A Simple new Improved Lattice Action [a]

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We introduce a new improved action, by adding a 2×2 plaquette to the Symanzik action, for which the gauge field propagator greatly simplifies. We present the tree-level parameters of this new action and its Lambda parameter.

1 Introduction

Lattice gauge theory [1] has proven to be a viable tool for non-perturbative studies in field theories and non-Abelian gauge theories in particular. Despite the tremendous increase of computer power since the first Monte Carlo calculations [2], it remains a technical challenge to make the lattice spacing a small and the physical volume large enough. Instead one can use the increased computer power of today to study alternative lattice actions, which are chosen to remove as much as possible the scaling violations introduced by a finite lattice cutoff. The computational overhead in using more complicated lattice actions is usually not more than a few to ten times the cost for the Wilson action.

2 Symanzik improvement

The notion of improved actions was introduced more than a decade ago [3,4]. The last few years have seen a surge in applying these ideas in actual simulations [5]. When considering a lattice action, here restricting ourselves to pure gauge theories, the dynamical variables live on the (directed) links of the lattice and are elements of the gauge group $U_\mu(x)$. The connection to a continuum configuration is provided in terms of parallel transport of the vector potential $A_\mu(x)$ along the link $x \to x+\hat{\mu}$ ($\hat{\mu}$ the directional vectors of the lattice)

$$U_\mu(x) = P\exp(\int_0^a A_\mu(x+s\hat{\mu})ds) \ . \tag{1}$$

This allows us to express any of the lattice actions in powers of the lattice spacing. Terms that are of order a^4 (in four dimensions), being the volume of a unit cell on the lattice, correspond to the naive continuum limit. Note that apart from the corrections to the action density, sums need to be converted to

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integrals (which for smooth fields would only give exponential corrections, but for rough fields is part of the renormalization procedure) and the measure for integration over the fields needs to be converted from (compact) link variables to vector potentials. We note that modifications in the integration measure can, in principle, be absorbed in the action.

There is a large redundancy in the Symanzik improvement scheme. At tree-level there are many ways of choosing the action in terms of the gauge invariant Wilson loops (formed by multiplying the link variables along the loop), that extend further than a single plaquette, such that the relative order a^2 correction has a vanishing coefficient. The class of actions we will consider is defined by ($<>$ implies averaging over orientations [6])

$$\begin{aligned}
S(\{c_i\}) \;\equiv\;& \sum_x \mathrm{Tr}\{c_0 \left\langle 1- \right\rangle + 2c_1 \left\langle 1- \right\rangle + \frac{4}{3}c_2 \left\langle 1- \right\rangle \\
&+ 4c_3 \left\langle 1- \right\rangle + c_4 \left\langle 1- \right\rangle \} \\
=\;& -\frac{a^4}{2}(c_0 + 8c_1 + 8c_2 + 16c_3 + 16c_4) \sum_{x,\mu,\nu} \mathrm{Tr}(F^2_{\mu\nu}(x)) \\
&+\frac{a^6}{12}(c_0 + 20c_1 - 4c_2 + 4c_3 + 64c_4) \sum_{x,\mu,\nu} \mathrm{Tr}(\mathcal{D}_\mu F_{\mu\nu}(x))^2 \\
&+a^6(\frac{c_2}{3} + c_3) \sum_{x,\mu,\nu,\lambda} \mathrm{Tr}(\mathcal{D}_\mu F_{\mu\lambda}(x)\mathcal{D}_\nu F_{\nu\lambda}(x)) \\
&+a^6\frac{c_2}{3} \sum_{x,\mu,\nu,\lambda} \mathrm{Tr}((\mathcal{D}_\mu F_{\nu\lambda})^2) + \mathcal{O}(a^8)\ . \qquad (2)
\end{aligned}$$

We have added a 2×2 plaquette to the parametrization of the lattice action employed in ref. 4. Note that the invariance under field redefinitions allows one to choose $c_3 \equiv 0$, as was shown by Lüscher and Weisz [7].

3 The new improved action

To study perturbation theory on the lattice we write $U_\mu(x) = \exp(q_\mu(x))$ and expand the lattice action to second order in q. At tree level we have, as for the Lüscher-Weisz Symanzik action, $c_2 = c_3 = 0$, cf. Eq. 2. To remove the remaining $\mathcal{O}(a^2)$ irrelevant operator we have the three constants c_0, c_1 and c_4 at our disposal. One is eliminated by requiring the appropriate continuum limit without the need of rescaling the coupling constant. For the LW Symanzik action, corresponding to $c_4 = 0$, one finds $c_0 = 5/3$ and $c_1 = -1/12$. The disadvantage of this choice in perturbation theory becomes clear when writing the quadratic part of the action

$$\begin{aligned} S_2 &= \sum_{x,\mu,\nu} -\tfrac{1}{2}\mathrm{Tr}[c_0(\partial_\mu q_\nu(x)-\partial_\nu q_\mu(x))^2 + 2c_1\{(2+\partial_\mu)(\partial_\mu q_\nu(x)-\partial_\nu q_\mu(x))\}^2 \\ &\qquad + c_4\{(2+\partial_\nu)(2+\partial_\mu)(\partial_\mu q_\nu(x) - \partial_\nu q_\mu(x))\}^2] \;, \end{aligned} \tag{3}$$

where the lattice derivative ∂_μ is the difference operator

$$\partial_\mu \varphi(x) \equiv \varphi(x+\hat{\mu}) - \varphi(x) \;. \tag{4}$$

There is no "covariant" gauge condition that will make the gauge field propagator diagonal in the space-time indices. In analytic calculations this makes perturbative computations cumbersome. If we choose

$$c_4 \cdot c_0 = c_1^2 \;, \tag{5}$$

we can complete a square, from which we read off a simple gauge condition

$$\begin{aligned} S_2 = &-\sum_{x,\mu,\nu} c_0 \mathrm{Tr}[\partial_\mu q_\nu(x) \left(1+z(2+\partial_\mu^\dagger)(2+\partial_\mu)\right) \left(1+z(2+\partial_\nu^\dagger)(2+\partial_\nu)\right) \partial_\mu q_\nu(x)] \\ &+ \sum_x \mathrm{Tr}\mathcal{F}_{gf}^2(x), \quad \mathcal{F}_{gf}(x) \equiv \sqrt{c_0}\sum_\mu \partial_\mu^\dagger \left(1+z(2+\partial_\mu^\dagger)(2+\partial_\mu)\right) q_\mu(x), \end{aligned} \tag{6}$$

where $z \equiv c_1/c_0$. At tree level we find $c_0 = 16/9$, $c_1 = -1/9$, $c_2 = c_3 = 0$ and $c_4 = 1/144$. We call this new action the *square* Symanzik action or square action for short. The propagators are now easily seen to be diagonal in colour and space-time indices.

$$\begin{aligned} \text{Ghost :} \qquad & P(k) = \frac{1}{\sqrt{c_0}\sum_\lambda \left(4\sin^2(k_\lambda/2) + 4z\sin^2 k_\lambda\right)} \;, \\ \text{Vector :} \qquad & P_{\mu\nu}(k) = \frac{P(k)\delta_{\mu\nu}}{\sqrt{c_0}\,(1+4z\cos^2(k_\mu/2))} \;. \end{aligned} \tag{7}$$

To determine the Lambda parameter for the new improved action, it is sufficient to compute the effective action for a constant background field and study the finite difference in the renormalization as compared to (say) the Wilson action [8] in the limit of zero lattice spacing. In addition one can follow the alternative determination of the Lambda parameter through the heavy quark potential [9].

$$\begin{aligned} \Lambda_{S^2}/\Lambda_W &= 4.0919901(1) \quad \text{for SU(2)}, \\ \Lambda_{S^2}/\Lambda_W &= 5.2089503(1) \quad \text{for SU(3)}. \end{aligned} \tag{8}$$

The index S^2 denotes the square Symanzik action and W the Wilson action.

4 Outlook

From the practical point of view we presented an alternative action, motivated by the requirement to simplify the perturbative calculations, rather than to minimize the lattice artefacts. Nevertheless, comparisons will give a clue on how big the higher order corrections are. Of course at tree level it is trivial to write down many types of improved actions, so we have set out to compute the one-loop improvement coefficients too, to bring our new action to the same footing as the Lüscher-Weisz Symanzik improved action. For a more extended discussion see ref. 10. We did not discuss the independent development based on the classically perfect action, which also aims at correcting for lattice artefacts and yields promising results [11]. The method is based on Wilson's renormalization group approach [12], constructing the fixed-point action in the limit of zero coupling as an approximation to the renormalized trajectory.

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