Instanton Moduli for $T^3 \times \mathbb{R}$

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We review the specific problems that arise when studying instantons on a torus. We discuss how the Nahm transformation shows that no exact charge one instanton on T^4 can exist. However, taking one of the directions (the time) to infinity, it can be shown that vacuum to vacuum tunnelling solutions exist. A precise description of the moduli space for $T^3 \times \mathbb{R}$, studied numerically using lattice techniques, remains an interesting open problem. New is an explicit application of the Nahm transformation to (anti-)selfdual constant curvature solutions on T^4 and a discussion of its properties relevant to instantons on $T^3 \times \mathbb{R}$.

1. Introduction

Instantons and monopoles have become important mathematical tools to study invariants of four dimensional manifolds [1,2]. This is due to the fact that their moduli space (the parameter or solution space) depends on the space-time on which the self-duality equations are studied. To some extent we can compare [3] the situation with the study of harmonic forms on a manifold and its relation to the DeRham cohomologies. Like in that case, also here there are certain spaces for which there are no solutions to the equations studied.

The torus T^4 turns out to be an example where the moduli space for instantons of unit charge is empty. For higher charges existence was proved by Taubes [4] quite some time ago. The general proof for existence takes a small localized instanton from $\mathbb{R}^4$ that is matched smoothly to the flat connection of the trivial bundle. In particular if the moduli space of flat connections has non-zero dimension, there can be obstructions against this procedure. This generally occurs when $H_2(M, \mathbb{Z})$ is non-trivial, as the trace of the Wilson loop for a flat connection can be non-trivial when the loop cannot be contracted to a point.

Using the Nahm transformation [5], it can be proven that indeed charge one instantons do not exist for T^4. The Nahm transformation will be reviewed in the next section (see also refs. [6]). It maps self-dual solutions on T^4 to self-dual solutions on the dual torus [7,8]. This map is an involution, i.e. its square is the identity, and it preserves metric and hyperKähler structures of the moduli spaces [9].

From the physical point of view it is somewhat disturbing, in particular when scaling up the volume to sizes much bigger than $1/\Lambda_{\rm QCD}$, that the existence would depend on the geometry of space-time. The point is of course that one can get arbitrarily close to a solution, for sufficiently large volumes, but an exact solution is only achieved in the singular limit. Nevertheless, the obstruction against the existence of a solution is relatively mild. For twisted boundary conditions [10], the existence of charge one instantons can be understood from the fact that any twist removes the continuous degeneracy in the moduli space of (four dimensional) flat connections, removing the obstruction to glueing in a localized instanton [11]. Twisting the boundary conditions in the time direction provides the proper framework for understanding the existence of charge one instanton solutions on $T^3 \times \mathbb{R}$, as was confirmed by numerical lattice studies [12].

The physical picture is most simply explained at infinite time. To keep the action of the imaginary time solution finite, the magnetic (and elec-

[*]Based on a talk presented at the 29th International Symposium on the Theory of Elementary Particles (Buckow, 29 August - 2 September, 1995)

tric) energy should vanish at either end. This means that the connections at these ends are flat connections on T^3, whose moduli space is an orbifold (for SU(2) it is T^3/Z_2), most conveniently parametrized by the trace of the Wilson loops along the three generators of the three-torus. When twisting in the time direction, at least one of these Wilson loops is required to have opposite signs at the two ends, somewhat misleadingly this can be compared to anti-periodic boundary conditions. Apparently, instanton solutions are not compatible with periodic boundary conditions. From the Hamiltonian point of view, in the $A_0 = 0$ gauge, there is no difference between the twisted and periodic case, except that in the periodic case the trace of the Wilson loops at $t = \pm\infty$ are identical and for the twisted case opposite in sign. In particular the sphaleron giving the saddle point, or minimal barrier that separates two vacua, is obtained by solving the static Yang-Mills equations on the (untwisted) three-torus [13]. This sphaleron would not exist in an infinite volume, but the finite volume breaks the classical scale invariance, which also implies that the scale parameter of the instanton is no longer associated to a symmetry, but it is still a non-trivial moduli parameter.

An interesting open problem remains if there are solutions on $T^3 \times \mathbb{R}$ that can not be compactified to T^4, with twist in the time.

2. The Nahm transformation

It is convenient to view T^4 as $\mathbb{R}^4/\Lambda$, where $\Lambda = \oplus_{\mu=1}^4 \mathbb{Z} e_\mu$ is a four dimensional lattice. The connection one-forms $\omega(x) = A_\mu(x)dx_\mu$ are invariant up to a gauge transformation under translation over a lattice vector. These gauge transformations, also called cocycles, satisfy cocycle conditions so as to assure one has an appropriate principal fiber bundle over the torus:

$$\begin{aligned} \omega(x+\lambda) &= g_\lambda(x)(\omega(x)+d)g_\lambda^{-1}(x), \\ g_{\lambda+\mu}(x) &= g_\lambda(x+\mu)g_\mu(x), \quad \lambda,\mu\in\Lambda. \end{aligned} \tag{1}$$

We will consider anti-selfdual connections with gauge group SU(N) whose curvature satisfies

$$\Omega = d\omega + \omega\wedge\omega = -{}^*\Omega. \tag{2}$$

In local coordinates the curvature is given in terms of the field strength by $\Omega = \frac{1}{2}F^{\mu\nu}dx_\mu\wedge dx_\nu$, whereas ${}^*\Omega$ is defined similarly in terms of the dual of the field strength, $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}$. The topological charge k is given by the integral over the second Chern class, $k = \frac{1}{2}\int \mathrm{Tr}\left(\frac{\Omega}{2\pi i}\wedge\frac{\Omega}{2\pi i}\right)$.

The Nahm transformation, $\hat{\omega} \equiv \mathcal{N}\omega$, will define a connection on the dual torus $\hat{T}^4 = \mathbb{R}^4/\hat{\Lambda}$, where

$$\hat{\Lambda} = \{\mu\in\mathbb{R}^4 | <\mu,\lambda>\in\mathbb{Z},\ \forall\lambda\in\Lambda\} \tag{3}$$

is the lattice dual to Λ. We will show that $\hat{\omega}$ is a SU(k) anti-selfdual connection with topological charge N. The rank of the gauge group and the topological charge are therefore interchanged under the Nahm transformation. If we denote by $\mathcal{M}_{N,k}$ the moduli space of SU(N) charge k instantons, the Nahm transformation induces a map between moduli spaces, $\mathcal{N}:\mathcal{M}_{N,k}\to\mathcal{M}_{k,N}$, which is an involution that preserves the natural metric and hyperKähler structure of the moduli space [9]. The dimension of the moduli space, $4Nk$, is indeed symmetric under interchanging k and N.

We will now explain the essential ingredients of the Nahm transformation. The starting point is that a charge k (anti-)instanton has k (negative) positive chirality zero-modes for the massless Dirac equation, which is most conveniently written in the Weyl format. We introduce the four unit quaternions σ_μ, with $\sigma_0 = 1_2$ and $\sigma_i = -i\tau_i$, where τ_i are the usual Pauli-matrices. The Weyl operators are given (somewhat unconventionally) by $D^- \equiv D$ and $D^+ \equiv -D^\dagger$, with

$$D \equiv \sigma_\mu D_\mu(A) = \sigma_\mu(\partial_\mu + A_\mu). \tag{4}$$

Hence, in the background of a charge k instanton there are k independent solutions to $D\Psi = 0$. For $\Psi(x)$ to be defined as a two-spinor on the torus one requires $\Psi(x+\lambda) = g_\lambda(x)\Psi(x)$. One now adds a spectral parameter $z_\mu\in\mathbb{R}^4$ in the form of a flat abelian connection

$$\omega_z = \omega + 2\pi i z_\mu dx^\mu, \tag{5}$$

which leaves the curvature unchanged, $\Omega_z = \Omega$, and in particular anti-selfdual. Hence there is a smooth family of k fermionic zero-modes

$$D_z\Psi_z^{(i)}(x) = \sigma_\mu(\partial_\mu + A_\mu + 2\pi i z_\mu)\Psi_z^{(i)}(x) = 0. \tag{6}$$

From this family one construct a connection $\hat{\omega}$ by

$$\hat{A}^{ij}_\mu(z)=\int_{T^4}d^4x\ \Psi^i_z(x)^\dagger\frac{\partial}{\partial z_\mu}\Psi^{(j)}_z(x). \qquad (7)$$

This can be seen to form a connection on the dual torus using the fact that the $\Psi^{(i)}_z(x)$ form a complete orthogonal set of solutions of the Weyl equation and the observation that $e^{-2\pi ix\cdot\hat{\lambda}}\Psi^{(i)}_z(x)\in$ ker $D_{z+\hat{\lambda}}$, such that

$$\Psi^{(i)}_{z+\hat{\lambda}}(x)=e^{-2\pi ix\cdot\hat{\lambda}}\Psi^{(j)}_z(x)S^{ji}_{\hat{\lambda}}(z), \qquad (8)$$

with $S_{\hat{\lambda}}(z)$ a unitary $k\times k$ matrix, which defines the (inverse) of the cocycle for $\hat{\omega}\equiv\hat{A}_\mu(z)dz_\mu$ as a connection on $\hat{T}^4$

$$\hat{A}_\mu(z+\hat{\lambda})=S^{-1}_{\hat{\lambda}}(z)(\hat{A}_\mu(z)+\frac{\partial}{\partial z_\mu})S_{\hat{\lambda}}(z). \qquad (9)$$

We can use the Atiyah-Singer family index theorem [14] to relate the Chern character of the bundle E, associated with the connection ω to the Chern character of the bundle $\hat{E}$, associated with the connection $\hat{\omega}$. For this it is necessary to assume that ω is 1-irreducible or WFF (without flat factors [1]), which is equivalent to stating that coker $D_z=$ker $D^+_z=0$, implying that $\hat{E}_z\equiv$ker D_z is smooth (remember that the index theorem states that $k=$ dim ker D^- $-$ dim ker D^+). One can view ω_z as a connection over $T^4\times\hat{T}^4$, where the abelian part has a curvature $2\pi idz_\mu\wedge dx_\mu$ (forming the so-called Poincaré bundle $\mathcal{P}$ [9]). It now follows that

$$\mathrm{ch}(\hat{E})=\int_{T^4}\mathrm{ch}(E)\wedge\mathrm{ch}(\mathcal{P}). \qquad (10)$$

This is easily seen to interchange the rank and topological charge between the original and Nahm bundles. Also the first Chern classes will be related, but we will assume that E is a SU(N) bundle, for which the first Chern class vanishes. The family index theorem guarantees that the Nahm bundle $\hat{E}$ also has vanishing first Chern class, from which it follows that $\hat{E}$ is a SU(k) bundle with topological charge N. If, however, the first Chern class does not vanish (in which case the topological charge should be determined from the first Pontryagin class), one has [9]

$$c_1(\hat{E})=-\int_{T^4}(dz_\mu\wedge dx_\mu)^2\wedge c_1(E). \qquad (11)$$

A direct corollary is now that charge 1 instantons cannot exist on T^4. Suppose they would exist. The Nahm bundle would give rise to a U(1) bundle of charge N, which is impossible as the first Chern class vanishes and U(1) bundles have always vanishing second Chern class.

The family index theorem only provides topological information on the Nahm bundle obtained from an anti-selfdual connection ω. To demonstrate that $\hat{\omega}$ is also an anti-selfdual connection, a little more work is needed. As some of the ingredients will be important for the formulation of the Nahm transformation on $T^3\times\mathbb{R}$ we provide a few details necessary to understand this beautiful result due to Nahm, who originally introduced his transformation for the study of monopoles, as a generalization [5,6] of the ADHM construction [15]. A crucial ingredient is formed by the Weitzenböck formula

$$D^-_zD^+_z=D^2_\mu(A_z)+\tfrac{1}{2}\sigma_{[\mu}\sigma^\dagger_{\nu]}F_{\mu\nu}, \qquad (12)$$

using $\sigma_\mu\sigma^\dagger_\nu=\delta_{\mu\nu}+\sigma_{[\mu}\sigma^\dagger_{\nu]}$. Since $\sigma_{[\mu}\sigma^\dagger_{\nu]}\equiv\sigma_i\eta^i_{\mu\nu}$ is a selfdual tensor, we see that $D^-_zD^+_z=D^2_\mu(A_z)$. Its kernel is trivial for ω WFF and the Greens function $G_z=(D^-_zD^+_z)^{-1}$ commutes with the quaternions σ_μ.

It is now remarkably simple to show that $\hat{\omega}$ is also anti-selfdual

$$\hat{\Omega}^{ij}(z)=\hat{d}\hat{\omega}+\hat{\omega}\wedge\hat{\omega}=<\hat{d}\Psi^{(i)}_z|1-P|\hat{d}\Psi^{(j)}_z>, \qquad (13)$$

where

$$\hat{d}\equiv dz_\mu\frac{\partial}{\partial z_\mu},\quad P\equiv|\Psi^{(k)}_z><\Psi^{(k)}_z|. \qquad (14)$$

One easily shows that

$$\begin{aligned}P&=1-D^+_zG_zD_z, \qquad (15)\\ D_z\hat{d}\Psi^{(i)}_z&=[D_z,\hat{d}]\Psi^{(i)}_z=-2\pi i\sigma_\mu\Psi^{(i)}_zdz_\mu,\end{aligned}$$

such that

$$\hat{F}^{ij}_{\mu\nu}(z)=-8\pi^2<\Psi^{(i)}_z|\sigma^\dagger_{[\mu}\sigma_{\nu]}G_z|\Psi^{(j)}_z>. \qquad (16)$$

There are two essential ingredients that enter these manipulations. First, on T^4 we can perform partial integrations without picking up boundary terms. Second, $G_z=(D^-_zD^+_z)^{-1}$ commutes with the quaternions. Anti-selfduality immediately follows from the fact that $\sigma^\dagger_{[\mu}\sigma_{\nu]}\equiv\sigma_i\bar{\eta}^i_{\mu\nu}$ is an anti-selfdual tensor. When one or more space-time directions become non-compact, boundary terms complicate the construction [5,6]. For T^4, applying the Nahm transformation the second time (in case of non-compact directions this requires modification), it can be shown that $\mathcal{N}^2\omega = \omega$, and the explicit form of $\hat{\Psi}_x(z)$ in terms of $\Psi_z(x)$ allows one to show that metric and hyperKähler structures of the moduli spaces are preserved under $\mathcal{N}$ [9].

We note that in the case of twisted boundary conditions [10], it is not possible to construct the Nahm transformation, as the fields need to be invariant under the center of the gauge group, which is not the case for the fermionic zero-modes required for the construction of $\hat{\omega}$. Nevertheless, for gauge theories on $T^3\times\mathbb{R}$, where we have periodic boundary conditions in the space directions and none in the time directions, we can attempt to construct a variant of the Nahm transformation, as will be discussed in the last section.

3. Explicit example

In general no explicit (anti-)selfdual connections on T^4 are known. However, for some choices of the periods Λ (or equivalently for some choices of metrics), constant curvature solutions exist [16]. A complete classification for SU(2) was given in ref. [17]. Under deformations of Λ these connections remain solutions, but are no longer (anti-)selfdual. In absence of twist, that is as proper SU(2) (rather than SU(2)/Z_2=SO(3)) bundles, their topological charge is always even. They have U(1) holonomy, as there exists a gauge in which they are abelian. Consequently they are examples of so-called reducible connections, giving rise to singular points in the moduli space [1]. The term reducible derives from the fact that these bundles decompose in the direct sum of U(1) line bundles $E = L\oplus L^{-1}$, see ref. [1]. As the Nahm transformation preserves the metric structure of the moduli spaces, reducible connections should be mapped to reducible connections. This implies that the Nahm transformation of a (anti-)selfdual constant curvature connection is expected to be also a (anti-)selfdual constant curvature connection, to be illustrated in this section with the help a simple explicit example for SU(2) and topological charge 2.

We consider on $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ the connection

$$A_\mu(x) = -\tfrac{1}{2}\pi i n_{\mu\nu}x_\nu\tau_3, \quad n_{03}=n_{21}=2, \qquad (17)$$

with the cocycles given by

$$g_\lambda(x) = \alpha(\lambda)\exp(\lambda_\mu A_\mu(x)). \qquad (18)$$

The so-called bi-characters α are given by

$$\alpha(\lambda) \equiv \exp(-\tfrac{1}{2}\pi i\sum_{\mu<\nu}\lambda_\mu n_{\mu\nu}\lambda_\nu). \qquad (19)$$

As in the study of the fluctuations around the constant curvature connections [17], the zero-modes (and for that matter the whole spectrum) of the Weyl operators can be expressed in terms of theta functions. This is most simply seen by introducing complex coordinates

$$y_1 = \frac{x_3+ix_0}{\sqrt{2}} \quad , \quad y_2 = \frac{x_1+ix_2}{\sqrt{2}}$$
$$u_1 = \frac{z_3+iz_0}{\sqrt{2}} \quad , \quad u_2 = \frac{z_1+iz_2}{\sqrt{2}} \qquad (20)$$

and suitable creation and annihilation operators

$$\begin{aligned} a_i &= -i\sqrt{2}\left(\frac{\partial}{\partial\bar{y}_i}+\pi(y_i+2iu_i)\right), \\ b_i &= -i\sqrt{2}\left(\frac{\partial}{\partial y_i}+\pi(\bar{y}_i+2i\bar{u}_i)\right), \end{aligned} \qquad (21)$$

which satisfy the commutation relations

$$[a_i,a_j^\dagger] = 4\pi\delta_{ij}, \quad [b_i,b_j^\dagger]=4\pi\delta_{ij}, \qquad (22)$$

with all other commutators trivial. Hence all eigenfunctions can be constructed, as for a harmonic oscillator, in terms of the functions $\chi_z(x)$ and $\chi^*_{-z}(x)$, annihilated by respectively a_i and b_i,

$$\chi_z(x) = \exp(-\tfrac{1}{2}\pi(x+nz)^2)\theta_u(y), \qquad (23)$$

where $\theta_u(y)$ is holomorphic in the complex coordinates y_i. Introducing isospin projection operators $I_\pm \equiv \frac{1}{2}(1\pm\tau_3)$ one easily finds that

$$D_z = \begin{pmatrix} a_1 & a_2^\dagger \\ a_2 & -a_1^\dagger\end{pmatrix}\otimes I_+ + \begin{pmatrix} b_1^\dagger & b_2 \\ b_2^\dagger & -b_1\end{pmatrix}\otimes I_-. \quad (24)$$

With $D_zD_z^\dagger = (a_i^\dagger a_i+4\pi)\otimes I_+ + (b_i^\dagger b_i+4\pi)\otimes I_-$, one finds that the cokernel of D_z is trivial such that the Nahm transformation is well defined. The splitting of D_z in isospin-up and down components is a direct consequence of the fact that A is a reducible connection. The associated line bundle L has a section $s(x)=\chi_z(x)$ and the cocycle condition $\Psi_z(x+\lambda)=g_\lambda(x)\Psi_z(x)$ reduces to

$$\chi_z(x+\lambda)=\alpha(\lambda)\exp(-\tfrac{1}{2}\pi i\lambda_\mu n_{\mu\nu}x_\nu)\chi_z(x), \quad (25)$$

which implies that $\theta_u(y+q_c)$ equals $\theta_u(y)$ up to a q and z-dependent holomorphic factor, as required for holomorphic $\theta_u(y)$. With q_c we indicate the complex two-vector constructed form $q\in\mathbb{Z}^4$ as in eq. (19) (cmp. $x_c=y$ and $(nz)_c=2iu$). From the general theory on θ-functions one finds that $\dim\ker a_i=1$, such that $\chi_z(x)$ is unique up to a (z-dependent) factor. For other choices of $n_{\mu\nu}$ it can be easily shown that there are $\sqrt{\det(\frac{1}{2}n)}$ such functions, as required by the index theorem, see ref. [17] and references therein. Explicitly

$$\begin{aligned}\chi_o(x) &= \sum_{q\in\mathbb{Z}^4}\alpha(q)e^{-\frac{1}{2}\pi\{ixnq+(x+q)^2\}}, \\ \theta_o(y) &= \sum_{q\in\mathbb{Z}^4}\alpha(q)e^{2\pi y\bar{q}_c-\frac{1}{2}\pi q^2},\end{aligned} \quad (26)$$

from which we can construct a smooth family of zero-modes

$$\chi_z(x)=e^{-\pi ixz}\chi_o(x+\tfrac{1}{2}nz). \quad (27)$$

Its norm is independent of z and found to be [17] $\chi_o(0)=(1.66925368)^2$, obtained by working out the sum over $\mathbb{Z}^4\times\mathbb{Z}^4$ in $\int_{T^4}\chi_z^*(x)\chi_z(x)$, such that one of the sums allows one to extend the integral over the unit cell to an integral over $\mathbb{R}^4$, and evaluate the gaussian integral over x.

We therefore find two normalized zero-modes $\Psi_{ab}^{(i)}(x;z)=<a,b,x|\Psi_z^{(i)}>$ for the Weyl operator D_z. The spinor and isospin indices are denoted by a and b. The only non-zero components are

$$\begin{aligned}\Psi_{11}^{(1)}(x;z) &= h(z)\chi_z(x)/\sqrt{\chi_o(0)}, \\ \Psi_{22}^{(2)}(x;z) &= h^*(z)\chi_{-z}^*(x)/\sqrt{\chi_o(0)},\end{aligned} \quad (28)$$

where $h(z)$ is an arbitrary phase. This phase ambiguity is equivalent to a gauge transformation in the subgroup generated by σ_3 (assigning a different phase factor to $\Psi_z^{(2)}$ will introduce in addition a U(1) gauge component, generated by $i\sigma_0$). We will choose $h(z)=1$.

One easily shows that eq. (25) holds[2], and that furthermore (note that $\alpha(\frac{1}{2}nq)=\alpha(q)$)

$$\chi_{z+q}(x)=\alpha(q)e^{-2\pi iqx}e^{-\frac{1}{2}\pi iqnz}\chi_z(x). \quad (29)$$

From this one easily deduces that (see eq. (8))

$$S_\lambda(z)=\alpha(\lambda)\exp(-\tfrac{1}{2}\pi i\lambda_\mu n_{\mu\nu}z_\nu\tau_3). \quad (30)$$

As $S_\lambda(z)=\hat{g}_\lambda^{-1}(z)$, we see that the cocycle for the Nahm bundle is the inverse of the original cocycle (see eq. (18)). The topological charge is not affected by this inversion. This can also be seen from the explicit computation of the connection $\hat{\omega}$ and its curvature $\hat{\Omega}$. Using the fact that

$$\partial_{\bar{y}}\chi_o(x)=-\pi y\chi_o(x)\Rightarrow\partial_{\bar{u}}\chi_z(x)=-\pi u\chi_z(x), \quad (31)$$

it is not hard to show that

$$\hat{A}_\mu(z)=-A_\mu(z), \quad \hat{F}_{\mu\nu}(z)=\tilde{F}_{\mu\nu}(z). \quad (32)$$

We could have anticipated the fact that the curvature of the Nahm bundle is associated to the dual of (and hence minus) the curvature of the original reducible bundle. The line bundle L of its reducible U(1) component has topological charge 1, determined from the square of the first Chern class, $c_1(L)=\frac{1}{4}n_{\mu\nu}dx_\mu\wedge dx_\nu$. The above construction has given us the Nahm transformation for this line bundle. Therefore we can compute $c_1(\hat{L})$ from eq. (11), which is seen to give the desired result, $c_1(\hat{L})=\frac{1}{4}\tilde{n}_{\mu\nu}dz_\mu\wedge dz_\nu$.

The generalization to arbitrary constant curvature connections is now straightforward and is left to the reader.

[2] It should be noted that there are many representations of the θ-function by a lattice sum. Any holomorphic function $r(y)$, under some mild conditions that guarantee convergence of the lattice sum, gives a proper solution by defining $\tilde{\theta}(y)=\sum_{q\in\mathbb{Z}^4}\alpha(q)e^{2\pi y\bar{q}_c-\frac{1}{2}\pi q^2}r(y+q_c)$. For example, this leads to the following alternative choice for a family of solutions, $\tilde{\chi}_z(x)=\sum_{q\in\mathbb{Z}^4}\alpha(q)e^{-\frac{1}{2}\pi\{ixnq+(x+q+nz)^2\}}$. The normalized solution $\tilde{\chi}_z(x)/\tilde{\chi}_z(-nz)^{\frac{1}{2}}$ is necessarily related to $\chi_z(x)/\chi_o(0)^{\frac{1}{2}}$ by a phase $h(z)$. As $\tilde{\chi}_z(-nz)$ vanishes whenever $z_0=z_3=\frac{1}{2}$ or $z_1=z_2=\frac{1}{2}$, $h(z)$ represents a singular gauge transformation, also signaled by the fact that $\tilde{\chi}_z$ gives rise to a trivial cocycle, $\tilde{S}_\lambda(z)=1$. Yet, with $\hat{\tilde{\omega}}=\frac{1}{4}i\partial_\mu\log\tilde{\chi}_z(-nz)n_{\mu\nu}dz_\nu$ one does retrieve the correct curvature, using that $\partial_{u_i}\partial_{\bar{u}_i}\log\tilde{\chi}_z(-nz)=-2\pi\delta_{ij}$.

4. Numerical results

To warm up for our discussion of instantons on $T^3 \times \mathbb{R}$ it is useful to first consider the case of instantons for the O(3) non-linear sigma model in two dimensions. The model is described by a field that lives on S^2, parametrized through stereographic projection by a complex function u(x+iy). Instantons [18] on $\mathbb{R}^2$ are meromorphic functions $u(z) = c\prod_i(z-a_i)/(z-b_i)$, where the topological charge equals the number of poles. A standard result in complex function theory says that a periodic meromorphic function with only one pole cannot exist, ruling out the existence of charge 1 instantons on T^2. Higher charged instantons are constructed from the Weierstrass σ-functions, see ref. [19].

From the Hamiltonian point of view, the classical vacuum is degenerate along the constant functions on the circle S^1. Its moduli space (the equivalent of the space of flat connections on T^3) is equal to S^2, parametrized by a complex number. Charge 1 instantons, corresponding to vacuum to vacuum tunnelling events do, however, exist for $S^1 \times \mathbb{R}$. Taking $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and introducing $z \equiv t + ix$ one finds [20,21]

$$u(z) = -(c + de^z)/(a + be^z), \quad ad - bc \neq 0. \tag{33}$$

It follows that $u(t\to\infty) = -d/b$ and $u(t\to-\infty) = -c/a$, two *different* points in the classical vacuum. As for $\mathbb{R}^2$ the charge 1 instanton has six parameters, e.g. the complex ratios b/a, c/a and d/a, two of which are associated to translations, and three to O(3) rotations. The scale parameter is no longer associated to a symmetry (as S^1 has a fixed size). A suitable parametrization of charge one instantons [21] is found in terms of the polar angle $\varphi \in [0, \pi]$

$$u(z) = \mathrm{ctg}(\tfrac{1}{2}\varphi) - e^z/\sin(\tfrac{1}{2}\varphi), \tag{34}$$

chosen such that $u(t\to\infty) = \infty$ (the north pole) and $u(t\to-\infty) = \mathrm{ctg}(\frac{1}{2}\varphi)$, whereas $u(0)$ is real and the maximum of the energy occurs at $t = 0$. This is seen to fix all symmetries. Other solutions can be found by applying symmetry transformations to this basic solution. Apparently the angle φ is related to the size of the instanton. On the other hand, φ is the geodesic distance between the points $u(t \to \pm\infty)$ in the classical vacuum. A suitable definition for the size ρ of the instanton is given by [21] $\rho(\varphi) = \sin(\frac{1}{2}\varphi)$. We explicitly see how the instanton is becoming singular if one tries to impose periodic boundary conditions, $\varphi \to 0$. On the other hand for antipodal $u(t \to \pm\infty)$, which are like anti-periodic boundary conditions, the instanton has maximal size. In this limit the energy profile is constant on S^1 and the configuration at $t = 0$ corresponds to a static solution of the equations of motion with one unstable mode in the direction of tunnelling. This configuration is therefore a sphaleron, which gives the minimal energy barrier to be taken along the tunnelling path.

The situation for $T^3 \times \mathbb{R}$ is seen to be quite similar. In the absence of exact results we have used the lattice approximation to study the instanton solutions numerically. Within the theory of finite volume gauge theories [22] these instanton solutions provide an essential ingredient to go to larger volumes. Typically instantons become dynamically relevant for the low-lying glueball spectrum in volumes around one half to one cubic fermi. The finite volume sphaleron [13] sets the energy scale beyond which tunnelling effects are no longer exponentially suppressed. The tunnelling path through the sphaleron [12] gives important information on the degrees of freedom that need to be accounted for non-perturbatively. For a recent review of the dynamical aspects see ref. [23].

The standard Wilson action for the lattice is a discrete approximation to the continuum action. This discretization breaks the scale invariance, such that the action of approximate instanton solutions will weakly depend on the moduli parameters. In particular the action is decreasing as the scale parameter of the instanton is lowered. As a consequence, looking for solutions by lowering the action towards a local minimum (called cooling [25]), moves the configuration to (approximate) instantons of ever smaller size, at some point "falling through the lattice". To reverse the shrinking of instantons under cooling we have introduced over-improved cooling [12]. This is cooling with a modified lattice action, which reverses the size of the scaling violations (rather than removing them to second order in the lattice spacing). This is achieved by adding to the standard Wilson action plaquettes of size 2×2, with a relative coefficient of $-1/40$. As a consequence, the cooling now increases the size of the instanton, until it cannot grow further due to the finite volume. Even on a lattice this provides an exact solution. By scaling-up the number of lattice points a rather good approximation to the continuum solution can be found, because lattice artefacts are small for large instantons.

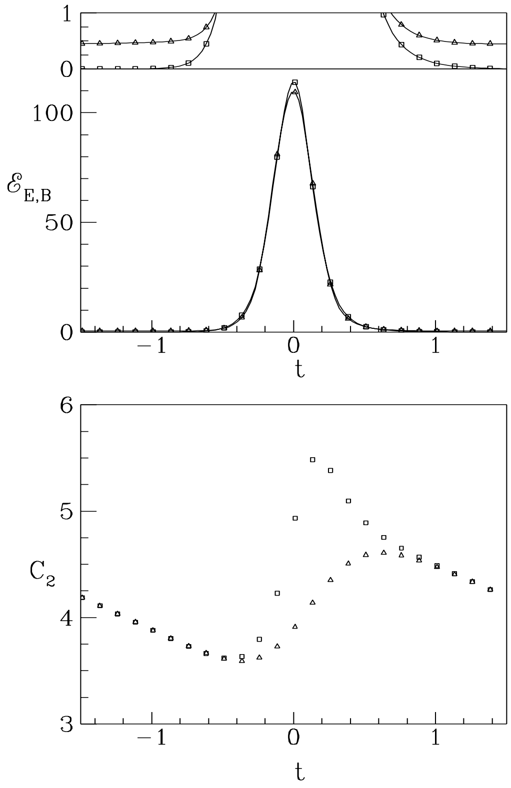

Figure 1. Numerical results [12] (after scaling appropriately with N_s) for the case of an $8^3 \times 24$ lattice with periodic boundary conditions, obtained from over-improved cooling. In the top figure electric ($\mathcal{E}_E(t)$ triangles) and magnetic ($\mathcal{E}_B(t)$ squares) energies are plotted. The tails are plotted at an enlarged scale in the inset. In the lower figure we plot $C_2(t)$ through two distinct spatial points on the lattice.

We have clearly observed that the instanton does not like to have periodic boundary conditions in the time direction. As we discussed before, the obstruction against the existence of charge one instantons on T^4 seems to imply that the configuration becomes peaked in the limit that it tends to a solution. For the O(3) model this persists even when time has an infinite extent. With the over-improved action this effect is countered by the tendency of instantons to grow under cooling. Indeed we found these effects to balance each other at some point. Nevertheless, the resulting lattice solution clearly shows deviations from self-duality, see fig. 1.

As we already described in the introduction, the classical vacua on T^3 are given by flat connections. They can be parametrized by abelian constant vector potentials $A_i^{\mathrm{flat}} = iC_i\tau_3/2L$ for a three-torus of size L. Not all of these are gauge inequivalent, and it is most convenient to parametrize these flat connections by the Wilson loops that close because of the periodic boundary conditions (also called Polyakov loops) for each of the three generating circles of the torus

$$\begin{aligned} P_i(x) &\equiv \tfrac{1}{2}\,\mathrm{Tr}\,\mathrm{Pexp}\{\int_0^L A_i(x+s\hat{e}_i)ds\} \\ &\equiv \cos(C_i(x)/2). \end{aligned} \tag{35}$$

One easily finds that $P_i^{\mathrm{flat}}(x) = \cos(C_i/2)$. For large extensions T in the time direction, the finite action of an instanton requires the potential energy to go to zero at both ends, such that A_i will approach a flat connection. As cooling brings the configuration to a solution of the equations of motion, one easily finds that restricted to the set of flat connections, the electric field ($\partial_t C_i$) has to vanish too. Self-duality would require C_i to become time-independent. From the lower part of figure 1 it is clear that this is not the case for our lattice solution. Indeed, it can be verified that the slope of $C_i(t)$ (only shown for $i = 2$) is entirely responsible for the electric energy in the tail regions where the magnetic energy vanishes to a high accuracy. It should be noted that for increasing T (N_t lattice units) this so-called electric tail will go down as $1/T^2$ and from these numerical studies we can not rule out if instantons do exist when $T{\to}\infty$.

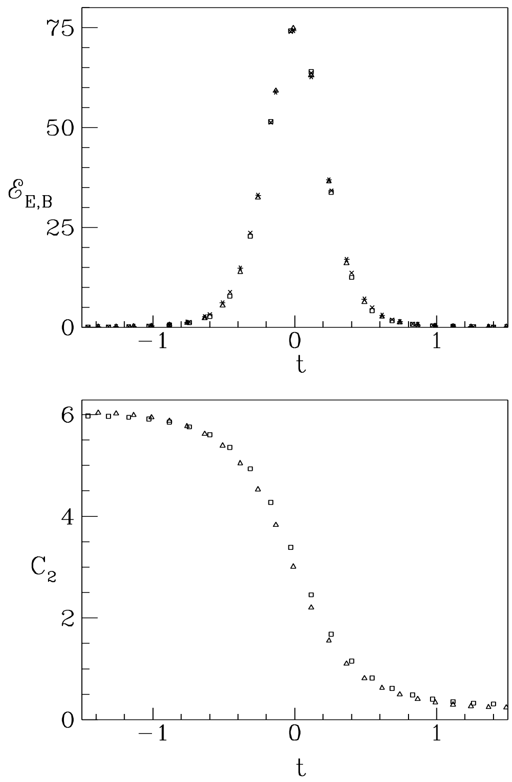

Figure 2. Numerical results [12] for lattices with $N_s = 7$ and 8 (and $N_t = 3N_s$), properly scaled with N_s and using twisted boundary conditions ($n_{0i} = 1$). In the top figure electric ($\mathcal{E}_E(t)$ squares at $N_s = 7$ and triangles at $N_s = 8$) and magnetic ($\mathcal{E}_B(t)$ crosses at $N_s = 7$ and stars at $N_s = 8$) energies are plotted. In the lower part we plot $C_2(t)$ through a given spatial point.

Nevertheless, the situation is dramatically different when twisting the boundary conditions in the time-direction, as demonstrated in fig. 2. Lattice artefacts are seen to be very small by comparing results on lattices with $N_s = 7$ and $N_s = 8$. The electric tail has completely disappeared, as can also be seen from the behaviour of $C_i(t)$, and the solution is perfectly self-dual. Twisting the boundary conditions can be most simply achieved in the $A_0 = 0$ gauge by applying an anti-periodic gauge transformation [10],

$$g(x) = \exp\{\pi i(x_1+x_2+x_3)\tau_3/L\} \tag{36}$$

at $t = T$, properly transposed to the lattice. It has the effect of changing the sign of all Wilson loops, $P_i(t=T) = -P_i(t=0)$. One can also apply a twist such that, say, only P_1 is (anti-)periodic. In all these cases instantons can be shown to exist. Assuming that solutions persist under taking the limit $T \to \infty$, this shows existence of instanton solutions on $T^3 \times \mathbb{R}$.

It is tempting to expect that solutions will exist for any value of $P_i^{\rm flat}$. It would give a natural way of counting the moduli parameters, adding a scale parameter and the four translation parameters. It should be said that this is not the way one counts the parameters when glueing in a localized solution from $\mathbb{R}^4$ to the flat four-dimensional connection. In this case there are three so-called attachment parameters [4,11]. Assuming the conjecture to be true, however, we can prove the existence of an instanton solution on $T^3 \times \mathbb{R}$ with $P_i = 0$ at *both* ends. We have shown that this solution cannot be compactified, as it is also compatible with periodic boundary conditions. It might, however, show the same behaviour on $T^3 \times \mathbb{R}$ in approaching $P_i(t\to\pm\infty) = 0$ as the O(3) instanton for $\varphi\to 0$. Over-improved cooling with twisted boundary conditions will lead to a solution with a fixed value of $P_i^{\rm flat}$, driven there by the well defined (but hard to control) scaling violations.

Fixing $P_i^{\rm flat}$ at the two ends is easily implemented on the lattice. For the continuum, in the $A_0 = 0$ gauge, one would need to apply a gauge transformation with unit winding number to the constant abelian representation of the flat connection at $t = T$ to admit an instanton. On the lattice a smooth configuration can only be specified by requiring gauge invariant quantities to be smooth. A quantity like the vector potential (on the lattice encoded in link variables) is in general not smooth, unless one imposes in addition a "smoothing" gauge condition, like the Coulomb gauge. It implies that configurations can be in a singular gauge, with the singularity "hidden" between the meshes of the lattice. In this way the periodic boundary conditions of the lattice are no obstacle for having smooth instanton configurations.

Initially, fixed boundary conditions were introduced to search for instantons with minimal energy barrier to locate the sphaleron[3]. We can, however, now also study solutions with $|P_i|$ different at both ends. Our results indicate that, like for the case of periodic boundary conditions, solutions exist that become self-dual when T is chosen sufficiently large. If this is also true in the continuum remains to be seen, as we have insufficient control over interchanging the limits N_t and $N_s \to \infty$.

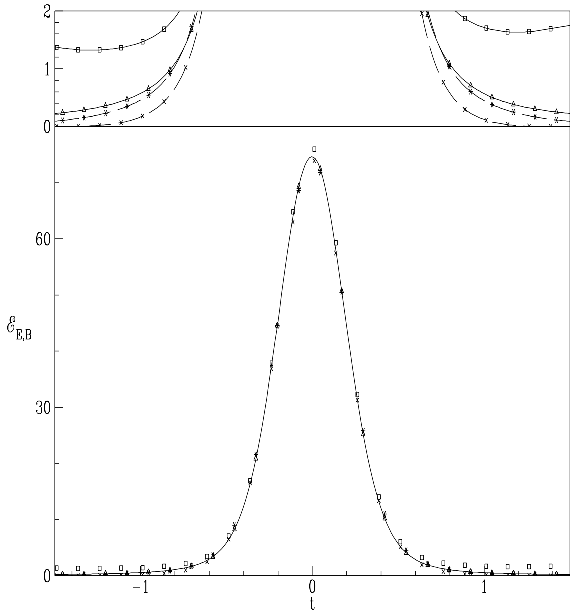

Figure 3. The electric and magnetic energies obtained after over-improved cooling for a lattice of spatial size $N_s = 8$ with boundary conditions fixed in the time direction to $P_i = -1$ at one and $P_i = 1$ at the other end [13]. Results are for $N_t = 24$ (squares for $\mathcal{E}_E$ and crosses for $\mathcal{E}_B$, respectively the upper and lower curves in the inset) and $N_t = 48$ (triangles for $\mathcal{E}_E$ and stars for $\mathcal{E}_B$, respectively the middle upper and lower curves in the inset).

[3]The sphaleron was also constructed more directly using saddle-point cooling [13]. It was found to have the remarkable property that the gauge invariant part of the magnetic field, $\mathrm{Tr}\, B_k^2(\vec{x})$, is independent of the direction k for each point $\vec{x}$ separately. Cubic invariance only requires this to be the case for the average. So far we have been unable to find an analytic expression for this sphaleron.

Since counting the number of moduli parameters (based on index theorems) in general requires compact four-dimensional manifolds, it is not ruled out that the moduli space of charge one instantons on $T^3 \times \mathbb{R}$ is of higher dimension than eight. We should point out, however, that despite of the similarities with the non-linear sigma model, there is a marked difference. Different points in the vacuum valley are not related by a symmetry. Indeed, the shape of the static potential $V(A)$ depends strongly on the position within the vacuum valley. Our analysis has, however, established that the instanton that corresponds to tunnelling through the sphaleron is associated to $P_i(t \to \pm\infty) = \pm 1$. These are rather special points of the vacuum valley, with directions in field space where $V(A)$ shows quartic, rather than quadratic behaviour. This also means that the approach of the tunnelling path to these points in the vacuum valley is not guaranteed to be exponential in time, as seems to be confirmed by our numerical results illustrated in fig. 3. Note that by forcing the configuration to be in the vacuum valley at either end (with $P_i^{\rm flat} = \pm 1$) we also destroy the self-duality, in spite of the fact that the configuration in fig. 3 is compatible with twisted boundary conditions. At finite T, instantons do no longer correspond to vacuum-to-vacuum tunnelling solutions (it takes an infinite time to exactly reach the vacuum).

It might very well be that in the continuum only instantons with $P_i(t \to \pm\infty) = \pm 1$ will be exact solutions. At the other extreme there seems to be the possibility of having three additional moduli parameters. From the physical point of view the most urgent problem would be to get some analytic hold on the instanton solution that tunnels through the sphaleron, so as to be able to take these effects into account beyond a semiclassical approximation, mandatory when the wave functionals of states that appear in the low-lying spectroscopy will no longer be exponentially suppressed near the sphaleron. For $S^3 \times \mathbb{R}$ such a study [26] has been recently shown to be feasible [27]. Also from the mathematical point of view, the properties of the moduli space for $T^3 \times \mathbb{R}$ present an interesting challenge. In the next section we make some general observations about the Nahm transformation in this context.

5. The Nahm transformation on $T^3\times\mathbb{R}$

When time extends to infinity, the partial integration involved, going from eq. (13) to (16) can pick up a boundary term,

$$\hat F^{ij}_{\mu\nu}(z)=-8\pi^2<\Psi^{(i)}_z|\sigma^\dagger_{[\mu}\sigma_{\nu]}G_z|\Psi^{(j)}_z> \qquad (37)$$
$$+\,4\pi i\oint\frac{\partial\Psi^{(i)}_z(x)^\dagger}{\partial z_{[\mu}}\sigma^\dagger_\lambda\sigma_{\nu]}\Big(G_z\Psi^{(j)}_z\Big)(x)\,d^3_\lambda x.$$

This boundary term in general destroys the anti-selfduality of $\hat\Omega$. It is useful to review the situation for $\mathbb{R}^4$, so as to get some insight in how the Nahm transformation needs to be modified in the presence of these boundary terms [5,6]. If $\Psi(x)$ is a normalized zero-mode of the Weyl operator D, also $e^{-2\pi izx}\Psi(x)$ is normalized and easily seen to be a zero-mode of D_z. Hence

$$\begin{aligned}\Psi_z(x)&=e^{-2\pi izx}\Psi(x),\\ G_z(x,y)&=e^{2\pi iz(y-x)}G(x,y).\end{aligned}\qquad(38)$$

Compactifying to S^4 and using conformal invariance allows for a regular description of the gauge field at infinity, after a suitable gauge transformation $g(x)$ (with unit winding number). From this a solution of the Weyl equation is seen to have the following asymptotic behaviour [5,6]

$$\begin{aligned}\Psi(x)&=\sigma^\dagger_\mu x_\mu g(x)\frac{\alpha}{\pi}|x|^{-4}+\mathcal{O}(|x|^{-4}),\\ (G\Psi)(x)&=-\tfrac14\sigma^\dagger_\mu x_\mu g(x)\frac{\alpha}{\pi}|x|^{-2}+\mathcal{O}(|x|^{-3}),\end{aligned}\qquad(39)$$

where α, associated to each zero-mode, is a constant factor with spinor and isospin (i.e. group) indices. We easily deduce from this that the Nahmtransformed connection is z-independent and that its curvature is no longer anti-selfdual

$$\begin{aligned}\hat A^{ij}_\mu&=-2\pi i<\Psi^{(i)}|x_\mu|\Psi^{(j)}>,\\ \hat F^{ij}_{\mu\nu}&=-8\pi^2<\Psi^{(i)}|\sigma^\dagger_{[\mu}\sigma_{\nu]}G|\Psi^{(j)}>+\pi^2\alpha^{i\dagger}\sigma_{[\mu}\sigma^\dagger_{\nu]}\alpha^j.\end{aligned}\qquad(40)$$

The boundary term in this case has added a self-dual contribution to the anti-selfdual part that was already present. Nevertheless, one can proceed to attempt to perform the Nahm transformation a second time. The Weyl operator now becomes algebraic, $\hat D^-_x=\sigma_\mu(\hat A_\mu+2\pi ix_\mu)$, but $\hat D^-_x\hat D^+_x$ no longer commutes with the quaternions, since $\hat F_{\mu\nu}=[\hat A_\mu,\hat A_\nu]$ has a self-dual component. In this way, applying the Nahm transformation again will not give an anti-selfdual connection. However, one can modify the Weyl operator as follows [5,6]

$$\hat{\hat D}{}^-_x\equiv\Big(-2\pi i\lambda,\hat D^-_x\Big)=\Big(-2\pi i\lambda,\,\sigma_\mu(\hat A_\mu+2\pi ix_\mu)\Big).\qquad(41)$$

Putting $\lambda^{iI}_m\equiv\sigma_2^{IJ}\alpha^i_{mJ}$, this leads to the desired commutation with the quaternions

$$\begin{aligned}\hat{\hat D}{}^-_x\hat{\hat D}{}^+_x&=-4\pi^2\lambda\,\lambda^\dagger+\sigma_0(\hat A_\mu+2\pi ix_\mu)^2+\tfrac12\sigma_{[\mu}\sigma^\dagger_{\nu]}\hat F_{\mu\nu}\\ &=\big((\hat A_\mu+2\pi ix_\mu)^2-2\pi^2\alpha^\dagger\alpha\big)\sigma_0,\end{aligned}\qquad(42)$$

using $\sigma_{[\mu}\sigma^\dagger_{\nu]}\hat F_{\mu\nu}/\pi^2=\sigma_{[\mu}\sigma^\dagger_{\nu]}\alpha^\dagger\sigma_{[\mu}\sigma^\dagger_{\nu]}\alpha=-4\tau_i\alpha^\dagger\tau_i\alpha$ and the completeness relation for the Pauli matrices, $\frac12\tau^{ab}_i\tau^{cd}_i=\delta_{ad}\delta_{bc}-\frac12\delta_{ab}\delta_{cd}$. The Nahm transformation can be shown to be complete and coincides with the algebraic ADHM [15] construction.

It is also worthwhile to discuss the construction for monopoles [5], which are time independent self-dual solutions on $\mathbb{R}^3$. Here A_0 plays the role of the Higgs field, required to approach a fixed length at infinity, which can always be scaled to one. For SU(2) we therefore can choose $A_0\to -\frac12 ig(x)\tau_3g^{-1}(x)$, for $r=|\vec x|\to\infty$, with $g(x)$ a non-trivial gauge transformation. As for the instantons, one easily sees that $\hat A_\mu(z)$ is independent of z_i. The remaining parameter $z=z_0$ would be expected to have infinite range. But for $\Psi\in\ker D_z$, the asymptotic behaviour of the Weyl equation is seen to require $\det(A_0+2\pi iz)>0$, in order for it to admit normalizable zero-modes [5,29], thus $z\in(-1/4\pi,1/4\pi)$. Amazingly, the BPS [28] monopole solution can be retrieved by putting $\hat A_\mu(z)=0$ as a solution to the one dimensional duality equations on this interval and by performing the Nahm transformation [5].

For $T^3\times\mathbb{R}$, eq. (37) will only give a non-trivial boundary term from the $\lambda=0$ contribution. For convenience we will choose $T^3=\mathbb{R}^3/\mathbb{Z}^3$, i.e. $L=1$. Like for $\mathbb{R}^4$, the z_0 dependence associated with the non-compact time parameter, becomes trivial, $\Psi_z(x)=\exp(-2\pi iz_0t)\Psi_{\vec z}(x)$. The Weyl equation, with $D_{\vec z}$ its spatial part, reduces to

$$D_z\Psi_z(x)=e^{-2\pi iz_0t}(\partial_0+D_{\vec z})\Psi_{\vec z}(x),\qquad(43)$$

in the $A_0=0$ gauge, and we can study the zero-modes in terms of the spectral decomposition of the spatial part of the Weyl equation, which for a flat connection reduces, up to a gauge, to

$$D_{\vec{z}}(A^{\rm flat})e^{2\pi i\vec{p}\cdot\vec{x}}\Psi^s_{\vec{z}}(\vec{p})=e^{2\pi i\vec{p}\cdot\vec{x}}E^s_{\vec{z}}(\vec{p})\Psi^s_{\vec{z}}(\vec{p}), \quad (44)$$

where $s=\pm\frac{1}{2}$ is the isospin of the solution and $\vec{p}\in\mathbb{Z}^3$ gives the momentum. The momentum and isospin eigenstates satisfy the spinor equation

$$\vec{\tau}\cdot(2\pi\vec{p}+s\vec{C}+2\pi\vec{z})\Psi^s_{\vec{z}}(\vec{p})=E^s_{\vec{z}}(\vec{p})\Psi^s_{\vec{z}}(\vec{p}), \quad (45)$$

with positive and negative energy eigenvalues given by

$$E^s_{\vec{z}}(\vec{p})=\pm\|2\pi\vec{p}+s\vec{C}+2\pi\vec{z}\|. \quad (46)$$

When $A=A^{\rm flat}$, the zero-modes can be decomposed according to

$$\Psi_{\vec{z}}(x)=\sum_{\vec{p},s,v}a^{s,v}_{\vec{p}}(t)e^{2\pi i\vec{x}\cdot\vec{p}}\Psi^{s,v}_{\vec{z}}(\vec{p}), \quad (47)$$

where v labels the positive or negative energy states[4].

At fixed $\vec{C}$, for all but a finite number of values of $\vec{z}$ within one unit cell of $\hat{T}^3=\mathbb{R}^3/\mathbb{Z}^3$, the spectrum of $D_{\vec{z}}(A^{\rm flat})$ has a gap, in which case the zero-modes of the Weyl equation are expected to decay exponentially in time. There would in this case be no boundary term contributing to eq. (37). As long as $\vec{C}\neq\vec{0}\,{\rm mod}\,2\pi$, any instanton solution will approach $A^{\rm flat}$ exponentially in time and the conclusion that $\Psi_{\vec{z}}$ decays exponentially in time will hold. If, however, $A^{\rm flat}$ is associated to one of the quartic points in the vacuum valley, $\vec{C}=\vec{0}\,{\rm mod}\,2\pi$, this might require more care. Assuming for the moment that this will cause no problems for the behaviour of the zero-modes of the Weyl equation, also in this case boundary terms will be absent for $\vec{z}\neq\vec{0}\,{\rm mod}\,\frac{1}{2}$. As a consequence eq. (16) will remain valid almost everywhere, with a correction that has support at a finite number of points only. From the theory of distributions, this correction term should be expressible in terms of delta functions. This provides the interesting suggestion to study the BPS equations, $B_k=D_kA_0$, in the presence of point-like sources. Performing the Nahm transformation away from these finitely many point-sources might provide us with a method of constructing instanton solutions on $T^3\times\mathbb{R}$. Future work will tell if we require more detailed information on these singularities in order to successfully construct non-trivial solutions.

[4]In the background of an instanton configuration which connects the flat connections at either end, each eigenstate of $D_{\vec{z}}(\vec{A}(t,\vec{x}))$ evolves smoothly as a function of t from one state in the spectrum determined by eq. (46) at $t=-\infty$ to another at $t=\infty$, of course with the appropriate values of $\vec{C}$ inserted at either end. From the relation between the index theorem and spectral flow, we still expect that generically there is exactly one such state Ψ_t (with eigenvalue E_t) that moves from negative energy at $t=-\infty$ to positive energy at $t=\infty$. Eq. (43) would imply that in an adiabatic approximation $\Psi_{\vec{z}}\sim\exp(-E_tt)\Psi_t$ and the spectral flow provides a natural explanation for the existence of a normalizable zero-mode. We will not attempt to make this more precise here.

6. Conclusion

We have reviewed the role of the Nahm transformation in constructing solutions of the self-duality equations for gauge theories, with an emphasis on the applications to T^4 and in particular $T^3\times\mathbb{R}$. On T^4 we explicitly illustrated this transformation for the class of (reducible) constant curvature solutions, which form special (singular) points in the moduli space. The quest for explicit instanton solutions on a torus is motivated by the dynamical study of glueball spectroscopy in small to intermediate volumes. Numerical lattice studies of these instantons have therefore been developed recently and have been reviewed here too. These numerical studies have led to interesting conjectures on the moduli space of charge one instantons on $T^3\times\mathbb{R}$. It provides the motivation for a renewed attack on the study of the Nahm transformation for this setting. We showed how singularities arise due to certain boundary terms (like for $\mathbb{R}^4$), related to the asymptotic behaviour of the chiral zero-modes of the Dirac-Weyl equation. The study of the BPS equations on T^3 with finitely many sources is hoped to give us an analytic technique to construct the charge one instanton on $T^3\times\mathbb{R}$, using the Nahm transformation.

Acknowledgements

I am grateful to Gerhard Weigt and Dieter Lüst for their generous invitation to the Buckow meeting. I thank Andreas Wipf for discussions concerning instantons for the O(3) model. I would like to also thank my many collaborators for their interest in these problems. In particular I am grateful to Peter Braam and Margarita García Pérez for many stimulating and helpful discussions over the years. The numerical aspects of this work were supported in part by a grant from "Stichting Nationale Computer Faciliteiten (NCF)" for use of the CRAY C98 at SARA.

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