MASSLESS FERMIONS IN FINITE VOLUMES

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We show that in the finite volume perturbation theory no zero-energy modes for the fermions appear and that chiral symmetry is unbroken in small volumes. We discuss the influence of the massless fermions on the glueball spectrum in a finite cubic volume for an arbitrary number of flavours in both SU(2) and SU(3). We also analyse the SU(2) group valued variables recently introduced by Kripfganz and Michael and show that a minor but essential modification of their effective Lagrangian for the zero momentum modes is exactly equivalent to the more conventional Lüscher-type effective Hamiltonian. This effective Lagrangian opens the way to intermediate volume calculations in the presence of massless fermions. Some of the limitations of the method especially in relation to the possible decay of glueballs into mesons will also be discussed.

1. INTRODUCTION

The last few years have seen a considerable progress in our understanding of pure gauge theories, especially in small to intermediate volumes with an extension of up to five scalar glueballs [1-4]. Particularly fruitful was the comparison between lattice Monte Carlo and analytic calculations in these volumes. Considering the difficulty of incorporating dynamical fermions in lattice Monte Carlo calculations [5], in particular for small masses, it should be useful to have analytic results in small to intermediate volumes, including the dynamical effects of massless fermions. In these volumes the present day algorithms might be sufficient to do a Monte Carlo analysis at reasonably small coupling.

There has, however, also been a considerable folklore concerning perturbative calculations in the presence of fermions. For example, if one chooses periodic boundary conditions in the spatial directions for the fermions, the perturbative expansion is to be performed *away* from $A = 0$ [6]. As a consequence there are no zero-energy modes for the fermions and there is *no* chiral symmetry breaking in small volumes.

In section 2 we will set up the boundary conditions, identify the quantum vacua and discuss the chiral pitfalls. Section 3 will contain the effective Hamiltonian and discuss some remarks on the non-perturbative analysis. In section 4 we will analyse the effective Lagrangian proposed by Kripfganz and Michael [8]. We end with some discussions. Sections 1 to 3 are a review of refs. 6-8, whereas section 4 contains new material inspired by ref. 8.

2. BOUNDARY CONDITIONS, QUANTUM VACUA AND CHIRAL PITFALLS

The finite volume will be a cube with sides of length L. This will allow for direct comparison with lattice Monte Carlo calculations on *finite* lattices. The reason that (semi-)analytic calculations are possible is based on asymptotic freedom, which implies that in small volumes the effective coupling constant is small. We will use dimensional regularization with minimal subtraction at the scale $\mu = 1/L$. This means that the renormalized coupling constant is L dependent:

$$\begin{aligned} g^{-2}(L) &= -2b_0\left\{\ell n(L\Lambda_{MS}) + \frac{b_1}{2b_0^2}\ell n[-2\ell n(L\Lambda_{MS})]\right\} \quad , \\ b_0 &= \frac{1}{(4\pi)^2}(\frac{11N}{3} - \frac{2n_f}{3}) \quad , \\ b_1 &= \frac{1}{(4\pi)^4}(\frac{-34N^2}{3} + \frac{10Nn_f}{3} + \frac{(N^2-1)n_f}{N}) \quad , \end{aligned} \tag{1}$$

where eq. (1) holds for SU(N) gauge theory with n_f families of Dirac fermions.

The gauge fields can be chosen periodic, whereas for the fermions we either have periodic or antiperiodic boundary conditions in the spatial directions. There is an interesting theorem which states that periodic fermions are gauge equivalent to antiperiodic fermions for SU(2). The gauge tranformation in question is given by:

$$g(\mathbf{x}) = \exp(\frac{2\pi i}{L}(x_1 + x_2 + x_3)\frac{\sigma_3}{2}) \quad . \tag{2}$$

This is an antiperiodic gauge function. It leaves the boundary conditions on the gauge fields unchanged, whereas it is easily seen to interchange periodic and antiperiodic fermions (which are of course assumed to be in the fundamental representation of SU(2)). From this equivalence it is clear that it is no longer obvious whether $A = 0$ is the appropriate vacuum to expand around. It can only be such for either periodic or antiperiodic fermions. Only in the first case can there be zero-energy modes for the fermions. It has been argued that this can lead to spontaneous chiral symmetry breaking, even in small volumes [9]. We will see, however, that zero-energy fermion modes do not occur in perturbation theory. However, even if they would occur it would *not* lead to chiral symmetry breaking through the formation of a condensate [9]. To see this we calculate $\langle\bar{\Psi}\Psi\rangle$ in a background of zero-momentum gauge fields, without averaging over the gauge fields:

$$\langle\bar{\Psi}\Psi\rangle_A = \frac{mNn_f}{L^3\sqrt{m^2 + 2\mathrm{Tr}(A^2)}} + O(m/L^2) \quad , \tag{3}$$

where m is the fermion mass term. For $A = 0$ one finds in the limit $m \to 0$ that $\langle\bar{\Psi}\Psi\rangle_{A=0} = Nn_f/L^3$. However, one should take the limit $m \to 0$ only after having averaged over the gauge fields in which case $\langle\bar{\Psi}\Psi\rangle = 0$ and no condensation of zero-energy fermion modes will occur.

As we said before, no zero-energy fermion modes are occurring in perturbation theory anyhow, making the above analysis somewhat academic. To show that indeed perturbation theory for periodic fermions has to be performed away from $A = 0$, we will give the effective potential for a background of abelian zero-momentum gauge fields, which is the set of classical vacua:

$$A_i(\mathbf{x}) = B_i = \frac{C_i \sigma_3}{2L} \quad . \tag{4}$$

$V_{eff}(\mathbf{C})$ is half the sum of the transverse frequencies or equivalently:

$$\int_0^T dt V_{eff}(C_i^a) = \ell n \left[\frac{\det'(-D_\mu^2(B)) \det(\not{D}(B))^{n_f}}{\det'(W_{\mu\nu}(B))^{1/2}} \right] \quad , \tag{5}$$

where:

$$\begin{aligned} D_\mu &= \partial_\mu + i \mathrm{ad} B_\mu \quad , \\ \not{D}(B) &= \gamma_\mu(\partial + i B_\mu) \quad , \\ W_{\mu\nu}(B) &= -\delta_{\mu\nu} D_\lambda^2(B) \quad . \end{aligned} \tag{6}$$

Fig. 1: The graph for the effective potential $V = L \cdot V_{eff}^{+}(C, 2\pi, 2\pi)$ for (a) $n_f = 0$ and (b) $n_f = 3$. In the latter case we added 24.79793 to normalize the energy of the quantum vacuum to zero.

After some calculation one finds for SU(2):

$$\begin{aligned} V_{eff}^{\pm}(\mathbf{C}) &= \hat{V}_1(\mathbf{C}) - 2n_f \hat{V}_1(I(\mathbf{C} + \mathbf{C}_\pm)) \quad , \\ \hat{V}_1(\mathbf{x}) &= \frac{4}{\pi^2 L} \sum_{\mathbf{n} \neq \mathbf{0}} \frac{\sin^2(\frac{1}{2}\mathbf{n} \cdot \mathbf{x})}{(\mathbf{n}^2)^2} \quad , \\ &= \frac{2}{L} \sum_{\mathbf{n}} |\mathbf{x} + 2\pi \mathbf{n}| + \text{constant} \quad , \end{aligned} \tag{7}$$

where I is the isospin of the fermion, generally taken to be 1/2, $\mathbf{C}_+ = \mathbf{0}$ which is relevant for periodic fermions and $\mathbf{C}_- = 2\pi(1,1,1)$ which is to be used for antiperiodic fermions. For a similar result in the case of SU(3) we refer to ref. 6. From the properties [10] that $\hat{V}_1(\mathbf{x})$ is minimal at $\mathbf{x} = \mathbf{0}$, maximal at $\mathbf{x} = \pi(1,1,1)$ and periodic with periods 2π, it follows that only for antiperiodic fermions the quantum vacuum is situated at $\mathbf{C} = 0$, whereas for periodic fermions the quantum vacuum is at $\mathbf{C} = 2\pi(1,1,1)$, see fig. 1. In both cases no zero-energy fermion modes occur.

It is instructive to verify that for supersymmetric QCD the effective potential vanishes. Since the superpartner of the gluon has isospin one, satisfies periodic boundary conditions and is a Majorana fermion, effectively with $n_f = 1/2$, eq.(7) indeed yields zero for V^{+}_{eff}.

The periodicity of the effective potential is related to a gauge symmetry

$$\begin{aligned} &g_{\mathbf{k}}(\mathbf{x}) = \exp(-\pi i \mathbf{k}\cdot\mathbf{x}\sigma_3/L) \quad , \\ &[g_{\mathbf{k}}]\,\mathbf{C} = \mathbf{C} + 2\pi\mathbf{k} \quad . \end{aligned} \tag{8}$$

For $\mathbf{k}$ even, $g_{\mathbf{k}}(\mathbf{x})$ is periodic and therefore an allowed gauge transformation, leading to:

$$V_{eff}(\mathbf{C} + 4\pi\mathbf{n}) = V_{eff}(\mathbf{C}) \quad , \tag{9}$$

whereas for any $\mathbf{k}$, $g_{\mathbf{k}}(\mathbf{x})$ is a symmetry for the pure gauge sector, explaining:

$$\hat{V}_1(\mathbf{C} + 2\pi\mathbf{k}) = \hat{V}_1(\mathbf{C}) \quad . \tag{10}$$

We end this section by discussing yet another way of finding the quantum vacua, which will be useful for studying an interesting phenomenon for the gauge group SU(3) with periodic fermions. For this we consider the Polyakov line (without taking the trace) evaluated at a quantum vacuum configuration $A_i(\mathbf{x})$:

$$P_j = \mathrm{Pexp}(i\int_0^L A_j(\mathbf{x} + t\mathbf{e}_j)dt) \quad . \tag{11}$$

This should take its value in the center of the gauge group in order not to spontaneously break gauge invariance. For SU(2) we therefore find $P_i = \exp(C_i^{vac}\sigma_3/2) = \pm 1$. To decide which of the two inequivalent configurations corresponds to the quantum vacuum, we still need to substitute $\mathbf{C}^{vac}$ in to V_{eff} and one easily finds that for periodic fermions $C_i^{vac} = 2\pi(\mathrm{mod} 4\pi)$ and $P_i = -1$, whereas for antiperiodic fermions $C_i^{vac} = 0(\mathrm{mod} 4\pi)$ and $P_i = 1$.

For SU(3) one has $P_i \in Z_3$, but for antiperiodic fermions one easily shows that the quantum vacuum is unique and corresponds to $P_i = 1$ and $\mathbf{A}^{vac} = \mathbf{0}$. However, using periodic fermions leads to the surprising situation that the true quantum vacuum has $P_i \neq 1$. Using the following discrete symmetries of the Hamiltonian:

$$\begin{aligned} &C: P_i \to P_i^* \quad ; \quad P: P_i \to P_i^* \quad ; \\ &\pi_i : P_j \to P_j^* \quad , \quad i = j \quad ; \quad P_j \to P_j \quad , \quad i \neq j \quad , \end{aligned} \tag{12}$$

leads to an eightfold degenerate vacuum, spontaneously breaking charge conjugation and parity but preserving CP. This therefore means that for SU(3) the small volume spectrum dramatically depends on the choice of boundary conditions. For some of the consequences for the glueball spectrum, we refer to ref. 6. In the remainder of this paper we will restrict ourselves to SU(2) in the formulation with antiperiodic fermions.

3. THE EFFECTIVE HAMILTONIAN

Naively the low-lying spectrum would follow from the effective Hamiltonian:

$$\frac{-g^2}{2L}\frac{\partial^2}{\partial C_i^2}+V_{eff}^-(\mathbf{C}) \quad , \tag{13}$$

but the conic singularities of $\hat{V}_1(\mathbf{x})$ at $\mathbf{x}=\mathbf{0}(\mathrm{mod}2\pi)$ signal the breakdown of the one-loop approximation for integrating out the transverse degrees of freedom. The trouble is caused (at $\mathbf{C}=\mathbf{0}$) by zero-momentum gauge fields for which the Yang-Mills potential is quartic:

$$\begin{aligned} V &= -\frac{1}{2g^2L}\mathrm{Tr}(\left[c_i,c_j\right]^2) \quad , \\ A_i(\mathbf{x}) &= \frac{c_i}{L}=\frac{c_i^a\sigma_a}{2L} \quad . \end{aligned} \tag{14}$$

At $\mathbf{C}=2\pi(1,1,1)(\mathrm{mod}4\pi)$ there are also singularities due to the zero-energy fermion modes. However, these occur at a finite energy ($2n_f\times 4.132988/L$) above the vacuum and will therefore not be excited in small enough volumes, i.e. they will not cause a breakdown of the adiabatic approximation. The recipe to fix the quartic mode problem is as in the pure gauge case [11]. One derives an effective Hamiltonian which includes the quartic modes as slow modes. The fermions simply contribute through their polarization effects.

We will only give the final result for the effective Hamiltonian and refer for the derivation to refs. 6 and 7.

$$H_{eff}=-\frac{1}{2L}(\frac{1}{g^2}+\alpha_1(n_f))^{-1}\frac{\partial^2}{\partial c_i^{a2}}+V_T(c)+V_\ell(c)+\cdots \quad , \tag{15}$$

where:

$$\begin{aligned} V_\ell(c) &= \hat{V}_1(\mathbf{r})-2|\mathbf{r}|/L-2n_f\hat{V}_1(\mathbf{r}/2+(\pi,\pi,\pi)) \quad ; \\ r_i &= \sqrt{2\mathrm{Tr}(c_i^2)} \quad , \end{aligned} \tag{16}$$

$$\begin{aligned} L\cdot V_T(c) =& \frac{1}{4}(\frac{1}{g^2}+\alpha_2(n_f))F_{ij}^aF_{ij}^a+\alpha_3(n_f)F_{ij}^aF_{ij}^ac_k^bc_k^b+\alpha_4(n_f)F_{ij}^aF_{ij}^ac_j^bc_j^b+ \\ & \alpha_5(n_f)(\mathrm{det}c)^2+\cdots \quad ; \quad F_{ij}^a=-\varepsilon^{abd}c_i^bc_j^d \quad . \end{aligned} \tag{17}$$

The numerical values of $\alpha_i(n_f)$ are

$$\begin{aligned} \alpha_1(n_f) &= 2.1810429\times 10^{-2}-2n_f\cdot 3.098211\times 10^{-5} \\ \alpha_2(n_f) &= 7.5714590\times 10^{-3}-2n_f\cdot 1.7211922\times 10^{-3} \\ \alpha_3(n_f) &= -1.1130266\times 10^{-4}-2n_f\cdot 3.0178786\times 10^{-5} \\ \alpha_4(n_f) &= -2.1475176\times 10^{-4}-2n_f\cdot 3.2156523\times 10^{-5} \\ \alpha_5(n_f) &= -1.2775652\times 10^{-3}+2n_f\cdot 3.2271736\times 10^{-5} \end{aligned} \tag{18}$$

We note that the n_f dependence is rather weak, and that in general the higher order corrections in V_T have very small coefficients. Furthermore, the set of classical vacua is contained within the set of zero momentum gauge fields by restricting to $V_T(c) = 0$, which is equivalent to $F_{ij} = 0$.

The perturbative results for the glueball masses can be summarized by the formula

$$L \cdot E(n_f, g) = \rho^{-1} L \cdot E(0, \rho^{2/3} g) + O(g^{8/3}) \quad , \tag{19}$$

where $\rho = (1 - 0.14132 \times n_f)$, which means in particular that mass-ratios will only be weakly depending on the number of flavours. This remains true for SU(3) with antiperiodic fermions (with $\rho = (1 - 0.09421 \times n_f)$). However, as we have seen before, with periodic fermions there is a strong dependence on the number of flavours, especially comparing $n_f = 0$ and $n_f \neq 0$ [6]. Consequently, this would be more useful for testing fermion algorithms, but unfortunately intermediate volume results are not easy to obtain for SU(3). In the next section we will see that for SU(2) in intermediate volumes there is in principle a strong flavour dependence for the glueball masses, but mass ratios seem to be less dependent.

We end this section by discussing how to go to intermediate volumes for SU(2). The idea is quite similar to the pure gauge case, where we decompose the wave function in a vacuum-valley and transverse wave function and demand the vacuum-valley wave function to be continuous and differentiable, which should be a good approximation, at least for the positive parity states [4]. To be more precise, we will have eight coordinate patches, labelled by:

$$c_i(\mathbf{k}) = \frac{1}{L^2} \int d^3x \left[g_{(\mathbf{k})} \right] A_i(\mathbf{x}) \quad , \tag{20}$$

with $k_i \in \{0, 1\}$ and $g_{\mathbf{k}}$ as in eq. (8). We choose the radial coordinates as

$$r_i(\mathbf{k}) = 2\pi k_i + (-1)^{k_i} \sqrt{2\mathrm{Tr}(c_i(\mathbf{k})^2)} \tag{21}$$

and identify the angular coordinates at the overlap for each pair of patches, i.e. at $r_i = \pi$. In each coordinate patch the effective Hamiltonian is as in eqs. (15-18), with c_i replaced by $c_i(\mathbf{k})$, except that in the expression for $V_\ell(c)$ we rather have:

$$\begin{aligned} V_\ell(c) &= \hat{V}_1(\mathbf{s}(\mathbf{k})) - 2|\mathbf{s}(\mathbf{k})|/L - 2n_f \hat{V}_1(\mathbf{r}(\mathbf{k})/2 + (\pi, \pi, \pi)) \quad . \\ s_i(\mathbf{k}) &= \sqrt{2\mathrm{Tr}(c_i(\mathbf{k})^2)} \quad , \end{aligned} \tag{22}$$

It is important to observe that $V_\ell(c)$ still has a conic singularity at $\mathbf{r} = 2\pi(1, 1, 1)$ due to the zero-energy mode for the fermions. We assume that the adiabatic approximation holds, which will be true if the wave function is small near this conic singularity. For example, if we take three flavours the effective potential is at its maximum exactly at this point, with an energy of $24.797928/L$ above the vacuum, which is considerably higher than the energies of the typical low-lying states one would like to consider. Therefore, in not too big a volume the adiabatic approximation for integrating out the zero-energy fermion mode at $\mathbf{r} = 2\pi(1, 1, 1)$ should be a valid one.

There is a slight imperfection in our derivation for the non-perturbative effective Hamiltonian. We have partially ignored the fact that the one-loop contribution of the fermions does depend on $\mathbf{k}$. This dependence is properly incorporated in eq. (22) for V_ℓ. But it is in principle incorrect to take V_T in each patch as in eq. (17) with c replaced by $c(\mathbf{k})$. One way to fix this is to redo the fermionic one-loop calculation by expanding around $c(\mathbf{k})=0$ (all that changes are the momentum sums in eq. (19) of ref. [6], which will only modify the coefficients linear in n_f). This would, however, give a problem in the patch $\mathbf{k}=(1,1,1)$ due to the zero-energy fermion mode. The adiabatic approximation amounts to ignoring the zero momentum contribution in those momentum sums. With this procedure, H_{eff} would be discontinuous at $r_i=\pi$, due to the fact that we expand only to a finite order in the fields when computing V_T. We should realize, however, that in the region where these higher order corrections can become appreciable the wave function will in general be small. With this in mind and the fact that the coefficients linear in n_f are already very small, we will be making only a very small error if we ignore the fact that the terms linear in n_f, occurring in V_T and in the kinetic term, depend on the coordinate patch.

It is now relatively straightforward to perform a Rayleigh-Ritz analysis for this effective Hamiltonian, with only a minor modification with respect to the pure gauge case, which was described in full detail in ref. 3. We introduce the spherical coordinates:

$$\begin{aligned} c_i^a(\mathbf{k}) &= s_i(\mathbf{k})n_i^a \quad , \\ (n_i^1,n_i^2,n_i^3) &= (\cos\phi_i\sin\theta_i,\sin\phi_i\sin\theta_i,\cos\theta_i) \quad , \\ r_i(\mathbf{k}) &= 2\pi k_i+(-1)^{k_i}s_i(\mathbf{k}) \end{aligned} \tag{23}$$

and take as a complete basis:

$$\langle c|\ell,\mathbf{n};\mathbf{e}\rangle=(-1)^{\mathbf{k}\cdot\mathbf{e}}\sum_{\mathbf{m}}W(\ell,\mathbf{m})\prod_{i=1}^{3}\chi_{n_i\ell_i}^{(e_i)}(s_i(\mathbf{k}))Y_{\ell_i m_i}(\theta_i,\phi_i) \tag{24}$$

where $\chi_{n_i\ell_i}^{(e_i)}$ is exactly the same radial one-particle wave function, satisfying the same boundary conditions as for the pure gauge case. However, notice that due to the lack of symmetry it is impossible to block-diagonalize H_{eff} with respect to $\mathbf{e}$. Nevertheless, most of the matrix of H_{eff} can be directly copied from the pure gauge result. The only modification is the matrix elements for the terms linear in n_f. For the above described simplification the only newly to evaluate matrix elements are those for the fermionic contribution to V_ℓ; it is the only part where H_{eff} is off-diagonal in $\mathbf{e}$.

In conclusion, a Rayleigh-Ritz analysis for the glueball spectrum in the presence of massless dynamical fermions is straightforward to perform. In the next section we describe an effective Lagrangian proposed by Kripfganz and Michael [8] which avoids working, at least explicitly, with coordinate patches and allows a Monte Carlo analysis for the low-lying glueball spectrum in intermediate volumes.

4. THE EFFECTIVE LAGRANGIAN

As is clear from the discussion in the previous section, the effective Hamiltonian is defined on a topologically nontrivial manifold, because we are using different coordinate patches. It is instructive to find the topology of this manifold. Each coordinate patch is a three-ball B_3 with boundary S^2, and for each spatial direction the two balls are joined at this boundary S^2, which forms the equator of the total space S^3, i.e. the three-sphere. On the other hand the group SU(2) is also topologically S^3. Since furthermore c_i is in the Lie-algebra of SU(2) it is only natural to consider

$$U_j = \exp(ic_j(\mathbf{k}))(-1)^{k_j} = \cos(r_j(\mathbf{k})/2) + i\sigma^a n_j^a \sin(r_j(\mathbf{k})/2) \quad , \tag{25}$$

as the new dynamical variables, rather then c_i. This gives the standard spherical parametrization of S^3 in terms of the angular variables $(r_j/2, \theta_j, \phi_j)$, with indeed $r_j = \pi$ being the equator of S^3. Note that the coordinate patch is simply encoded through: $\mathrm{sign}(\mathrm{Tr}(U_j)) = (-1)^{k_j}$. This thus motivates the Lagrangian proposed by Kripfganz and Michael [8]:

$$\begin{aligned}
\mathcal{L}_{eff} &= \mathcal{L}_{kin} + \mathcal{L}_{FF} + \mathcal{L}_V \quad , &(26)\\
\mathcal{L}_{kin} &= -2(\frac{1}{g^2} + \alpha_1(n_f)) \sum_i \mathrm{Tr}(U_i(t)U_i^\dagger(t+\varepsilon))/\varepsilon^2 \quad , &(27)\\
\mathcal{L}_{FF} &= -2(\frac{1}{g^2} + \alpha_2(n_f)) \sum_{i<j} \{\frac{4}{3}\mathrm{Tr}(U_i(t)U_j(t)U_i^\dagger(t)U_j^\dagger(t)) \\
&\quad - \frac{1}{12}\mathrm{Tr}(U_i(t)^2U_j(t)^2U_i^\dagger(t)^2U_j^\dagger(t)^2)\} \quad , &(28)\\
\mathcal{L}_V &= \sum a_{ijk}t_1^{(2i)}t_2^{(2j)}t_3^{(2k)} - 2n_f b_{ijk}t_1^{(i)}t_2^{(j)}t_3^{(k)} \quad , &(29)\\
t_i^{(n)} &= \mathrm{Tr}(U_i^n)/2 - 1 \quad .
\end{aligned}$$

Let us analyse each term separately. The kinetic term (eq. (27)) is the discretized version of the non-linear sigma model Lagrangian:

$$\begin{aligned}
\lim_{\varepsilon\to 0} 2\mathrm{Tr}(U_i(t)U_i^\dagger(t+\varepsilon) - 1)/\varepsilon^2 &= -\mathrm{Tr}(\{\dot{U}_i(t)U_i^\dagger(t)\}^2) \\
&= (\dot{r}_i(t)^2 + 4\sin^2(r_i(t)/2)\dot{\mathbf{n}}_i(t)^2)/2 \quad , \qquad (30)
\end{aligned}$$

which is nothing but the standard S^3 kinetic term. Eq. (28) is such that for small c_i it coincides with $L\cdot V_T(c)$ in eq. (17), but the terms of $O(c^6)$ have coefficients of order one, rather than of $O(10^{-3})$. Finally in eq. (29) the coefficients a_{ijk} and b_{ijk} were chosen such as to reproduce eq. (22) as accurately as possible [8]. In fig. 2 we show the deviation for the bosonic part of V_ℓ, whereas the fermionic contribution is reproduced very well (except near $\mathbf{r} = 2\pi(1,1,1)$). The glueball masses are calculated through time-time correlation functions. The operators used are the appropriate combinations of $\mathrm{Tr}(U_i)$ (Polyakov lines in the fundamental representation) to form irreducible representation of the cubic group. In the pure gauge case these operators couple to the various electric flux sectors. In that case, if one is interested in the glueball spectrum for zero electric flux, one should use the trace in the adjoint representation for the operators. The measure in the path integral is taken to be the Haar measure (which is the standard measure of S^3: $dU_i \propto \sin(\theta_i)\sin^2(r_i/2)d\theta_i dr_i d\phi_i$).

Fig. 2: A comparison of (a) $L \cdot \hat{V}_1(C,0,0) - 2(\pi - |\pi - C|)$ with (b) the approximation used in ref. 8 for $V_\ell(C,0,0)$ in the pure gauge case.

In fig. 3 we compare for the pure gauge case ($n_f = 0$) results of Kripfganz and Michael [8] (dashed lines), those obtained by using the Rayleigh-Ritz technique on H_{eff} [3,4,12] (solid lines; for $z_{A_1^+} < 1$ we indicated the perturbative result [11]) and those obtained from the lattice Monte Carlo calculations [2] (data points). We consider the mass ratios: $\sqrt{K}/m_{A_1^+}$, $m_{T_{11}^+}/m_{A_1^+}$, $m_{E^+}/m_{A_1^+}$ and $m_{T_2^+}/m_{A_1^+}$, where K is the finite volume string tension, which is the energy of one unit of electric flux divided by L and T_{11}^+ is the state with two units of electric flux and the T_2^+ quantum numbers. We see that $\mathcal{L}_{eff}$ certainly describes the main features correctly, but deviates in particular for T_2^+ and T_{11}^+ considerably from what H_{eff} and the lattice Monte Carlo results give. The reasons for these deviations were already partly analysed, namely the deviation of $\mathcal{L}_{FF}$ from $L \cdot V_T(c)$ and the deviation of $\mathcal{L}_V$ from V_{eff}. But equaly important is that the kinetic term and the measure should be those of flat three-space (in each coordinate patch). Fortunately, flat three-space is conformally equivalent to S^3 and the conformal transformation can be described in terms of $\mathrm{Tr}(U_i)$, allowing for a simple modification of $\mathcal{L}_{eff}$, which is exactly equivalent to H_{eff} (provided we also transform the Haar measure used in the path integral to the flat measure).

To be more precise we can use stereographic coordinates for S^3 to show its conformal equivalence with the flat metric:

$$\dot{r}_i^2 + 4\sin^2(r_i/2)\dot{\mathbf{n}}_i^2 = \frac{16\pi^2(\dot{s}_i^2 + s_i^2\dot{\mathbf{n}}_i^2)}{(\pi^2 + s_i^2)^2} \quad , \quad s_i = \pi tg(r_i/4) \quad ,$$

$$dU_i \propto \sin^2(r_i/2)\sin\theta_i dr_i d\theta_i d\phi_i = \frac{s_i^2}{16\pi^3(\pi^2 + s_i^2)^3}\sin\theta_i ds_i d\theta_i d\phi_i \quad . \tag{31}$$

This transformation maps the three-ball with radius π on to one hemisphere of S^3. For the other hemisphere we replace the tangent in eq. (31) by a cotangent. One now easily verifies that in the new flat coordinates we have:

$$U_j = (-1)^{k_j}\frac{(\pi + i(-1)^{k_j}c_j(\mathbf{k}))^2}{\pi^2 + s_j(\mathbf{k})^2} \quad . \tag{32}$$

In this new parametrization we find

$$r_j = \pi + \pi(\sqrt{2 - \mathrm{Tr}(U_i)} - \sqrt{2 + \mathrm{Tr}(U_i)})/\sqrt{K_i} \quad , \tag{33}$$

where

$$K_i = 2 + |\mathrm{Tr}(U_i)| \quad . \tag{34}$$

Fig. 3: Comparison of the Rayleigh-Ritz results [3,10,11] (solid lines), the Monte Carlo results [8] obtained with the effective Lagrangian of eqs. (26-29) (dashed lines) and the lattice Monte Carlo results [2] (data points) for the pure gauge theory. $z_{A_1^+} = m_{A_1^+} \cdot L$.

The algebraic relation between U_i and c_i also considerably simplifies the expression for F^2:

$$\frac{1}{4}F^a_{ij}F^a_{ij} = -\frac{1}{2}\mathrm{Tr}(\left[c_i, c_j\right]^2) = \frac{(\pi^2+s_i^2)(\pi^2+s_j^2)}{128\pi^4}(1-\frac{1}{2}\mathrm{Tr}(U_iU_jU_i^\dagger U_j^\dagger)) \quad . \tag{35}$$

Combining all the information we can now easily write down a path integral in terms of the SU(2)-group variables which is exactly equivalent to what one would obtain from the effective Hamiltonian:

$$Z = \int \prod_{i=1}^{3} \frac{dU_i}{K_i^3} \exp(\int dt \quad 2\pi^2(\frac{1}{g^2}+\alpha_1(n_f))\sum_i \frac{\mathrm{Tr}(U_i(t)U_i^\dagger(t+\varepsilon)-1)}{(\varepsilon K_i(t))^2} \tag{36}$$
$$+ 2\pi^4(\frac{1}{g^2}+\alpha_2(n_f))\sum_{i,j}\frac{\frac{1}{2}\mathrm{Tr}(U_i(t)U_j(t)U_i^\dagger(t)U_j^\dagger(t))-1}{(K_i(t)K_j(t))^2} - V_{eff}(\mathbf{r}(t)) \quad)$$

There are very rapidly converging sums for V_{eff}, using the error function [10], such that this theory can be easily simulated with the Monte Carlo method (we left the ε term in the Lagrangian to indicate the obvious discretization in time to be taken). The only complication is the modification of the Haar-measure, which requires a change in the updating procedure. But this is easily incorporated by rescaling the updating interval with the U_i dependent factor K_i.

We wish to end this section with a few remarks concerning the flavour dependence at intermediate volumes. Kripfganz and Michael [8] compute for various representations the glueball masses as a function of n_f and found that when extrapolating the result to $n_f = 0$, one would not retrieve the relevant pure gauge glueball mass. For example the T_2^+ mass would become in the limit $n_f \to 0$ the T_{11}^+ mass of the pure gauge theory (this is the state with two units of electric flux and the T_2^+ quantum numbers), whereas both the A_1^+ and E^+ masses would converge in this limit to the energy of one unit of electric flux. On itself this is hardly surprising since they used as operators Polyakov lines in the fundamental representation, which at $n_f = 0$ couple to electric flux. What is surprising though is the limiting behaviour. The operators based on adjoint Polyakov lines will give the same masses (but smaller signals) as those based on the fundamental Playkov lines, as long as $n_f \neq 0$. So the behaviour of the masses as a function of n_f is inherent to the dynamics of the theory. In fig. 4 we exhibit a very simple toy-model which has the same behaviour. It is the double harmonic oscillator, with a build-in asymmetry, parametrized by b which is the equivalent of n_f (compare fig. 1).

Fig. 4 is only meant to be schematic in illustrating the dependence of the energy levels on b. We have sketched the behaviour of the first four levels for $b = 0$, where even and odd states alternate. We also sketched the first two levels for relatively large b (the dashed lines) and for rather small b (the dotted lines). For a given value of g we see that the lowest mass (the difference in energy between the two dashed lines) does approach in the limit $b \to 0$ the energy difference between the even and odd ground states. The reason is of course that the wave function for the first excited state has only one node. But at $b=0$, when we wish to restrict ourselves to the symmetric sector, the first excited wave function has two nodes. Hence, the discontinuity is simply a consequence of identifying the appropriate states. However, in our perturbative analysis we saw that the spectrum depended smoothly on n_f.

Fig. 4: The toy model of the asymmetric anharmonic oscillator to illustrate schematically the n_f dependence of the glueball masses. The full lines give the energy levels for the symmetric double-well ($b=0$). The dashed lines are for relatively large b, whereas the dotted lines give the energies for small b.

This is because as long as the energy of the excited state is smaller than b the wave function will be restricted to the left well in fig. 4, which explains why the dotted line joins up with the even excited state at weak coupling and thus exhibiting a singularity in interchanging $b \to 0$ and $g \to 0$. This instability has been studied extensively in the literature in the context of a semiclassical analysis [13], here it simply illustrates the rather distinct n_f dependence in small and intermediate volumes.

5. DISCUSSIONS

In conclusion we have seen that in intermediate volumes the glueball masses do strongly depend on the number of flavours for obvious reasons. But the analysis of ref. 8 with the effective Lagrangian as given in eqs. (26-29) indicates that nevertheless the mass ratios are far less sensitive to the number of flavours. Probably the main advantage of using a Lagrangian formulation is that it seems possible to include the zero-energy fermion mode in the degrees of freedom for the effective Lagrangian (I thank Peter Lepage for mentioning this option). This should be possible with the same background field type calculations as were used before [3,6,10]. One now evaluates also diagrams with external fermion lines. The finite number of fermion degrees of freedom might make a Monte Carlo analysis feasible. It is much harder to include the zero-energy fermion mode in an effective Hamiltonian description.

This brings us to the question where the effective theory will breakdown. First of all, like in the pure gauge case, if the coupling becomes too strong one needs to include tunnelling over a classical potential barrier. The instanton configurations associated with this are well known to be resposible for some form of chiral symmetry breaking [14]. On the other hand, it might be that as soon as the zero-energy fermion mode becomes appreciably excited, it will be resposible for the formation of a non-zero chiral condensate. We consider this, however, unlikely in the light of the discussion in section 2.

The other interesting physical phenomenon to be addressed is the possibility that the low-lying glueballs can decay in to pions. In a small volume the glueball is kinematically forbidden to decay. This is because, due to the absence of zero fermion modes the pion will have a mass $O(1/L)$, whereas the glueball mass is $O(g^{2/3}(L)/L)$. In larger volumes we will ultimately hit the pion-pion threshold where the effective Hamiltonian will develop an imaginary part. It is possible that this will in some sense be incorporated by including the zero-energy fermion mode in the effective Lagrangian. It is clear that there is a large number of interesting dynamical questions, which are worthwhile investigating. In the meantime, the intermediate volume results should provide a useful testing ground for the dynamical fermion algorithms, as for example employed in ref. 15.

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