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Gauge Theory in a Finite Volume[†]

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ABSTRACT

In these lecture notes we discuss an analytic calculational scheme for SU(N) gauge theories in a finite volume. We will mainly concentrate on pure SU(2) gauge theory. The method relies on using an effective Hamiltonian for the zero momentum modes. The the notorious problem of Gribov horizons is evaded by encoding the topological nontrivial nature of configuration space into boundary conditions for the zero momentum modes. This system then allows us to compute the low-lying energy spectrum in volumes up to about five times the size of the scalar glueball. These continuum results agree in general well with the lattice Monte Carlo results. We discuss in some detail the resolution of a discrepancy with the Monte Carlo results for the T_2^+ glueball.

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1. Introduction

This will be an exploration of the femto-universe [1]. A universe with the topology of a torus in which we will study pure non-abelian gauge theories. Asymptotic freedom will guarantee that perturbation theory is accurate in the early femto-universe (small volumes). But we will follow its expansion (volume dependence) to the point where the interactions become strong in the hope to be able to peek beyond the edge of this universe.

One might wonder why we choose the volume to be a torus (a cube with sides of length L and periodic boundary conditions on the fields). The main theoretical reason is that on a flat background no mass term for the gluon is generated, due to the background curvature. It is therefore expected that more of the infrared behaviour, which is so characteristic for non-abelian gauge theories, will survive in a small volume. One indication that there might be some truth in this statement is that the finite volume "string tension" will be close to its value in larger volumes. A more technical reason is related to how 't Hooft [2] introduced gauge fields on the torus, relaxing the periodic boundary conditions to have a twist. This allowed him to define electric and magnetic flux quanta and enables one to calculate electric and magnetic energies without having to use singular operators.

Employing perturbation theory was hampered by a technical obstacle which is unavoidable if one wants to evade a dynamical mass for the gluon. This problem was spelled out explicitly first in the context of lattice gauge theory [3], but it was Lüscher's [4] analysis in the Hamiltonian framework which provided the basis for further development. The problem occurs only in the sector with zero magnetic flux and although, in the presence of magnetic flux (twist in the spatial directions), perturbation theory is much easier [5,6], it also dynamically introduces a gluon mass term. Nevertheless, a detailed calculation of the spectrum in the presence of magnetic flux [6] is important in deciding in how far the spectral properties are specific for the finite volume, the reason being that one expects (if confinement is realized [2]) that in the large volume limit both cases will yield the same spectrum.

However, we will restrict ourselves to zero magnetic flux, in which case the vector fields can be chosen periodic, not only for the reasons mentioned above. Equally important is that this enabled us to compare our results with existing lattice Monte Carlo data, pioneered in the finite volume and Hamiltonian context by Berg and Billoire [7].

In section 2 we discuss the intricacies of the classical vacuum, which forms the basis for the expansion in the coupling constant. Section 3 discusses how the definition of electric flux is related to the gauge symmetries. In section 4 we display the extension of Lüscher's effective Hamiltonian as a preparation for the non-perturbative analysis in section 5, where we address the issue of Gribov horizons. This leads to a detailed discussion in section 6 of our choice of boundary conditions in configuration space and we show how a discrepancy with Monte Carlo results for the T_2^+ glueball is resolved. Section 7 concludes with a discussion.

2. The classical vacuum

The general principle of the calculation is based on a Born-Opppenheimer type of approximation, integrating out the "fast" fluctuations to be left with an effective theory for the "slow" fluctuations. These "slow" fluctuations will include the set of the minimal classical energy configurations, which for pure gauge theories on a torus with periodic boundary conditions ($\vec{f}_i$ is the unit vector in the i-direction):

$$A_k(\vec{x}+L\vec{f}_i)=A_k(\vec{x}) \tag{1}$$

forms a $Z-$ labelled set of connected $3r-$ dimensional subspaces [8] (r is the rank of the gauge group, for SU(N) $r=(N-1)$). To be precise, one component is described by the spatially constant abelian vector potentials. The other components are related to this one by a gauge transformation $g:T^3\rightarrow G$ where G is the gauge group and

$$g(\vec{x}+L\vec{f}_i)=g(\vec{x}) \quad , \tag{2}$$

$$[g]A_k(\vec{x})=g(\vec{x})A_k(\vec{x})g(\vec{x})^{-1}-ig(\vec{x})\partial_k g(\vec{x})^{-1} \quad , \tag{3}$$

which is homotopically nontrivial, i.e. $P\in Z, P\neq 0$:

$$P=\frac{1}{24\pi^2}\int_{T^3}\mathrm{Tr}((g(\vec{x})dg(\vec{x})^{-1})^3) \quad . \tag{4}$$

The various components which we will call vacuum-valleys [9] (also called toron-valleys [3]) are thus labelled by P and separated by a classical potential barrier of order $1/(Lg^2)$, where g is the coupling constant. The classical potential energy is given by

$$V(A)=\frac{1}{2g^2}\int_{T^3}d_3x\mathrm{Tr}(F_{ij}^2(\vec{x})) \quad , \tag{5}$$

$$F_{ij}=\partial_i A_j-\partial_j A_i+i[A_i,A_j] \quad . \tag{6}$$

Since $V(A)$ is positive semi-definite with the lower bound saturated by $A=0$, a vacuum-valley is specified by $V(A)=0$, or $F_{ij}(\vec{x})=0$. One then easily shows [8] that up to a periodic gauge transformations, eq.(2), $A_i(\vec{x})$ is spatially constant and takes values in the Cartan subalgebra (i.e. is abelian). However, this space is a multiple cover of the vacuum-valley. One can show [8] that gauge transformations are on the vacuum-valley represented by the Weyl group $\mathcal{W}$ and the translations over 4π with respect to the standard basis of the dual weight lattice $\tilde{\Lambda}$. Hence, each connected component $\mathcal{V}_{\mathcal{P}}$ of the vacuum-valley is isomorphic to the orbifold $(R^r/\tilde{\Lambda})^3/\mathcal{W}$, i.e. three copies of the torus $R^r/\tilde{\Lambda}$, with the action of the Weyl group acting simultaneously at each copy, divided out.

For SU(2) we can do without all this formal terminology. The abelian spatially constant configurations are

$$A_i(\vec{x})=\frac{C_i}{L}\frac{\sigma_3}{2} \tag{7}$$

and this set is invariant under abelian periodic and non-abelian constant gauge transformations:

$$\begin{aligned} g_{\vec{k}}(\vec{x}) &= \exp(-2\pi i\vec{x}\cdot\vec{k}\sigma_3/L) \quad , \\ g_-(\vec{x}) &= \sigma_1 \quad . \end{aligned} \tag{8}$$

These gauge transformations imply the following transformations of $\vec{C}$:

$$\begin{aligned} [g_{\vec{k}}]\vec{C} &= \vec{C}+4\pi\vec{k} \quad , \\ [g_-]\vec{C} &= -\vec{C} \end{aligned} \tag{9}$$

and hence each vacuum-valley component is the orbifold T^3/Z_2.

3. The electric flux quanta

Clearly we should take the parameters of the vacuum-valley as "slow" parameters, and integrate out the degrees of freedom transverse to the vacuum-valley. However, the fact that $\mathcal{V}$ is an orbifold should make us cautious to what happens at the fixed points of the Z_2 action. One easily finds these fixed points to be given by $2\vec{C}=4\pi\vec{k}$, hence there are eight inequivalent fixed points. They correspond to configurations invariant under a larger subgroup of the gauge group than neighbouring configurations: $A=0$ is invariant under all constant gauge transformations, whereas neighbouring points are only invariant under constant abelian gauge transformations. This leads to zero energy modes in the transverse fluctuations and prevents one from simply integrating out the transverse degrees of freedom in a quadratic (one-loop) approximation. Indeed if we expand around $A = 0$, the potential is quartic in the spatially constant gauge fields:

$$A_i(\vec{x}) = \frac{c_i}{L} = \frac{c_i^a}{L}\frac{\sigma_a}{2} \quad , \tag{10}$$

$$V_{\rm quartic} = -\frac{1}{2g^2 L}{\rm Tr}([c_i, c_j]^2) \quad . \tag{11}$$

We consequently have to take the additional degrees of freedom of eq.(10) over eq.(7) into account as "slow" variables, which is exactly what Lüscher did to derive his effective Hamiltonian [4]. Furthermore it is clear that at the fixed points the potential valley is widest and consequently, in perturbation theory wave functionals will peak at these points. However, there were eight fixed points and we shall now illustrate how their presence is related to the electric flux quanta.

There is a symmetry of the Yang-Mills Hamiltonian which maps these fixed points into each other:

$$h_{\vec{k}}(\vec{x}) = \exp(-2\pi i \frac{\vec{x}\cdot\vec{k}}{L}\frac{\sigma_3}{2}) \quad , \tag{12}$$

$$[h_{\vec{k}}]\vec{C} = \vec{C} + 2\pi\vec{k} \quad . \tag{13}$$

Despite the fact that $h_{\vec{k}}(\vec{x})$ is not periodic (note however that $g_{\vec{n}} = h_{2\vec{n}}$), it does preserve the periodicity of the gauge potentials, the reason being that $h_{\vec{k}}(\vec{x})$ is periodic up to an element of the centre Z_G of the gauge group G:

$$h_{\vec{k}}(\vec{x} + L\vec{f}_i) = (-1)^{k_i} h_{\vec{k}}(\vec{x}) \quad . \tag{14}$$

It is sometimes advantageous to consider, in the absence of fields in the fundamental representation the gauge group as G/Z_G (i.e. for SU(2) we rather consider SO(3) as gauge group). Then $h_{\vec{k}}(\vec{x})$ is strictly periodic, however, it is a homotopically nontrivial gauge transformation, due to the fact that $\pi_1(G/Z_G) = Z_G$ and the fact that the torus is multiply connected. The homotopy of a gauge transformation is therefore classified by $Z_G^3 \times Z$ (Z corresponding to the Pontryagin index P in eq.(4)). Wave functionals are representations of this homotopy group. For SU(N), $Z_G = Z_N$ and the representations are labelled by the electric flux $\vec{e} \in Z_N^3$ and the theta parameter $\theta \in [0, 2\pi]$:

$$\Psi_{\vec{e},\theta}([g]A) = \exp(iP\theta)\Psi_{\vec{e},\theta}(A) \quad ,$$
$$\Psi_{\vec{e},\theta}([h_{\vec{k}}]A) = \exp(2\pi i\vec{k}\cdot\vec{e}/N)\Psi_{\vec{e},\theta}(A) \quad . \tag{15}$$

In the following we will neglect the $\theta-$dependence, which amounts to neglecting tunnelling through the classical potential barrier separating two components of the vacuum-valley. But the $\vec{e}$ dependence can be addressed within one connected component of the vacuum-valley as we see from eq.(13). The reason that $\vec{e}$ corresponds to electric flux can be found in 't Hooft's original paper [2].

As we mentioned before, the perturbative wave function peaks at the fixed points of the Z_2 action, which we will call the quantum vacua. This is not obvious since the classical potential vanishes along the vacuum-valley. However, one easily shows [4,9] that (away from the fixed points) the transverse fluctuations induce an effective potential barrier, which prevents the wave functional to spread to the other quantum vacua. Hence, to all orders in perturbation theory there is a degeneracy in the electric flux $\vec{e}$, which will be lifted by tunnelling through the quantum induced potential barrier [8,9,10]. (There remains a degeneracy in θ, lifted by tunnelling through the classical potential barrier separating two components of the vacuum-valley.)

4. The effective Hamiltonian

We will first discuss Lüscher's effective Hamiltonian [4] relevant for perturbation theory around $A = 0$, obtained by integrating out the spatially non-constant modes. In the Hamiltonian formulation [11] used by Lüscher, one introduces gauge fixed coordinates for configuration space in the Coulomb gauge

$$\partial_k A_k = 0 \quad . \tag{16}$$

This leads to a nontrivial integration measure, closely related to the Faddeev-Popov determinant, given by:

$$\rho(A) = \det{}'(-\partial_k D_k(A)) \quad , \tag{17}$$

where $D_k(A)$ is the covariant derivative. This measure is absorbed in the wave functional

$$\hat{\Psi}(A) = \rho(A)^{1/2}\Psi(A) \quad . \tag{18}$$

In many respects this is similar to using spherical coordinates for the $\vec{L}=\vec{0}$ sector, albeit in an infinite dimensional setting.

Since the non-zero momentum modes (q) are in lowest order occurring quadratically, the wave functional is approximately given by

$$\hat{\Psi}(A)=\Phi(c)\chi_{[c]}(q) \quad , \tag{19}$$

where $\chi_{[c]}(q)$ is the product of the ground state wave functions for the non-zero momentum modes q. The effective wave function $\Phi(c)$ satisfies an effective Hamiltonian (H_{eff}) equation and there exists a method developed by Bloch [12] which allows one to construct H_{eff} to all orders in perturbation theory [4]. In general, the effective wave function $\Phi(c)$ will no longer have the simple relation to $\hat{\Psi}(A)$ given in eq.(19), but to the order we are interested in, this relation is still assumed to hold (with $\chi_{[c]}(q)$ now the **exact** ground state wave function for the non-constant fields, or the ground state for the Hamiltonian where the c degrees of freedom are frozen).

We will now give the result [13,14] for H_{eff}, referring for the derivation to ref. 4 or for a Lagrangian approach to refs. 8,13 (the latter is much easier, but hides the information on the wave functional, which we will make use of further on).

$$L\cdot H_{eff}(c)=-\frac{1}{2}(g^{-2}(L)+\alpha_1)^{-1}\frac{\partial^2}{\partial c_i^{a\,2}}+V_T(c)+V_\ell(c)+\cdots \tag{20}$$

$$V_{\ell,1}(c)=\frac{4}{\pi^2}\sum_{\vec{n}\neq\vec{0}}\frac{\sin^2(\vec{n}\cdot\vec{r}/2)}{(\vec{n}^2)^2}-2|\vec{r}| \tag{21}$$

$$V_{\ell,2}(c)=\frac{g^2(L)}{32}\left[(\Delta V_{\ell,1}(c))^2+2\Delta V_{\ell,1}(c)\Delta V_{\ell,1}(0)\right] \tag{22}$$

$$r_i=\left[\sum_{a=1}^{3}c_i^a c_i^a\right]^{1/2} \quad ,\Delta=\sum_{i=1}^{3}\frac{\partial^2}{\partial {r_i}^2} \quad ,F_{ij}^a=-\epsilon_{abd}c_i^b c_j^d$$

$$V_T(c)=\frac{1}{4}(g^{-2}(L)+\alpha_2)(F_{ij}^a)^2+\alpha_3(F_{ij}^a c_k^b)^2+\alpha_4(F_{ij}^a c_j^b)^2+\alpha_5(\det c)^2 \tag{23}$$

$$g^{-2}(L)=-\frac{11}{12\pi^2}\ln(L\Lambda_{MS})-\frac{51}{121}\ln\left[-2\ln(L\Lambda_{MS})\right]+\cdots \tag{24}$$

where the numerical constants are given by:

$$\alpha_1=2.1810429\cdot 10^{-2},\alpha_2=7.5714590\cdot 10^{-3},\alpha_3=1.1130266\cdot 10^{-4},$$
$$\alpha_4=-2.1475176\cdot 10^{-4},\alpha_5=-1.2775652\cdot 10^{-3} \tag{25}$$

There are a few things worthwhile noting. $g(L)$ is the running coupling constant at $\mu = 1/L$, $V_T(c)$ vanishes whenever $F_{ij}^a = 0$ or equivalently whenever c is an element of the vacuum-valley. Therefore $V_{\ell,1}(c) + 2|\vec{r}|$ is the one-loop effective potential along the vacuum-valley [4,9], with $2|\vec{r}|$ coming from integrating out the transverse spatially constant modes, which is possible for $|\vec{r}| \gg g^{2/3}$ [8].

The low-lying spectrum (the energy levels for which $LE \to 0$ as $g \to 0$) is now directly determined by the spectrum of H_{eff} and to lowest non-trivial order in g we have:

$$L \cdot H_{eff}(c) = -\frac{g^2}{2} \frac{\partial^2}{\partial c_i^{a\,2}} - \frac{1}{2g^2} \mathrm{Tr}([c_i, c_j]^2) \quad . \tag{26}$$

A simple rescaling of the fields with $g^{2/3}$ shows that $LE = \mathcal{O}(g^{2/3})$ (due to the quartic nature of the potential) and LE has a perturbative expansion in powers of $g^{2/3}$. Since eq.(26) is non-integrable [15], Lüscher and Münster [16] used Rayleigh-Ritz perturbation theory to determine the spectrum. The various states are classified according to the irreducible representations of the cubic group $O(3, Z)$. There are ten irreducible representations, five for each parity: The singlets $A_1^\pm, A_2^\pm$, the doublet $E^\pm$ and the triplets $T_1^\pm, T_2^\pm$. The mass of a particular state is given by the energy difference of the appropriate level with the ground state (which is the lowest A_1^+ level). A remarkable result was that the E^+ state was lower in mass than the A_1^+ glueball.

In the infinite volume the states are to combine into angular momentum multiplets. The decompositions into the irreducible representations of the cubic group are as follows [17]: $0^\pm = A_1^\pm, 1^\pm = T_1^\pm, 2^\pm = E^\pm \oplus T_2^\pm, \cdots$. Hence there has been some confusion whether or not the tensor glueball would be lighter than the scalar glueball. For this we clearly have to go to larger volumes and we will next study the level-splitting as a function of the electric flux as a next step towards that goal.

5. Beyond the Gribov horizon

For increasing volume, which through eq.(25) is equivalent with increasing coupling, at some point the energy levels will approach the height of the induced potential barrier separating the eight quantum vacua in one component of the vacuum-valley. Beyond this point the perturbative approximation breaks down and energy levels will split into four different levels. Because of the cubic symmetry two of these are threefold degenerate and are labelled by $\vec{e}=(1,0,0),(0,1,0),(0,0,1)$, resp. $\vec{e}=(1,1,0),(1,0,1),(0,1,1)$. The two other levels are labelled by $\vec{e}=(0,0,0)$, resp. $\vec{e}=(1,1,1)$. The semiclassical evaluation of these level slittings was the subject of refs. 8,9,10, but will not concern us here, since we intend to go beyond where this analysis would be valid.

Since the vacuum-valley is contained within the set of constant vector potentials, one would, as a first attempt, try to incorporate the other quantum vacua by simply expanding H_{eff} around the other fixed points. A first sign that this is impossible is that one easily verifies that $V_{\ell,1}(\vec{r})$ has a conic singularity at $\vec{r}=2\pi\vec{k}$, exactly the one which was subtracted in eq.(21). Secondly, at exactly these quantum vacua, $\rho(A)$ will be zero because this is where the covariant derivative has zero-modes (note that in eq.(17) the zero-modes were not taken into account, which is what the prime in this equation stands for). This means that the other quantum vacua lie on the Gribov horizon [18] on which $\hat{\Psi}$ vanishes due to the rescaling in eq.(18). One easily shows that ρ not only vanishes at the quantum vacua, but on the whole planes $r_i = 2\pi$. However, this singularity is similar to a coordinate sigularity one would get in choosing local coordinates on the south pole of a sphere, trying to extend them to the north pole. In other words, the Gribov horizons occur just because configuration space is topologically non-trivial [19] and we simply avoid these singularities by choosing local coordinates around each quantum vacuum with appropriate transition functions [20]. The coordinate patches are centred at $A_k^{(\vec{n})}=\frac{2\pi\vec{n}}{L}\frac{\sigma_3}{2}$, with the gauge fixing $\partial_k A_k + i[A_k^{(\vec{n})}, A_k]=0$, which is equivalent to $\partial_k([h_{\vec{n}}^{-1}]A_k)=0$. Thus the local coordinates in each patch are related by the (homotopically non-trivial) gauge transformation $h_{\vec{n}}$ and consequently the Hamiltonian in each patch is of the same form. The transition functions connecting two patches simply describe a change of gauge. Yet, the infinite dimensional setting has prevented us up to now to implement this rigorously. The patches described so far would for example not provide a complete covering of configuration space, for this also coordinate patches around $[g]A_k^{(\vec{n})}$, where $P(g)\neq 0$, have to be included. But since we are at this point concerned with the case where $\theta-$dependence is neglected we will assume the wave functional to be localized along the vacuum-valley.

6. Boundary conditions in configuration space

The method we will follow amounts in the strongly simplified example of the double-well anharmonic oscillator to the following. The equivalent of the gauge transformations $h_{\vec{n}}$ is the parity $x \to -x$ and the equivalent of the electric flux quantum is the parity eigenvalue of the wave function. The Hilbert space with definite parity can now alternatively be specified by the space of the square integrable wave functions on $R^+ = \{x|x \geq 0\}$ (the half-line) supplemented with Dirichlet or Von Neumann boundary conditions: $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-, \mathcal{H}^+ = \{\Psi \in L^2(R^+)|\partial_x \Psi(0) = 0\}, \mathcal{H}^- = \{\Psi \in L^2(R^+)|\Psi(0) = 0\}$.

Since we know how the symmetries act on the vacuum-valley coordinates it would be most natural to derive an effective vacuum-valley wave function. As we discussed before this is not possible near the quantum vacua, but it is to a good approximation possible in the overlap regions of two coordinate patches around $C_i = \pm\pi$. To consider this effective vacuum-valley wave function we parametrize the vacuum-valley by eq.(7) and choose the gauge [4]

$$D_k(\vec{C})A_k = 0 \quad , \tag{27}$$

with $D_k(\vec{C})$ the covariant derivative in the background of a vacuum-valley configuration. One easily finds the eigenmodes for the transverse fluctuations to be given by:

$$q_j^a(\vec{x}) = v_{j;\vec{n}}^a \exp(2\pi i \vec{n} \cdot \vec{x}/L) \quad , \tag{28}$$

with the gauge conditions ($v_k^{\pm} = (v_k^1 \pm i v_k^2)/\sqrt{2}$):

$$\begin{aligned} (2\pi n_k \pm C_k) v_{k;\vec{n}}^{\pm} &= 0 \quad , \\ 2\pi n_k v_{k;\vec{n}}^3 &= 0 \quad , \quad (\vec{n} \neq \vec{0}) \quad . \end{aligned} \tag{29}$$

In lowest order the Hamiltonian is quadratic in the transverse fluctuations:

$$\frac{1}{L}\sum_{\vec{n}} \left\{ -\frac{1}{2}\frac{\partial^2}{\partial {v_{k;\vec{n}}^a}^2} + \frac{1}{2}(2\pi\vec{n} + \vec{C})^2 |v_{k;\vec{n}}^+|^2 + \frac{1}{2}(2\pi\vec{n} - \vec{C})^2 |v_{k;\vec{n}}^-|^2 + \frac{1}{2}(2\pi\vec{n})^2 (v_{k;\vec{n}}^3)^2 \right\} \quad . \tag{30}$$

Hence, to lowest non-trivial order (cmp. eq.(19))

$$\Psi(A) = \phi(\vec{C})\chi_{[\vec{C}]}(v) \quad , \tag{31}$$

with $\chi_{[\vec{C}]}$ a product of **ground state** harmonic oscillator eigenfunctions. It is now imperative to note the $\chi_{[\vec{C}]}(v)$ is invariant under the gauge transformation $h_{\vec{n}}$ and is even under the one-particle parity transformations:

$$\pi_i A_k(\vec{x}) = (-1)^{\delta_{ik}} A_k(\pi_i \vec{x}) \quad ,$$
$$\pi_i x_k = (-1)^{\delta_{ik}} x_k \quad . \tag{32}$$

Consequently, as long as the wave function decomposes as in eq.(31) (for which we do **not** have to insist $\chi_{[\vec{C}]}$ to be purely harmonic), the symmetries induced by $h_{\vec{n}}$ and π_i will be represented on the reduced vacuum-valley wave functional:

$$\phi(\vec{C} + 2\pi\vec{k}) = (-1)^{\vec{k}\cdot\vec{e}} \phi(\vec{C}) \quad ,$$
$$\phi(\pi_i \vec{C}) = p_i \phi(\vec{C}) \quad , \tag{33}$$

where $p_i = \pm 1$ are the one-particle parities.

We now look for subgroups of order two, which leave a hypersurface of codimension one fixed. The direction normal to the hypersurface will provide a situation analogous to the double-well anharmonic oscillator. For these subgroups we take the Z_2's generated by:

$$T_i = [h_{\vec{f}_i}] \circ \pi_i \quad , \tag{34}$$

which map $C_i \to 2\pi - C_i$ and $C_j \to C_j$ for $i \neq j$. The associated boundary is hence at $C_i = \pi$ which leads to:

$$\phi(\vec{C} + t\vec{f}_i) = p_i (-1)^{e_i} \phi(\vec{C} - t\vec{f}_i) \tag{35}$$

at $C_i = \pi$ or:

$$\phi(C_i = \pi) = 0 \quad \text{for} \quad p_i(-1)^{e_i} = -1 \quad ,$$
$$\frac{\partial \phi}{\partial C_i}(C_i = \pi) = 0 \quad \text{for} \quad p_i(-1)^{e_i} = +1 \quad . \tag{36}$$

This choice of boundary conditions is due to Vohwinkel [21] and implies that the boundary conditions used in ref. 13 are only correct for states with all one-particle parities positive, that is for the state A_1^+, A_2^+, E^+ and e_i^+ considered in [13,14].

The states T_1^+ and T_2^+ have two out of the three one-particle parities negative, (where $T_{(1)2}^+$ is (anti-)symmetric under interchanging the two negative parity directions). In view of eq.(36), the states we labelled $T^+_{1,2}$ in refs. 13,14 have effectively electric flux in the two directions of negative parity and it is therefore misleading to call them glueball states. Recent Monte Carlo results in intermediate volumes [22,23] have impressively confirmed the choice of boundary conditions in eq.(36) for both the T_2^+ (zero units of electric flux) and what was called the T^+_{11} state in ref. 23 (the state, with electric flux in both of the negative parity directions, which is symmetric under interchanging these two directions. This is the state which was labelled T_2^+ in refs. 13,14, the analytic results follow hence from these references). In fig. 1 we exhibit these results by comparing mass ratios as a function of $z = m_{A_1^+}L$ coming from the Monte Carlo and the analytical calculations. In this figure K is the finite volume "string tension", which is defined as the energy of electric flux divided by the length L.

We would intuitively think that adding electric flux to a state will increase the energy. This is why we generalized the boundary conditions in refs. 13,14 without further thought to be independent of the one-particle parities. However, we now see that adding electric flux to a T_2^+ state will drastically lower the energy, which is surely an artifact of the intermediate volume. The T_2^+ state therefore continues to surprise us, but our original claim that in intermediate volumes the T_2^+ has the lowest mass of all glueball states is false. We do confirm that the E^+ state is slightly lower than the scalar glueball up to $z \sim 5$ as was first observed in a Monte Carlo study [24,25]. However, it remains true that in these intermediate volumes the E^+ and T_2^+ are far apart and thus rotational invariance is still badly broken. Recent Monte Carlo results [22] provide evidence that for $z > 5$ these two states rapidly merge, with E^+ having the largest variation (see fig. 1).

Let us now consider the negative parity states. Here we immediately run into a problem, since parity cannot be implemented on the vacuum-valley as is obvious from eq.(9). Gauge invariance forces the effective vacuum-valley wave function to have positive parity [21]. This implies that negative parity states have a transverse wave funtional which behaves non-trivially under a parity transformation. Hence, at least one of the transverse harmonic oscillators is in an excited state. There are infinitely many possibilities and clearly we have to take the ones with the lowest transverse energy. However, within the approximation of eq.(30), where these various transverse states are non-interacting, this choice is discontinuous when crossing $C_i = \pi$. The reason is that the two transverse energies $|\vec{C}|$ and $|2\pi\vec{f}_i -$

$\vec{C}|$ cross at $C_i = \pi$. Including the interactions between the crossing levels will remove the discontinuity. But things are even further complicated by the fact that the excitation spectrum of eq.(30) is degenerate (not only at the crossing points). We should also not forget the possibility that the transverse wave functional could be odd under the gauge transformations $h_{\vec{f}_i}$, although we do not expect this to be the case. If, however, one can still decompose the wave functional as $\Psi(A) = \phi'(\vec{C})\chi'_{[\vec{C}]}(v)$, with a single transverse state dominating, then $\chi'_{[\vec{C}]}$ carries the negative one-particle parities and we would have for $\phi'(\vec{C})$ the same boundary conditions as in the positive parity case:

$$\begin{aligned} \phi'(C_i = \pi) = 0 \quad &\text{for} \quad p_i(-1)^{e_i} = +1 \quad , \\ \frac{\partial \phi'}{\partial C_i}(C_i = \pi) = 0 \quad &\text{for} \quad p_i(-1)^{e_i} = -1 \quad . \end{aligned} \tag{37}$$

For A_1^-, A_2^- and E^- these are the boundary conditions used in refs. 13,14,26. For $T_{1,2}^-$ the boundary conditions used in ref. 26 would correspond again to a state with two units of electric flux. In table I we compare with the Monte Carlo results of ref. 22. Especially the E^- results agree well with eq.(37). However this would make one expect the A_1^- results to agree better than they actually do. Furthermore, T_2^- agrees better with the boundary conditions of eq.(36). It should be remarked though, that the interaction between the crossing levels can lead to appreciable corrections of the effective potential used in the analytic calculations, certainly at the larger values of g, which are employed. In an appendix we analyse a simple model for crossing transverse energy levels to illustrate some of the issues involved.

We should also not forget that these calculations are expected to have only a limited range of applicability, which for higher mass states will be restricted to a smaller range in z. For the states considered in fig. 1, the method seems to be valid up to $z = 5$. But, integrating out non-zero momentum modes is expected to yield a good approximation only for those states for which the mass is smaller than the typical momentum separation $2\pi/L$ (i.e. the mass ratio should be small w.r.t $2\pi/z$). We agree with Vohwinkel [21] that this is the most likely explanation for the predictions of the T_1^+ and the A_2^+ states (where there is **no** ambiguity in the boundary condition) to fail. From $z = 1.5$ onwards $m/m_{A_1^+}$ surpasses $2\pi/z$. This, however, is testable by doing Monte Carlo calculations at these small values of z. The other cause for the analytic method to break down was discussed previously in refs. 13,26 and is related to the energy levels flowing over the potential barrier separating two vacuum-valleys. This is where the degeneracy in θ will be lifted. We expect though, that these two mechanisms are intimately connected.

We conclude this section by displaying how we implemented the boundary conditions of eqs.(36) and (37) on the wave function of Lüscher's effective Hamiltonian (eqs.(20)-(24)). For this it is useful to point out that $\Phi(c)$ has a decomposition along the vacuum-valley, similar to eq.(31):

$$\Phi(c)=\phi(\vec{C})\tilde{\chi}_{[\vec{C}]}(v^{\pm}_{k;\vec{0}}) \tag{38}$$

and one can show, that at least to some order, $\phi(\vec{C})$ in eq.(31) and eq.(38) do coincide [8,13]. This motivates the following choice of boundary conditions:

$$\begin{aligned}\Phi(c)=0 \quad \text{at} \quad r_i=\pi \quad \text{if} \quad p_i(-1)^{e_i}=-p \quad ,\\ \frac{\partial}{r_i}(r_i\Phi(c))=0 \quad \text{at} \quad r_i=\pi \quad \text{if} \quad p_i(-1)^{e_i}=+p \quad ,\end{aligned} \tag{39}$$

where p is the overall parity ($p=\prod p_i$). The additional factor r_i in the second line of eq.(39) is due to the radial nature of the coordinates r_i, giving rise to a Jacobian factor. The choice of boundary conditions in eq.(39) allowed an efficient way of implementing a Rayleigh-Ritz analysis, providing both upper and lower bounds on the energies. We refer to ref. 13 for further details. Since the boundary conditions are formulated gauge invariantly, it is possible that their validity is also guaranteed away from the vacuum-valley, but we have presently no convincing argument for this, except for pointing to the remarkable agreement with Monte Carlo results.

7. Discussion

We have seen that the effective Hamiltonian for the zero-momentum gauge fields gives a detailed description of the non-perturbative dynamics of pure SU(2) gauge theory in a finite cubic volume, smaller than five times the size of the scalar glueball, which is about 0.7 to 1 fermi. In fig. 2 we summarize the structure of the spectrum in a schematic fashion, illustrating the lifting of the various degeneracies. In the first step this leads to the glueball masses, in the second it gives us the energy of electric flux and the final hurdle, waiting to be taken, will yield the $\theta-$dependence of the energy levels.

Especially in the positive parity sector for states with masses smaller than three scalar glueball masses the agreement between the analytic and the Monte Carlo data is rather satisfying (see fig. 1). Even the excited A_1^+ and E^+ masses give a fair agreement (see table I). Good agreement was also found for the ratios of the energies of different units of electric flux [24,27], which provide a rather sensitive test for the formation of electric flux tubes. Any state, say with two units of electric flux (e.g. $\vec{e} = (1,1,0)$) is expected to have a mass $\sqrt{2}\sigma L$, (where $\sigma = \lim_{L\to\infty} K(L)$) in sufficiently large volumes [2]. In fig. 1 the dashed line corresponds to the large volume prediction $m_{T_{11}^+}/m_{A_1^+} = \sqrt{2}z/4$. Up to now, this expected string behaviour has not been observed and it forms a clear challenge for the Monte Carlo calculations. Further recent Monte Carlo data, largely for $z > 5$, can be found in ref. 28. It confirms the near degeneracy of the E^+ and T_2^+ states, which combine to form the tensor glueball. Furthermore, these data confirm the near constant behaviour of $\sqrt{K}/m_{A_1^+}$, which together with the small volume results is therefore almost constant from $z = 1$ onwards. This we believe can hardly be accidental (although there is certainly no flux tube formation below $z = 7$[27]), for some speculations see refs. 13,26. One might now also be reaching volumes which are large enough to test the large volume predictions of Lüscher for the string tension[29] and the glueball masses[30], which are quite crucial for establishing full control over any infinite volume mass estimates. However, much remains to improve in the accuracy of the Monte Carlo data before a realistic comparison becomes feasible.

An obvious generalization of the analytic work is to go to the more realistic gauge group of color SU(3). Analytic work in small volumes exists [31,32], but the intermediate volume calculation remains to be done. Finally including dynamical fermions would bring us one further step closer to reality. For massless fermions and small volumes a first analysis in this direction has recently been completed [23,33] indicating that the glueball spectrum for the mass ratios is remarkably independent of the number of fermion flavours.

Remarkable is also the result obtained for asymmetric tori [34]. Taking two sides much larger than the third and studying the energy of electric flux in the third short direction as a function of the renormalized coupling constant at the scale set by the short direction, seems to yield a critical value of the coupling in the limit of infinite asymmetries, which is then related to the deconfining temperature by the two-loop beta-function. This method gives an approach to the deconfining temperature from within the finite volume deconfining phase, in which the described analytic techniques are used. These techniques breakdown in the actual limit to be taken, but the evidence gathered is tempting.

In conclusion I hope to have convinced you that finite volume gauge theories is fun and that it is physics. It would be nice if somebody like Casimir[35] would come along to think of a real experiment, but in the meantime we are more than happy to compare with that other "experiment" called Monte Carlo. Of course, what we should do is look beyond the edge of the femto universe and use lattice Monte Carlo calculations as our telescope to map out carefully "where no one has gone before".

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Appendix

In this appendix we discuss a simple model for what can happen when two transverse energy levels cross. We study the Hamiltonian

$$\begin{aligned} H(x) &= -\frac{g^2}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}(x^2+1) + \vec{n}(x)\cdot\vec{\sigma} \quad , \\ \vec{n}(x) &= (\epsilon, 0, x) \quad . \end{aligned} \tag{A.1}$$

This Hamiltonian is invariant under the Z_2 group generated by $\sigma_1 \circ \pi$, where $\pi(x) = -x$. When $\epsilon = 0$, the Hamiltonian splits into two identical harmonic oscillators, one shifted by one unit to the right, the other shifted an equal amount to the left and each level is twofold degenerate.

This result is quite different from a naive application of the adiabatic approximation. For $\pm x < 0$ the transverse ground state has an energy $(x \pm 1)^2/2$ such that the effective potential would be given by $V(x) = (|x|-1)^2/2$, which is the double harmonic oscillator. Its spectrum is for $g \neq 0$ clearly non-degenerate. The reason for this discrepancy is of course the crossing of the transverse energy levels for $\epsilon = 0$ at $x = 0$, where the adiabatic approximation breaks down.

This can be nicely illustrated in an explicit calculation. We find the following results for the transverse eigenfunctions and eigenvalues (see fig. 3):

$$\chi^{\pm}_{[x,\epsilon]} = g_\epsilon(x)v^{\pm} \quad , \quad g_\epsilon(x) = \exp(-i\alpha_\epsilon(x)\sigma_2/2) \quad ,$$
$$\lambda^{\pm}_\epsilon(x) = \frac{1}{2}(x^2+1) \pm \sqrt{x^2+\epsilon^2} \quad , \tag{A.2}$$

where v^+ is the spin-up and v^- the spin-down spinor and $\alpha_\epsilon(x)$ is the angle of $\vec{n}(x)$ with the z-axis. We fix the 2π ambiguity of α by demanding $\alpha_\epsilon(\infty)=0$, from which it follows that $\alpha_\epsilon(-\infty)=\mathrm{sign}(\epsilon)\pi$ and more importantly

$$\alpha_\epsilon(x) = \mathrm{sign}(\epsilon)\pi - \alpha_\epsilon(-\mathrm{x}) \quad , \tag{A.3}$$

or

$$g_\epsilon(x) = \mathrm{sign}(\epsilon)\sigma_1 \mathrm{g}_\epsilon(-\mathrm{x})\sigma_3 \quad . \tag{A.4}$$

This implies the following transformation properties of the transverse eigenfunctions under the Z_2 symmetry:

$$\chi^{\pm}_{[x,\epsilon]} = \pm\mathrm{sign}(\epsilon)\sigma_1\chi^{\pm}_{[-\mathrm{x},\epsilon]} \quad . \tag{A.5}$$

The Z_2 eigenvalue is therefore $-\mathrm{sign}(\epsilon)$ for the transverse ground state (and the opposite for the excited transverse state). The fact that this eigenvalue depends **only** on the sign of ϵ is a direct consequence of Berry's phase [36], since the adiabatic evolution from $x=-\infty$ to $x=\infty$ along a path with $\epsilon \geq 0$ and back along a path with $\epsilon \leq 0$ forms a closed path in configuration space (x,ϵ), encircling the point of level crossing at $x=\epsilon=0$ once.

We conclude that when $\epsilon \neq 0$, such that the adiabatic approximation is valid for sufficiently small values of g, the symmetry of the effective wave function $\phi(x)$ in $\Psi(x) = \phi(x)\chi^{-}_{[x,\epsilon]}$ is determined by the sign for the off-diagonal coupling. The effective wave function satisfies the equation:

$$-\frac{g^2}{2}\frac{d^2\phi(x)}{dx^2} + \lambda^{-}_\epsilon(x)\phi(x) = E\phi(x) \tag{A.6}$$

and in the case of negative parity states in the SU(2) finite volume calculation we have substituted for $\lambda^{-}_\epsilon(x)$ its limit for $\epsilon \to 0$, because of our ignorance of the precise value for $\epsilon(g)$.

Finally, just for the fun of it, let us show how the limit $\epsilon \to 0$ is to be taken properly. Let $|\eta>$ and $|\xi>$ be two spinors. We wish to evaluate $< x_T| \otimes < \xi| \exp(-iHT)|\eta> \otimes |x_0>$, which can be written as the path integral

$$\int_{x_0}^{x_T} \mathcal{D}x(t) < \xi|g_\epsilon(x_T) \exp(\frac{i}{2g^2}\int_0^T \dot{x}(t)^2 dt) P \exp[-i\int_0^T \{\Lambda_\epsilon(x(t)) + iA_\epsilon(x(t))\}dt] g_\epsilon(x_0)|\eta> \tag{A.7}$$

where $\Lambda(x)$ is the diagonalized transverse Hamiltonian and $A(x)$ is the U(1) "adiabatic gauge field" [36]:

$$\begin{aligned} \Lambda_\epsilon(x) &= \mathrm{diag}(\lambda_\epsilon^+(x), \lambda_\epsilon^-(x)) \quad , \\ A_\epsilon(x) &= i < \chi_{[x,\epsilon]}^+ | \frac{d}{dx} | \chi_{[x,\epsilon]}^- > \sigma_2 \quad . \end{aligned} \tag{A.8}$$

One easily shows that

$$\lim_{\epsilon \to 0} A_\epsilon(x) = -i\frac{\pi}{2}\delta(x)\sigma_2 \quad , \tag{A.9}$$

which will cause the spin to flip whenever $x(t)$ passes through zero. A straightforward computation gives:

$$\int_{x_0}^{x_t} \mathcal{D}x(t) < \xi| \exp[i\int_0^T \{\frac{\dot{x}(t)^2}{2g^2} - \frac{1}{2}(x(t) + \sigma_3)^2\}dt]|\eta> \tag{A.10}$$

for the path integral of eq.(A.7) in the limit $\epsilon \to 0$, which is easily seen to be what one would obtain by putting $\epsilon = 0$ from the start.

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Table I

$a \quad z$	$m_{A_1^-}/m_{A_1^+}$	$m_{E^-}/m_{A_1^+}$	$m_{T_2^-}/m_{A_1^+}$	$m'_{A_1^+}/m_{A_1^+}$	$m'_{E^+}/m_{A_1^+}$
3.85	$2.19^{+0.10}_{-0.70}$	2.95 ± 0.16	$3.02^{+0.06}_{-0.13}$	$1.56^{+0.05}_{-0.29}$	$1.59^{+0.02}_{-0.16}$
4.20	1.94 ± 0.06	$2.89^{+0.13}_{-0.62}$	$2.95^{+0.14}_{-0.27}$	$1.49^{+0.02}_{-0.06}$	$1.79^{+0.02}_{-0.19}$
5.27	$1.74^{+0.08}_{-0.35}$	$2.48^{+0.14}_{-1.38}$	$2.54^{+0.06}_{-1.40}$	1.11 ± 0.08	$1.87^{+0.03}_{-0.13}$
b					
3.701	1.673	2.966	4.018	1.691	1.651
4.037	1.547	2.849	3.905	1.612	1.622
5.134	1.260	2.592	3.650	1.422	1.502
c					
3.701	2.969	5.126	2.827		
4.037	2.914	5.044	2.708		
5.134	2.794	4.853	2.452		

Comparison of the Monte Carlo [22] data (a) and the analytic results for the boundary conditions of eq.(39) (b) and the boundary conditions of eq.(36) applied to the negative parity states (c) (both of type IA [13]). $m'_{A_1^+}$ and m'_{E^+} are the first excited glueball masses.

Figure Captions

Figure 1: Comparison of the pure gauge SU(2) mass ratios as a function of $z = m_{A_1^+}L$ between the lattice Monte Carlo data [22] and the intermediate volume analytic calculations [13,14,21] for $\sqrt{K}/m_{A_1^+}$, $m_{E^+}/m_{A_1^+}$, $m_{T_2^+}/m_{A_1^+}$ and $m_{T_{11}^+}/m_{A_1^+}$. The dashed line gives the infinite volume prediction for the latter ratio, assuming a string tension of $(m_{A_1^+}/4)^2$.

Figure 2: A schematic overview of the spectrum for SU(2) pure gauge theories in a finite volume. In lowest order one has the "spin waves" with an infinite degeneracy, lifted by the interactions of the zero-momentum modes giving glueball masses of order $g^{2/3}/L$. The eightfold degeneracy due to electric flux is lifted by tunnelling through a quantum induced potential barrier, giving a further splitting of order $\exp(-12.4637\cdots/g)/L$. Finally, tunnelling through classical potential barriers separating two components of the vacuum-valley will lift the degeneracy in θ giving a splitting of order $\exp(-8\pi^2/g^2)/L$.

Figure 3: The transverse energies and the angle $\alpha_\epsilon(x)$ for the Hamiltonian in eq.(A.1) at $\epsilon = 0$ and $\epsilon = 0.5$, to illustrate the intricacies of the adiabatic approximation in the presence of crossing transverse energy levels.