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TOPOLOGY OF THE YANG-MILLS CONFIGURATION SPACE

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ABSTRACT

It will be described how to uniquely fix the gauge using Coulomb gauge fixing, avoiding the problem of Gribov copies. The fundamental modular domain, which represents a one-to-one representation of the set of gauge invariant degrees of freedom, is a *bounded* convex subset of the transverse gauge fields. Boundary identifications are the only remnants of the Gribov copies, and carry all the information about the topology of the Yang-Mills configuration space. Conversely, the known topology can be shown to imply that (on a set of measure zero on the boundary) some points of the boundary coincide with the Gribov horizon. For the low-lying energies, wavefunctionals can be shown to spread out "across" certain parts of these boundaries. This is how the topology of Yang-Mills configuration space has an essential influence on the low-lying spectrum, in a situation where these non-perturbative effects are *not exponentially suppressed.*

The write-up of my contribution to the International Symposium on Advanced Topics of Quantum Physics will be a short summary, with adequate references, where most of the material I have presented can be found. The observation concerning Henyey's gauge copies have not been published before.

Gauge fixing has remained an essential ingredient for studying non-abelian gauge theories[1], despite the many attempts of finding gauge invariant variables. This is not too surprising, as the Yang-Mills configuration space is topologically non-trivial[2]. There exists no choice of (affine) coordinates suitable for the whole manifold. Gauge fixing can be conveniently seen as just one particular choice of coordinates[3]. They are only locally defined, and one would need different coordinate patches and transition functions to describe the whole manifold[4]. Trying to extend the coordinate patch to the whole manifold has as a consequence that the gauge condition, beyond a certain distance in the configuration space, no longer uniquely fixes the gauge, as was first discovered by Gribov[5]. Analysing this in a finite spatial volume using the Hamiltonian formulation ($A_0 = 0$ and using the Coulomb gauge to fix the time-independent gauge parameters) has demonstrated the usefulness of this approach. In that case the transition functions were described by topologically non-trivial gauge transformations[6]. The reason for its success lies in the fact that asymptotic freedom in four dimensions guarantees that the effective coupling constant in a small volume is small[7], and that non-perturbative effects will only be of importance for a few low-energy modes.

More recently, a different way of describing the Yang-Mills configuration space was developed, by using a functional method to not only satisfy the Coulomb gauge condition $\partial_i A_i = 0$, but also pick from the different Gribov copies a unique configuration. The collection of these configurations should thus form a fundamental modular domain, in other words, it should form a one-to-one mapping with the Yang- Mills configuration space. The relevant functional is the L^2-norm of the gauge field: $\| A_i \|^2 = \int_M Tr(A_i^\dagger A_i)$. For each gauge invariant field configuration this gives a Morse function on the gauge orbit. One easily verifies that stationary points of this Morse function satisfy the Coulomb gauge condition and that the Hessian at the stationary point is precisely given by the Faddeev-Popov

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operator, whose determinant measures the volume of the gauge orbit. This procedure to fix the gauge has been known already for quite some time[8]], as well as the obvious conclusion that the absolute minimum provides the natural way of choosing a unique representative[9], whose recent rediscovery[10] has resulted in a recurrent interest in this problem.

Gribov's original conjecture[5] was that one could find a unique representative by demanding the Faddeev-Popov operator to be positive. This region in the space of connections can be shown to be convex and is called the Gribov region. In a finite volume one can show that in each direction of configuration space, the distance of its boundary to the origin is finite[11]. However, this would mean that this choice is only unique if the above discussed Morse function has all its minima to be absolute minima. The existence of a Gribov horizon, where the Faddeev-Popov determinant vanishes (we will be reserving the name horizon for the more stringent condition that the lowest eigenvalue of the Faddeev-Popov operator vanishes), does imply that points just outside the horizon have copies just inside, as Gribov showed. But more importantly, taking the global nature of the Morse function into account, implies two possibilities. Either near the horizon the points inside the Gribov region correspond to local minima, which therefore have to be copies of an absolute minimum, by definition also inside the Gribov region[9,10]. This happens if the third order term of the expansion of the Morse function around the stationary point, in the direction where the Hessian vanishes, is non-vanishing. Or, in the case this third order term vanishes, the (local) minimum bifurcates at the horizon in two local minima and one saddle point[12]. As the two local minima in the above bifurcation coalesce at the horizon, the relevant gauge transformation has to be topologically trivial. By definition, these gauge copies lie inside the Gribov region.

Care is required if the gauge configuration is reducible, as in that case the gauge group does not act freely. However, the gauge subgroup that leaves the reducible connection fixed, generally does not support topologically non-trivial maps. Nevertheless, reducible connections give rise to singularities in the physical configuration space[13], where the Morse theory arguments might not be valid. These singularities turn out to be particularly treacherous in the case of an abelian connection, which are the simplest examples of reducible connections. As the Coulomb gauge does not fix the constant gauge transformations, one has to still divide out these constant gauge transformations to obtain the fundamental modular domain.

With this in the back of our minds let us reconsider an old example due to Henyey[14], for gauge copies (inside the Gribov horizon) that are related by a homotopically trivial gauge transformation. As was explained in ref.[12], his was an example related to a bifurcation of local minima at the horizon. Explicitly Henyey's example was given by [14] an abelian $SU(2)$ connection on compactified $I\!R^3$:

$$\vec{A}(\beta) = a(\beta)\hat{\phi}\tau_3,$$

$$a(\beta) = \frac{1}{2r\sin\theta} - \frac{b + r^2\sin^2\theta(\frac{d^2b}{dr^2} + \frac{4}{r}\frac{db}{dr})}{\beta^{-1}\sin(2r\beta b\sin\theta)},$$

$$g(\beta) = \exp(i\beta r b\sin\theta(e^{i\phi}\tau_- + e^{-i\phi}\tau_+)).$$

where (r,θ,ϕ) are the spherical coordinates, τ_i the Pauli matrices, b an arbitrary function of r and $g(\pm\beta)$ defines the two gauge copies $[g(\pm\beta)]A(\beta)$, that will coalesce at $\beta = 0$, which is easily verified to coincide with the Gribov horizon for a suitable choice of the radial function b[12]. Unfortunately, close inspection shows that, since $g(-\beta) = \tau_3 g(\beta)\tau_3$ and since $A(\beta)$ commutes with τ_3, the two copies $[g(\pm\beta)]A(\beta)$ are related by a constant gauge transformation. As we still have to divide out these constant gauge transformations, Henyey's example does not really provide an example of two gauge copies inside the Gribov horizon, that are related by a topologically trivial gauge transformation.

Fortunately, however, the general Morse theory argument is sufficiently strong not to have to worry, that bifurcations of the above type do not occur. To find an example one simply should look for an irreducible connection on the Gribov horizon, where the third order term in the expansion around the stationary point of the Morse function vanishes in the direction of the zero eigenvector of the Hessian. In a recent paper, that analysed gauge fields on a three-sphere[15], such configurations featured prominently as the so-called sphaleron modes (u, v). Part of the Gribov horizon is given by the equation $u + v = 1.5$ (for u and $v \in [0, 1.5]$) and one easily verifies these configurations are not reducible and have a Morse function that vanishes to third order in the direction of the zero eigenvector of the Faddeev-Popov operator.

We now come back to the issue of the fundamental modular domain. It can be shown that the collection of absolute minima of the Morse function is also convex[10]. One would thus expect it to be contractable, which is in flagrant contradiction with the assumption that it is a in one-to-one correspondence with the physical configuration space. The catch is that the absolute minimum of the Morse function need not be unique. Using the convexity of the set of absolute minima (also called Λ) it is not hard to show[12] that A has a boundary, which if it does not coincide with the Gribov horizon, necessarily corresponds to a degenerate absolute minimum of its Morse function. The other absolute minimum is necessarily a gauge copy and also a point on the boundary. Thus the set of absolute minima will only become a fundamental modular domain if these appropriate boundary identifications are taken into account. In this way it will support the known non-trivial topology of the physical configuration space. On the other hand, the known non-trivial topology implies the existence of non-contractable n-spheres. Those can only arise through boundary identifications if all points of a suitable $n-1$-dimensional subspace of the boundary are identified (gauge equivalent). These points consequently coincide with the Gribov horizon. These "singular" boundary points, however, form a subset of the boundary of zero measure (which does not mean they are unimportant). Note that as the Gribov region is bounded in each direction, the same holds for the fundamental modular domain.

The dynamical consequence in the context of gauge theories in a finite volume is that at increasing volume (i.e. increasing coupling constant) the wavefunctional starts to spread out over configuration space, and unavoidably will start to overlap with the boundary of the fundamental modular domain. Important is that the size of the volume controls this process. At very small volumes the boundary is irrelevant (for a torus typically below $0.1 - 0.01 fm^3$), whereas at increasing volume first the lowest energy modes start to be sensitive to the boundary identifications. In this way the energy of electric flux and the low-lying glueball spectrum were calculated in volumes up to $1 fm^3$ in the geometry of a torus[12,16], which agreed with the lattice Monte Carlo results obtained in the same volume within the 2% statistical error of the latter[17]. For the torus geometry the zero energy modes are parameterized by the constant abelian modes, $A_i = C_i \tau_3/(2L)$ (where L is the length of the torus). The Gribov horizon and the boundary of the fundamental modular domain in C- space are given by cubes, centered at $C = 0$ and respectively with sides of length 4π and 2π[12]. A recent analysis on the three-sphere is aimed at going to larger volumes[15].

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