Constituent monopoles without gauge fixing

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We discuss the recent construction of new exact finite temperature instanton solutions with a non-trivial value of the Polyakov loop at infinity. They can be shown, in a precise and gauge invariant way, to be formed by the superposition of n BPS monopoles for an $SU(n)$ gauge group.

1. Introduction

Instantons at finite temperature (or calorons) are constructed on $\mathbb{R}^3 \times S^1$, taking a periodic array of instantons. For $SU(2)$ the five parameter Harrington-Shepard solution [1] can be formulated within the 't Hooft ansatz. New exact solutions with a non-trivial value of the Polyakov loop at infinity [2] were only constructed very recently, either using [3] results due to Nahm [4] or by using [5] the well-known ADHM construction [6], translated by Fourier transformation to the Nahm language. Thus mapped to an Abelian problem on the circle, the quadratic ADHM constraint is solved [5].

2. New caloron solutions

In the periodic gauge, $A_\mu(x+\beta)=A_\mu(x)$, the Polyakov loop at spatial infinity

$$\mathcal{P}_\infty = \lim_{|\vec{x}|\to\infty} P\,\exp(\int_0^\beta A_0(\vec{x},t)dt), \tag{1}$$

after a constant gauge transformation, is characterised by ($\sum_{m=1}^n \mu_m = 0$)

$$\mathcal{P}^0_\infty = \exp[2\pi i\mathrm{diag}(\mu_1,\ldots,\mu_n)], \tag{2}$$
$$\mu_1 < \ldots < \mu_n < \mu_{n+1} \equiv \mu_1 + 1.$$

A non-trivial value, $\mathcal{P} \notin Z_n$, acts like a Higgs field. We found [5c] a remarkably simple formula for the action density, valid for arbitrary $SU(n)$. Using the classical scale invariance to put $\beta = 1$,

$$\mathrm{tr}F^2_{\mu\nu} = \partial^2_\mu\partial^2_\nu \log\psi,\ \ \psi = -\cos(2\pi t) + \tfrac{1}{2}\mathrm{tr}\prod_{m=1}^n A_m,$$

$$A_m \equiv \frac{1}{r_m}\begin{pmatrix} r_m & |\vec{y}_m-\vec{y}_{m+1}| \\ 0 & r_{m+1}\end{pmatrix}\begin{pmatrix} c_m & s_m \\ s_m & c_m\end{pmatrix}, \tag{3}$$

with $r_m = |\vec{x}-\vec{y}_m|$ the center of mass radius of the $m^{\rm th}$ constituent monopole, which can be assigned a mass $8\pi^2\nu_m$, where $\nu_m \equiv \mu_{m+1} - \mu_m$. Also $r_{n+1} \equiv r_1$, $\vec{y}_{n+1} \equiv \vec{y}_1$, $c_m \equiv \cosh(2\pi\nu_m r_m)$ and $s_m \equiv \sinh(2\pi\nu_m r_m)$. The order of matrix multiplication is crucial here, $\prod_{m=1}^n A_m \equiv A_n \ldots A_1$.

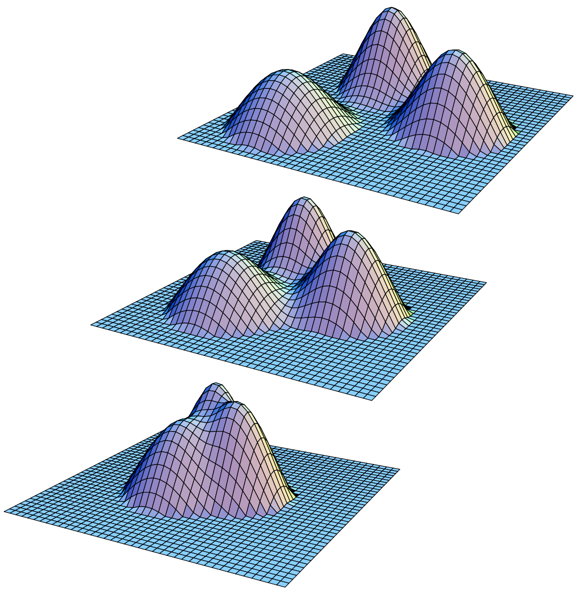

Figure 1. Action densities for the $SU(3)$ caloron on equal logarithmic scales, cut off at $1/(2e)$, for $t = 0$ in the plane defined by $\vec{y}_1 = (-\frac{1}{2}, -\frac{1}{2}, 0)$, $\vec{y}_2 = (0, \frac{1}{2}, 0)$ and $\vec{y}_3 = (\frac{1}{2}, -\frac{1}{4}, 0)$, in units of β, for $\beta = 1/4$, $1/3$ and $2/3$ from top to bottom, using $(\mu_1, \mu_2, \mu_3) = (-17, -2, 19)/60$.

*Presented by second author at Lattice '98, 13-18 July.

For $\mathcal{P}_\infty = \exp(2\pi i\omega\tau_3)$ the $SU(2)$ gauge field reads [5a], in terms of the anti-selfdual 't Hooft tensor $\bar\eta$ and Pauli matrices τ_a,

$$A_\mu(x) = \frac{i}{2}\bar\eta^3_{\mu\nu}\tau_3\partial_\nu \log\phi(x) + \frac{i}{2}\phi(x)\mathrm{Re}\left((\bar\eta^1_{\mu\nu} - i\bar\eta^2_{\mu\nu})(\tau_1 + i\tau_2)\partial_\nu\chi(x)\right), \tag{4}$$

where $\phi^{-1} = 1 - \frac{\pi\rho^2}{\psi}\left(\frac{s_1c_2}{r_1} + \frac{s_2c_1}{r_2} + \frac{\pi\rho^2 s_1 s_2}{r_1r_2}\right)$ and $\chi = \frac{\pi\rho^2}{\psi}\left(e^{-2\pi i t}\frac{s_1}{r_1} + \frac{s_2}{r_2}\right)e^{2\pi i\nu_1 t}$, with $\nu_1 = 2\omega$, $\nu_2 = 1 - 2\omega$ and $\pi\rho^2 = |\vec y_2 - \vec y_1|$. The solution is presented in the "algebraic" gauge, $A_\mu(x+\beta) = \mathcal{P}_\infty A_\mu(x)\mathcal{P}_\infty^{-1}$.

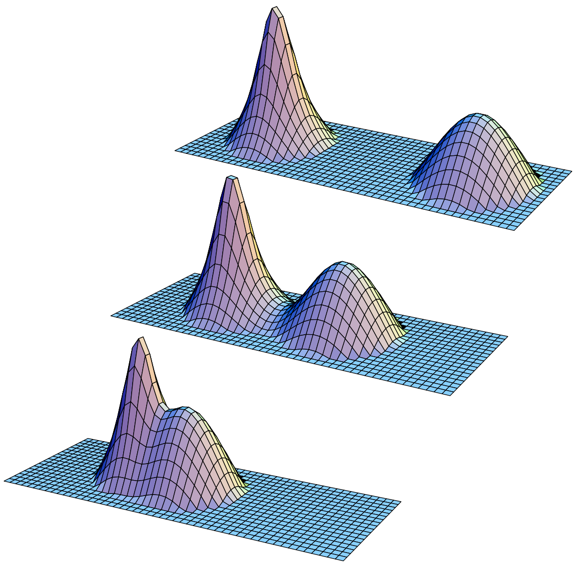

Figure 2. Action densities for the $SU(2)$ caloron on equal logarithmic scales, cut off below $1/(2e^2)$, for $t=0$, $\omega=0.125$, $\beta=1$ and $\rho=1.6$, 1.2 and 0.8 (from top to bottom).

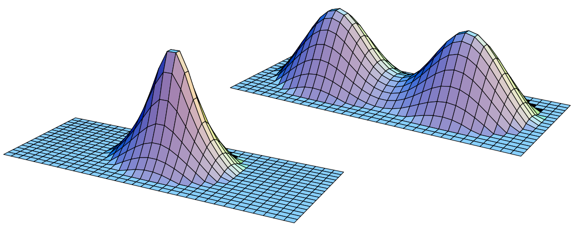

Figure 3. As in fig. 2, now cut off below $1/(2e)$, for $t=0$, $\rho=\beta=1$ with $\omega=\frac{1}{4}$ (top) and 0 (bottom).

For small ρ, equivalent to large β, the caloron approaches the ordinary single instanton solution, with no dependence on $\mathcal{P}_\infty$. Finite size effects set in roughly when $\rho = \frac{1}{2}\beta$. At this point, for $\nu_i \neq 0$, two lumps (n for $SU(n)$) are formed, whose separation grows as $\pi\rho^2/\beta$. When $\mathcal{P}_\infty = (-)1$ for $SU(2)$, one of the lumps disappears, as $\nu_{1(2)}=0$, and the spherically symmetric Harrington-Shepard solution is retrieved.

A non-trivial value of $\mathcal{P}_\infty$ will modify the vacuum fluctuations and thereby leads to a non-zero vacuum energy density [2] as compared to $\mathcal{P}\in Z_n$. A dilute, semi-classical instanton calculation is no longer considered a reliable starting point for QCD. Rather, it is the monopole constituent nature from which we should draw important lessons for QCD [5b].

3. Monopoles from instantons

At small β the solution becomes static and the lumps are well separated and spherically symmetric. As they are self-dual, they are necessarily BPS monopoles [7]. Also, when sending (at fixed β) one of the constituents to infinity, $|\vec y_m|\to\infty$, the solution becomes static and yields a simple way to obtain $SU(n)$ monopole solutions [5c]. Explicitly we find (assuming $\nu_n\neq 0$) in the limit $|\vec y_n|\to\infty$, which removes the n-th constituent,

$$A_n \to 2c_n\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}, \quad A_{n-1}\to\frac{|\vec y_n|}{r_{n-1}}\begin{pmatrix}s_{n-1} & c_{n-1}\\ s_{n-1} & c_{n-1}\end{pmatrix}, \tag{5}$$

implying $\psi(x)\to 2|\vec y_n|\exp(2\pi\nu_n|\vec y_n - \vec x|)\tilde\psi(\vec x)$, with

$$\tilde\psi(\vec x) = \tfrac{1}{2}\mathrm{tr}\left\{\frac{1}{r_{n-1}}\begin{pmatrix}s_{n-1} & c_{n-1}\\ 0 & 0\end{pmatrix}\prod_{m=1}^{n-2}A_m\right\}. \tag{6}$$

As was emphasised in ref. [5c], the energy density of the $SU(n)$ monopole is easily found from eq. (3) (for a detailed description of some special cases see ref. [8])

$$\mathcal{E}(\vec x) = -\tfrac{1}{2}\mathrm{tr}F^2_{\mu\nu}(\vec x) = -\tfrac{1}{2}\Delta^2\log\tilde\psi(\vec x). \tag{7}$$

4. Instantons from monopoles

The new caloron solutions provide examples of gauge fields with topological charge built out of monopole fields, a construction going back to

Taubes [9]. Non-trivial $SU(2)$ monopole fields are classified by the winding number of maps from S^2 to $SU(2)/U(1) \sim S^2$, where $U(1)$ is the unbroken gauge group. Isospin orientations for a configuration made out of monopoles with opposite charges behave as shown in fig. 4 (top), in a suitable gauge and sufficiently far from the core of both monopoles. Taubes constructed topologically non-trivial configurations by creating a monopole anti-monopole pair, bringing them far apart, rotating one of them over a full rotation and finally bringing them together to annihilate (cmp. fig. 5). We can describe this as a closed monopole line (or loop) with the orientation of the core defined by $SU(2)/U(1) \sim S^2$, "twisting" along the loop, thus describing a Hopf fibration [5b] (see fig. 4 (bottom)). The only topological invariant available to characterise the homotopy type of this Hopf fibration is the Pontryagin index. It prevents full annihilation of the "twisted" monopole loop.

For large ρ, eq. (4) gives up to exponential corrections, i.e. outside the cores of the constituents,

$$A_\mu = \frac{i}{2}\tau_3\bar{\eta}^3_{\mu\nu}\partial_\nu \log\phi_0, \quad \phi_0 \equiv \frac{r_1 + r_2 + \pi\rho^2}{r_1 + r_2 - \pi\rho^2}. \quad (8)$$

This describes two Abelian Dirac monopoles and one easily verifies $\log\phi_0$ is harmonic, as required by self-duality. Furthermore ϕ_0^{-1} vanishes on the line connecting the two monopole centers, giving rise to return flux, absent in the full theory. The relative phase $e^{-2\pi i t}$ in the expression for χ given

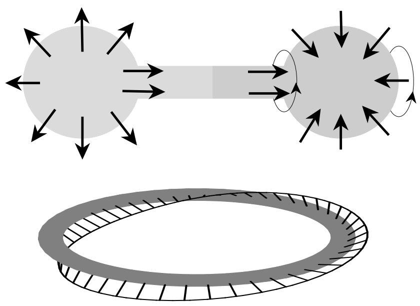

Figure 4. Topological charge constructed from oppositely charged monopoles by rotating one of them. For a closed monopole line, the embedding of the unbroken subgroup makes a full rotation.

before, describes the full rotation of the core of a constituent monopole, required so as to give rise to non-trivial topology.

A conjectured QCD application, in the form of a hybrid monopole-instanton liquid, was discussed in ref. [5b]. Abelian projection applied to our solutions was also discussed at this conference [10].

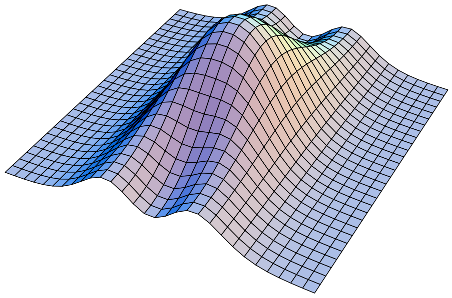

Figure 5. Action density in the z-t–plane for $x=y=0$, $\omega=\frac{1}{4}$, $\rho=\frac{1}{2}$ and $\beta=1$ on a linear scale. One can trace the constituent monopoles in the low field regions, "annihilating" to give an instanton.

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