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Nahm gauge fields for the torus

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Abstract: We present the exact expression for the Nahm gauge field associated to a $SU(N)$ charge one self-dual gauge field on $T^3 \times R$. The result implies that the size of the instanton is determined by the "distance" between its two flat connections at $t \to \pm\infty$.

1. A long standing problem is to find analytic self-dual solutions on a higher dimensional torus for non-Abelian gauge theories. For T^4 solutions with topological charges higher than one can be proven to exist [1]. However, it can be shown there is no regular solution with topological charge one [2], though existence is assured [3] in case one allows for twisted boundary conditions [4]. The main tool for studying self-dual solutions has become the Nahm duality transformation [5] that maps a charge k, $U(N)$ solution to a charge N, $U(k)$ solution on the dual torus. We consider gauge fields in four euclidean dimensions on spaces $T^d \times R^{4-d} = R^4/\Lambda$, where Λ is a d-dimensional lattice embedded in R^4, whose dual is denoted by the d-dimensional lattice $\hat{\Lambda}$, which we consider to be embedded in R^d. The dual torus on which the Nahm transformed gauge field lives is $\hat{T}^d = R^d/\hat{\Lambda}$. Hence, for $d \neq 4$ there is a reduction in dimension. The extreme case is when $d = 0$, which reduces to the algebraic ADHM [6] construction on R^4. The case $d = 1$ has also led to considerable progress in demonstrating that instantons at finite temperature (calorons) have monopoles as constituents [7, 8]. Important practical use of the Nahm transformation stems from the fact that the charge one solutions are mapped to self-dual Abelian fields.

Recently González-Arroyo [9] has constructed the Nahm transformation in the presence of twisted boundary conditions for T^4. One interprets twisted boundary conditions as "half-period" conditions (for $SU(2)$ a quite appropriate terminology), applying the Nahm transformation to the gauge fields on the (smallest) extended torus with periodic boundary conditions. One subsequently looks for "half-periods" on the dual torus. This elegant construction will lead to important new insights, but increases the topological charge of the periodic solutions. It forces one to deal with Nahm gauge fields that are non-Abelian, not giving an obvious simplification that is likely to lead to a handle on an analytic construction.

Here we will be concerned with $T^3 \times R$, for which the Nahm transformation was introduced a few years ago [10], and which is relevant for the Hamiltonian formulation of gauge theories in a finite volume with periodic boundary conditions (for a recent review addressing the dynamical issues see ref.[11]). In this Letter we will present the analytic solution for the gauge field obtained after applying the Nahm transformation to a charge one instanton, in terms of its flat connections at $t \to \pm\infty$. That the instanton has to approach a flat connection in these limits is a simple consequence of the requirement of finite action.

2. We first present the formalism, being rather brief and referring the interested reader to refs. [2, 5, 10] for details. A self-dual $U(N)$ gauge field $A(x) = A_\mu(x)dx_\mu$ (with $A^\dagger = -A$) is defined on R^4/Λ by

$$A(x+\lambda) = g_\lambda(x)(A(x)+d)g_\lambda^\dagger(x), \quad \lambda \in \Lambda. \tag{1}$$

It is made into a family of self-dual gauge fields by adding a (flat) constant Abelian connection, $A_z(x) = A(x) + 2\pi i d(z\cdot x)$. Note that the differential is with respect to x only, $d(z\cdot x) = z\cdot dx = z_\mu dx_\mu$, and an identity matrix, I_N, in the algebra of $U(N)$ is implicit. Even though the curvature (field strength) is independent of z, its dependence cannot be gauged away since the appropriate Abelian gauge transformation $g(x) = \exp(2\pi i z\cdot x)$ is not periodic, except when $z \in \hat{\Lambda}$. This shows that z can be considered to live on the dual space $R^d/\hat{\Lambda}$. The reduction in dimension alluded to above occurs since for non-compact directions the relevant components of z *can* be gauged away. Equivalently, non-compact directions can be interpreted as having infinite periods, which under the duality are mapped to zero periods, removing the dependence on the dual coordinate.

The Nahm transformation involves the zero-modes of the Weyl equation, of which there are as many as the charge (k) of the gauge field

$$\begin{aligned} D_z\Psi_z(x) &= \sigma_\mu D_\mu(A_z)\Psi_z(x) = \sigma_\mu(\partial_\mu + A_\mu(x) + 2\pi i z_\mu)\Psi_z(x) = 0, \\ \Psi_z(x+\lambda) &= g_\lambda(x)\Psi_z(x), \quad \lambda \in \Lambda, \end{aligned} \tag{2}$$

where σ_μ form a basis of unit quaternions ($\sigma_0 = I_2$ and $-i\sigma_j = \tau_j$ the Pauli matrices). As Ψ_z is in the fundamental representation of the gauge group we can not allow for twisted boundary conditions, which require the center of the gauge group to act trivially. As mentioned above one can enlarge the periods [9] to deal with twisted boundary conditions. Here we will instead consider only twisted boundary conditions in the non-compact directions, where the action of the center of the gauge group is trivialised [10] due to Weyl fermions vanishing asymptotically, so as to ensure normalisability.

The Nahm connection is given in terms of the normalised zero-modes by

$$\hat{A}_\mu^{ij}(z) = \int d^4x\ \Psi_z^{(i)}(x)^\dagger \frac{\partial}{\partial z_\mu}\Psi_z^{(j)}(x). \tag{3}$$

It is not difficult to show that this is a $U(k)$ connection on $R^d/\hat{\Lambda}$ and using the family index theorem one concludes [2] the topological charge of the Nahm gauge field to be N. The index theorem relates the difference of the number of zero-modes with opposite chirality to the topological charge, that is ker(D_z)–coker(D_z)=ker(D_z)–ker$(D_z^\dagger)$ has dimension k. Additional (i.e. non-generic) zero-modes are therefore detected as zero-modes of

$$D_z D_z^\dagger = -D_\mu^2(A_z) - \tfrac{1}{2}\bar\eta_{\mu\nu}F_{\mu\nu}(x), \tag{4}$$

where $\bar\eta_{\mu\nu} = \sigma_{[\mu}\sigma^\dagger_{\nu]}$ is the anti-self dual 't Hooft tensor. For self-dual fields we have that $D_z D_z^\dagger = -D_\mu^2(A_z)$, which commutes with σ_μ, from which Nahm derived the remarkable result that the Nahm gauge field is self-dual as well. For T^4 this is most easily demonstrated, as the manifold has no boundaries. Technically one requires the absence of flat factors [12] to ensure that ker$(D_z^\dagger)$ is trivial.

For a non-compact manifold, applying index theorems requires some care, but in principle $\hat A(z)$ is well defined as long as dim ker$(D_z^\dagger)$=0, and one finds [10]

$$\begin{aligned}\hat F^{ij}_{\mu\nu}(z) &= 8\pi^2\int d^4x d^4x'\ \Psi_z^{(i)}(x)^\dagger \eta_{\mu\nu} G_z(x,x')\Psi_z^{(j)}(x') \\ &+ \ 4\pi i \oint \frac{\partial \Psi_z^{(i)}(x)^\dagger}{\partial z_{[\mu}} \eta_{\nu]\alpha}\left(G_z\Psi_z^{(j)}\right)(x)\, d^3_\alpha x,\end{aligned} \tag{5}$$

where $\eta_{\mu\nu} = \sigma^\dagger_{[\mu}\sigma_{\nu]}$ is the self-dual 't Hooft tensor and G_z is the Greens function for $-D_\mu^2(A_z)$.

3. On T^4, applying the Nahm transformation again brings one back to the original solution. In other cases one needs to modify the second, or inverse, Nahm transformation. The boundary terms are particularly important in the case of instantons on R^4, leading to the ADHM construction [6] for reconstructing the original gauge field. The modification corrects for the fact that $\hat A(z)$ is no longer self-dual due to the boundary terms. However, boundary terms only occur at a finite number of isolated points, and can be expressed in terms of delta functions (excluding the situation of R^4 where the dual space is reduced to a single point). The singularities are fixed by the asymptotic holonomies (the Polyakov loops). For $T^3\times R$ there are two disconnected asymptotic regions and we specify

$$P_\pm(\vec\ell, z) \equiv \lim_{t\to\pm\infty} P\exp\left(\int_{C(\vec\ell)} A_z(x)\right), \tag{6}$$

where $\vec\ell$ is the number of windings for each direction on T^3, specifying the homotopy type of the curve $C(\vec\ell)$. The N eigenvalues of $P_\pm(\vec\ell,z)$ are given by $\exp(2\pi i(\vec\omega^j_\pm + \vec z)\cdot\vec\ell)$. For $SU(N)$ gauge fields one has in addition $\sum_j \vec\omega^j_\pm = \vec 0$.

It is now easily seen [10], when all eigenvalues are *unequal* to one, that a Weyl zero-mode decays as $\exp(\mp t M_\pm(z))$, where $M_\pm(z)$ is the mass-gap of the Weyl equation reduced to T^3 for $t\to\pm\infty$, with

$$M_\pm(z) = \min\{2\pi|\vec\omega^j_\pm + \vec z + \vec p|; j = 1,\cdots,N; \vec p\in Z^3\}. \tag{7}$$

Now, $M_\pm(z)$ vanishes whenever $P_\pm(\vec\ell,z)$ has a unit eigenvalue for all $\vec\ell$. Only for those cases the boundary terms arising from a partial integration in computing $\hat F$ can be non-vanishing.

Outside of the singularities the field is self-dual and $\hat{E}_i(z) = \hat{B}_i(z)$. As $\hat{A}(z)$ is independent of z_0 we have $\hat{E}_i(z) = i\partial_i \hat{A}_0(z)$. We extract a factor i, to define $\hat{E}_i(z)$ and $\hat{B}_i(z)(= -\frac{i}{2}\varepsilon_{ijk}\hat{F}_{jk}(z))$ as real fields. Assuming other than periodic boundary conditions (to avoid the contributions from $t \to \pm\infty$ to interfere), near one of the singularities integration by steepest descent yields $\hat{A}_0(z) \to \pm iq/M_\pm(z)$, where q is a positive constant. This means that the singularities act as point sources with charges $\pm q$. Since self-duality implies that the (Maxwell) field equations are satisfied, the exact solution is found by performing the sum over periods for these point charges. But we can say more. For the gauge field to be well-defined outside of the singularities, charge quantisation should be enforced and one concludes that q/π has to be an integer, but generically $q = \pi$. This ensures the magnetic sources are those of Dirac monopoles with unobservable Dirac strings. We will now demonstrate this on the basis of a Berry phase type argument.

4. We restrict ourselves to the case of twisted boundary conditions in the time direction. For ease of notation we take all three periods equal to one (generalisation to another torus is straightforward) and we consider $T^3 \times R$ as the limit for $T \to \infty$ of $T^3 \times [0, T]$. The twist can be implemented by choosing [4, 14] the gauge field to be periodic in the spatial directions, whereas the gauge field at $t = T$ is related to the one at $t = 0$ by a gauge transformation $g(\vec{x}) = g_k(\vec{x})g_{\vec{k}}(\vec{x})$. Here $g_k(\vec{x})$ is a periodic gauge transformation with winding number k and $g_{\vec{k}}(\vec{x}) = \exp(2\pi i \vec{x}\cdot\vec{k}\Theta)$, with Θ $(= \frac{1}{2}\tau_3$ for $SU(2))$ such that $g_{\vec{k}}(\vec{\ell}) = \exp(2\pi i \vec{k}\cdot\vec{\ell}/N) \in Z_N$ for $\vec{\ell} \in Z^3 (= \Lambda)$. Thus k_i are integers, defined modulo N. For finite T and $\vec{k} \neq \vec{0} \bmod N$ existence of a $4Nk$ parameter set of instantons is guaranteed [3]. Taking $T \to \infty$ yields solutions on $T^3 \times R$. With twisted boundary conditions $P_+(\vec{\ell}, z) = \exp(2\pi i \vec{k}\cdot\vec{\ell})P_-(\vec{\ell}, z)$, up to $g_k(\vec{x})$ gauge conjugation, relating the singularities discussed above.

Consider T finite and add an Abelian background gauge field, whose flux compensates for the twist [13]. The price one pays is that the $U(N)$ gauge field will in general no longer be self-dual. We introduce the antisymmetric twist tensor $n_{\mu\nu}$, with $n_{0i} = k_i$ and vanishing spatial components, $n_{ij} = 0$. One defines

$$\bar{A}_\mu = \pi i n_{\mu\nu} x_\nu I_N/(NT), \quad \bar{F}_{\mu\nu} = -2\pi i n_{\mu\nu} I_N/(NT). \tag{8}$$

In terms of the curvature two-form $\bar{\Omega} = \frac{1}{2}\bar{F}_{\mu\nu} dx_\mu \wedge dx_\nu$, the first Chern class is given by

$$c_1 = Tr\bar{\Omega}/(2\pi i). \tag{9}$$

The Pontryagin index P_t for the $U(N)$ bundle $A + \bar{A}$ is now

$$P_t = (8\pi^2)^{-1}\int Tr(\Omega + \bar{\Omega}) \wedge (\Omega + \bar{\Omega}) = (8\pi^2)^{-1}\int Tr\Omega\wedge\Omega + Tr\bar{\Omega}\wedge\bar{\Omega}, \tag{10}$$

where Ω is the curvature two-form of the original (self-dual) $SU(N)$ connection A, satisfying $Tr\Omega = 0$. For the Pontryagin index of A we find [14] $P = \nu - (N-1)\mathrm{pf}(n)/N$, with ν integer and $\mathrm{pf}(n) = \frac{1}{8}\varepsilon_{\mu\nu\alpha\beta} n_{\mu\nu} n_{\alpha\beta}$, such that $P_t = P + \bar{P} = P - \int c_1 \wedge c_1/(2N) = P - \mathrm{pf}(n)/N$.

Thus, $P_t = \nu - \mathrm{pf}(n)$ is always integer as required for $U(N)$ vector bundles. For the case at hand, with only twist in the time direction, $\mathrm{pf}(n) = 0$ and the topological charge is not affected by adding the twist compensating Abelian background field.

The Nahm transformation maps this to a bundle with rank P_t, charge N and first Chern class $\int_{T^4}(dz_\mu \wedge dx_\mu)^2 \wedge c_1$ (for the precise formulation see ref. [2]). Consider now the case of topological charge one. Assuming the Weyl cokernel to be trivial for all z, we get a (non-selfdual) $U(1)$ connection with charge $\hat{P} = N$. But for an Abelian connection we also have $\hat{P} = -\frac{1}{2}\int \hat{c}_1 \wedge \hat{c}_1 = -\mathrm{pf}(n) = 0$. So there must be values of z for which the Weyl cokernel is non-trivial. We have

$$D_z D_z^\dagger = -D_\mu^2(A_z + \bar{A}) + i\pi n_{\mu\nu}\bar{\eta}_{\mu\nu} I_N/(NT) = -D_\mu^2(A_z + \bar{A}) + \vec{H}\cdot\vec{\tau}, \qquad (11)$$

with $\vec{H} = 2\pi\vec{k}/(NT)$. We note that $D_z D_z^\dagger$ is positive, but it may vanish. Using that $\vec{H}\cdot\vec{\tau}$ has eigenvalues $\pm|\vec{H}|$, one easily finds this to be the case if and only if the positive function $f(z) \equiv \lambda_0(z) - |\vec{H}|$ vanishes, where $\lambda_0(z)$ is the lowest eigenvalue of $-D_\mu^2(A_z + \bar{A})$. With $\Phi_z(x)$ its normalised eigenvector, we have $f(z) = \int d^4x\ \Phi_z^\dagger(x) D_z D_z^\dagger \Phi_z(x)$ such that $\partial^2 f(z)/\partial z_i \partial z_j = 8\pi^2\delta_{ij}$ at the points where $f(z)$ vanishes. Considering the fact that $f(z)$ is a smooth and positive function of z, the zeros are generic and cannot bifurcate in zeros of lower order.

As long as T is finite we can use the index theorem and conclude that at exactly the same points where $\dim\ker(D_z^\dagger)$ jumps from zero to one, $\dim\ker(D_z)$ has to jump from one to two. Since D_z depends smoothly on z, this necessarily describes the case of a generic level crossing. The only unusual feature is that the "Hamiltonian" is arranged so as to vanish exactly for one of the "adiabatic" eigenstates. The resulting Berry potential [15] associated to the isolated crossing corresponds precisely to the spatial components of the Nahm connection, eq.(3). Due to the topological nature of the Berry phase we can immediately conclude that the level crossing acts as the source of a Dirac monopole with the appropriate charge quantisation, enforcing $q = \pi$. It is clear that this assignment is independent of T and this fixes the charges for the case of $T^3 \times R$.

Note that $\bar{A} \to 0$ for $T \to \infty$. We reiterate that under this limit the total action stays fixed to $8\pi^2$, as dictated by the unit topological charge, and the fields are forced to decay to flat connections at both ends, thereby dictating the location of singularities that act as point sources. We note that for twisted boundary conditions one can put $\vec{\omega}_+^j = \vec{\omega}_-^j + \vec{k}/N \bmod Z^3$. Generically there are N sources with charge $q = \pi$ and N sources with the opposite charge. Higher charges appear only in case some of the $\vec{\omega}_\pm^j$ coincide. The enlarged subgroup that leaves the holonomies invariant leads to appropriate additional zero-modes for $D_z^\dagger$.

5. The Nahm connection on $\hat{T}^3$ is uniquely determined by the point charges we described above,

$$\hat{A}_0 = \frac{i}{2}\sum_{\vec{\ell}\in Z^3}\sum_{j=1}^{N}\left(|\vec{\omega}_+^j + \vec{z} + \vec{\ell}|^{-1} - |\vec{\omega}_-^j + \vec{z} + \vec{\ell}|^{-1}\right). \qquad (12)$$

One difficulty is to evaluate the sum over the periods, as it formally diverges. This can be achieved in terms of lattice sum techniques based on resummations [16]. Quite fortunately this problem was already tackled long ago in evaluating the one-loop effective potential for constant Abelian gauge fields on T^3, $V_1(\vec{C}) = 2\sum_{\vec{\ell}\in Z^3} |2\pi\vec{\ell}+\vec{C}|$. In terms of $W(\vec{C}/2\pi) \equiv \frac{1}{2}\pi\partial^2 V_1(\vec{C})/\partial C_i^2$ one easily finds

$$\hat{A}_0(\vec{z}) = \frac{i}{2}\sum_{j=1}^{N}\left(W(\vec{z}+\vec{\omega}_+^j) - W(\vec{z}+\vec{\omega}_-^j)\right), \tag{13}$$

where the rapidly converging expression for $W(\vec{z})$ can be taken from eq. (A.10) of ref. [17],

$$W(\vec{z}) = -1 + \sum_{\vec{\ell}\neq\vec{0}} \frac{e^{-\pi\vec{\ell}^2}}{\pi\vec{\ell}^2}\cos(2\pi\vec{\ell}\cdot\vec{z}) + \sum_{\vec{\ell}} \frac{\mathrm{erfc}(|\vec{\ell}+\vec{z}|\sqrt{\pi})}{|\vec{\ell}+\vec{z}|} \tag{14}$$

It can be shown [17] that $\partial^2 W(\vec{z})/\partial z_i^2 = -4\pi(\delta(\vec{z})-1)$, where $\delta(\vec{z})$ is the periodic delta function, such that indeed

$$\partial_i \hat{E}_i(\vec{z}) = 2\pi\sum_{j=1}^{N}\left(\delta(\vec{z}+\vec{\omega}_+^j) - \delta(\vec{z}+\vec{\omega}_-^j)\right). \tag{15}$$

It is still a formidable task to reconstruct from this explicit expression for the Nahm connection the original non-Abelian gauge field on $T^3\times R$. This requires, like for the simpler case of the calorons [7], the formulation of a modified Nahm transformation, dealing with the singularities to which violations of self-duality are restricted. Nevertheless, given the existence of solutions with twisted boundary conditions interesting conclusions can be drawn. Up to an overall constant, related to the position of the instanton on $T^3\times R$, $\hat{A}(\vec{z})$ is determined uniquely by the eigenvalues of the holonomies. These holonomies, when taking the limit $T\to\infty$ (unlike in the case of the calorons) arise from the properties of the solutions on $T^4 = T^3\times[0,T]$, and are thus part of the gauge invariant moduli. Together with the position of the charge one instanton, these account for the $3(N-1)+4$ parameters of the gauge invariant moduli space. The holonomy breaks the gauge group to $U(1)^{N-1}$, accounting for $N-1$ additional parameters that are part of the moduli space of framed instantons, which has dimension $4N$, as is appropriate for the torus and this leaves no room for a scale parameter of the instanton. We thus conclude that the size of the instanton is related to the holonomies, something that was conjectured [10] on the basis of numerical studies [19] and in direct analogy with the situation for instantons on the cylinder for the $O(3)$ model [20].

For $SU(2)$ it was indeed observed that the largest instanton, the one that described tunnelling through the lowest barrier (sphaleron), is associated to $\vec{k}=(1,1,1)$ and all holonomies equal to ±1. This corresponds to $\vec{\omega}_-$ and $\vec{\omega}_+ (=\frac{1}{2}\vec{k}+\vec{\omega}_-)$ separated on $\hat{T}^3$ over the maximal distance possible. On the other hand, when the trace of the holonomy vanishes, $\pm\vec{\omega}=\frac{1}{4}\vec{k}$, the twisted boundary conditions are compatible with periodic boundary conditions, and indeed $\vec{\omega}_-$ and $\vec{\omega}_+$ become equal and $\hat{A}(\vec{z})$ will vanish (apart from a trivial constant). In that case, like in the analysis for the caloron [7], associating the Nahm and ADHM formulation by Fourier transformation the solution becomes expressible in terms of the 't Hooft ansatz [18], $A_\mu = \frac{1}{2}\bar{\eta}_{\mu\nu}\partial_\nu\log\phi(x)$, with $\phi(x) = \rho^{-2}+\sum_{\vec{\ell}}[t^2+(\vec{x}+\vec{\ell})^2]^{-1}$. Resummation of the lattice sum is easily performed, but positivity of $\phi(x)$ is seen to force the instanton scale ρ to zero.

6. It is tempting to conjecture on the basis of our results for the Nahm connection that charge 1 instantons will exist for open boundary conditions, in which case the holonomies at both ends are not related[1]. If true, we can obtain periodic boundary conditions as a limit from the one with open boundary conditions, implying $\hat{A}(\vec{z})$ to approach a constant, and one would as above conclude that this forces the size of the instanton to zero.

It is amusing in the light of this to note that, when extending the caloron construction [7] based on Fourier transformation of the ADHM formulation in an obvious way to $T^3 \times R$ (e.g. see eq. (40) in ref. [7b]), one finds

$$\tau_j(\hat{E}_j(\vec{z}) - \hat{B}_j(\vec{z})) = \vec{a} \cdot \vec{\tau}\,(\delta(\vec{z} - \vec{\omega}) - \delta(\vec{z} + \vec{\omega}))\,. \tag{16}$$

where for simplicity we only considered $SU(2)$. Here the direction of $\vec{a}$ is related to the common gauge orientation of the holonomies and its length is related to the square of the scale parameter that appears in the ADHM construction. Due to the vectorial nature of the singularity it is natural to assume the gauge field is described by an electric-magnetic dipole. Remarkably, it is well known in the theory of classical Electrodynamics (e.g. see ref. [21]) that for dipoles

$$\vec{E}(\vec{x}) = \frac{3\vec{x}(\vec{p}\cdot\vec{x}) - \vec{p}(\vec{x}\cdot\vec{x})}{|\vec{x}|^5} - \frac{4\pi}{3}\vec{p}\delta(\vec{x}), \quad \vec{B}(\vec{x}) = \frac{3\vec{x}(\vec{m}\cdot\vec{x}) - \vec{m}(\vec{x}\cdot\vec{x})}{|\vec{x}|^5} + \frac{8\pi}{3}\vec{m}\delta(\vec{x}). \tag{17}$$

These delta functions differ between electric and magnetic dipoles, as the first comes from two approaching point charges and the second from a shrinking current loop. Thus, with $\vec{p} = \vec{m}$, one finds $\vec{B}(\vec{x}) - \vec{E}(\vec{x}) = 4\pi\vec{m}\delta(\vec{x})$, precisely of the required form. The appropriate solution for $SU(2)$ is described by two "dyonic" dipoles of opposite strength located at $\vec{z} = \pm\vec{\omega}$. Outside the singularities this can easily be expressed in terms of the lattice sums we defined before,

$$\hat{B}_i(\vec{z}) = \hat{E}_i(\vec{z}) = \frac{a_j}{4\pi}\frac{\partial^2}{\partial z_i \partial z_j}\left(W(\vec{z} + \vec{w}) - W(\vec{z} - \vec{w})\right). \tag{18}$$

Indeed, resolving the quadratic ADHM constraint [6] and explicit Fourier transformation reproduces this result, including the appropriate delta functions that violate the self-duality [22].

Nevertheless, as we cannot argue for the existence of solutions with periodic boundary conditions on $T^3 \times R$, it may be that this dipole solution to the Nahm equations is not realised, except for $\vec{a} \to \vec{0}$, implying the size of the instanton to go to zero. The dipole approximation obtained from open boundary conditions, letting $\vec{\omega}_+$ tend to $\vec{\omega}_-$ to approach periodic boundary conditions, indeed leads to vanishing dipole moments (since the charge is fixed). This conclusion can only be avoided in case *no* solutions with open boundary conditions exist.

[1] The number of gauge invariant parameters describing such instantons would be $6(N-1)+4$. As these solutions cannot be compactified to T^4, there need be no conflict with the standard result on a compact manifold [12].

7. In conclusion, we have shown that one can extract, by solving Abelian Bogomolny equations with specified sources, analytic results for the Nahm transformation of the charge one instanton on $T^3 \times R$, which provides interesting information on the parameters that describe these instanton solutions. The situation is quite similar to that for the $O(3)$ non-linear sigma model on the cylinder. It will be interesting to be able to demonstrate the existence of solutions with open boundary conditions. However, the biggest challenge remains the formulation of the inverse Nahm transformation, which requires us to study the (modified) Weyl equation in a lattice of "dyonic" charges.

Numerical studies of the Nahm transformation [23] have been implemented, and may well play a role in analytically addressing these issues. Preliminary results [24] relevant for the case studied here are very encouraging and stimulating. Also the deformation of the Nahm transformation to the noncommutative torus [25] and its M-theory compactifications [26] could perhaps provide insight in the problem addressed here.

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