UNDERSTANDING SU(2) LATTICE ARTIFACTS IN INTERMEDIATE VOLUMES

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We will discuss the results of the semi-analytic calculation of the low-lying spectrum for pure SU(2) gauge theory in an intermediate volume with periodic boundary conditions as obtained from the Wilson lattice action. The results are for an arbitrary number of lattice sites in the three spatial directions and allows us to show what the influence of lattice artifacts on the glueball spectrum amounts to. The close agreement found between these semi-analytic calculations and the full lattice Monte Carlo simulations on lattices of spatial sizes 4^3 through 10^3 indirectly proves the accuracy of an adiabatic-type approximation previously employed in continuum calculations. The continuum results are actually reproduced as the scaling limit (the number of lattice sites in each direction to infinity) of these lattice results. These results also unambiguously demonstrate the almost "preposterous" statement that at the 10% level a 4^3 lattice is able to describe continuum physics in an intermediate volume.

1. INTRODUCTION

Progress in physics would probably be much slower if we wouldn't occasionally be blind for something which seems obvious. Expecting that a lattice of 4 to 6 sites in the space directions can give a good approximation of the low-lying spectrum for pure SU(2) gauge theories in volumes up to a size of 5 scalar glueball correlation lengths is maybe such an example, since in that case the correlation length is of the order of the cutoff and one should have anticipated large lattice artifacts. The accuracy of the Monte Carlo data for the intermediate volume calculations[1, 2] has improved to such an extent that one started to see systematic deviations between the lattice and the continuum results, especially in the smaller volume range, where the continuum approach is supposedly under control[3]. Nevertheless, the latter needs what was phrased as an adiabatic approximation, whose quality could not be quantified in all rigour. Thus the aim of the study of the lattice artifacts was twofold: understanding why the lattice artifacts are so small and testing the validity of this approximation.

It is maybe useful to recall here the motivation and the aim of the finite volume expansion which originated in work by 't Hooft[4] and Lüscher[5]. The finite volume serves both as an infrared cutoff (nevertheless retaining the zero-momentum modes) and as a control parameter for changing the strength of the interaction. Such a control parameter is lacking in an infinite volume, due to the classical scale invariance of the theory. In a finite volume, the renormalisation of the coupling constant relates the size of the volume to the strength of the interactions through the use of the renormalisation group, and asymptotic freedom allows us to do perturbation theory in sufficiently small volumes. In this perturbative expansion, obviously zero-momentum modes will dominate, but it is crucial that they remain interacting to lowest non-trivial order[6].

The aim of the finite volume expansion is to try to incorporate increasingly more complicated dynamical effects, which become important for larger volumes (i.e. stronger interactions). What essentially happens is that the wave functional starts to spread out over configuration space and sees more of the non-trivial topology of this configuration space and of the structure of the potential energy. Incorporating one of these effects, which can be largely (but not completely) formulated within the zero-momentum sector, allows one to understand the origin of the energy of electric flux and the low-lying glueball masses in volumes of up to 0.7 fermi (5 times the correlation length of the scalar glueball)[3]. The next and hopefully final step would be to incorporate effects that are responsible for the energy of the theta-vacua.

That this seems to become relevant only at volumes bigger than 0.7 fermi is indicated by the volume dependence of the topological susceptibility[7].

In the remainder we discuss a few of the ingredients of the analytic derivation of the effective Hamiltonian for the zero-momentum modes. By taking the Wilson lattice action as our starting point, we will include all the lattice artifacts. For more details and results we refer to[8] and a forthcoming publication. For more on the finite volume expansion see Michael's review in this volume.

2. THE LATTICE EFFECTIVE ACTION

As in the continuum we can calculate the effective action for the zero-momentum modes by integrating out the non-zero momentum modes. On the lattice they arise in the following expansion: $U_{x,x+\hat{\mu}} = \exp(ic_\mu(t)/N_\mu)\exp(iq_\mu(x)/N_\mu)$. The constraint of zero-momentum is most easily implemented on the lattice using a Fourier expansion $q_\mu^{(n)}(x+\hat{\nu}) = \exp(2\pi i n_\nu/N_\nu)q_\mu^{(n)}(x)$, with n_ν restricted to the Brillouin-zone (i.e. $n_\nu = 0,1,\cdots,N_\nu-1$) and $\mathbf{n}\neq\mathbf{0}$. The path integral represents the trace of the transfer-matrix to the power N_0 and the logarithm of this transfer-matrix is what we want to extract, giving us the lattice effective Hamiltonian for the zero-momentum modes. For the ease of the calculations we let the time direction extend to infinity.

First one derives the effective action, which depends only on the zero-momentum modes c_i (choosing the gauge $c_0=0$) and which still has time discrete. This, in principle, does not differ from what is done in the continuum calculation, except that for the lattice action the coefficients in the expansion in powers of the quantum fields contain arbitrary powers of the background field $\mathbf{c}$. For convenience we restrict ourselves here to a cubic spatial lattice, for which the result can be written as:

$$S_{eff}(c) = (1+g_0^2\hat{\alpha}_1(N))S_k + S_p + \sum_t\{V_\ell(c_i(t);N)+V_T(c_i(t);N)\}+\cdots \tag{2.1}$$

where S_k+S_p is the tree-level action, split in a kinetic and potential part. Explicitly, in terms of the background-field link variables $U_i^{(o)}(t) = \exp(ic_i(t)/N_i)$:

$$S_k = \frac{2N^3}{g_o^2}\sum_{t,i}\mathrm{Tr}(1-U_i^{(o)}(t+1)U_i^{(o)\dagger}(t)) \tag{2.2}$$

$$S_p = \frac{N^3}{g_o^2}\sum_{t,ij}\mathrm{Tr}(1-U_i^{(o)}(t)U_j^{(o)}(t)U_i^{(o)\dagger}(t)U_j^{(o)\dagger}(t)) \tag{2.3}$$

whereas the vacuum-valley and transverse potentials (resp. V_ℓ and V_T) are given by:

$$V_\ell(c;N) = \frac{1}{N}\{\sum_{i=1}^{3}[\gamma_1(N)r_i^2+\gamma_2(N)r_i^4+\gamma_4(N)r_i^6] + \sum_{i>j}\gamma_3(N)r_i^2r_j^2+\sum_{i\neq j}\gamma_5(N)r_i^2r_j^4+\gamma_6(N)\prod_{i=1}^{3}r_i^2+\cdots\}$$

$$V_T(c;N) = \frac{1}{N}\{\sum_{ij}\frac{1}{4}\hat{\alpha}_2(N)F_{ij}^2+\sum_{ijk}\alpha_3(N)r_k^2F_{ij}^2 + \sum_{(ij)}\alpha_4(N)r_i^2F_{ij}^2+\alpha_5(N)\mathrm{det}^2c+\cdots\} \tag{2.4}$$

with $F_{ij}^a = -\varepsilon_{abd}c_i^bc_j^d$ and $r_i^2 = 2\mathrm{Tr}(c_i^2)$.

3. RELATING Λ_L AND Λ_{MS}

In all cases except for $\hat{\alpha}_1$ and $\hat{\alpha}_2$, the continuum values are the respective scaling limits ($N\to\infty$) of $\alpha_i(N)$ and $\gamma_i(N)$. We also note that in lowest order in $1/N$ the tree-level potential term is given by $F_{ij}^2/(4Ng_o^2)$. Hence, to one-loop order $\hat{\alpha}_{1,2}(N)$ occur in the combination $1/g_o^2+\hat{\alpha}_{1,2}(N)$, which will therefore determine the renormalisation of the coupling constant, necessary due to the logarithmic divergence of $\hat{\alpha}_{1,2}$. We define

$$\tilde{\alpha}_{1,2}(N) = \hat{\alpha}_{1,2}(N)+\frac{11}{12\pi^2}\ln(N) \tag{3.1}$$

where $\tilde{\alpha}_{1,2}(N)$ has a finite scaling limit as can be seen in table 1. As usual, one absorbs the divergence in the renormalised coupling constant

$$1/g_R^2 = 1/g_o^2 - 11\ln(N)/12\pi^2 \tag{3.2}$$

It is worthwhile to point out that it is not necessary to introduce the lattice spacing a, as long as we are only interested in scale invariant quantities, like $z=mN$, where m is some mass in lattice units. The scaling limit is determined by keeping g_R fixed and letting N go to infinity. At fixed coupling we want

Table 1: The constants $\tilde{\alpha}_1(N)$ and $\tilde{\alpha}_2(N)$ for various cubic lattices of N sites. The scaling limit ($N \to \infty$) can be used to related Λ_L to Λ_{MS}, using the values of $\alpha_1 = 2.1810429 \times 10^{-2}$ and $\alpha_2 = 7.5714590 \times 10^{-3}$, which were computed with the dimensional regularisation in the minimal subtraction scheme.

N	$\tilde{\alpha}_1$	$\tilde{\alpha}_2$
2	-0.1520446897	-0.1489339448
4	-0.1706212159	-0.1631776638
8	-0.1671141882	-0.1741754180
16	-0.1654821183	-0.1778324170
32	-0.1650271986	-0.1787866263
64	-0.1649090195	-0.1790274104
128	-0.1648789899	-0.1790877426

the physical size $L = aN$ to remain fixed, which means that a scales as $1/N$, and therefore depends on the bare coupling constant g_o as dictated by the renormalisation group.

One *defines* Λ_L to be the constant of integration for the renormalisation group, which to one-loop order amounts to $1/g_o^2 = -11\ln(a\Lambda_L)/12\pi^2$. This allows us to relate Λ_L to Λ_{MS} in the finite volume expansion, because the scaling limit of the lattice effective action should be the same as the effective action derived in the continuum with dimensional regularisation in the minimal subtraction scheme. This is the case when

$$\frac{1}{g_R^2} + \tilde{\alpha}_{1,2}(\infty) = -\frac{11}{12\pi^2}\ln(L\Lambda_{MS}) + \alpha_{1,2} \quad (3.3)$$

which implies the relation

$$\Lambda_L/\Lambda_{MS} = \exp\{12\pi^2(\tilde{\alpha}_{1,2}(\infty) - \alpha_{1,2})/11\} \quad (3.4)$$

From table 1 we see that $\Lambda_L/\Lambda_{MS} = 0.1339958(2)$, and it is important that the two ways of calculating this ratio give the same result within the error (due to extrapolating $N \to \infty$). Important is also that this ratio agrees with the previous determination of this ratio[9] based on an infinite volume calculation (reading the literature one has to remember that Dashen&Gross corrected in their published paper an error detected by Weisz, but they were left over with an error in the calculation with the Pauli-Villars regulator which, however, did not affect Λ_L/Λ_{MS}).

4. FROM TRANSFER MATRIX TO EFFECTIVE HAMILTONIAN

We will now return to a finite number of lattice sites, since here we are particularly interested in the size of lattice artifacts. As remarked before, the effective action will still have time discrete and we can therefore interpret the effective theory in terms of a transfer matrix for a one-point lattice. To derive this transfer matrix $\mathcal{T} = \exp(-\mathcal{H})$, we have to convert the path integral representation to an operator representation. As usual one introduces the position and momentum eigenstates (here for $SU(2) \sim S^3$, which can be viewed as R^4 with the constraint that the radial distance is fixed and the radial momentum is zero). Then one writes the path integral in terms of products of matrix elements of the evolution operator, however, without taking the time step infinitesimal. Nevertheless, provided the action splits in purely kinetic ($\dot{x}_i^2$) and potential ($V(x)$) terms, the relevant matrix element can easily be seen to be given by $< p|e^{-K}e^{-V}|x >$. Here K is the kinetic operator, for which $|p>$ is an eigenstate (after imposing the constraints which reduce R^4 to S^3 this becomes the covariant Laplacian on S^3). We therefore deduce that the partition function Z can be written as $Z = \mathrm{Tr}(\{e^{-K}e^{-V}\}^{N_o})$, which by definition is also given by $Z = \mathrm{Tr}(\mathcal{T}^{N_o}) = \mathrm{Tr}(e^{-N_o\mathcal{H}})$. One therefore finds:

$$\mathcal{H} = \ln[\exp(-K/2)\exp(-V)\exp(-K/2)] \quad (4.1)$$

which is, however, only unique modulo unitary transformations. It is amusing to check this formula in the case of a harmonic oscillator ($V = \frac{1}{2}\omega^2 x^2$, $K = -\frac{1}{2}d^2/dx^2$), for which both the finite time step path integral and eq. (4.1) can be calculated exactly (the resulting spectrum is again harmonic, but with a frequency $2\mathrm{asinh}(\omega/2)$). The effective Hamiltonian is now obtained by expanding eq. (4.1) to fifth order in K and V (which are both of order $1/N$), and converting the kinetic term K for S^3, by a rescaling of the wave function, to the kinetic term of R^3.

Figure 1: Comparison of the Monte Carlo results on $4^3 \times N_o$ (dots) and on $6^3 \times N_o$ (squares) lattices[2] with the semi-analytic results for the mass ratio $m_{A_1^+}/m_{E^+}$ as a function of $z = m_{E^+}N$, indicated by the curves labelled by $N = 4$, 6 and ∞, where $N = \infty$ corresponds to the continuum limit.

5. COMPARISON WITH MONTE CARLO DATA

One now has the effective Hamiltonian in precisely the same form as for the continuum, but with quite a few additional terms that are important for finite N and which therefore carry the information of the lattice artifacts. The spectrum, including the non-perturbative effects, encoded in the boundary conditions for the wave functions, is determined as in the continuum[3]. In figure 1 we compare our results for the mass ratio $m_{A_1^+}/m_{E^+}$ as a function of $z = m_{E^+}N$ at $N = 4$, 6 and ∞ with the Monte Carlo data obtained using "fuzzed" operators[2], which are efficient in removing spurious couplings to higher lying states[1].

6. CONCLUSION

From the excellent agreement between the Monte Carlo and the semi-analytic results we can say with much more confidence than before that $N = 4$ is not too far from $N = \infty$, but it seems somewhat accidental (due to cancellations of different types of corrections). We are now also confident that the approximations made in the analytic calculations are accurate. Finally it is gratifying to see that the pure gauge Monte Carlo calculations can achieve the accuracy and reliability necessary for playing the games we played.

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