0/2)<\cos(\sqrt{\pi}L_t/L_s). \tag{6}$$

We see that these conditions cannot be satisfied when $L_t=L_s$ and the finite temperature situation ($b=L_t<L_s$) is essential for providing the opportunity of stability.

This stability was called marginal, because one can change C_0 without changing the classical action. Thus nothing prevents us to bring C_0 close to 0, where the lowest eigenvalue λ_- turns negative. Under the cooling there is no reason for C_0 to change, as one can easily show that the degeneracy of the action as a function of C_0 survives on the lattice. This then explains the stability of these constant gauge field configurations, provided the two conditions of Eq. (6) are satisfied.

IV. COMPARISON WITH LATTICE DATA

In Ref. [6] SU(2) gauge theory in four-dimensional Euclidean space was considered on an asymmetric lattice with periodic boundary conditions in all four directions. The respective ensembles of configurations have been created by heat-bath Monte Carlo using the standard Wilson plaquette action. The lattice size was $N_s^3\times N_t$ with the spatial extension $N_s=8,10,12,16,20$ and with temporal extension $N_t=4$, i.e., $b=4a$ and $L_s=aN_s$ with a the lattice spacing. For $N_t=4$ the model is known to undergo the deconfinement phase transi-

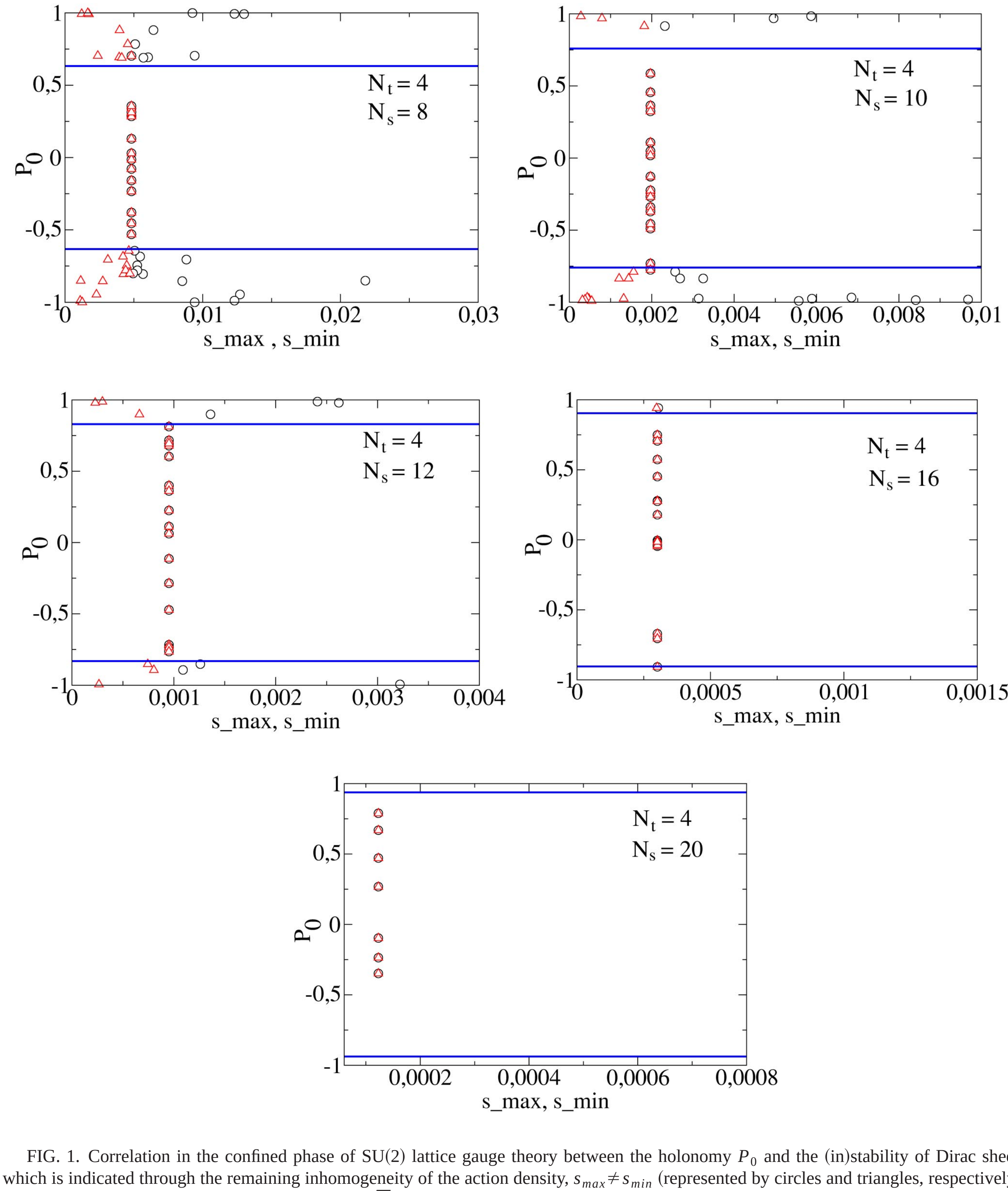

FIG. 1. Correlation in the confined phase of SU(2) lattice gauge theory between the holonomy P_0 and the (in)stability of Dirac sheets which is indicated through the remaining inhomogeneity of the action density, $s_{max} \neq s_{min}$ (represented by circles and triangles, respectively). The limiting values of the holonomy, $|P_0| = \cos[\sqrt{\pi}(N_t/N_s)]$, beyond which constant magnetic fluxes become unstable, are indicated by horizontal lines. The temporal size of the lattices is $N_t = 4$, and the DS events are shown for different spatial lattice size $N_s = 8,10,12,16,20$.

tion at the critical coupling $\beta_c \simeq 2.299$ [15]. In Ref. [6] two sets of ensembles with $\beta_1 = 2.2 < \beta_c$ and $\beta_2 = 2.4 > \beta_c$ were generated.

The equilibrium field configurations have been cooled by iterative minimization of the Wilson action with the focus on the structure of self-dual caloron solutions. In addition, Dirac sheet (DS) events were observed at the very last stages of cooling, applying a stopping criterion which selects action plateaux in the interval $S \leqslant 0.6\, S_{\text{inst}}$. In the confined phase, approximately 7% (at $N_s = 8,10,12$), 5% ($N_s = 16$), and 3% ($N_s = 20$) of equilibrium configurations have turned into these purely magnetic configurations, whereas in the deconfined phase the yield was 5–18% [6]. The action values were found close to $(N_t/N_s)S_{\text{inst}}$ characteristic for constant Abelian magnetic flux [16] of size 4π periodically closed along one of the spatial directions. Although the action showed the same dependence on the lattice extensions N_t and N_s, supporting the common interpretation as (almost) homogeneous magnetic flux, the configurations were unstable when derived from the deconfined phase and partly stable in the case of the confined phase. In the case of confinement, the issue of stability vs instability was strongly correlated to the value of the temporal Polyakov line (holonomy) P_0. This is shown in Fig. 1. It presents a set of scatter plots (each for another spatial size N_s) where each DS event is characterized by two entries: (s_{min}, P_0) and (s_{max}, P_0). The values s_{min} and s_{max} express the action density at sites where it is minimal and maximal, respectively. If these values differ, the configuration is bound to decay to the trivial vacuum. Provided the holonomy remains sufficiently far from trivial, we find only DS events which consist of a highly homogeneous Abelian magnetic flux signaled by $s_{min} = s_{max}$. This case is tantamont to absolute stability under further cooling. In contrast to this, when the holonomy was close to the trivial one ($P \approx \pm 1$) the Abelian magnetic fluxes happened not to be homogeneous ($s_{min} \neq s_{max}$) and proved to be unstable under further cooling. The critical value of the holonomy, $|P_0| = \cos(\sqrt{\pi} L_t / L_s)$, limiting the region of stability as given by the second condition in Eq. (6), is marked in Fig. 1 by horizontal lines. *No deviations* from the predicted (in)stability are seen.

V. CONCLUSIONS

Purely Abelian constant magnetic-field configurations were observed [6], randomly emerging from the process of cooling-down equilibrium lattice fields representing the confined and deconfined phases of SU(2) gluodynamics. In the confined phase they were found to be absolutely stable provided their Polyakov loop was sufficiently nontrivial. We have shown here that this fact is related to the notion of marginal stability of the appropriate constant magnetic-field configurations. We have found perfect agreement between the analytically predicted dependence of stability on the value of the Polyakov loop (the holonomy) for the set of spatial lattice sizes that were studied in Ref. [6] and the numerical observations made there, separating stable from unstable Dirac sheet configurations. The dependence on the geometry of the effect we found makes us believe we are dealing with a finite volume artifact. Nevertheless, it demonstrates that the influence of a background constant A_0 (as manifested by the Polyakov loop) on the dynamics of the gauge field should not be ignored. The physical significance lies in the fact that the Polyakov loop is the order parameter for the confinement/deconfinement phase transition. A similar conclusion can be drawn from the caloron solutions with nontrivial holonomy.

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