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# On the Algebraic Characterization of Witten's Topological Yang-Mills Theory

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## ABSTRACT

We interpret in terms of "basic" cohomology the recently proposed supersymmetric, supergauge invariant formulation of topological Yang-Mills theory. Our interpretation shows that this formulation leads to the correct observables.

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## 1. Introduction

In a recent series of articles [1], E. Witten investigates the expression of various topological invariants in terms of local field theory. The first examples of this sort we know of are due to A.S. Schwarz [2] who gives a field theory expression for the Ray-Singer analytic torsion [3], and are related to the quantization of differential forms [4]. The situations considered by E. Witten are of a more exotic type and lead to essentially non-linear field theories, to be treated in the weak coupling regime. In principle, to obtain the sort of results one expects, a rigorous treatment of the renormalized perturbation expansion ought to be sufficient for a rigorous mathematical construction. Here, we shall be concerned with gauge fields and the recently discovered Donaldson invariants [5].

The following construction owes much to seminars by I.M. Singer, L. Baulieu [6] and P. Braam [7]. However, since local field theory is to be used [8], we find it necessary to characterize the model by a complete set of Ward identities. We believe that ref.[6], as well as subsequent proposals [9] are incomplete in this respect. The solution is to be found in an article by T.H. Horne [10]. The purpose of this note is to explain why, in more geometrical terms.

## 2. The differential algebra

As suggested in E. Witten's paper [1a] (eq. (2.41)) and emphasized in ref.[6], one wishes to gauge-fix a topological invariant, e.g.

$$S_{inv}(A) = \int_M \mathrm{tr}(F(A) \wedge F(A)) \quad , \tag{1}$$

where $F(A)$ is the curvature of a connection $A$ on a principal $G-$bundle $P(M, G)$, over a compact 4-manifold $M$, without boundary, and tr is an invariant polynomial over Lie $G$. The group $G$ is assumed to be compact.

The action $S_{inv}$ is, by essence, invariant under arbitrary variations of $A$:

$$\delta A = \psi \quad . \tag{2}$$

From now on, all fields are differential forms on $M$, taking values in ad(Lie $G$). One insists on gauge-fixing $S_{inv}$, leaving the gauge freedom pending till the end (because of the known Gribov difficulty), localizing the system on the self-dual connections. The corresponding gauge fixing term is:

$$S^{(1)} = \int_M \mathrm{tr}(b \wedge b - b \wedge F^- - \bar{\psi} \wedge (D\psi)^-) \quad , \tag{3}$$

where $F^-$ is the antiself-dual part of $F(A)$ - for some metric $g$ on $M$ -, $b$ and $\bar{\psi}$ are antiself-dual two-forms and $(D\psi)^-$ is the antiself-dual part of $D\psi$, the covariant differential of the one-form $\psi$.

The new action $S_{inv} + S^{(1)}$ is invariant under the Slavnov symmetry:

$$\begin{aligned} s_1 A = \psi \quad &, \quad s_1 \psi = 0 \quad , \\ s_1 \bar{\psi} = b \quad &, \quad s_1 b = 0 \end{aligned} \tag{4}$$

and satisfies

$$S^{(1)} = s_1 \int_M \mathrm{tr}(\bar{\psi} \wedge (b - F^-)) \quad . \tag{5}$$

It is still gauge invariant, i.e. invariant under:

$$\begin{aligned} \delta A &= D\omega \quad , \\ \delta \psi &= [\psi, \omega] \quad , \\ \delta \bar{\psi} &= -[\omega, \bar{\psi}] \quad , \\ \delta b &= [b, \omega] \quad , \end{aligned} \tag{7}$$

where $\omega \in \mathrm{Lie}\, \mathcal{G}$, $\mathcal{G}$ being the gauge group of $P(M, G)$. This yields the nilpotent $s_2$ operation:

$$\begin{aligned} s_2 A &= \psi - D\omega \quad , \\ s_2 \psi &= [\psi, \omega] \quad , \\ s_2 \bar{\psi} &= -[\omega, \bar{\psi}] + b \quad , \\ s_2 b &= [b, \omega] \quad , \\ s_2 \omega &= -\frac{1}{2}[\omega, \omega] \quad . \end{aligned} \tag{8}$$

The action $S_{inv} + S^{(1)}$ is invariant under $s_2$ and does not depend on $\omega$ (the Faddeev-Popov ghost for Lie $\mathcal{G}$). Except for the $\int b \wedge b$ term, it is invariant under ($\varphi$ is odd):

$$\begin{aligned} \delta b &= [\bar{\psi}, \varphi] \quad , \\ \delta \psi &= D\varphi \quad , \end{aligned} \tag{9}$$

which yields the nilpotent operation $s$ (now with $\varphi$ even):

$$
\begin{aligned}
sA &= \psi - D\omega \quad , \\
s\psi &= [\psi, \omega] - D\varphi \quad , \\
s\bar{\psi} &= -[\omega, \bar{\psi}] + b \quad , \\
sb &= [b, \omega] - [\bar{\psi}, \varphi] \quad , \\
s\varphi &= [\varphi, \omega] \quad , \\
s\omega &= -\frac{1}{2}[\omega, \omega] + \varphi \quad .
\end{aligned} \tag{10}
$$

It is easy to modify $S^{(1)}$ in such a way that it is invariant under (10). Following eq. (5) we find:

$$
\hat{S}^{(1)} = s\int_M \mathrm{tr}(\bar{\psi} \wedge (b - F^-)) = \int_M \mathrm{tr}(b \wedge b - b \wedge F^- - \bar{\psi} \wedge (D\psi)^- + \bar{\psi} \wedge [\bar{\psi}, \varphi]) \tag{11}
$$

Notice that in eq.(10), $sb$ needs a $\varphi$ dependent term in order for $s$ to be nilpotent. The $\varphi$ invariance can be gauge fixed in a gauge invariant way using the gauge function $D^*\psi$:

$$
\hat{S}^{(2)} = \int_M \mathrm{tr}({}^*\beta \wedge D^*\psi + {}^*\bar{\varphi} \wedge (D^*D\varphi + [{}^*\psi, \psi]) \tag{12}
$$

Including $\hat{S}^{(2)}$ in the action, one gets the Slavnov symmetry defined by (10), together with:

$$
\begin{aligned}
s\bar{\varphi} &= \beta + [\bar{\varphi}, \omega] \quad , \\
s\beta &= -[\omega, \beta] + [\varphi, \bar{\varphi}] \quad ,
\end{aligned} \tag{13}
$$

such that we have the following expression:

$$
\hat{S}^{(1)} + \hat{S}^{(2)} = s\int_M \mathrm{tr}({}^*\bar{\varphi} \wedge D^*\psi + \bar{\psi} \wedge (b - F^-)) \quad . \tag{14}
$$

A few remarks are in order:

*i.* $S_{inv} + \hat{S}^{(1)} + \hat{S}^{(2)}$ is not quite the most general $\omega$ independent, gauge invariant, renormalizable action invariant under $s$ - actually of the form $S_{inv} + sS_{gf}$: one may add an extra term compatible with ghost number neutrality and renormalizability as in refs. [1,6,9,10] and of the form $s\int \mathrm{tr}(\beta[\varphi, \bar{\varphi}])$. Both $\omega$ independence and gauge invariance are essential.

*ii.* Changing generators according to

$$\begin{aligned}
\psi' &= \psi - D\omega \quad , \\
b' &= b - [\omega, \bar{\psi}] \quad , \\
\varphi' &= \varphi - \frac{1}{2}[\omega, \omega] \quad , \\
\beta' &= \beta + [\bar{\varphi}, \omega] \quad ,
\end{aligned} \tag{15}$$

the $s$-operation assumes the form:

$$\begin{aligned}
sA &= \psi' \quad , \quad s\psi' = 0 \quad , \\
s\bar{\psi} &= b' \quad , \quad sb' = 0 \quad , \\
s\omega &= \varphi' \quad , \quad s\varphi' = 0 \quad , \\
s\bar{\varphi} &= \beta' \quad , \quad s\beta' = 0 \quad .
\end{aligned} \tag{16}$$

It therefore has vanishing cohomology as well as vanishing cohomology mod $d$. The desired cohomology [1] is, however, not the local cohomology of $s$ mod $d$, but its restriction to $\omega$ independent, gauge invariant objects. What is involved is equivariant cohomology [7,11], or rather its original form, namely "basic" cohomology [12], which is exactly adapted to the present local field theory context, as we shall see.

*iii.* It is interesting to observe that $s$ can be split into a sum of two anticommuting differentials and that the algebra can be cast in a supersymmetric form, which is *distinct* from that of ref. [10], if we insist that, as we will demonstrate, $s$ generates the supersymmetry. However, the superfield content will be that of ref. [10] (without imposing the gauge condition $\omega = 0$). Details are given in section 4.

### 3. The "basic" cohomology of $s$

The differential algebra defined by the structure equations (10) and (13) has the following property, which makes it a differential algebra with an action of the gauge Lie algebra; for $\lambda \in \mathrm{Lie}\,\mathcal{G}$ define:

$$\begin{aligned}
\delta_\lambda \psi &= [\psi, \lambda] \quad , \quad \delta_\lambda A = D\lambda \quad , \\
\delta_\lambda b &= [b, \lambda] \quad , \quad \delta_\lambda \bar{\psi} = -[\lambda, \bar{\psi}] \quad , \\
\delta_\lambda \varphi &= [\varphi, \lambda] \quad , \quad \delta_\lambda \omega = -[\lambda, \omega] \quad , \\
\delta_\lambda \bar{\varphi} &= [\bar{\varphi}, \lambda] \quad , \quad \delta_\lambda \beta = -[\lambda, \beta] \quad .
\end{aligned} \tag{17}$$

Define also for $\lambda \in \text{Lie}\,\mathcal{G}$, $\iota_\lambda$ by

$$\iota_\lambda A = \iota_\lambda \psi = \iota_\lambda \bar{\psi} = \iota_\lambda b = \iota_\lambda \varphi = \iota_\lambda \bar{\varphi} = \iota_\lambda \beta = 0 \quad ,$$
$$\iota_\lambda \omega = \lambda \quad . \tag{18}$$

One can easily check that

$$\delta_\lambda = \iota_\lambda s + s \iota_\lambda \tag{19}$$

and one has the classical [12] commutation rules:

$$\begin{aligned} [\iota_\lambda, \iota_\mu]_+ &= 0 \quad , \\ [\delta_\lambda, \delta_\mu]_- &= \delta_{[\lambda,\mu]} \quad , \\ [\delta_\lambda, \iota_\mu]_- &= \iota_{[\lambda,\mu]} \quad , \\ [s, \iota_\lambda]_+ &= \delta_\lambda \quad , \\ [s, \delta_\lambda]_- &= 0 \quad . \end{aligned} \tag{20}$$

This makes $\{\delta_\lambda, \iota_\mu | \lambda, \mu \in \text{Lie}\,\mathcal{G}\}$ into a graded Lie algebra. Recall that $S_{tot} = S_{inv} + \hat{S}^{(1)} + \hat{S}^{(2)}$ fulfils:

$$s S_{tot} = \delta_\lambda S_{tot} = \iota_\mu S_{tot} = 0 \quad , \quad \lambda, \mu \in \text{Lie}\,\mathcal{G} \quad . \tag{21}$$

In technical terms $S_{tot}$ is a "basic" [12] local functional for the differential structure (10),(13), with the Lie $\mathcal{G}$ action defined by (17),(18).

Now let us turn $\lambda$ and $\mu$ into ghosts, in the usual fashion ($\lambda$ odd and $\mu$ even) and define

$$W = \delta + \iota \quad , \tag{22}$$

where $\delta$ and $\iota$ are obtained by (17),(18) on all fields except for $A$, $\omega$, $\lambda$ and $\mu$ for which

$$\begin{aligned} WA &= -D\lambda \quad , \\ W\omega &= -[\lambda, \omega] - \mu \quad , \\ W\lambda &= -\frac{1}{2}[\lambda, \lambda] \quad , \\ W\mu &= [\mu, \lambda] \quad . \end{aligned} \tag{23}$$

One easily shows that $W^2 = 0$. Adjoining $\lambda$ and $\mu$ as new generators to our differential algebra, we still have a choice to define $s\lambda$ and $s\mu$. In particular, if we define $s$ on $\lambda$ and $\mu$ by:

$$s\lambda = \mu \quad , \quad s\mu = 0 \quad , \tag{24}$$

we obtain

$$[s, W]_+ = 0 \quad . \tag{25}$$

The comparison with the supersymmetric formalism of ref. [10] is now straightforward. In terms of the primed variables defined in eq. (15), one may introduce the superfields:

$$\begin{aligned} \mathcal{A}_x &= A + \theta\psi' \quad , \\ \mathcal{A}_\theta &= \omega + \theta\varphi' \quad , \\ \bar{\Psi} &= \bar{\psi} + \theta b' \quad , \\ \bar{\Phi} &= \bar{\varphi} + \theta\beta' \quad . \end{aligned} \tag{26}$$

Then one has

$$s = \frac{\partial}{\partial\theta} \quad . \tag{27}$$

The supergauge transformation ghost

$$\Lambda = \lambda + \theta\mu \quad , \tag{28}$$

fulfils $W\Lambda = -\frac{1}{2}[\Lambda, \Lambda]$ and $s$ still acts on $\Lambda$ by $\partial/\partial\theta$. W acts on all fields by supergauge transformations, with $\mathcal{A}_x$, $\mathcal{A}_\theta$ a superconnection and $\bar{\Psi}$, $\bar{\Phi}$ transforming under the adjoint representation.

In terms of the unprimed variables the action and local cohomology mod $d$ are characterized by $\omega$ independence and gauge invariance, as we have already remarked. In terms of the primed variables and the supersymmetric formulation of ref. [10], this is equivalent to supersymmetry (invariance under $\partial/\partial\theta$) and supergauge invariance. So, this equivalence proves in particular that the supersymmetric supergauge invariant cohomology is identified with the "basic" cohomology, which is known to be correct ([1,6,7]). We refer to ref. [10] for the $s$-invariant gauge fixing of $W$.

**4. An alternative supersymmetry**

In this section we discuss the alternative supersymmetry mentioned at the end of section 2. One may split $s$ in eqs. (10,13) as

$$s = \sigma + w \quad , \tag{29}$$

with:

$$
\begin{aligned}
\sigma A = \psi \quad &, \quad \sigma\psi = 0 \quad , \\
\sigma\bar{\psi} = b \quad &, \quad \sigma b = 0 \quad , \\
\sigma\omega = \varphi \quad &, \quad \sigma\varphi = 0 \quad , \\
\sigma\bar{\varphi} = \beta \quad &, \quad \sigma\beta = 0 \quad .
\end{aligned}
\tag{30}
$$

and

$$
\begin{aligned}
w\psi = [\psi,\omega] - D\varphi \quad &, \quad wA = -D\omega \quad , \\
wb = [b,\omega] - [\bar{\psi},\varphi] \quad &, \quad w\bar{\psi} = -[\omega,\bar{\psi}] \quad , \\
w\varphi = [\varphi,\omega] \quad &, \quad w\omega = -\frac{1}{2}[\omega,\omega] \quad , \\
w\bar{\varphi} = [\bar{\varphi},\omega] \quad &, \quad w\beta = -[\omega,\beta] + [\varphi,\bar{\varphi}] \quad .
\end{aligned}
\tag{31}
$$

One can easily check that $\sigma^2 = w^2 = [\sigma, w]_+ = 0$. This structure suggests the use of a supersymmetric formalism. Let:

$$
\begin{aligned}
\mathbf{A} &= A + \theta\psi \quad , \\
\mathbf{\Omega} &= \omega + \theta\varphi \quad , \\
\bar{\mathbf{\Psi}} &= \bar{\psi} + \theta b \quad , \\
\bar{\mathbf{\Phi}} &= \bar{\varphi} + \theta\beta \quad .
\end{aligned}
\tag{32}
$$

Then, in terms of the superfields:

$$
\sigma = \frac{\partial}{\partial\theta} \quad , \tag{33}
$$

and $w$ is a supergauge transformation:

$$
\begin{aligned}
w\mathbf{A} &= -\mathbf{D}(\mathbf{A})\mathbf{\Omega} \quad , \\
w\mathbf{\Omega} &= -\frac{1}{2}[\mathbf{\Omega},\mathbf{\Omega}] \quad , \\
w\bar{\mathbf{\Psi}} &= -[\mathbf{\Omega},\bar{\mathbf{\Psi}}] \quad , \\
w\bar{\mathbf{\Phi}} &= [\bar{\mathbf{\Phi}},\mathbf{\Omega}] \quad ,
\end{aligned}
\tag{34}
$$

where the covariant differential $\mathbf{D}(\mathbf{A})$ is given by

$$
\mathbf{D}(\mathbf{A})\mathbf{\Omega} = d\mathbf{\Omega} + [\mathbf{A},\mathbf{\Omega}] \quad . \tag{35}
$$

Eq. (34) defines a differential superalgebra with a super Lie algebra action in terms of:

$$\mathbf{\Lambda} = \lambda + \theta\mu \quad , \quad \lambda, \mu \in \mathrm{Lie}\,\mathcal{G} \quad . \tag{36}$$

We define $\delta_{\mathbf{\Lambda}}$ according to

$$\begin{aligned} \delta_{\mathbf{\Lambda}}\mathbf{A} &= \mathbf{D}(\mathbf{A})\mathbf{\Lambda} \quad , \\ \delta_{\mathbf{\Lambda}}\mathbf{\Omega} &= [\mathbf{\Omega}, \mathbf{\Lambda}] \quad , \\ \delta_{\mathbf{\Lambda}} &= [\bar{\mathbf{\Psi}}, \mathbf{\Lambda}] \quad , \\ \delta_{\mathbf{\Lambda}} &= [\bar{\mathbf{\Phi}}, \mathbf{\Lambda}] \end{aligned} \tag{37}$$

and $\iota_{\mathbf{\Lambda}}$ according to

$$\iota_{\mathbf{\Lambda}}\mathbf{A} = \iota_{\mathbf{\Lambda}}\bar{\mathbf{\Psi}} = \iota_{\mathbf{\Lambda}}\bar{\mathbf{\Phi}} = 0 \quad , \iota_{\mathbf{\Lambda}}\mathbf{\Omega} = \mathbf{\Lambda} \quad . \tag{38}$$

Then we have:

$$\delta_{\mathbf{\Lambda}} = [w, \iota_{\mathbf{\Lambda}}] \quad . \tag{39}$$

## 5. Concluding remarks

The algebraic set-up proposed in ref. [10] has been shown to describe the "basic" cohomology adapted to the characterization of a perturbative treatment [8] of the situation described by E. Witten [1] in terms of equivariant cohomology [7,11]. There are two heavy technical problems to be dealt with:

*i*. Perturbative renormalization theory for a field theory associated with an arbitrary compact manifold without boundary in a particular topological sector.

*ii*. The proper treatment of different vacua and the inclusion in the $s-W$ operation of global zero modes, that ought to make the theory not completely empty.


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