ANALYTIC METHODS FROM FINITE TO INFINITE VOLUMES

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We shall review the analytic methods that have been developed over the past few years to understand the volume dependence of field theories on a torus, which is the geometry relevant for lattice gauge theories. First we start in small to intermediate volumes, below one cubic fermi, and discuss the results for the low-lying spectrum of pure SU(3) gauge theory. Then we assume a mass gap is formed in the infinite volume and show how this leads to a precise asymptotic expression for the volume dependence of the stable one-particle masses, due to polarization effects and for the two-particle masses, due to scattering processes in the finite volume. The latter is first illustrated in the two dimensional O(N) model, using the exact S-matrix. The generalization to four (or any) dimensions for the relation between the two-particle masses in a finite volume and the scattering phase shifts, based on a very recent analysis by Lüscher, will be discussed. In the near future this will make a careful study of resonances in a finite volume feasible. Finally we discuss how the Bethe Ansatz for the two dimensional O(N) model can be used to calculate the infinite volume mass gap, providing an important test for lattice Monte Carlo studies to extract this quantity.

1. INTRODUCTION

This review will discuss the volume dependence in field theories. The geometry of the finite volume will be that of a torus, which is relevant for almost all lattice Monte Carlo studies. As lattice artifacts are unphysical most of the analysis will be described in the continuum. It should also be needless to point out that the only way one will be able to reliably predict infinite volume quantities, like the mass-spectrum, is to match the volume dependence to analytic predictions that are parametrized by infinite volume quantities.

In sect. 2 we will first review the analytic work of Vohwinkel[1] for the low-lying mass-spectrum in SU(3) pure gauge theory. It is through the comparison between the continuum and lattice calculations in volumes up to one cubic volume for mass-ratios as a function of L, that one has been able to find a good quantitative measure for the size of lattice artifacts. Especially for SU(2) gauge theory this has allowed even for an analytic control over the lattice artifacts[2].

The remainder of this review will be concerned with the large volume domain. There are basically two approaches to study the large volume asymptotics of field theories. One is based on chiral perturbation theory and was reviewed by Leutwyler three years ago[3]. The Goldstone bosons (associated to a spontaneously broken symmetry) will strongly dominate the low-energy properties of the theory. These can be captured in an effective Lagrangian, restricted by the symmetries. Relatively few parameters will thus determine the leading finite volume dependence. An example is of course the non-linear sigma-model, both as an effective description of the pions in QCD and as a model for a heavy Higgs particle in the standard model. Especially in the context of the latter, chiral perturbation theory has been used rather successfully in recent years. We refer to[4] for the most recent analytic results and for further references, including references that compare with lattice Monte Carlo results.

Here we will, however, emphasize techniques that have a wider range of applicability. The only assumption is that the theory has a non-vanishing mass gap. Sect.3 will review Lüscher's analysis[5] for the volume dependence of stable one-particle masses. The volume dependence here arises due to polarization effects. This is typically the formula one will want to employ to extrapolate (necessarily) finite volume lattice Monte Carlo estimates to their infinite volume values for, say the glueball mass in pure gauge theory. We will also review Neuberger's

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analysis[6] of the volume dependence for the one-particle irreducible vertex functions at zero external momenta, which can be analysed with almost identical techniques and has important implications for the analysis of the Higgs-model in the unbroken phase.

Sect.4 will discuss the volume dependence for two-particle masses. If one neglects the polarization effects, which are exponential in L, the volume dependence here arises because one particle can propagate around the volume and scatter off the other particle, which in leading order will give a $1/L^3$ correction on the two-particle mass, as was first conjectured correctly by Parisi e.a.[7]. With techniques similar to those developed for the one-particle masses Lüscher had previously[8] developed an expansion in powers of $1/L$. However, very recently[9] he found a way to extract a relation between the two-particle masses in a finite volume and the phase-shifts to all orders in $1/L$. This result is quite similar to what can be derived in two dimensions (one space and one time dimension), to be illustrated for the O(N) model. We will give a very short review of the exact S-matrix found with the bootstrap method by A.B. and Al.B. Zamolodchikov[10]. The S-matrix was verified by measuring the two-particle masses in a finite volume[11], illustrating the power of the formalism. We then derive in a simple setting the all order result in four dimensions, using a technique different from (but related to) Lüscher's[9] analysis. This technique, based on the pseudo-potentials, was used by Huang and Yang[12] to derive the volume dependence for the groundstate energy of two hard spheres.

Sect.5 will describe a method, originally proposed in work by Polyakov and Wiegmann[13] and recently successfully employed by Hasenfratz, Maggiore and Niedermayer[14] to calculate the mass gap for the O(N) model in two dimensions, directly in an infinite volume. This method is based on the Bethe Ansatz, and we will discuss some of the intricate issues, related to the validity of the Bethe Ansatz for the O(N) model on one hand and the analysis of a highly singular integral equation on the other hand. We also review the present status of the Monte Carlo results, especially based on simulations by Wolff[15] using the cluster algorithm and the large-N expansion[16, 17].

2. SU(3) PURE GAUGE SPECTRUM IN INTERMEDIATE VOLUMES

One defines a gauge theory on the torus by imposing periodic boundary conditions with period L in each of the three spatial directions. The principle of the calculation of the low-lying spectrum in small and intermediate volumes is based on deriving an effective Hamiltonian[18] for the zero-momentum gauge fields ($A_i^a(\mathbf{x}) = c_i^a/L$), by integrating out the non-zero momentum modes in perturbation theory.

$$\mathcal{H}_{eff}(c) = -\frac{g^2}{2L}\frac{\partial^2}{\partial c_i^{a2}} + \frac{1}{4g^2L}(F_{ij}^a)^2 + V_1(c)$$
$$V_1(c) = \frac{1}{L}\sum_{\mathbf{n}\neq\mathbf{0}} \mathrm{Tr}_{\mathrm{ad}}\sqrt{(2\pi\mathbf{n}+\mathbf{c}^a T_a)^2}\ . \quad (2.1)$$

Here, T_a are the generators and $F_{ij}^a = -c_i^b c_j^d f_{abd}$ is the field strength in terms of the zero-momentum gauge field and the structure constants f_{abc}. Furthermore, g is the renormalized coupling constant at the scale $\mu = 1/L$. The effective potential $V_1(c)$, being invariant under constant gauge transformations, only depends on the Casimir invariants

$$r_i^2 = 2\mathrm{Tr}[(c_i^a T_a)^2], \quad s_i = 4\mathrm{Tr}[(c_i^a T_a)^3], \cdots, \quad (2.2)$$

where the sum over the colour indices is implicit. There are as many independent Casimir invariants as the rank of the gauge group. For SU(2) only r_i will be non-trivial. These coordinates are uniquely related to those obtained by restricting the zero-momentum gauge field to be abelian. For SU(2), $c_i^a T_a = X_i\sigma_3/2$ yields $r_i^2 = X_i^2$ whereas for SU(3), $c_i^a T_a = (X_i\lambda_8 + Y_i\lambda_3)/2$ gives $r_i^2 = X_i^2 + Y_i^2$ and $s_i = X_i\sqrt{3}(Y_i^2 - X_i^2/3)$. It implies that if we know the effective potential restricted to the abelian configurations there is a unique and minimal way to extend it to all constant gauge fields. In this way one can show that eq. (2.1) is accurate to O($g^{8/3}$) (using that c is O($g^{2/3}$)). To this order the calculated spectrum[19] is valid for $L \ll 0.1 fm$ and the wavefunction is localized around $c = 0$.

Bij adding the zero-momentum ($\mathbf{n} = \mathbf{0}$) contribution to the one-loop effective potential one gets

$$\hat{V}_1(c) = V_1(c) + L^{-1}\mathrm{Tr}_{\mathrm{ad}}\sqrt{(\mathbf{c}^a T_a)^2}\ , \quad (2.3)$$

whose symmetries are a consequence of the gauge invariance. A crucial role is played by the twisted gauge transformations $g_{\mathbf{k}}(\mathbf{x})$ introduced by 't Hooft[20] and which are only periodic up to an element of the center of the gauge group. In this case Gauss's law does not guarantee invariance of the wavefunctional. Instead, for SU(N)

$$\Psi([g_{\mathbf{k}}]A) = \exp(\frac{2\pi i}{N}\mathbf{e}\cdot\mathbf{k})\Psi(A), \tag{2.4}$$

where $\mathbf{e}$ (defined modulo N) is the gauge invariant definition of electric flux. For SU(N) the center Z_N is generated by $\exp(2\pi i \mathbb{T})$ (for SU(2), $\mathbb{T} = \sigma_3/2$ and for SU(3), $\mathbb{T} = \lambda_8/\sqrt{3}$). Explicitly one has

$$g_{\mathbf{k}}(\mathbf{x}) = \exp(\frac{2\pi i \mathbf{k}\cdot\mathbf{x}}{L}\mathbb{T}). \tag{2.5}$$

This gauge transformation, when applied to an abelian zero-momentum gauge field (as specified above) will amount to a shift in the X_i coordinates. For SU(2) $\mathbf{X} \to \mathbf{X} + 2\pi\mathbf{k}$ and for SU(3) $\mathbf{X} \to \mathbf{X} + 4\pi\mathbf{k}/\sqrt{3}$. Furthermore, $\hat{V}_1$ is invariant under constant gauge transformations, which when expressed in terms of the abelian coordinates amounts to the invariance under the Weyl group (the subgroup that leaves the set of abelian generators invariant). Wavefunctionals are of course invariant under these constant gauge transformations. For SU(3), figure 1 gives a cross section for the effective potential $\hat{V}_1$. The hexagonal symmetry is related to the Weyl invariance (which for SU(3) is isomorphic to the permutation group S_3) and the periodic structure is due to the invariance under twisted gauge transformations. For SU(2) the Weyl group is isomorphic to the permutation group S_2, which coincides with parity, restricted to the abelian zero-momentum gauge fields ($X \to -X$).

To implement these symmetries on the wavefunctional we perform a change of coordinates $c_i^a \to (X_i, Y_i, \Omega_i)$, where Ω_i stands for the collection of SU(N)-angular coordinates and (X_i, Y_i) are restricted to a fundamental Weyl chamber, where the relation between (X,Y) and (r,s) is one to one. For SU(2) these new coordinates are the spherical coordinates. The Jocobian, $J^2 = \prod_i J_i^2$, of this transformation is given by $J_i = X_i$ for SU(2) and $J_i = Y_i(Y_i^2 - X_i^2)$ for SU(3). The wavefunctions can be decomposed as

$$\Psi(c) = \prod_i J_i^{-1}\chi_i(X_i, Y_i)\mathcal{Y}_i(\Omega_i). \tag{2.6}$$

Figure 1: Two dimensional cross-section of $\hat{V}_1$, for $c_1^a T_a = X\lambda_8 + Y\lambda_3$ and $c_{2,3} = 0$. The circles indicate the equipotential lines. The dashed hexagon represents the unit cell, the fat hexagon the Gribov horizon and the shaded area is the fundamental Weyl chamber.

The "radial" part χ will be antisymmetric with respect to Weyl reflections, so as to cancel the zero's of the Jacobian. For SU(2) the angular wavefunctions $\mathcal{Y}(\Omega)$ are nothing but the spherical harmonics, whereas in general they are irreducible representations of the gauge group.

We note that a combination of the twisted gauge transformation and a constant gauge transformation leaves invariant the line $X = \pi$ for SU(2) and $X = 2\pi/\sqrt{3}$ for SU(3). This implies that alternatively the properties of χ can be described in terms of boundary conditions at this line. Applying a Weyl transformation leaves the wavefunction invariant and maps this line into another line where the wavefunction will satisfy the same boundary conditions. In this way one obtains the unit cell indicated by the dashed line for SU(3) in figure 1 (or for SU(2) defined by $r_i \leq \pi$), to which the theory can be restricted by imposing appropriate boundary conditions. This restriction to one unit cell is quite essential if one realizes that the effective Hamiltonian of eq. (2.1) can not be extended beyond the Gribov horizon, which is indicated by the fat hexagon indicated in figure 1. For more details on this we refer to[21]. Carefully working out the consequence of the symmetries one can show[1, 21] that a complete basis for χ in the case

Figure 2: For SU(3) pure gauge theory the mass ratios $m(A_1^{++})/m(E^{++})$ and $\sqrt{\epsilon}/m(E^{++})$ are given as a function of $z = m(E^{++})L$. The masses are labeled by the representations of the cubic group and ϵ is the string tension (energy of electric flux per unit length). The Monte Carlo data is from[23], the drawn curve below $z = 1$ from[19] and above from[1].

of SU(2) is given by $\chi(X) = \sin(nX/2)$, whereas for SU(3) one finds

$$\begin{aligned}\chi(X,Y) &= \sin(mY/2)e^{inX\sqrt{3}/2} \\ &+ \sin(m(\sqrt{3}X - Y)/4)e^{in\sqrt{3}(\sqrt{3}X+Y)/2} \\ &- \sin(m(\sqrt{3}X + Y)/4)e^{in\sqrt{3}(\sqrt{3}X-Y)/2},\end{aligned} \tag{2.7}$$

for each of the three coordinate directions. The quantum numbers n and m will be restricted by the electric flux and irreducible representations of the Weyl and cubic groups. Finally if one restricts $\prod_i \mathcal{Y}_i$ to transform as a singlet under SU(N) one obtains a complete and gauge invariant basis for the effective Hamiltonian, that through the boundary conditions carries the information of electric flux. It is clear that for SU(3)[1] the computations of all the relevant matrix elements is rather more cumbersome than the corresponding calculations in SU(2)[22]. However, once the matrix of the Hamiltonian for this basis is computed one performs a simple Rayleigh-Ritz analysis to determine the spectrum. The intermediate volume range sets in when tunnelling trough the quantum induced barriers of $\hat{V}_1$ becomes frequent (below that, the results of ref.[19] are valid and the energy of electric flux is exponentially suppressed). Figure 2 (from[1]) compares Monte Carlo data[23] (for a more complete presentation see[24]) with Vohwinkel's[1] analytic results.

We conclude this section by mentioning that alternative boundary conditions have been used to calculate the spectrum in a small volume. Notably twisted boundary conditions, reviewed two years ago[25], is of continued interest[26]. Very recently Kronfeld and Wiese[27] have devised so-called C-periodic boundary conditions, which for SU(3) provides an interesting alternative over periodic and twisted boundary conditions.

3. LARGE VOLUME EXPANSION FOR STABLE PARTICLE MASSES

The method of extracting the volume dependence at large L is based on universal properties of *euclidean* one-particle irreducible vertex functions in massive field theories. The expression of a Feynman diagram in a finite volume is exactly of the same form as in an infinite volume, except that integrating over the vertex positions is restricted to the finite volume and the infinite volume propagator $\Delta(x)$ is replaced by the finite volume propagator $\Delta_L(x) = \sum_{n\in\mathbb{Z}^{d+1}} \Delta(x + nL)$. The latter sum will converge for massive propagators $\Delta(x) \sim e^{-m|x|}$ (d is the space-dimension). Physically $\Delta(x + nL)$ can be identified with propagation from $x = 0$ to x, going in addition n times around the "universe". Going only once around the "universe" should thus give the leading finite volume correction of the order e^{-ML}, where M is a suitable mass-parameter.

A general Feynman diagram $\mathcal{D}$ will correspond to the amplitude

$$\begin{aligned}\mathcal{J}_L(\mathcal{D}) = \prod_{v\in\mathcal{V}\setminus v_0}\prod_{\mu=0}^{d}\int_0^{L_\mu} \mathrm{d}x_\mu(v)\exp(i\sum_e p(e)\cdot x(e)) \\ \times\mathbf{V}\prod_{\ell\in\mathcal{L}}\Delta_L(x(f(\ell)) - x(i(\ell))),\end{aligned} \tag{3.1}$$

where $p(e)$ are external momenta, v are vertices, ℓ propagators, $\mathcal{V}$ and $\mathcal{L}$ the sets of all vertices and propagators, v_0 is a vertex connected to an external line and $\mathbf{V}$ is the product of all vertex factors (coupling constants). To define precisely what is meant by propagating once around the "universe" (that this requires care is obvious when, say $x_1 = L_1/2$) it is useful to define the $\mathbb{Z}^{d+1}$ gauge invariance $x(v) \to x(v) + m(v)L,\quad n(\ell) \to n(\ell) + m(f(\ell)) - m(i(\ell))$, where $m_\mu(v) \in \mathbb{Z}$ and $n(\ell)$ occurs in the expansion of $\Delta_L(x)$. With graph theory[5] one can show that the sum over $n(\ell)$ splits in a sum over the gauge orbits $[n]$ and a sum over the integers $m(v)$. Obviously one has $\sum_m \int_0^L dx = \int_{-\infty}^{\infty} dx$ and only the sum over the gauge orbits remains, $\mathcal{J}_L(\mathcal{D}) = \sum_{[n]} \mathcal{J}_L(\mathcal{D},[n])$,

$$\mathcal{J}_L(\mathcal{D},[n]) = \prod_{v\in\mathcal{V}\backslash v_0} \int \mathrm{d}x^{d+1}(v) \exp(i\sum_e p(e)\cdot x(e)) \times \mathbf{V} \prod_{\ell\in\mathcal{L}} \Delta(x(f(\ell)) - x(i(\ell)) + n(\ell)L). \quad (3.2)$$

The advantage of this decomposition is clearly that for the pure gauge orbit $[n]=[0]$, this is precisely the infinite volume expression, i.e. $\mathcal{J}_L(\mathcal{D},[0]) = \mathcal{J}_\infty(\mathcal{D})$. One can prove[5] that

$$\mathcal{J}_L(\mathcal{D},[n]) \sim \exp[-mL\kappa(\{p(e)\})W([n])], \quad (3.3)$$

where $W([0])=0$ (corresponding to a pure gauge), $W([1])=1$ for a "simple" orbit $[1]$ (where $|n(\ell)|=1$, for one and 0 for all other links) and $W([n])\geq\sqrt{2}$ for all other cases. Two particular cases of interest will be where $\mathcal{D}$ is a self-energy graph with the external momenta on-shell $(p_k(e)=0, \quad p_0(e)=im)$ for which[5] $\kappa(\{p(e)\})=\frac{\sqrt{3}}{2}$ and the case where $\mathcal{D}$ is a 1PI vertex function with vanishing external momenta for which[6] $\kappa(\{p(e)\})=1$. The leading finite volume correction is clearly given by the contribution due to the simple gauge orbit $[1]$, which can be analysed using the Schwinger-Dyson equation, graphically represented by figure 3.

The 1PI vertex functions that appear on the right hand side of the Schwinger-Dyson equation are strictly infinite volume vertex functions and general theorems allow one to extract the domain of analyticity. Sufficient for our purposes will be that $\Gamma^{(3)}(p,-q-p/2,q-p/2)$ and $\Gamma^{(4)}(p,q,-p,-q)$ are analytic for $(\mathrm{Im}p_\mu \pm \mathrm{Im}q_\mu)^2 < 4m^2$, whereas any n-point function $\Gamma^{(n)}(0,\cdots,0,q,-q)$ is analytic for $(\mathrm{Im}q_\mu)^2 < 4m^2$. Let us first analyse the finite volume corrections for the lightest stable one-particle state. In that one case takes $L_0=\infty$ and $L_i=L, \quad \forall i$. The mass is measured using $<\phi(t)\phi(0)>_L = \exp(-M_L t)$, which coincides with the pole in the finite volume two-point function, $G_L^{-1}(iM_L,0)=0$. The two-point function $G_L(p)$ is defined by $G_L^{-1}(p)=p^2+m^2-\Sigma_L(p)$, where the normalization of the self-energy Σ is such that

Figure 3: The Schwinger-Dyson equation for the leading finite volume correction of an 1PI vertex function with m external lines. The full two-point function $G(p)$ is denoted by the open blob, whereas the open blob with the letter L stands for $\sum_{L_\mu<\infty} 2\exp(ip_\mu L_\mu)G(p)$.

$\Sigma_\infty(im,\mathbf{0}) = \frac{\partial^2}{\partial p^2}\Sigma_\infty(im,\mathbf{0})=0$, which implies that in leading order $\delta M_L = -\frac{1}{2m}(\Sigma_L-\Sigma_\infty)(im,\mathbf{0}) = -\frac{1}{4m}\Gamma^{(2)}((im,\mathbf{0}),[1])$. This is expressed in terms of the full four-point Greens function by $\Gamma^{(2)}(p,[1]) = 2d\int\frac{\mathrm{d}^{d+1}q}{(2\pi)^d}e^{iq_1L}G(q)G_E^{(4)}(p,q,-p,-q)$ which is related to the forward scattering amplitude through $F(\frac{p\cdot q}{4m}) \equiv G^{(4)}(p,q,-p,-q)$.

To extract the leading order corrections we can shift the contours of integration[5], using our knowledge of the analyticity domains of the various 1PI vertex functions. For the rhs of figure 3 we can shift the q_1 contour of integration to $q_1+im\sqrt{7}/2$. This is within the analyticity domains of the relevant 1PI functions. For the last two diagrams we only have to keep track of the contribution coming from the meson-pole at $q_1 = i\sqrt{m^2+q_0^2+q_2^2+q_3^2}$ for $q_0^2+q_2^2+q_3^2 \leq 3m^2/4$. This is because the integral over the shifted contour can be neglected w.r.t. to $\exp(-\sqrt{3/2}mL)$, which is the leading order contribution due to the gauge orbits $[n]\neq[0],[1]$ (see above). On the other hand, for the first diagram there are meson poles at $q_1 = i\sqrt{m_2+q_2^2+q_3^2+(q_0\pm im/2)^2}$, which furthermore coincide for $q_0=0$. To deal with this, one first[5] shifts $q_0\to q_0\pm im/2$ and then shifts the q_1 integration contour. The extra contribution from $q_0=0$

corresponds to having both $q-p/2$ and $-q-p/2$ on-shell, such that the residue is proportional to λ^2, where λ equals the 1PI three-point vertex function with all three external momenta on-shell (the physical coupling constant). The remaining single pole contributions combined with the contributions from the last two diagrams in figure 3 can be rewritten in terms of the forward scattering amplitude and one obtains[5] up to $\mathcal{O}(e^{-\sqrt{\frac{3}{2}}mL})$

$$\frac{\delta M_L}{m} = -\frac{d\lambda^2 m^{d-5}}{8\pi}(\frac{4\pi mL}{\sqrt{3}})^{1-d/2}K_{d/2-1}(\frac{\sqrt{3}mL}{2}) - \frac{d}{4m^2}\int\frac{\mathrm{d}^d q}{(2\pi)^d}\frac{\exp(-\sqrt{\mathbf{q}^2+m^2}L)}{\sqrt{\mathbf{q}^2+m^2}}f(iq_1), \quad (3.4)$$

where K_ν is the modified Bessel-funtion. In leading order for three spatial dimensions this amounts to $\delta M_L = -\frac{3\lambda^2}{8\pi L}\exp(-\sqrt{3}mL/2)+\mathcal{O}(e^{-mL})$. For an estimate of the glueball coupling constant in strong coupling see[28].

For a scalar ϕ^4 theory in the unbroken phase all odd-point 1PI vertex functions vanish and in particular $\lambda = 0$, which implies rather complicated finite volume corrections, as they depend in leading order on the analytically continued forward scattering amplitude. Instead Neuberger[6] considered 1PI vertex functions at vanishing external momenta. They are relevant as an alternative to the so-called constrained effective potential for the magnetization, that has a powerlike volume dependence[6]. For zero-momentum 1PI vertex functions the meson poles in the first diagram of figure 3 coincide for all q_0. Now we shift the q_1 contour of integration to $q_1 + im\sqrt{3}$ and the residue of the meson-pole only contributes for $q_0^2+q_2^2+q_3^2 < 2m^2$. At this pole $\Gamma^{(n)}(0,\cdots,0,q,-q)$ is independent of $q_0^2+q_2^2+q_3^2$, such that the first diagram is proportional to a one-loop graph with two mass insertions, whereas the last diagram is proportional to a one-loop graph with only one mass insertion (the second diagram vanishes). Thus, for the symmetric case $L_0 = L_i$ the volume dependence up to $\mathcal{O}(e^{-mL\sqrt{2}})$ is given by

$$\Gamma_L^{(n)}(0)-\Gamma_\infty^{(n)}(0) = m^{4-n}\{\alpha\Delta(mL)+m\beta\frac{d\Delta(mL)}{dm}\}. \quad (3.5)$$

Note that for $d = 3$ (four euclidean dimensions) $\Delta(z) = K_1(z)/z$ and $\Delta'(z) = K_2(z)$. Even the subleading contribution is of a simple form now.

4. VOLUME DEPENDENCE AND SCATTERING PHASE SHIFTS

4.1. Two dimensions

Let us first consider the case of quantum mechanics for two interacting particles. In the center of mass we have the reduced Hamiltonian $H = -m^{-1}\partial^2/\partial x^2 + V(x)$ and the Bose symmetry dictates the reduced wavefuntion to be even $\psi(x) = \psi(-x)$. If the potential has a finite range λ ($V(x) = 0$ for $|x| > \lambda$), then the wavefunction is a plane-wave outside of the interaction region. From scattering theory in 1+1-dimensions one gets a unique relation between the incoming wave $e^{-ip|x|}$ and the outgoing wave $e^{ik|x|}$ in terms of the scattering phase shift $\delta(k)$

$$\psi(x) = e^{-ik|x|} + e^{2i\delta(k)}e^{ik|x|}, \quad |x| > \lambda. \quad (4.1)$$

These states form a complete basis of scattering states. In a finite volume with periodic boundary conditions, $\psi(x+L/2) = \psi(x-L/2)$ implies the following implicit equation for the momenta

$$e^{2i\delta(k)}e^{ikL} = 1, \quad (4.2)$$

which holds as long as $L > 2\lambda$. Indeed, for a vanishing potential (which of course implies a vanishing phase shift, $\delta(k) = 0$) one obtains the standard discretization of the momenta, $k = 2\pi n/L$.

In field theory it can be shown[8, 11] that the reduced Hamiltonian is replaced by an effective Schrödinger equation, that can be derived from the Bethe-Salpeter equation for the four-point function,

$$-\frac{1}{m}\frac{\partial^2\psi(x)}{\partial x^2} + \frac{1}{2}\int \mathrm{d}x' U_E(x,x')\psi(x') = E\psi(x). \quad (4.3)$$

The energy-dependent "potential" $U_E(x,x')$ is proportional to the Bethe-Salpeter kernel, its range is determined by the polarization cloud (which is why we restrict our analysis to field theories with a mass gap) and $E = k^2/m$ (with k the momentum of the two particles in the center of mass). The fully relativistic two-particle energy W is then given by

$$W = 2\sqrt{k^2+m^2} = 2\sqrt{m(m+E)}. \quad (4.4)$$

This generalizes to arbitrary dimensions and allows one to derive a relation between the energy of two-particle states in a finite volume and the scattering phase shifts in the context of non-relativistic quantum mechanics. The obtained relation is universal and extends to the relativistic case using eq. (4.4)

4.2. The two dimensional O(N) model

The two dimensional non-linear σ-model with O(N) symmetry is defined by the action

$$\mathcal{S}=\frac{1}{2g_0}\int \mathrm{d}_2x\partial_\mu n^a(x)\partial^\mu n_a(x), \quad n_a(x)n^a(x)=1. \tag{4.5}$$

Due to an infinite set of conserved charges one can prove[10] that the scattering is purely elastic, that the particle number (per particle type) is conserved and that the set of momenta before and after the scattering is the same ("soliton behaviour"). The particular kinematics in one space-dimension then leads to factorization of the S-matrix

$$S(k_1,k_2,\cdots,k_n)=\prod_{i<j}S(k_i,k_j). \tag{4.6}$$

Using the Bose symmetry, the S-matrix has to be independent of the ordering of the momenta. If the particles have internal degrees of freedom (in this case the O(N) degrees of freedom), $S(k_1,k_2)$ is a matrix and this independence on the ordering imposes a non-trivial constraint, which will be satisfied for an arbitrary number of particles, as soon as it is satisfied for three particles (the latter defines the multiplication rules for a suitable non-commutative algebra[10]). This equation is known as the Yang-Baxter (or sometimes star-triangle) equation. In figure 4 we give the graphical representation of the matrix equation $S_{13}S_{12}S_{23}=S_{23}S_{12}S_{13}$, where the index only indicates the relevant momentum.

Figure 4: The consistency condition for three-particle scattering to be independent of the ordering of the particles (the Yang-Baxter equation). The O(N) indices are suppressed.

Remembering that one cannot have (massless) Goldstone bosons in two dimensions it is natural to assume that the particles will form massive O(N) vectors of mass m. Elastic unitarity, crossing symmetry and analyticity will then almost uniquely determine the scattering matrix. In terms of the rapidity θ defined by $k=k^1=m\sinh(\theta)$ and $W=k^0=m\cosh(\theta)$ one writes the two-particle S-matrix $S(k_1,k_2)$ as $S(\theta_1-\theta_2)$ and for the O(N)-model one can make a decomposition in "isospin" eigenstates[11]

$$\begin{aligned}
S_{ab}^{cd}(\theta)&=\sum_{I=0}^{2}s_I(\theta)P_I(a,b;c,d),\\
P_0(a,b;c,d)&=\frac{1}{N}\delta_{ab}\delta_{cd},\\
P_1(a,b;c,d)&=\frac{1}{2}(\delta_{ac}\delta_{bd}-\delta_{bc}\delta_{ad}),\\
P_2(a,b;c,d)&=\frac{1}{2}(\delta_{ac}\delta_{bd}+\delta_{bc}\delta_{ad})-\frac{1}{N}\delta_{ab}\delta_{cd}.
\end{aligned} \tag{4.7}$$

The phase shift for each isospin channel is given by $s_I(\theta)=\exp(2i\delta_I(\theta))$. In terms of $\Delta=(N-2)^{-1}$ and $x=\frac{i\theta}{2\pi}$ one finds explicitely[10]

$$\begin{aligned}
\frac{s_2(\theta)}{f(\theta)}&=-\frac{\Gamma(1+x)\Gamma(\frac{1}{2}-x)\Gamma(\frac{1}{2}+\Delta+x)\Gamma(\Delta-x)}{\Gamma(1-x)\Gamma(\frac{1}{2}+x)\Gamma(\frac{1}{2}+\Delta-x)\Gamma(\Delta+x)},\\
s_1(\theta)&=\frac{(x-\Delta)}{(x+\Delta)}s_2(\theta),\\
s_0(\theta)&=\frac{(x-\frac{1}{2})(x-\Delta)}{(x+\frac{1}{2})(x+\Delta)}s_2(\theta).
\end{aligned} \tag{4.8}$$

The only ambiguity is in the so-called CDD-factor, which describes the presence of bound states or resonances, $f(\theta)=\pm\prod_k(\sinh(\theta)+iz_k)(\sinh(\theta)-iz_k)^{-1}$. In the limit $N\to\infty$ it can be shown[10] that $f(\theta)=1$ and this is assumed to hold for finite N as well. For O(3) this leads to the following particularly simple expression of the S-matrix

$$\begin{aligned}
s_0(\theta)=\frac{x-1}{x+1},\quad s_2(\theta)&=\frac{x+\frac{1}{2}}{x-\frac{1}{2}},\\
s_1(\theta)&=\frac{(x-1)(x+\frac{1}{2})}{(x+1)(x-\frac{1}{2})}.
\end{aligned} \tag{4.9}$$

Actually, measuring the energy of a two-particle state in a finite volume and using eq. (4.4), will give the momentum k in the center of mass frame and through eq. (4.2) will determine the scattering phase shift (at the momentum k). Figure 5, from[11], illustrates this for δ_2. (The deviations, increasing with

Figure 5: The scattering phase shift $\delta_2(k)$ obtained from eq. (4.2) for the O(3) model on lattices[11] of size 128×64 at $\beta = 1.5$ (bars) and 1.4 (open squares) and 256×128 at $\beta = 1.54$ (filled square). The full curve indicates the exact result, eq. (4.9).

the momentum k, are most likely due to lattice artifacts.) It confirms for O(3) that the CDD-factor equals 1. It also illustrates the feasibility of extracting phase shifts this way.

4.3. Four dimensions

As we discussed in sect. 4.1, it is sufficient to analyse the two-particle states in the non-relativistic situation. Again we assume the interaction potential to have a finite range, $V(\mathbf{x}) = 0$ for $|\mathbf{x}| > \lambda$ and the Hamiltonian in the center of mass frame is now given by $H = -m^{-1}\partial^2/\partial\mathbf{x}^2 + V(\mathbf{x})$. A complete set of scattering wavefunctions is given by

$$\psi^{(\mathbf{k})}_{\ell m}(\mathbf{x}) = [j_\ell(kr) - \tan(\delta_\ell(\mathbf{k}))n_\ell(kr)]Y_{\ell m}(\hat{\mathbf{k}}), \tag{4.10}$$

where obviously $k = |\mathbf{k}|$, $\hat{\mathbf{k}} = \mathbf{k}/k$, $r = |\mathbf{x}|$ and its energy is $E = \mathbf{k}^2/m$. It appears that the spherical nature of the scattered waves makes it impossible to impose periodic boundary conditions.

However, this problem is quite similar to the analysis of the hard-sphere bose gas by Huang and Yang[12]. The essential idea is equally simple as powerful. Replace $V(\mathbf{x})$ by a simple pseudo potential $V_\delta(\mathbf{x})$ such that for $|\mathbf{x}| > \lambda$, eq. (4.10) is still an exact solution of the relevant Schrödinger equation. Then one solves in the finite volume the Schrödinger equation with V_δ as its (energy dependent) potential. Let us illustrate this by making the simplified assumption that all phase shifts vanish, except for δ_0. In that case one easily verifies that the pseudopotential is given by

$$V_\delta(\mathbf{x}) = -\frac{4\pi}{mk}\tan(\delta_0(\mathbf{k}))\hat{\delta}_3(\mathbf{x}), \tag{4.11}$$

where one defines[12] $\hat{\delta}_3(\mathbf{x})\psi(\mathbf{x}) = \frac{\partial}{\partial r}(r\psi(\mathbf{x}))\delta_3(\mathbf{x})$. Alternatively, since we have to allow for wavefunctions that are singular at the origin, we can extend our class of functions to those that, when averaged over the angles, have a Laurent expansion $\sum_{n=-N}^{\infty} c_n r^n$. In that case we can define $\hat{\delta}_3(\mathbf{x})\psi(\mathbf{x}) = c_0\delta_3(\mathbf{x})$. For the torus we expand the wavefunction in plane waves $\psi(\mathbf{x}) = L^{-3}\sum b_{\mathbf{n}}\exp(2\pi i\mathbf{x}\cdot\mathbf{n}/L)$ and substitute this in the relevant Schrödinger equation

$$(-\frac{1}{m}\frac{\partial^2}{\partial\mathbf{x}^2} + V_\delta(\mathbf{x}))\psi(\mathbf{x}) = \frac{\mathbf{k}^2}{m}\psi(\mathbf{x}) \tag{4.12}$$

which is easily seen to give

$$b_{\mathbf{n}} = -\frac{4\pi\tan(\delta_0(\mathbf{k}))}{|\mathbf{k}|(\mathbf{k}^2 - (2\pi\mathbf{n}/L)^2)}c_0 \tag{4.13}$$

Substituting this back into the expression for the wavefunction one can calculate c_0 and find it to be proportional to c_0. The constant of proportionality therefore has to be 1, which yields the desired relation between the momentum in the center of mass and the scattering phase shift *at this momentum*

$$\frac{\tan(\delta_0(\mathbf{k}))\mathcal{Z}_{00}(1;\mathbf{q})}{2\pi^2|\mathbf{q}|} = 1, \quad \mathbf{q} = \frac{\mathbf{k}L}{2\pi}. \tag{4.14}$$

The zeta-function $\mathcal{Z}_{00}(s;\mathbf{q}) \equiv \sum_{\mathbf{n}\in\mathbb{Z}^3}(\mathbf{n}^2 - \mathbf{q}^2)^{-s}$ is defined[8, 9] through analytic continuation in s.

There are also "singular" solutions at momenta $\mathbf{k} = 2\pi\mathbf{n}/L$, if there exists $\mathbf{n}' \in \mathbb{Z}^3$, such that $|\mathbf{n}| = |\mathbf{n}'|$. In that case $\psi(\mathbf{x}) = \exp(2\pi i\mathbf{n}\cdot\mathbf{x}/L) - \exp(2\pi i\mathbf{n}'\cdot\mathbf{x}/L)$ is a solution that vanishes (and is regular) at the origin (in other words these wavefunctions correspond to the case $c_0 = 0$). If ψ is in the scalar representation (A_1) of the cubic group, this singular behaviour only occurs when $\mathbf{n}'$ is not related to $\mathbf{n}$ by a cubic transformation, which will make the momentum where this occurs quite large. Furthermore, restricting to the A_1 sector, eq. (4.14) will also be valid if the phase shifts only vanish for angular momenta $\ell \geq 3$. This is because a spin 2 wavefunction decomposes in the E and T_2 representations of the cubic group and hence does not couple to the scalar sector (note that due to the Bose symmetry all odd angular momentum phase shifts will vanish).

Lüscher's[9] analysis is based on studying the solutions of the Helmholtz equation $(\partial^2/\partial\mathbf{x}^2 + \mathbf{k}^2)\psi(\mathbf{x}) = 0$ in a finite volume, allowing for power-like singularities at the origin. For the simplified case that only δ_0 is non-vanishing, this is precisely what we achieved above. Lüscher[9] works out the general case in great detail and shows that truncating to a finite number of angular momenta ($\ell < \Lambda$) in general converges rapidly with Λ. This is important for practical applications, where one in general has only a finite number of phase shifts available. One of the most important future applications will be to analyse what will happen to the energy of a resonance (like the ρ) as a function of L. The presented results, valid to all orders in $1/L$, will allow for a more accurate analysis then was possible in the two channel resonance model of Wiese[29]. A most trivial application will be a simple derivation of the large L expansion[8]. One easily shows that

$$\mathcal{Z}_{00}(1;\mathbf{q}) = -\frac{1}{\mathbf{q}^2} + Z_{00}(1;\mathbf{0}) + \mathbf{q}^2 Z_{00}(2;\mathbf{0}) + \mathcal{O}(q^4),$$
$$\tan(\delta_0(\mathbf{k})) = a_0|\mathbf{k}| + \mathcal{O}(k^3), \quad (4.15)$$

where[8] $Z_{00}(s;\mathbf{q}) = \mathcal{Z}_{00}(s;\mathbf{q}) - (-\mathbf{q}^2)^{-s}$ and a_0 is called the scattering length. One can now straightforwardly iterate eq. (4.14) and find

$$k^2 = -\frac{4\pi a_0}{L^3}\left(1 + \frac{c_1 a_0}{L} + \frac{c_2 a_0^2}{L^2}\right) + \mathcal{O}(L^{-6}), \quad (4.16)$$
$$c_1 = \frac{Z_{00}(1;\mathbf{0})}{\pi}, \quad c_2 = \frac{Z_{00}^2(1;\mathbf{0}) - Z_{00}(2;\mathbf{0})}{\pi^2}.$$

The numerical values[8, 12] are $c_1 = -2.837297$ and $c_2 = 6.375183$.

Finally, let us mention that many of the aspects of the large volume expansions for the one-particle and two-particle masses have been verified successfully for the four dimensional Ising model by Montvay and Weisz and for the four dimensional O(4) ϕ^4 model in the symmetric phase by Frick, e.a.[30].

More ambitious has been the attempts to extract the pion scattering length $a_{\pi\pi}$ from the volume dependence of the two particle masses. For both Wilson and staggered fermions Ape[31] presented some results. Also new results with staggered fermions and a comparison with Weinberg's formula (adjusted for the number of flavours) were presented by Sharpe[32], to which we refer for further discussions. Also see the plenary talk by Kilcup[33], who furthermore discussed the method of Maiani and Testa[34] to extract the scattering length from suitable infinite volume euclidean correlation functions.

5. THE INFINITE VOLUME O(N) MASS GAP IN TWO DIMENSIONS

Like for the two-particle case, when $|x_i - x_j| \gg \lambda$ for all i and j, the M-particle wavefunction will be a plane wave

$$\psi(x_1,\cdots,x_M) = \sum_{\pi\in S_M} b_\pi \exp(i\sum_j k_j x_{\pi(j)}). \quad (5.1)$$

Here we used that the set of momenta will be conserved under any scattering process; the momenta can only be mutually permuted. There are $M!$ such so-called free regions (defined by the different orderings of x_i), which are separated by regions where at least two particles are within each others polarization cloud (which is by definition of the size λ). Going from one to another such region (generically) involves the collision of two particles and the wavefunction in the two free regions differ precisely by a factor which is the S-matrix for the relevant scattering process. There is a nice and compact way to write the wavefunction in a Fock-space notation that captures all the above features and satisfies the Bose symmetry[35]

$$|k_1,\cdots,k_M> = \int \prod_i \mathrm{d}x_i \prod_{i<j}(S(k_i,k_j)\eta_\lambda(x_j - x_i)$$
$$+\eta_\lambda(x_i - x_j)) \prod_i (e^{ik_i x_i}\phi^\dagger(x_i))|0> . \quad (5.2)$$

where $\eta_\lambda(x)$ is an operator valued, smeared-out representation of the Heaviside function, coinciding with it for $|x| \gg \lambda$, i.e. outside the polarization cloud. This is called the Bethe Ansatz[36]. For the non-linear Schrödinger model, the massive Thirring model, the sine-Gordon model or any model that can be solved by the quantum inverse scattering method, the Bethe Ansatz is exact (i.e. one can replace η_λ by the Heaviside function). Although it is likely that this is also the case for the O(N) model[13] there is no rigorous proof available in that case. The analysis of Polyakov and Wiegmann uses the equivalence with a fermionic model. Although the Bethe Ansatz is exact for this fermionic model, the equivalence between the models is not sufficiently rigorous, to conclude the exactness of the Bethe Ansatz for the O(N) model. Showing equivalence by proving that the Bethe Ansatz for the two models gives the same results is in this context unfortunately a circular argument.

Let us nevertheless make the assumption that the Bethe Ansatz is exact. In that case, imposing periodicity on the wavefunction (see eq. (5.2)) $\psi(x_1,\cdots,x_M) = < 0|\prod_i \phi(x_i)|k_1,\cdots,k_M >$, one easily finds as for the two-particle case (eq. (4.2))

$$e^{ik_iL}\prod_{i\neq j} S(k_i,k_j)=1, \quad \forall i. \qquad (5.3)$$

We now wish to take the thermodynamic limit, L and $M\to\infty$, while $\rho=M/L$ remains finite. The application we have in mind[14] will only deal with the isospin 2 channel for the O(N) model, such that S is given by s_2 in eq.(4.8), which has the property $s_2(\theta_i-\theta_j)=-1$ for $\theta_i=\theta_j$. Wavefunctions with $k_i=k_j$ will therefore vanish identically ("local" Pauli exclusion, similar to the hard-sphere repulsion). Thus, each momentum state is at most occupied only once. It is now easy to compute the vacuum energy $\mathcal{E}$ at a fixed density ρ. Eq. (5.3) can be written as $mL\sinh(\theta_i)-i\sum_{i\neq j}\ln S(\theta_i-\theta_j) = 2\pi n_i$, which enumerates with n the consecutive momentum states. Defining $dn = Lg(\theta)d\theta/2\pi$ one easily derives the following thermodynamic limit for the above equation

$$g(\theta)-\int_{-\theta_F}^{\theta_F}K(\theta-\theta')g(\theta')\mathrm{d}\theta' = m\cosh\theta,$$
$$K(\theta)\equiv -i\frac{d}{d\theta}\ln S(\theta), \qquad (5.4)$$

where θ_F is the "Fermi level", up to which all momentum states are occupied and which is implicitly defined by

$$\rho=\frac{1}{2\pi}\int_{-\theta_F}^{\theta_F}g(\theta)\mathrm{d}\theta. \qquad (5.5)$$

The groundstate energy $\mathcal{E}$ at fixed density ρ is therefore

$$\mathcal{E}=\frac{m}{2\pi}\int_{-\theta_F}^{\theta_F}\cosh(\theta)g(\theta)\mathrm{d}\theta. \qquad (5.6)$$

At low densities one basically has a free Fermi gas, where $g(\theta)\mathrm{d}\theta = dk$, such that $\rho=\frac{m}{\pi}\sinh\theta_F = k_F/\pi$ and $\mathcal{E} = m\rho+\pi^2\rho^3/6m+\cdots$. But at high densities one might object that in case the Bethe Ansatz is not exact, there will not be a single free region where one can impose the periodic boundary conditions, in which case the vacuum energy could depend in quite a complicated manner on the details of the theory. However, it is not ruled out that even when the Bethe Ansatz is not exact, in the thermodynamic limit one will still recover the above equations (an example of this occurs in the class of Toda-lattice models[37]). This might well be the case since one can prove (Niedermayer's rebuttal) that when M particles are distributed over N cells (say $N=0.1L/\lambda$) the probability that no cell is empty will vanish exponentially in the thermodynamic limit. One empty cell, i.e. free region, will be sufficient to derive eq.(5.3). Nevertheless, there are examples of wavefunctions having different boundary conditions at a set of measure zero, that have different energies.

Assuming that the thermodynamic limit is described by eq. (5.4,5.5,5.6) one can obtain $m/\Lambda_{\overline{MS}}$ in an infinite volume by the following steps[14]. First couple the non-linear σ-model to the conserved current $j^{12}_\mu(x) = (n_1\partial_\mu n_2 - n_2\partial_\mu n_1)/g_0$ by adding $-h\int \mathrm{d}x\; j^{12}_\mu(x)$ to the Hamiltonian. This will create M isospin 2 particles (where $M=\int\mathrm{d}x\; j^{12}_\mu(x)$) and the free energy $f(h)$ will be given by the Legendre transform

$$f(h)=\mathrm{min}_\rho[\mathcal{E}(\rho)-h\rho]. \qquad (5.7)$$

On the other hand, one can also calculate $f(h)$ in perturbation theory for large h, using asymptotic freedom of the O(N) non-linear σ-model in two dimensions. There are $N-2$ degrees of freedom that acquire a mass h^2 and one finds

$$f(h)-f(0) = -\frac{h^2}{2g_0}+{\scriptstyle (N-2)}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\sqrt{k^2+h^2}+\cdots$$
$$= -{\scriptstyle (N-2)}\frac{h^2}{4\pi}\ln(\frac{h}{\sqrt{e}\Lambda_{\overline{MS}}})+\cdots \qquad (5.8)$$

(although the scale is already fixed by the one-loop equation, the two-loop contribution is also required in a systematic expansion that goes up to, and including, the constant term in $f(h)/h^2$. From the O(N) β-function the relevant two-loop contribution amounts to $4\pi f_2(h)h^{-2} = -\ln\ln(h/\Lambda_{\overline{MS}})$.) As the free energy determined with the Bethe Ansatz is a function of h/m, whereas the perturbative result gives the free energy as a function of $h/\Lambda_{\overline{MS}}$, equating the two expressions will give $m/\Lambda_{\overline{MS}}$. What therefore remains, is to determine the expansion of the free energy in eq. (5.7). This equation can again be cast in the form of an integral equation[14]

$$\varepsilon(\theta) - \int_{-\theta_F}^{\theta_F} K(\theta-\theta')\varepsilon(\theta')d\theta' = h - m\cosh\theta, \qquad \varepsilon(\pm\theta_F) = 0, \tag{5.9}$$

as can be seen from the fact that (using eq. (5.4))

$$\begin{aligned} h\rho - \mathcal{E} &= \frac{1}{2\pi}\int_{-\theta_F}^{\theta_F} d\theta g(\theta)(h - m\cosh(\theta)) \\ &= \frac{1}{2\pi}\int_{-\theta_F}^{\theta_F} d\theta g(\theta)\{\varepsilon(\theta) - \int_{-\theta_F}^{\theta_F} d\theta' K(\theta-\theta')\varepsilon(\theta')\} \\ &= \frac{m}{2\pi}\int_{-\theta_F}^{\theta_F} d\theta\varepsilon(\theta)\cosh\theta. \end{aligned} \tag{5.10}$$

We can interpret $-\varepsilon(\theta)/2\pi$ (noting that $d\theta\cosh\theta = dk$) as the energy one gains by adding a state with rapidity θ (with all lower levels occupied). The free energy is clearly minimal if one keeps on adding states until ε changes sign, which implicitly defines the Fermi level. It is important to observe that eq. (5.9) incorporates the effect of the shift in the energies of the occupied one-particle states when adding an additional particle. This is of course since the theory is interacting, despite the fact that particles propagate freely (but with a mass m) in the free regions.

It is highly non-trivial to extract from eq. (5.9) the asymptotic expansion of $f(h)$ for large h, as is already obvious from eq. (5.8). It is amusing to observe that for O(3), where the integral Kernel takes the simple form $K(\theta) = (\pi^2+\theta^2)^{-1}$, eq. (5.9) is related to the capacity of a circular plate condensor, if we put $m = 0$. That particular problem has been around for very long and it is illustrative for the complexity of the singular integral equation that only in the early sixties Hutson[38] was able to rigorously extract the asymptotic expansion. Unfortunately, eq. (5.9) is even more cumbersome to analyse if $m \neq 0$. It would lead us too far to discuss any of the details here (I thank Peter Hasenfratz for providing some of the details not presented in ref.[14]). The result, *which is valid for arbitrary* N, is[14]

$$\frac{m}{\Lambda_{\overline{MS}}} = \frac{b^\Delta}{\Gamma(1+\Delta)}\,, \quad \Delta = \frac{1}{N-2}. \tag{5.11}$$

The factor b is determined by a numerical analysis, but agrees to eight digits with $b = 8/e$. This is consistent with the large N result[16, 17],

$$m = [1 + \frac{1}{N-2}(\ln 8 - 1 + \gamma_E) + \mathcal{O}(N^{-2})]\Lambda_{\overline{MS}}. \tag{5.12}$$

The large N analysis was extended[16] to $\mathcal{O}(N^{-3})$, giving good agreement at larger N with Monte Carlo data by Wolff[15] and at large correlation length with eq. (5.11) as is illustrated in figure 6 for $N = 8$. However, in particular for $N = 3$ the impressively large correlation length that can be achieved using the cluster algorithm to beat critical slowing down[15], is not yet quite sufficient to see asymptotic scaling (based on the perturbative three-loop β-function[39]). Using Monte Carlo renormalization group methods allows one in principle to go to even much larger correlation length, giving[40] $m = (3.3 \pm 0.1)\Lambda_{\overline{MS}}$. As systematic errors due to lattice artifacts are not quite under control, it is not clear whether the three-σ difference with eq.(5.11) should be taken serious.

Figure 6: Monte Carlo data[15] and $1/N$ results[16] (dotted curve to $\mathcal{O}(N^{-1})$, dashed curve to $\mathcal{O}(N^{-2})$ and the full curve to $\mathcal{O}(N^{-3})$) at $N = 8$ (kindly made available by Flyvbjerg). The correlation length ξ is in lattice units. The mass gap m was converted to physical units using the three-loop β-function[39] and[41] $\Lambda_{\overline{MS}} = \Lambda_L\sqrt{32}\exp(\pi/[2(N-2)])$. m_{ex} is the infinite volume mass gap[14], eq. (5.11).

6. CONCLUSION

The importance of analytic methods for lattice field theories is twofold. First it can provide important checks on the reliability of the Monte Carlo technique to extract physical quantities from lattice field theories. Issues like the magnitude of lattice artifacts can be studied with much more confidence then is otherwise possible. In sect. 2 we discussed Vohwinkel's[1] analysis of the low-lying spectrum for pure SU(3) gauge theories. It is important that in these calculations no free parameters are introduced. The method is, however, only valid for volumes below one cubic fermi (the scale is set by a string tension of $(420 MeV)^2$). Beyond this volume the spectrum will become sensitive to the θ-parameter[21, 42], which has not yet been taken into account. Our understanding of finite volume gauge theories is now relatively complete. For SU(3) it might still be useful to include lattice artifacts and massless fermions in the intermediate volume analysis, similar to what was achieved for SU(2)[2, 43]. The finite volume studies have made it clear that Gribov horizons in the Hamiltonian formulation dominate the long range dynamics of gauge theories[21]. The volume provides a parameter that allows us to control how far the wavefunctional will spread out over configuration space and thus how many Gribov horizons are being crossed. Going to volumes larger than one cubic fermi involves crossing a new class of Gribov horizons and it is hoped that in the future we will be able to venture out beyond these additional horizons.

Another example of comparison with lattice gauge results was discussed for the O(N) non-linear σ-model in two dimensions (sects. 4.2 and 5). The evidence, although not conclusive, is nevertheless in favour of the correctness of the thermodynamic limit of the Bethe Ansatz, which gives a highly singular integral equation, from which the infinite volume mass gap could be extracted[14]. It allows one to address the issue of asymptotic scaling in the lattice Monte Carlo data on the mass gap. For O(3) this has been a notorious problem and even the most recent Monte Carlo results at very large correlation length are not sufficient to resolve this issue[15].

The second reason for the usefulness of analytic studies lies in the fact that lattice Monte Carlo results are always restricted to a finite volume, due to limited computer resources. Thus one needs to know when a volume is big enough to be able to extract with confidence infinite volume quatities. The most reliable method is based on analytic expressions for the volume dependence, usually a large volume expansion, parametrized by infinite volume quantities. We discussed in sect. 3 the situation for stable one-particle masses in massive field theories. The volume corrections are in that case exponential[5] and these formula were successfully applied to the Ising and ϕ^4 models in four dimensions[30]. A reliable application for the pure gauge glueball calculations[44] will, however, require better statistics at larger volumes. In sect. 4.3 we discussed the case of two-particle energies, that have power-like volume dependence and provides a relation with the scattering phase shifts. Also here the Ising and ϕ^4 models[30] were used to test these ideas, but ultimately one would like to use these results to calculate the pion scattering length[31, 32] providing an important test for the dynamical fermion algorithms to reproduce chiral behaviour. Even more important will be a full understanding of the volume dependence of resonances like the ρ or glueballs in the presence of light fermions[9]. For a preliminary study based on a two channel resonance model see[29] (see also[45]). Much activity is likely to develop in that area in the near future.

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