GRIBOV's HORIZON - A VISTA ON NON-PERTURBATIVE GAUGE THEORY

Pierre van Baal
Instituut-Lorentz for Theoretical Physics, University of Leiden,
P.O. Box 9506, Nl-2300 RA Leiden, The Netherlands

This special talk is dedicated to the memory of Vladimir Gribov. It explains in the simplest possible terms why one of Gribov's many important contributions and insights - namely that of gauge copies - is a decisive factor in understanding non-perturbative gauge theory.

TO THE MEMORY OF VLADIMIR GRIBOV

1 Introduction

Volodja Gribov was a lightning and thundering example of scientific integrity and honesty. The first time I heard him speak was at a meeting in Eger in August 1988. My first interactions with him stem from a visit to my friends here in Minneapolis in May of 1995. At that time I gave two talks, both on the global issue of gauge fixing and its applications to the dynamics in gauge theories. I did get to the end of those talks and had a few private discussions with Volodya. It is not that I believe to have convinced him this was the way to go - our results are furthermore only very modest. Later I learned [1] that he considered having a "picture" to be very important for approaching a problem. In February 1996 I managed to convince Volodya to come and present a colloquium Eherenfestii, a visit none of us in Leiden and his wife Julia are likely to forget. With sadness we have to bear the loss of a great and unique physicist who relentlessly looked for the truth and forcefully reminded us that we have the duty to do the same.

It is with great modesty that I present this special talk. I will have nothing new to report, but hope to explain to you in the simplest possible terms why one of Gribov's many important contributions and insights - namely that of gauge copies [2] - is a decisive factor in understanding non-perturbative gauge theory.

I restrict myself to gauge theory without matter - even though Gribov's believe has become that this might be irrelevant to the real world - confinement being due to the fortunate presence of massless or light quarks [3]. I will use the Hamiltonian point of view. With the highest respect for Feynman, path integrals have not yet developed into a tool - except for the brute force method of lattice gauge theory - that can truly treat non-perturbative physics, where quantum corrections (in the low-energy modes) are *not* small.

The price I pay is that I can only show for intermediate volumes, where its size can be used as a control parameter, that gauge copies, when treated properly, are all important.

2 The Landscape

Due to the non-Abelian nature of the theory the physical configuration space, formed by the set of gauge orbits $\mathcal{A}/\mathcal{G}$ ($\mathcal{A}$ is the collection of connections, $\mathcal{G}$ the group of local gauge transformations) is non-trivial [4]. Most frequently, coordinates of this orbit space are chosen by picking a representative gauge field on the orbit in a smooth and preferably unique way. It is known since the seminal work of Vladimir Gribov [2] that linear gauge conditions like the Landau or Coulomb gauge suffer from what is now known as Gribov ambiguities.

We will pursue the issue of finding a fundamental domain for non-Abelian gauge theories [5] and its consequence for the glueball spectrum in intermediate volumes. The finite volume context allows us to make reliable statements on the non-perturbative contributions, because asymptotic freedom guarantees that at small volumes the effective coupling constant is small, such that high-momentum states can be treated perturbatively. Only the lowest (typically zero or near-zero momentum) states will be affected by non-perturbative corrections. We emphasize that it is essential that gauge invariance is implemented properly at all stages. We will describe the results mainly in the context of a Hamiltonian picture [6] with wave functionals on configuration space. Although rather cumbersome from a perturbative point of view, where the covariant path integral approach of Feynman is vastly superior, it provides more intuition on how to deal with non-perturbative contributions to observables that do not vanish in perturbation theory. An essential feature of the non-perturbative behaviour is that the wave functional spreads out in configuration space to become sensitive to its non-trivial geometry. If wave functionals are localised within regions much smaller than the inverse curvature of the field space, the curvature has no effect on the wave functionals. At the other extreme, if the configuration space has non-contractible circles, the wave functionals are drastically affected by the geometry, or topology, when their support extends over the entire circle. Instantons are of course the most important examples of this. Not only the vacuum energy is affected by these instantons, but also the low-lying glueball states and this is what we are after to describe accurately, albeit in sufficiently small volumes.

3 Complete Gauge Fixing

An (almost) unique representative of the gauge orbit is found by minimising the L^2 norm of the vector potential along the gauge orbit [5,7]

$$F_A(g) \equiv \|{}^g A\|^2 = -\int_M d^3x \ \mathrm{tr}\left(\left(g^{-1}A_i g + g^{-1}\partial_i g\right)^2\right), \tag{1}$$

where the vector potential is taken anti-hermitian. Expanding around the minimum of eq. (1), writing $g(x) = \exp(X(x))$, one easily finds:

$$\begin{aligned}\|{}^g A\|^2 = \|A\|^2 + 2\int_M \mathrm{tr}\,(X\partial_i A_i) + \int_M \mathrm{tr}\,(X^\dagger FP(A)X) \\ +\frac{1}{3}\int_M \mathrm{tr}\,(X\,[[A_i, X], \partial_i X]) + \frac{1}{12}\int_M \mathrm{tr}\,([D_i X, X][\partial_i X, X]) + \mathcal{O}\left(X^5\right).\end{aligned} \tag{2}$$

Where $FP(A)$ is the Faddeev-Popov operator ($\mathrm{ad}(A)X \equiv [A, X]$)

$$FP(A) = -\partial_i D_i(A) = -\partial_i^2 - \partial_i \mathrm{ad}(A_i). \tag{3}$$

At any local minimum the vector potential is therefore transverse, $\partial_i A_i = 0$, and $FP(A)$ is a positive operator. The set of all these vector potentials is by definition the Gribov region Ω. Using the fact that $FP(A)$ is linear in A, Ω is seen to be a convex subspace of the set of transverse connections Γ. Its boundary $\partial\Omega$ is called the Gribov horizon. At the Gribov horizon, the *lowest* eigenvalue of the Faddeev-Popov operator vanishes, and points on $\partial\Omega$ are associated with coordinate singularities. Any point on $\partial\Omega$ can be seen to have a finite distance to the origin of field space and in some cases even uniform bounds can be derived [8,9].

The Gribov region is the set of *local* minima of the norm functional (3) and needs to be further restricted to the *absolute* minima to form a fundamental domain, which will be denoted by Λ. The fundamental domain is clearly contained within the Gribov region. To show that also Λ is convex we note that

$$\begin{aligned}\|{}^g A\|^2 - \|A\|^2 &= \int \mathrm{tr}\,\left(A_i^2\right) - \int \mathrm{tr}\,\left(\left(g^{-1}A_i g + g^{-1}\partial_i g\right)^2\right) \\ &= \int \mathrm{tr}\,\left(g^\dagger FP_f(A)\, g\right) \equiv \langle g, FP_f(A)\, g\rangle\,,\end{aligned} \tag{4}$$

where $FP_f(A)$ is the Faddeev-Popov operator generalised to the fundamental representation. At the critical points $A \in \Gamma$ of the norm functional (recall

$\Gamma = \{A \in \mathcal{A} | \partial_i A_i = 0\}$) $FP_f(A)$, like $FP(A)$, is a hermitian operator. We can define Λ in terms of the absolute minima over $g \in \mathcal{G}$ of $\langle g, FP_f(A)\ g \rangle$

$$\Lambda = \{A \in \Gamma | \min_{g \in \mathcal{G}} \langle g, FP_f(A)\ g \rangle = 0\}. \tag{5}$$

Using that $FP_f(A)$ is linear in A, the convexity of Λ is automatic: A line connecting two points in Λ lies within Λ.

If we would not specify anything further, as a convex space is contractible, the fundamental region could never reproduce the non-trivial topology of the configuration space. This means that Λ should have a boundary [10]. Indeed, as Λ is contained in Ω, this means Λ is also bounded in each direction. Clearly $A = 0$ is in the interior of Λ, which allows us to consider a ray extending out from the origin into a given direction, where it will have to cross the boundary of Λ and Ω. For any point along this ray in Λ, the norm functional is at its absolute minimum as a function of the gauge orbit. However, for points in Ω that are not also in Λ, the norm functional is necessarily at a relative minimum. The absolute minimum for this orbit is an element of Λ, but in general not

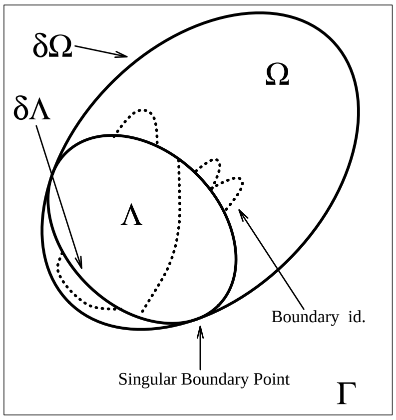

Figure 1: Sketch of the Gribov and fundamental regions. The dotted lines indicate the boundary identifications.

along the ray. Continuity therefore tells us that at some point along the ray, this absolute minimum has to pass the local minimum. At the point they are exactly degenerate, there are two gauge equivalent vector potentials with the same norm, both at the absolute minimum. As in the interior the norm functional has a unique minimum, again by continuity, these two degenerate configurations have to both lie on the boundary of Λ. We should note that the norm functional is degenerate along the constant gauge transformations and indeed the Coulomb gauge does not fix these gauge degrees of freedom. We simply demand that the wave functional is in the singlet representation under the constant gauge transformations and we have $\Lambda/G = \mathcal{A}/\mathcal{G}$. Here Λ is assumed to include the non-trivial boundary identifications that restore the non-trivial topology of $\mathcal{A}/\mathcal{G}$.

If a degeneracy at the boundary is continuous, other than by constant gauge transformations, one necessarily has at least one non-trivial zero eigenvalue for $FP(A)$ and the Gribov horizon will touch the boundary of the fundamental domain at these so-called singular boundary points. We sketch the general situation in figure 1. In principle, by choosing a different gauge fixing in the neighbourhood of these points one could resolve the singularity. If singular boundary points would not exist, all that would have been required is to complement the Hamiltonian in the Coulomb gauge with the appropriate boundary conditions in field space. Since the boundary identifications are by gauge transformations the boundary condition on the wave functionals is simply that they are identical under the boundary identifications, possibly up to a phase in case the gauge transformation is homotopically non-trivial.

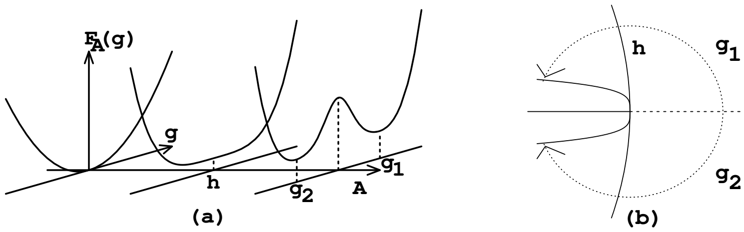

Figure 2: Sketch of a singular boundary point due to a bifurcation of the norm functional. It can be used to show that there are homotopically trivial gauge copies inside the Gribov horizon (h).

Unfortunately, one can argue that singular boundary points are to be expected[10]. Generically, at singular boundary points the norm functional undergoes a bifurcation moving from inside to outside the fundamental (and Gribov) region. The absolute minimum turns into a saddle point and two local minima appear, as indicated in figure 2. These are necessarily gauge copies of each other. The gauge transformation is homotopically trivial as it reduces to the identity at the bifurcation point, evolving continuously from there on. This is one way to see that within the Gribov region there are copies, not even related by a topologically non-trivial gauge transformation!

Also Gribov's original arguments for the existence of gauge copies[2] (showing that points just outside the horizon are gauge copies of points just inside) can be easily understood from the perspective of bifurcations in the norm functional. It describes the generic case where the zero-mode of the Faddeev-Popov operator arises because of the coalescence of a local minimum with a saddle point with only one unstable direction. At the Gribov horizon the norm functional locally behaves in that case as X^3, with X the relevant zero eigenfunction of the Faddeev-Popov operator. The situation sketched in figure 2 corresponds to the case where the leading behaviour is like X^4.

As the Gribov region is associated with the local minima, and since the space of gauge transformations resembles that of a spin model, the analogy with spin glasses makes it unreasonable to expect that the Gribov region is free of further gauge copies. Unfortunately restrictions to a subset of the transverse gauge fields is a rather non-local procedure. This cannot be avoided since it reflects the non-trivial topology of field space.

4 Gauge Fields on the three-Torus

Homotopical non-trivial gauge transformations are in one to one correspondence with non-contractible loops in configuration space, which give rise to conserved quantum numbers. The quantum numbers are like the Bloch momenta in a periodic potential and label representations of the homotopy group of gauge transformations. On the fundamental domain the non-contractible loops arise through identifications of boundary points. Although slightly more hidden, the fundamental domain will therefore contain all the information relevant for the topological quantum numbers. Sufficiently accurate knowledge of the boundary identifications will allow for an efficient and natural projection on the various superselection sectors (i.e. by choosing the appropriate "Bloch wave functionals"). All these features were at the heart of the finite volume analysis on the torus [11] and we see that they can in principle be naturally extended to the full theory, thereby including the desired θ dependence. In the next section this will be discussed in the context of the three-sphere. For the low-lying states in a finite volume, both on the three-torus and the three-sphere, singular boundary points will not play an important role in intermediate volumes.

Probably the most simple example to illustrate the relevance of the fundamental domain is provided by gauge fields on the torus in the abelian zero-momentum sector. For definiteness let us take $G = SU(2)$ and $A_i = i\frac{C_i}{2L}\tau_3$ (L is the size of the torus). These modes are dynamically motivated as they form the set of gauge fields on which the classical potential vanishes. It is called the vacuum valley (sometimes also referred to as toron valley) and one can attempt to perform a Born-Oppenheimer-like approximation for deriving an effective Hamiltonian in terms of these "slow" degrees of freedom. To find the Gribov horizon, one easily verifies that the part of the spectrum for $FP(A)$ that depends on $\vec{C}$, is given by $\lambda^{gh}_{\vec{n}}(\vec{C}) = 2\pi\vec{n}\cdot(2\pi\vec{n}\pm\vec{C})$, with $\vec{n}\neq\vec{0}$ an integer vector. The lowest eigenvalue therefore vanishes if $C_k = \pm 2\pi$. The Gribov region is therefore a cube with sides of length 4π, centred at the origin, specified by $|C_k| \leq 2\pi$ for all k, see figure 3.

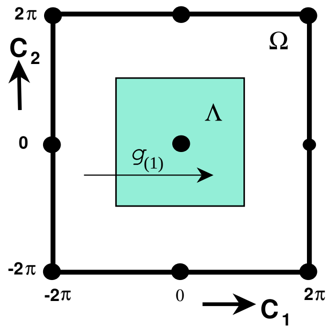

Figure 3: A two dimensional slice of the vacuum valley along the (C_1, C_2) plane. The fat square give the Gribov horizon, the grey square is the fundamental domain. The dots at the Gribov horizon are Gribov copies of the origin.

The gauge transformation $g_{(k)} = \exp(\pi i x_k \tau_3/L)$ maps C_k to $C_k + 2\pi$, leaving the other components of $\vec{C}$ untouched. As $g_{(k)}$ is anti-periodic it is homotopically non-trivial (they are 't Hooft's twisted gauge transformations [12]). We thus see explicitly that gauge copies occur inside Ω, but furthermore the naive vacuum $A = 0$ has (many) gauge copies under these shifts of 2π that lie on the Gribov horizon. It can actually be shown for the Coulomb gauge that for any three-manifold, any Gribov copy by a homotopically non-trivial gauge transformation of $A = 0$ will have vanishing Faddeev-Popov determinant [10]. Taking the symmetry under homotopically non-trivial gauge transformations properly into account is crucial for describing the non-perturbative dynamics and one sees that the singularity of the Hamiltonian at Gribov copies of $A = 0$, where the wave functionals are in a sense maximal, could form a severe obstacle in obtaining reliable results.

To find the boundary of the fundamental domain we note that the gauge copies $\vec{C} = (\pi, C_2, C_3)$ and $\vec{C} = (-\pi, C_2, C_3)$ have equal norm. The boundary of the fundamental domain, restricted to the vacuum valley formed by the abelian zero-momentum gauge fields, therefore occurs where $|C_k| = \pi$, well inside the Gribov region, see figure 3. The boundary identifications are by the homotopically non-trivial gauge transformations $g_{(k)}$. The fundamental domain, described by $|C_k| \leq \pi$, with all boundary points regular, has the topology of a torus. To be more precise, as the remnant of the constant gauge transformations (the Weyl group) changes $\vec{C}$ to $-\vec{C}$, the fundamental domain Λ/G restricted to the abelian constant modes is the orbifold T^3/Z_2. Generalisations to arbitrary gauge groups were considered in ref.[13].

Formulating the Hamiltonian on Λ, with the boundary identifications implied by the gauge transformations $g_{(k)}$, avoids the singularities at the Gribov copies of $A = 0$. "Bloch momenta" associated to the 2π shift, implemented by the non-trivial homotopy of $g_{(k)}$, label 't Hooft's electric flux quantum numbers [12] $\Psi(C_k = -\pi) = \exp(\pi i e_k)\Psi(C_k = \pi)$. Note that the phase factor is not arbitrary, but ± 1. This is because $g^2_{(k)}$ is homotopically trivial.

In weak coupling Lüscher [14] showed unambiguously that the wave functionals are localised around $A = 0$, that they are normalisable and that the spectrum is discrete. In this limit the spectrum is insensitive to the boundary identifications (giving rise to a degeneracy in the topological quantum numbers). This is manifested by a vanishing electric flux energy, defined by the difference in energy of a state with $|\vec{e}| = 1$ and the vacuum state with $\vec{e} = \vec{0}$. Although there is no classical potential barrier to achieve this suppression, it comes about by a quantum induced barrier, in strength down by two powers of the coupling constant. This gives a suppression [15] with a factor $\exp(-S/g)$ instead of the usual factor of $\exp(-8\pi^2/g^2)$ for instantons [16]. Here $S = 12.4637$ is the action computed from the effective potential. At stronger coupling the wave functional spreads out over the vacuum valley and the boundary conditions drastically change the spectrum [11]. At this point the energy of electric flux suddenly switches on.

Integrating out the non-zero momentum degrees of freedom, for which Bloch degenerate perturbation theory provides a rigorous framework [14], one finds an effective Hamiltonian. Near $A = 0$, due to the quartic nature of the potential energy $(F^a_{ij})^2$ for the zero-momentum modes (the derivatives vanish and the field strength is quadratic in the field), there is no separation in time scales between the abelian and non-abelian modes. Away from $A = 0$ one could further reduce the dynamics to one along the vacuum valley, but near the origin this would be a singular decomposition (the adiabatic approximation breaks down). However, as long as the coupling constant is not too large, the wave functional can be reduced to a wave function on the vacuum valley near $\partial\Lambda$ where the boundary conditions can be implemented. These boundary conditions are formulated in a manner that preserves the invariance under constant gauge transformation and the effective Hamiltonian is solved by Rayleigh-Ritz. The influence of the boundary conditions on the low-lying glueball states is felt as soon as the volume is bigger than an inverse scalar glueball mass.

After correcting for lattice artefacts [17], the (semi-)analytic results agree extremely well with the best lattice data [18] (with statistical errors of 2% to 3%) up to a volume of about .75 fermi, or about five times the inverse scalar glueball mass. In figure 4 we present the comparison for a lattice of spatial size 4^3. Monte Carlo data [18] are most accurate for this lattice size. For more detailed comparisons see ref.[17]. The analytic results below $z = 0.95$ are due to Lüscher and Münster [19], which is where the spectrum is insensitive to the identifications at the boundary of Λ.

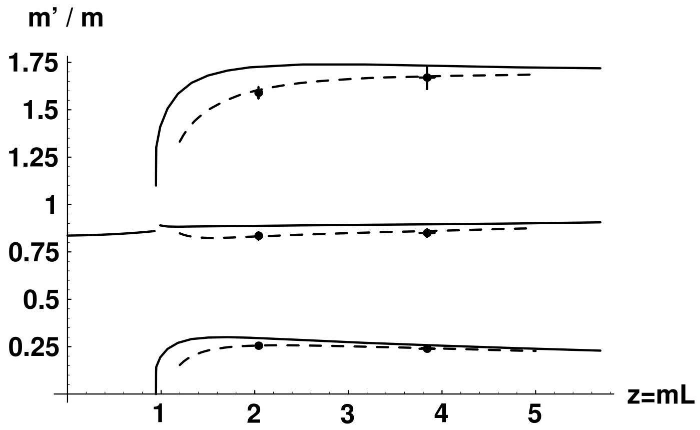

Figure 4: Mass ratios m'/m as a function of L in units of the inverse scalar glueball mass m^{-1}, $z = mL$. The analytic results are given by the full (continuum) and dashed (4^3 lattice) curves. Represented are the the square root of the energy of electric flux per unit length (bottom), the E tensor mass (middle) and the T_2 tensor mass (top). The latter two are (almost) degenerate below $z = 0.95$, which is also where the electric flux energy is exponentially suppressed. Where no error bars on the Monte Carlo data are visible they are smaller than the size of the data points.

Most conspicuously the tensor state in finite volumes is split in a doublet E, with a mass that is roughly 0.9 times the scalar A_1 mass and a triplet T_2 with a mass of roughly 1.7 times the scalar mass. Note that the multiplicity weighted average is approximately 1.4 times the scalar mass, agreeing well with what was found at large volumes from lattice data [18], where the rotational symmetry should be restored. Similar results have been obtained for $SU(3)$ by Vohwinkel [20].

At large volumes extra degrees of freedom will behave non-perturbatively. To demonstrate this, the minimal barrier height that separates two vacuum valleys that are related by gauge transformations with non-trivial winding number was found to be $72.605/Lg^2$, using the lattice approximation and carefully taking the continuum limit [21]. As long as the states under consideration have energies below this value, the transitions over this barrier can be neglected and the zero-momentum effective Hamiltonian provides an accurate description. One can now easily find for which volume the energy of the level that determines the glueball mass (defined by the difference with the groundstate energy) starts to be of the order of this barrier height. This turns out to be the case for L roughly 5 to 6 times the correlation length set by the scalar glueball mass. The situation is sketched in figure 5.

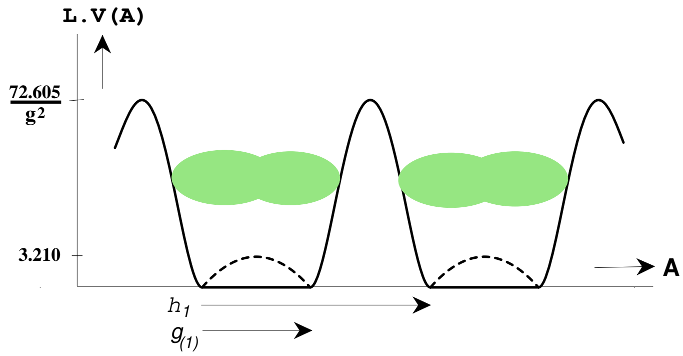

Figure 5: Sketch of the potential for the torus. Shown are two vacuum valleys, related to each other by a winding number 1 gauge transformation h_1. The induced one-loop effective potential, of height $3.210/L$, has degenerate minima related to each other by the anti-periodic gauge transformations $g_{(k)}$. The classical barrier, separating the two valleys, has a height $72.605/Lg^2$.

We expect, as will be shown for the three-sphere, that the boundary of the fundamental domain along the path in field space across the barrier (which corresponds to the instanton path if we parametrise this path by Euclidean time t), occurs at the saddle point (which we call a finite volume sphaleron) in between the two minima. The degrees of freedom along this tunnelling path go outside of the space of zero-momentum gauge fields and if the energy of a state flows over the barrier, its wave functional will no longer be exponentially suppressed below the barrier and will in particular be non-negligible at the boundary of the fundamental domain. Boundary identifications in this direction of field space now become dynamically important too. The relevant "Bloch momentum" is in this case obviously the θ parameter, as wave functionals pick up a phase factor $e^{i\theta}$ under a gauge transformation with winding number one. On the three-torus we have therefore achieved a self-contained picture of the low-lying glueball spectrum in intermediate volumes from first principles with *no free parameters*, apart from the overall scale.

5 Gauge Fields on the three-Sphere

The reason to consider the three-sphere lies in the fact that the conformal equivalence of $S^3 \times \mathbb{R}$ to $\mathbb{R}^4$ allows one to construct instantons explicitly [22]. This greatly simplifies the study of how to formulate θ dependence in terms of boundary conditions on the fundamental domain, and indeed we will see that for S^3 simple enough results can be obtained to address this question [23,24]. The disadvantage of the three-sphere is that in large volumes the corrections to the glueball masses are no longer exponential [14].

We will summarise the formalism that was developed in ref.[22]. We embed S^3 in $\mathbb{R}^4$ by considering the unit sphere parametrised by a unit vector n_μ. It is particularly useful to introduce the unit quaternions σ_μ and their conjugates $\bar{\sigma}_\mu = \sigma_\mu^\dagger$ by $\sigma_\mu = (id, i\vec{\tau})$ and $\bar{\sigma}_\mu = (id, -i\vec{\tau})$. They satisfy the multiplication rules $\sigma_\mu \bar{\sigma}_\nu = \eta^\alpha_{\mu\nu}\sigma_\alpha$ and $\bar{\sigma}_\mu \sigma_\nu = \bar{\eta}^\alpha_{\mu\nu}\sigma_\alpha$, where we used the 't Hooft η symbols [16], generalised slightly to include a component $(\alpha = 0)$ symmetric in μ and ν. We can use η and $\bar{\eta}$ to define orthonormal framings [25] of S^3, which were motivated by the particularly simple form of the instanton vector potentials in these framings. The framing for S^3 is obtained from the framing of $\mathbb{R}^4$ by restricting in the following the four-index α to a three-index a (for $\alpha = 0$ one obtains the normal on S^3): $e^\alpha_\mu = \eta^\alpha_{\mu\nu} n_\nu$ and $\bar{e}^\alpha_\mu = \bar{\eta}^\alpha_{\mu\nu} n_\nu$. Note that e and $\bar{e}$ have opposite orientations. The instanton describes tunnelling from $A = 0$ to $A_a = -\sigma_a$ ($A_\mu = A_a e^a_\mu$) over a potential barrier corresponding to a sphaleron, i.e. the vector potential $A_a = -\frac{1}{2}\sigma_a$ is a saddle point of the energy functional with one unstable mode, corresponding to the direction of tunnelling. We observe that $A_a = -\sigma_a$ has zero energy and is a gauge copy of $A_a = 0$ by a gauge transformation $g = n \cdot \bar{\sigma}$ with winding number one.

We consider the modes that are degenerate in energy to lowest order with the modes that describe tunnelling through the sphaleron and "anti-sphaleron". The latter is a gauge copy by a gauge transformation $g = n \cdot \sigma$ with winding number -1 of the sphaleron. The two dimensional space containing the tunnelling paths through these sphalerons is parametrised by u and v through $A_\mu(u,v) = \frac{1}{2}\left(-u e^a_\mu - v \bar{e}^a_\mu\right)\sigma_a$. The gauge transformation with winding number -1 is easily seen to map $(u,v) = (w,0)$ into $(u,v) = (0, 2-w)$. The relevant (18 dimensional) space is defined by

$$A_\mu(c,d) = \left(c^a_i e^i_\mu + d^a_j \bar{e}^j_\mu\right)\frac{\sigma_a}{2} = A_i(c,d) e^i_\mu. \qquad (6)$$

The c and d modes are mutually orthogonal and satisfy the Coulomb gauge condition $\partial_i A_i(c,d) = 0$. Note that $c^a_i = -u\delta^a_i$ and $d^a_i = -v\delta^a_i$ yields $A_\mu(u,v)$. The significance of this 18 dimensional space is that these degenerate modes, to second order in c and d, are the *lowest* energy modes.

To understand our success in constructing the fundamental region in this case, we note that $FP_f(A)$ in eq. (4) is defined as a hermitian operator acting on the vector space $\mathcal{L}$ of functions g over S^3 with values in the space of the quaternions $\mathbb{H} = \{q_\mu \sigma_\mu | q_\mu \in \mathbb{R}\}$. The gauge group $\mathcal{G}$ is contained in $\mathcal{L}$ by restricting to the *unit* quaternions. For arbitrary gauge groups $\mathcal{L}$ is defined as the algebra generated by the identity and the (anti-hermitian) generators of the algebra. When minimising the same functional over the larger space $\mathcal{L}$ one obviously should find a smaller space $\tilde{\Lambda} \subset \Lambda$. Since $\mathcal{L}$ is a linear space, $\tilde{\Lambda}$ can be specified as $\tilde{\Lambda} = \{A \in \Gamma | \langle g, FP_f(A)\, g\rangle \geq 0,\ \forall g \in \mathcal{L}\}$. As for the Gribov horizon, the boundary of $\tilde{\Lambda}$ is therefore determined by the location where the lowest eigenvalue vanishes. For the (c,d) space it can be shown [23] that the boundary $\partial\tilde{\Lambda}$ will touch the Gribov horizon $\partial\Omega$. This establishes the existence of singular points on the boundary of the fundamental domain due to the inclusion $\tilde{\Lambda} \subset \Lambda \subset \Omega$. By showing that the fourth order term in eq. (2) is positive (see app. A of ref.[23]) this is seen to correspond to the situation as sketched in figure 2.

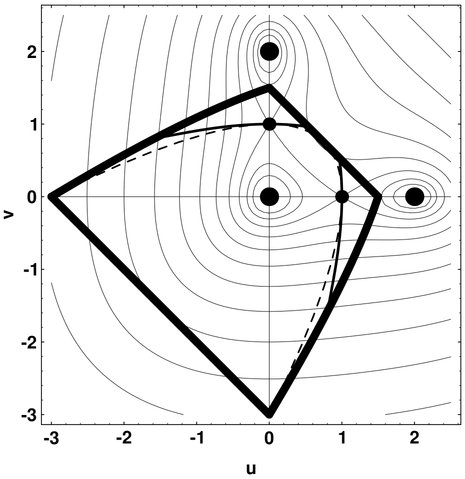

Figure 6: Location of the classical vacua (large dots), sphalerons (smaller dots), the Gribov horizon (fat sections), the boundary of $\tilde{\Lambda}$ (dashed curves) and part of the boundary of the fundamental domain (full curves). Also indicated are the lines of equal potential in units of 2^n times the sphaleron energy.

Figure 6 summarises the results for the (u,v) plane and also shows the equal-potential lines as well as the multiple vacua and the sphalerons. As the two sphalerons are each others gauge copies with equal norm, they lie on $\partial\Lambda$, which can be extend by perturbing around these sphalerons [26].

For $d = 0$ one can bring c to a diagonal form $c_i^a = x_i \delta_i^a$. The rotational and gauge symmetry reduce to permutations of the x_i and simultaneous changes of the sign of two of the x_i. In figure 7 we present the results for Λ and Ω in these coordinates. In this particular case Λ coincides with $\tilde{\Lambda}$, a consequence of the convexity and the fact that both the sphalerons (indicated by the dots) and the edges of the tetrahedron lie on $\partial\Lambda$, the latter also lying on $\partial\Omega$. It is essential that the sphalerons do not lie on the Gribov horizon and that the potential energy near $\partial\Omega$ is relatively high. This is why we can take the boundary identifications near the sphalerons into account without having to worry about singular boundary points, as long as the energies of the low-lying states will be not much higher than the energy of the sphaleron. It allows one to study the glueball spectrum as a function of the CP violating angle θ, but more importantly it incorporates for $\theta = 0$ the noticeable influence of the barrier crossings, i.e. of the instantons.

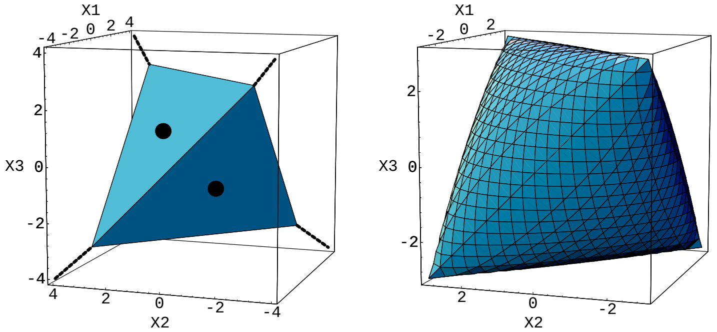

Figure 7: The fundamental domain (left) for constant gauge fields on S^3, with respect to the "instanton" framing e_μ^a, in the diagonal representation $A_a = x_a \sigma_a$ (no sum over a). By the dots on the faces we indicate the sphalerons, whereas the dashed lines represent the symmetry axes of the tetrahedron. To the right we display the Gribov horizon, which encloses the fundamental domain, coinciding with it at the singular boundary points along the edges of the tetrahedron.

An effective Hamiltonian for the c and d modes is derived from the one-loop effective action [24]. The boundary conditions are chosen so as to coincide with the appropriate boundary conditions near the sphalerons, but such that the gauge and (left and right) rotational invariances are not destroyed. Projections on the irreducible representations of these symmetries turned out to be essential to reduce the size of the matrices to be diagonalised in a Rayleigh-Ritz analysis. Remarkably all this could be implemented in a tractable way [24]. Results are summarised in figure 8. One of the most important features is that the 0^- glueball is (slightly) lighter than the 0^+ in perturbation theory, but when including the effects of the boundary of the fundamental domain, setting in at $f \equiv g^2/2\pi^2 \sim 0.2$, the $0^-/0^+$ mass ratio rapidly increases. Beyond $f \sim 0.28$ it can be shown that the wave functionals start to feel parts of the boundary of the fundamental domain which the present calculation is not representing properly [24]. This value of f corresponds to a circumference of roughly 1.3 fm, when setting the scale as for the torus, assuming the scalar glueball mass in both geometries at this intermediate volume to coincide.

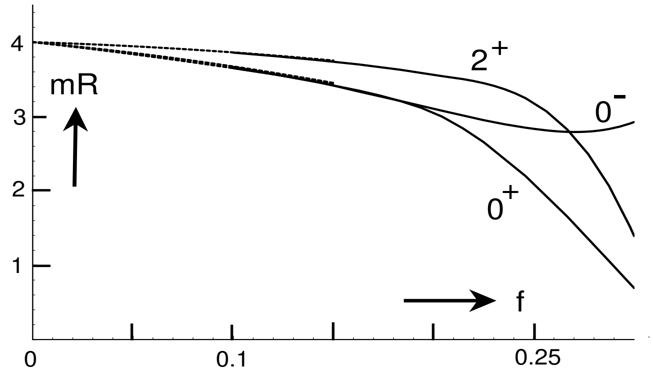

Figure 8: The full one-loop results for the masses of scalar, tensor and odd glueballs on S^3 as a function of $f = g^2(R)/2\pi^2$ for $\theta = 0$. The dashed lines correspond to the perturbative result.

6 The View

We have analysed in detail the boundary of the fundamental domain for SU(2) gauge theories on the three-torus and three-sphere. We had to divide $\mathcal{A}$ by the set of *all* gauge transformations, including those that are homotopically non-trivial, to get the physical configuration space. All the non-trivial topology is then retrieved by the identifications of points on the boundary of the fundamental domain. Boundary identifications become important when the wave functionals spread out in configuration space to such an extent that they touch it. This happens at large volumes, whereas at very small volumes the wave functional is localised around $A = 0$ and one need not worry about these non-perturbative effects. That these effects can be dramatic, even at relatively small volumes (above a tenth of a fermi across), was demonstrated for the case of the torus. However, for that case the structure of the fundamental domain (restricted to the abelian zero-energy modes) is a hypercube and deviates considerably from the fundamental domain of the three-sphere. Nevertheless, the spectrum for the sphere is compatible with that for the torus in volumes around one fermi across[27], with $m(2^+)/m(0^+) \sim 1.5$ and $m(0^-)/m(0^+) \sim 1.7$.

It should be noted that the shape of Λ is independent of L if the gauge field is expressed in units of $1/L$. Suppose that the coupling constant will grow without bound. This would make the potential irrelevant and makes the wave functional spread out over the whole of field space (which could be seen as a strong coupling expansion). If the kinetic term would have been trivial the wave functionals would be "plane waves" on a space with complicated boundary conditions. In that case it seems unavoidable that the infinite volume limit would depend on the geometry (like T^3 or S^3) that is scaled-up to infinity. Due to the non-triviality of the kinetic term this conclusion cannot be readily made and our present understanding only allows comparison in volumes around one cubic fermi. However, one way to avoid this undesirable dependence on the geometry is that the vacuum is unstable against domain formation. As periodic subdivisions are space filling on a torus, this seems to be the preferred geometry to study domain formation. In a naive way it will give the correct string tension (flux conservation tells us to "string" the domains that carry electric flux) and tensor to scalar mass ratio (averaging over the orientations of the domains is expected to lead to a multiplicity weighted average of the T_2 and E masses). Furthermore, the natural dislocations of such a domain picture are gauge dislocations. The point-like gauge dislocations in four dimensions are instantons and in three dimensions they are monopoles. Their density is expected to be given roughly as one per domain (with a volume of around 0.5 cubic fermi). Also the coupling constant will stop running at the scale of the domain size. See ref.[28] for further details.

None of this understanding would have been possible without Gribov having recognised the significance of gauge copies for the infrared sector of gauge field dynamics.

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