CERN-TH.5359/89

The Size of SU(2) Lattice Artifacts

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ABSTRACT

We compute analytically the effective Hamiltonian (i.e. the logarithm of the transfer matrix) in the zero-momentum modes on a finite lattice. We present the results for the low-lying spectrum obtained from a Rayleigh-Ritz analysis for lattices of spatial sizes $4^3, 6^3$ and ∞^3, and compare with lattice Monte Carlo results for intermediate volumes. We discuss the implications and limitations of the good agreement found.

CERN-TH.5359/89
April 1989

1. Introduction

Everyone who has ever looked into lattice perturbation theory must have realized that it would almost be miraculous if the Monte Carlo results were actually to agree with continuum results at the relatively large couplings employed. Still, within the limited range of varying lattice sizes being used, one does seem to find a behaviour close to scaling. Yet statistical and systematic errors are often still quite large, leaving more than enough room for those who wish to remain sceptical. Nevertheless, the last two years have seen considerable progress in the accuracy and reliability of the pure-gauge spectrum calculations in intermediate volumes [1,2]. These results seem to imply the almost "preposterous" claim that a 4^3 lattice is able to describe with some confidence the continuum behaviour of the low-lying spectrum in volumes of sizes between one and five scalar glueballs.

This was not only substantiated by Monte Carlo calculations of up to 10^3 lattices, but also by the comparison with our continuum calculations [3,4] based on Lüscher's effective Hamiltonian for the zero-momentum modes [5]. However, the accuracy of the Monte Carlo calculations has progressed for this volume range to the point where deviations from the analytic results become significant [2]. For the lattice one can obviously suspect mainly the lattice artifacts to be responsible for this deviation. However, the continuum calculation does contain a so-called adiabatic approximation, which can be considered essential for the reduction to the relevant degrees of freedom (the zero-momentum modes in this volume range), for which we had to look beyond the Gribov horizon [6]. A considerable effort was made on pinning down the accuracy of this approximation, nevertheless leaving some room for doubt [3].

Since our method of looking beyond the Gribov horizon (i.e. having different coordinate patches in configuration space with a "prescription" of glueing these patches together) is intended to be applied to more general and complex cases, we wish to give some weight to demonstrating the viability of the method in a situation which is under relatively accurate control. Rather than spending more effort on calculating the theoretical error due to the non-adiabatic behaviour, which we suspect not to lead to more insight in the dynamics and physics involved (although it would lead to a more rigorous formulation of the "patching"), we decided on the easier, more experimental route of calculating analytically the effect of the finite lattice size on the low-lying spectrum. In demonstrating the close agreement of the thus corrected data, we indirectly demonstrate the (expected, but unproven) accuracy of the approximations involved. As a bonus, we remove the mystery of why lattice Monte Carlo (with Wilson's original action [7]) works so well for the domain considered. We wish to emphasize, however, that we do not claim to have demonstrated the same miracles to occur for SU(3), or in larger volumes.

This Letter is intended to outline roughly the main points of the calculation and to give the most important results. Details and a more extensive discussion **will** be published elsewhere.

2. From lattice to effective action

We start with Wilson's action [7] for a lattice of dimensions $N_0 \times N_1 \times N_2 \times N_3$ with periodic boundary conditions

$$S = \frac{1}{g_o^2} \sum_{\mu,\nu,x} \mathrm{Tr}(1 - U_{x,x+\hat{\mu}} U_{x+\hat{\nu},x+\hat{\mu}+\hat{\nu}} U^{\dagger}_{x+\hat{\mu},x+\hat{\mu}+\hat{\nu}} U^{\dagger}_{x,x+\hat{\nu}}) \quad , \tag{1}$$

and split the gauge field in the spatially constant "background field" and the non-zero momentum "quantum fields":

$$U_{x,x+\hat{\mu}} = \exp(ic_\mu(t)/N_\mu)\exp(iq_\mu(x)/N_\mu) = U_\mu^{(o)}(t)\hat{U}_{x,x+\hat{\mu}} \quad . \tag{2}$$

We fix the gauge, similarly to what was done in the continuum [3], with a non-local "background field" gauge fixing

$$\sum_\mu q_\mu(x-\hat{\mu}) - U_\mu^{(o)}(t) q_\mu(x) U_\mu^{(o)\dagger}(t) = 0 \quad . \tag{3}$$

For a readable account of lattice perturbation theory we refer to ref. [8]. The lattice background field calculation to which our approach has some resemblance can be found in ref. [9].

As in the continuum, one next integrates out the non-zero momentum quantum modes to be left with an effective Lagrangian in the zero-momentum modes. We have done this calculation explicitly to one-loop order and to the same order in the background fields c_i ($c_0 = 0$) as in the continuum, taking N_0 to infinity but otherwise having arbitrary spatial size for the lattice (later on we specialize to the cubic case $N_1 = N_2 = N_3 = N$ and the asymmetric case $N_1 = N_2 \neq N_3$).

As compared to the continuum, there are four additional complications in evaluating the effective Lagrangian from the lattice action. First, instead of the infinite continuum momentum sums, we have finite lattice momentum sums, where the lattice momenta are defined by the Fourier decomposition of the fields:

$$q_\mu^{(n)}(x+\hat{\nu}) = \exp(2\pi i n_\nu / N_\nu) q_\mu^{(n)}(x) \quad , \tag{4}$$

with n_ν restricted to the Brillouin-zone (i.e. $n_\nu = 0, 1, 2, \cdots, N_\nu - 1$). In the time direction we want the lattice to extend to infinity, so one of the sums will actually be infinite (see below). Second, the tree-level action is non-polynomial. Likewise, the part of the action quadratic in the quantum fields has a more complicated structure than in the continuum (see ref. [9]). Third, the effective action is still discrete in time. Its path-integral represents the trace of the transfer-matrix to the power N_0 and the logarithm of this transfer-matrix is what we want to extract, giving us the lattice effective Hamiltonian for the zero-momentum modes. Finally the kinetic term is that of SU(2) (or the standard kinetic term on S^3).

Let us discuss the relevant "tricks" of working our way around each difficulty. The finite momentum sums can hardly be a problem. In a sense it is a blessing to be able to work for a change in a rigorous context (on a **finite** lattice). The interest is ultimately of course in the scaling limit of all $N_\mu \to \infty$, but for the comparison with the actual Monte Carlo results we do have only a **finite** number of degrees of freedom. As a little thought will reveal, all the infinite n_0 sums at the one-loop level can be computed in terms of the following heat-kernel:

$$\begin{aligned} f_s(t) =& \frac{1}{N} \sum_{n=1}^{N} \exp(-4s \sin^2(\pi(n+t)/N)) \\ =& \sum_{k \in Z} \exp(-2s - 2\pi i k t) I_{Nk}(2s) \quad , \end{aligned} \tag{5}$$

where the latter expression follows from a simple Poisson resummation (I_ν is the modified Bessel function of order ν [10]). The one-loop "integral" now consists of a sum over the (spatial) Brillouin zone (excluding the origin) of the expression

$$\begin{aligned} -\int_0^\infty \frac{ds}{s} \exp(-s\omega^2) f_s(0) =& \frac{1}{2}\Omega(\omega) + \frac{1}{N} \ln(1 - \exp(-N\Omega(\omega))) \\ =& \frac{1}{N} \ln[\sum_{n=0}^{\infty} \exp(-N(n+\frac{1}{2})\Omega(\omega))] \quad , \end{aligned} \tag{6}$$

with $\Omega(\omega)$ the "effective frequency"

$$\Omega(\omega)=2\mathrm{asinh}(\omega/2) \quad , \tag{7}$$

where ω is a function of n_i, N_i and c_i. This of course goes back to Feynman [11] and is nothing but the exact solvability of the harmonic oscillator path integral with a finite step size.

For an Abelian zero-momentum background field this dependence of ω on c_i "collapses" to a very simple expression and leads to the so-called vacuum-valley effective potential [3,12] ($c_i=C_i\frac{\sigma_3}{2}$)

$$\hat{V}_\ell(C;N)=4\sum_{n_i=1}^{N_i}\mathrm{asinh}(\sqrt{\sum_{i=1}^{3}\sin^2(\frac{2\pi n_i-C_i}{2N_i})}\quad) \quad , \tag{8}$$

which (for the cubic case) has the same properties as its well-studied scaling limit [12]

$$\hat{V}_\ell(C)=\frac{4}{\pi^2 L}\sum_{\vec{n}\in Z^3-\vec{0}}\frac{\sin^2(\vec{n}\cdot\vec{C}/2)}{(\vec{n}^2)^2} \quad . \tag{9}$$

By studying the deviation of $L\cdot\hat{V}_\ell(C)$ from $N\cdot\hat{V}_\ell(C;N)$, one can already get a decent impression for what values of N the lattice results might be close to the continuum.

3. The "flow" of the effective action

The remainder of the calculation is an order of magnitude more complicated than in the continuum, but tricks that are partly special to SU(2), allowed us to do the full calculation by hand. Needless to say that an independent check is not a luxury, for which we used the new symbolic manipulation programme FORM [13]. The final result looks much like in the continuum:

$$\begin{aligned}S_{eff}(c)=\sum_t[\prod_{k=1}^{3}N_k]\{2\sum_i(\frac{1}{g_o^2}+\hat{\alpha}_1^{(i)}(N))\mathrm{Tr}(1-U_i^{(o)}(t+1)U_i^{(o)\dagger}(t))&\\+\frac{1}{g_o^2}\sum_{ij}\mathrm{Tr}(1-U_i^{(o)}(t)U_j^{(o)}(t)U_i^{(o)\dagger}(t)U_j^{(o)\dagger}(t))\}&\\+V_\ell(c_i(t);N)+V_T(c_i(t);N)+\cdots \quad ,&\end{aligned} \tag{10}$$

where the vacuum-valley and transverse potentials (resp. V_ℓ and V_T) are given by:

$$
\begin{aligned}
V_\ell(c;N) =&\hat{V}_\ell(r;N) - 4\mathrm{asinh}\Big(\sqrt{\sum_{i=1}^{3}\sin^2(r_i/(2N_i))}\;\Big)\\
=&\frac{1}{N_1}\{\sum_{i=1}^{3}[\gamma_1^{(i)}(N)r_i^2+\gamma_2^{(i)}(N)r_i^4+\gamma_4^{(i)}(N)r_i^6]+\sum_{i>j}\gamma_3^{(ij)}(N)r_i^2r_j^2\\
&+\sum_{i\neq j}\gamma_5^{(ij)}(N)r_i^2r_j^4+\gamma_6(N)\prod_{i=1}^{3}r_i^2+\cdots\}\\
V_T(c;N) =&\frac{1}{N_1}\{\sum_{ij}\frac{1}{4}\hat{\alpha}_2^{(ij)}(N)F_{ij}^2+\sum_{ijk}\alpha_3^{(ijk)}(N)r_k^2F_{ij}^2\\
&+\sum_{(ij)}\alpha_4^{(ij)}(N)r_i^2F_{ij}^2+\alpha_5(N)\mathrm{det}^2c+\cdots\} \quad ,
\end{aligned} \tag{11}
$$

with the curvature F_{ij}^a and the radial coordinate r_i given by:

$$
F_{ij}^a = -\varepsilon_{abd}d_i^bc_j^d \quad , \qquad r_i^2 = 2\mathrm{Tr}(c_i^2) \quad . \tag{12}
$$

For later purposes we treated the most general case, but we will be mostly interested in the cubic situation $N = N_1 = N_2 = N_3$, for which all coefficients are independent of the coordinate index. Figure 1 gives the ratio of each coefficient with its continuum value as quoted in Table 1 (this corrects a thus far undetected sign error in α_3; it is now indeed given by eq.(2.17) of ref. [3]). In all cases except for α_1 and α_2, these continuum values are the respective scaling limits of $\alpha_i(N)$ and $\gamma_i(N)$.

For $\hat{\alpha}_1$ and $\hat{\alpha}_2$ the renormalization group dictates the following behaviour to one-loop order:

$$
\begin{aligned}
\hat{\alpha}_1(N) =& -\frac{11}{12\pi^2}\ln N + \tilde{\alpha}_1(N) \quad ,\\
\hat{\alpha}_2(N) =& -\frac{11}{12\pi^2}\ln N + \tilde{\alpha}_2(N) \quad ,
\end{aligned} \tag{13}
$$

where $\tilde{\alpha}_{1,2}(N)$ have finite scaling limits. This is of course such that the renormalized flow of the effective action to one-loop order is defined by keeping $\frac{1}{g_R^2} = \frac{1}{g_o^2} - \frac{11}{12\pi^2}\ln N$ fixed, where g_R is the lattice renormalized coupling. Let us remind the reader how $\tilde{\alpha}_{1,2}(\infty)$ are related to $\alpha_{1,2}$ obtained from the continuum calculation in the minimal subtraction scheme [9,14]. One has (by definition) to one-loop order:

$$
\frac{1}{g_o^2} = -\frac{11}{12\pi^2}\ln(a\Lambda_L) \quad , \tag{14}
$$

where a is the lattice spacing and Λ_L the lattice scale parameter. Therefore one has

$$
\begin{aligned}
\frac{1}{g_o^2}-\frac{11}{12\pi^2}\ln(N)+\tilde{\alpha}_{1,2}(N) &= -\frac{11}{12\pi^2}\ln(aN\Lambda_L)+\tilde{\alpha}_{1,2}(N)\\
&= -\frac{11}{12\pi^2}\ln(L\Lambda_{MS})+\tilde{\alpha}_{1,2}(N)-\frac{11}{12\pi^2}\ln(\Lambda_L/\Lambda_{MS})\\
&= \frac{1}{g^2(L)}+\alpha_{1,2} \quad ,
\end{aligned}
\tag{15}
$$

where $L=aN$ is the physical size of the volume, $g(L)$ is the renormalized coupling at the scale L for minimal subtraction and Λ_{MS} is its scale parameter. Equation (15) confirms independently from $\alpha_1-\tilde{\alpha}_1(\infty)$ and $\alpha_2-\tilde{\alpha}_2(\infty)$ to a high accuracy the result for Λ_L/Λ_{MS} [9,14] which is an important consistency check. In Figure 1 we have plotted

$$
\alpha_{1,2}(N)=\tilde{\alpha}_{1,2}(N)-11/12\pi^2\ln(\Lambda_L/\Lambda_{MS}) \quad , \tag{16}
$$

with $11\ln(\Lambda_L/\Lambda_{MS})/(12\pi^2)=-0.1866792$ [9].

Next we discuss the tree-level action, where both the kinetic and the potential parts differ from the continuum result. These can be considerably simplified for SU(2), by using the expression $U_i^{(o)}=\cos(r_i/(2N_i))+ic_i^a\sigma_a\sin(r_i/(2N_i))/r_i$, which gives:

$$
\sum_{ij}\mathrm{Tr}[1-U_i^{(o)}U_j^{(o)}U_i^{(o)\dagger}U_j^{(o)\dagger}]=\sum_{ij}\frac{\sin^2(r_i/(2N_i))\sin^2(r_j/(2N_j))}{r_i^2r_j^2}F_{ij}^2 \quad . \tag{17}
$$

This gives an important simplification for the Rayleigh-Ritz analysis of the effective Hamiltonian. The kinetic term will be discussed after we have dealt with the issue of discrete time.

4. The effective Hamiltonian

There is an elegant way of incorporating the discrete time when one realizes that the path integral is given by [8,11]

$$
\mathcal{Z}=\mathrm{Tr}((e^{-K}e^{-V})^{N_0})=\mathrm{Tr}(\mathcal{T}^{N_0}) \quad , \tag{18}
$$

where $\mathcal{T}$ is the transfer matrix, and the trace is taken over a complete set of quantum states. Here K is the (SU(2) or standard S^3) kinetic and V the potential term. The masses obtained from lattice Monte Carlo calculations, in the ideal situation [4], are the logarithm of the eigenvalues of the transfer matrix. Equivalently they are the eigenvalues of the effective Hamiltonian:

$$H_{eff} = \ln(\exp(-K/2)\exp(-V)\exp(-K/2)) \quad . \tag{19}$$

This choice is to ensure that the effective Hamiltonian is Hermitian. An alternative choice would have been $\ln(\exp(-V/2)\exp(-K)\exp(-V/2))$, but we leave it to the reader to verify that this is related to eq. (19) by a unitary transformation, and it has therefore the same spectrum. It is also some fun to play with the harmonic oscillator, for which H_{eff} is again harmonic, but with ω replaced by the effective frequency $\Omega(\omega)$, see eq. (7). From eq. (10) we see that K and V are proportional to $1/N$ and $N \cdot H_{eff}$ will be a power-series in $1/N^2$, which we computed (both by hand and with the algebraic manipulation programme FORM [13]) to eighth order in the background field c and to second order in $1/N^2$. This generates, for example, terms quartic in the curvature, but details will be postponed to the long write-up of this work.

Finally, the kinetic term in the Hamiltonian can be converted to that of flat three space by a rescaling of the wave function.

$$\begin{aligned}\left(-\frac{1}{\sin^2(r_i/(2N_i))}\frac{\partial}{\partial r_i}\sin^2(r_i/(2N_i))\frac{\partial}{\partial r_i}+\frac{1}{4\sin^2(r_i/(2N_i))}\vec{L}_i^{\,2}\right)\frac{r_i}{\sin(r_i/(2N_i))}\Psi \\ =\frac{r_i}{\sin(r_i/(2N_i))}\left(-\frac{\partial^2}{\partial c_i^{a\,2}}+(\frac{(r_i/(2N_i))^2}{\sin^2(r_i/(2N_i)}-1)\frac{\vec{L}_i^{\,2}}{r_i^2}-\frac{3}{4N_i^2}\right)\Psi \quad .\end{aligned} \tag{20}$$

This again leads to only a simple correction in the way we implement the Rayleigh-Ritz analysis for the effective Hamiltonian [3]. But it is important for the issues concerning the non-perturbative part incorporated in our analysis. As was discussed at great length for the continuum [3], a discussion which carries over directly to the lattice, there are coordinate singularities at $r_i = 2\pi$, related to the Gribov horizon [6] and signalled by the conic singularity in the vacuum-valley effective potential. This problem was resolved by employing an adiabatic approximation, which together with the symmetries of the effective Hamiltonian lead to implementing boundary conditions on the wave function at $r_i = \pi$. We restrict ourselves to the positive parity sector for which these boundary conditions are by now well established [3,4,15]. By the conversion of the SU(2) metric to the flat metric these boundary conditions are **identical** to those used for the continuum [3]. In conclusion, the effective Hamiltonian as obtained from the transfer matrix can be transformed to be exactly of the same form as in the continuum, except that its coefficients depend on the lattice size through N_i and that quite a few additional, and for small N_i relevant, eighth-order terms are generated. In the full analysis we have also expanded $V_\ell(c;N)$ to eighth order in c, but as in the continuum, their effect remains small for cubic volumes (this situation will dramatically differ in asymmetric volumes as we will see further on).

5. Comparison with Monte Carlo results

In Figure 2 we compare the Monte Carlo data of refs. [1,2] with our results obtained from the effective Hamiltonian, using a Rayleigh-Ritz calculation [3]. Note the highly blown-up scale. Horizontally we plot the scale-independent quantity $z_{E^+} = N \cdot m_{E^+}$ and vertically the ratio of the square root of the string tension with the E^+ mass. The full lines give the results from H_{eff} as described above for lattices with spatial sizes 4^3, 6^3 and ∞^4, where $N = \infty$ is equivalent to the continuum. We also have results for 8^3, which lie in between the $N = 6$ and $N = \infty$ curves, but were not drawn to keep the graphs legible. The dashed lines correspond to incorporating the two-loop correction to the vacuum-valley effective potential, which for $N = \infty$ was described in ref. [3] (the corresponding curve in Figure 2 corrects the earlier mentioned sign-error in α_3). For the finite lattice the equivalent two-loop analysis is complicated, to put it mildly and will be considered in the future. Instead, we have used the continuum expression for this two-loop correction, and expect that the error thus made should not be more than 25% of the correction due to this two-loop contribution (therefore being smaller than 0.5-1% in the final result).

We see that the agreement is excellent, except for the 8^3 data point of ref. [2] at $z_{E^+} = 3.1$, $\beta = 2.7$. In Table 3a of ref. [2] this point was labelled as having a "not so good" asymptotic estimate for the string tension. Nevertheless, we included the data point to avoid personal bias. Instead we imposed on the data of ref. [2] the rigorous cut that N_0 should be bigger than or equal to 64. The reason is that it was clearly demonstrated in ref. [2] that data with a shorter time extent suffer from finite temperature effects and mixing with excited states. The data of ref.[1], which was obtained using the "fuzzing" procedure, should be relatively free from these unwanted contaminations. Where $\beta = 4/g_o^2$ and lattice size agree, the two groups have the same results (for the data point at $z_{E^+} = 3.65$, even the blown-up scale cannot resolve the difference), except for the scalar A_1^+ mass at $\beta = 3.0, L = 4$

($z_{E^+} = 1.7$). We believe that the evidence is such that at smaller volumes the scalar glueball masses of ref. [2] are still overestimated by up to 10%, if the time extent is not taken large enough. This should explain any remaining discrepancies visible in Figure 3 for the mass-ratio $m_{A_1^+}/m_{E^+}$. Note that the data point with the smallest value for z_{E^+} was sampled for $\beta = 4.5$ and $N_0 = 256$ (which is quite an impressive achievement) and is therefore expected to be less affected by these contaminations. Finally table 2 collects additional intermediate volume results from the T_{11}^+ and the T_2^+ states considered in ref. [1]. Also here the agreement is excellent, taking into account that the effective Hamiltonian is expected to breakdown for $z > 5$, or even earlier for heavier states [3,4,15].

The main reason we embarked on this project (which got slightly out of hand) was to understand why the value of z_{E^+} where tunnelling sets in as found in the Monte Carlo data [2] (i.e. where the lines of Figure 2 cross the horizontal axis around $z_{E^+} = 1$) was 10% off from our prediction [3]. However, in this region (small values for g) our methods, including the adiabatic approximation, should be more accurate. We believe we have now demonstrated indirectly that they are. We could not have done it without the Monte Carlo results and this is maybe a good place to thank all of those involved for spending some of their efforts in probing the intermediate volume regime more accurately. We have explained in the introduction why this is not a purely academic exercise.

6. Problems in asymmetric volumes

We end this Letter with a discussion of some naively unexpected but nevertheless well-understood problems with calculating the low lying spectrum on an asymmetric lattice. This was introduced in ref. [16] to study the deconfining transition. We think the principles outlined in that paper are correct, but unfortunately have to conclude that the analytic evidence as provided there should be reconsidered. The geometry relevant for H_{eff} is now $N_1 = N_2 = N = z_T N_3$, where z_T is the asymmetry parameter considered in ref. [16]. In Figure 4 (the lower curve a) we reproduce the analytic results for $\sqrt{\kappa_t}/m$ at $z_T = 1.5$ and the Monte Carlo results of ref. [2] for $N_1 = N_2 = 6, N_3 = 4$. Here m is the mass gap and κ_t is the string tension for electric flux in the third (short) direction. Their curve coincides with what we phrased as results coming from the "minimal Hamiltonian" [3], which takes no two-loop and eighth-order terms into account and puts $\alpha_3 = \alpha_4 = \alpha_5 = 0$.

For the cubic geometry, corrections due to the higher order contributions are at most 3%, but already for the mild asymmetry of $z_T = 1.5$, including these higher order terms has a dramatic effect, as can be seen in Figure 4. The curves labelled by b include the effect of the sixth-order terms in the transverse potential, whereas the curves labelled by c include on top of that the eighth-order contribution coming from the vacuum-valley effective potential. One of the main reasons for the sensitivity of the spectrum on the higher order terms is that the polynomial approximation for the vacuum-valley potential is bad along the r_3 axis. The potential in this direction is mainly responsible for the string tension in the short third direction. To test this idea, we have also given the (analytic and Monte Carlo [2]) results for the string tension in the other directions, which is denoted by κ_s. We see that (at least in larger volumes) the sensitivity is much less.

Even including the eighth-order terms is far from sufficient to approximate the effective potential accurately (our programme would allow us to go up to twelfth order without much difficulty). As an example we give the reader the following situation at $z_T = 2$ to play with. With techniques identical to those employed in the cubic case [12] one can show that the following expansion of the effective potential is very accurate:

$$L_1 \cdot \hat{V}_\ell(r_1 = r_2 = 0) = 2r_3(2\pi - r_3)/\pi + \frac{16}{\pi}\sum_{n=1}^{\infty} a_n \sin^2(nr_3/2) \quad , \tag{21}$$

with $a_1 = 0.6156595, a_2 = 6.763187 \times 10^{-3}, a_3 = 1.467451 \times 10^{-4}, a_4 = 4.012186 \times 10^{-6}$ and the convergence is determined by the general estimate $a_n < \frac{5\pi^2}{2n}\exp(-\pi n)$. From this expression one easily deduces the bad convergence of the Taylor expansion up to eighth order (for $r_3 < \pi$). One can show that the mismatch grows roughly linearly with the asymmetry. Considering the bad behaviour for the still small value $z_T = 1.5$, this does not promise much good for the study of the deconfining transition based on these calculations. In principle, since one can easily construct rapidly converging sums for the vacuum-valley effective potential, valid for a much larger range of asymmetries, it is likely that one can correct for part of the above problem relatively easy. It would be more alarming if the expansion of the transverse potential V_T were to be badly converging. There are signs for this to occur, since α_3 increases by respectively the factors 6, 11 and 47 for the asymmetries 1.5, 2.0 and 4.0. This is far more difficult to correct for and would basically require a construction of V_T to all orders in c. We intend to look into these matters in the future.

In Figure 4 we have also given, with the curves labelled d, our results as obtained from the lattice effective Hamiltonian by taking $N_1 = N_2 = 6$ and $N_3 = 4$, to compare with the Monte Carlo data of ref. [2]. In the light of the above discussion the agreements and disagreements are relatively well understood.

7. Conclusion

In this Letter we studied the size of lattice artifacts in the low-lying spectrum of intermediate volume SU(2) pure gauge theory with the Wilson lattice action. We confirm that indeed a lattice of spatial size 4^3 is able to approximate continuum physics at the 10% level. We used the correction for the lattice artifacts to argue the accuracy of the approximations involved in the analytic approach based on the zero-momentum effective Hamiltonian. For asymmetric spatial volumes we indicated well-understood difficulties with truncating the effective Hamiltonian. Hopefully this can be corrected for in the near future.

Acknowledgements

I thank Jos Vermaseren for making his symbolic manipulation programme FORM available to me and Chris Michael for supplying not yet published numbers of the Monte Carlo data for a $4^3 \times 99$ lattice at $\beta = 3.0$.

Table I

$$\begin{aligned}
\gamma_1 &= -0.30104661 & \alpha_1 &= 2.1810429 \cdot 10^{-2} \\
\gamma_2 &= -1.4488847 \cdot 10^{-3} & \alpha_2 &= 7.5714590 \cdot 10^{-3} \\
\gamma_3 &= 1.2790086 \cdot 10^{-2} & \alpha_3 &= 1.1130266 \cdot 10^{-4} \\
\gamma_4 &= 4.9676959 \cdot 10^{-5} & \alpha_4 &= -2.1475176 \cdot 10^{-4} \\
\gamma_5 &= -5.5172502 \cdot 10^{-5} & \alpha_5 &= -1.2775652 \cdot 10^{-3} \\
\gamma_6 &= -1.2423581 \cdot 10^{-3} & &
\end{aligned}$$

Coefficients for the continuum effective Hamiltonian (eq. (10)).

Table II

	Monte Carlo [1]				Hamiltonian $m_R/m_{A_1^+}$					
R	β	$N^3 \times N_o$	$z_{A_1^+}$	$m_R/m_{A_1^+}$	4^3	$*4^3$	6^3	$*6^3$	∞^3	$*\infty^3$
T_{11}^+	3.0	$4^3 \times 99$	2.00	0.67 ± 0.02	0.66	0.65	0.61	0.60	0.60	0.58
	2.4	$4^3 \times 32$	3.84	0.53 ± 0.01	0.54	0.53	0.52	0.51	0.51	0.50
T_2^+	3.0	$4^3 \times 99$	2.00	1.60 ± 0.08	1.58	1.60	1.67	1.69	1.70	1.73
	2.4	$4^3 \times 32$	3.84	1.67 ± 0.06	1.66	1.68	1.70	1.72	1.71	1.73
	2.5	$6^3 \times 32$	4.20	1.64 ± 0.05	1.67	1.68	1.70	1.71	1.71	1.73
	2.4	$6^3 \times 32$	5.28	1.63 ± 0.04	1.68	1.69	1.70	1.71	1.71	1.72

Comparison of the Monte Carlo results of ref. [1] with our results from the effective Hamiltonian for the glueball state T_2^+ and the state T_{11}^+ which has two units of electric flux. The $*$ for the Hamiltonian results indicate the use of the two-loop contribution discussed in the text. The value of $z_{A_1^+}$ was used as input for the Hamiltonian calculation.

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Figure Captions

Fig. 1: The ratio of the coefficients of the effective action (eq. (10)), as a function of N, with their value in the continuum. The lines are drawn to guide the eye and $1/N^2$ is plotted on a linear scale to indicate the way the scaling limit is reached as a function of N.

Fig. 2: Comparison of the Monte Carlo results (a from ref. [2] and b from ref. [1]) with the Hamiltonian results. The full curves give for a 4^3, 6^3 and ∞^3 spatial volume the results from the effective Hamiltonian (for 8^3 the curve lies midway between those for $N = 6$ and $N = \infty$). The dashed curves include the continuum two-loop correction. Horizontally is plotted $z_{E^+} = N \cdot m_{E^+}$ and vertically the mass ratio $\sqrt{\kappa}/m_{E^+}$, where κ is the finite volume string tension, i.e. the energy of one unit of electric flux divided by N.

Fig. 3: As for Fig. 2, but now for the mass ratio $m_{A_1^+}/m_{E^+}$. Here the dashed curves would almost coincide with the full curves and are consequently not drawn.

Fig. 4: Various results for the spectrum in an asymmetric volume with asymmetry $z_T = 1.5$. The Monte Carlo data [2] are for a lattice of spatial size $6 \times 6 \times 4$ and the curves d are results from the effective lattice Hamiltonian with the same size. Curves a, b and c are results from the continuum effective Hamiltonian with an asymmetry of 1.5, where a corresponds to the truncation of the vacuum-valley potential to sixth and the transverse potential to fourth order, b includes the sixth-order terms of the transverse potential and c on top of that incorporates the eighth-order terms of the vacuum-valley potential. The upper part of the figure and the square data points correspond to $\sqrt{\kappa_s}/m$, the lower part of the figure and the circular data points correspond to $\sqrt{\kappa_t}/m$, where κ_s is the string tension in the long and κ_t the string tension in the short direction. Horizontally is plotted $z = N_3 \cdot m$.