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An Introduction to Topological Yang-Mills Theory†

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ABSTRACT

In these lecture notes we give a "historical" introduction to topological gauge theories. Our main aim is to clearly explain the origin of the Hamiltonian which forms the basis of Witten's construction of topological gauge theory. We show how this Hamiltonian arises from Witten's formulation of Morse theory as applied by Floer to the infinite dimensional space of gauge connections, with the Chern-Simons functional as the appropriate Morse function(al). We therefore discuss the De Rham cohomology, Hodge theory, Morse theory, Floer homology, Witten's construction of the Lagrangian for topological gauge theory, the subsequent BRST formulation of topological quantum field theory and finally Witten's construction of the Donaldson polynomials.

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1. Introduction

Topological quantum field theories are field theories that have at most a finite number of degrees of freedom (in particular there are no propagating physical states), so a legitimate question is then, why one should be interested in them. Probably a more than adequate justification comes from the way these theories were first discovered by Witten [1] in the context of Yang-Mills gauge theories. This justification is, however, mathematical in nature and is a prime example of the extremely fruitful interactions between physics and mathematics. Crudely stated, because there is no dynamics, the quantum field theory can only be sensitive to invariants of the basis manifold on which the theory is defined. As we will see, a topological quantum field theory can be defined as a field theory on a smooth manifold which is independent of additional structures, such as a metric, on the basis manifold. Thus appropriate observables will have expectation values independent of the metric, and will hence give invariants. As will be discussed in the last section, for Yang-Mills gauge theory over a four manifold, these will lead to the invariants considered by Donaldson [2], who used them to distinguish different differential structures. Hence these are called differential invariants, although one also encounters the name topological invariants (which are of course, strictly speaking, weaker).

By now, many types of topological field theories have been constructed [3,4]. There are two reasons why some of them might be interesting from the physical point of view. One argument by Witten [1] is that these theories have general covariance built in without having to integrate over the space of all metrics. It is this integration over all metrics which forms the obstacle to finding a quantum theory of gravity. The hope [1] is now that one might be able to find a suitable topological quantum field theory in which this general covariance is spontaneously broken, and a dynamical theory which includes gravity might arise. However, one should realize that since topological field theories have more or less by definition no dynamics, it will be very hard to find a mechanism for this spontaneous breaking of the general covariance, unless one "tinkers by hand" with the theory. The other reason for its physical significance lies in the beautiful connection of the pure Chern-Simons theories in three dimensions with the conformal field theories in two dimensions [4]. Hence, one might envisage this as a means to classify conformal field theories [5]. Whether this will give a complete classification is not clear yet. From the mathematical side, this connection between three and two dimensions is relevant for the knot invariants (Jones polynomials) [6]. The description of knots in three dimensions is more natural than its traditional two-dimensional formulation. This higher ventage point might resolve also many riddles related to the connection between Yang-Baxter equations, braid groups and conformal field theories [7]. It is therefore not surprising that this corner of topological quantum field theories is attracting most attention. However, we will concentrate ourselves in these lecture notes on topological Yang-Mills theory, because this is where the development started.

It is maybe instructive to sketch the history of topological Yang-Mills theory. The development of these ideas grew out of the study of harmonic forms in increasingly complex situations. Traditionally, harmonic forms have played a very important role both in mathematics and in physics. For example in three dimensions, they form the solutions to the Laplace equations, which are essential in the study of electrostatics and as we will show, the solutions to the Maxwell equations can be seen as harmonic 2-forms. From the mathematics point of view, harmonic forms are important for the study of topological invariants. Hodge theory relates the number of independent harmonic p-forms to the cohomology $H^p(M)$ of the compact and smooth manifold M, which in turn is dual to the homology $H_p(M)$. The homology can be defined for any orientable manifold by a topological construction. We will review homology and De Rham cohomology in section 2, to make these lectures more or less self contained. The reason is that (co-)homology is the central theme in all these developments. In five different settings we will introduce operators whose square is zero (they are called (co-)boundary operators). This is all one needs to define a (co-)homology.

Morse theory gives another way of studying the topology of a manifold, by studying the critical points for a generic function (called a Morse function) on the manifold M. For example, one can easily determine the Euler number of a manifold from Morse theory, but in general it gives only inequalities for the Betti numbers B_p, where B_p is the dimension of $H^p(M)$ (or by Hodge theory the number of harmonic p-forms on M). This is where physics re-entered mathematics, due to Witten's [8] analysis of supersymmetry breaking. Especially his study of a supersymmetric non-linear sigma model led to a formulation of Morse theory, based on studying harmonic forms constructed from an exterior algebra intertwined with a

Morse function. This exterior algebra was basically equivalent to the supersymmetry algebra (in the zero-momentum sector) and the harmonic forms are the zero-energy states for the Hamiltonian. The Witten index, which is a measure for supersymmetry breaking (to break supersymmetry the Witten index needs to be zero), can thus be shown to be the Euler characteristic. Topology was therefore in the way for supersymmetry breaking. However, Witten [9] realized that his formulation of Morse theory, which was based on tunnelling in supersymmetric quantum mechanics, was a powerful mathematical tool. Instead of bounds (weak Morse inequalities), it allowed one to obtain the Betti numbers directly from Morse theory (related to the strong Morse inequalities). This was based on constructing a homology based on tunnelling, as we will review in some detail in section 3. Another instance where physical questions have strongly stimulated mathematical development has been the study of instantons [10]. These are solutions to the (anti-)self-duality equations for non-Abelian gauge theories on four manifolds. Many powerful mathematical ideas were bundled in algebraically constructing the set of all solutions on S^4 for a given topological charge k (Pontryagin or Chern class), called the moduli space $\mathcal{M}_k$. This is the Atiyah-Drinfeld-Hitchin-Manin [11] construction. The instanton moduli spaces were used by Donaldson [2] to construct powerful differential invariants for four-manifolds. Four-dimensional manifolds are particularly notorious for their difficulty in classifying differential structures. For example, in five or more dimensions, fixing the topology will fix the differential structure up to finitely many possibilities (see for a review [12]). Note that the self-duality equations, in a sense, generalize the study of harmonic two-forms and it is therefore natural (maybe with some hindsight) that the added non-Abelian group structure will lead to more refined invariants.

Since in the Hamiltonian formulation of gauge theories, one has a three-manifold as a basis manifold, Floer (after Taubes) [13] asked himself whether one could similarly construct invariants for three manifolds. His answer was affirmative, beautifully combining the Yang-Mills gauge theories with Witten's analysis of Morse theory. We will outline this development in section 4. Basically, it amounts to considering the exterior algebra on the infinite dimensional manifold of gauge equivalence classes of connections (gauge potentials) on the three-manifold. Then he intertwines this, as in Witten's finite dimensional analysis, with a Morse function(al), for which he chooses the Chern-Simons functional. The resulting homology amounts to studying the zero-energy solutions of a "supersymmetric" Hamiltonian, whose bosonic part is nothing but the pure Yang-Mills Hamiltonian. Not surprisingly, the tunnelling analysis in this infinite dimensional formulation of Morse theory is precisely described by the (anti-)self-duality equations. For a review see also [14].

Atiyah and Donaldson now realized that there was a connection between this Floer homology and the Donaldson invariants, for particular four-manifolds. Since the Floer homology is naturally connected to field theory and since instantons also play a natural role, Atiyah [15] asked the question whether it would be possible to find the Lorentz invariant formulation of the Floer Hamiltonian, that is to find a Lagrangian, whose Hamiltonian will be the one which comes from the Floer theory. The euclidean formulation on closed four manifolds is then likely to lead to Donaldson invariants. This question was answered by Witten [1] in the affirmative (as well as [4] Atiyah's question whether there was a three-dimensional field theory, which would lead to the formulation of the Jones polynomials). Thus were born topological field theories. One thing is for sure, despite the fact that many details will still need to withstand the test of mathematical rigour, it will have a large impact on various areas of mathematics. The excitement is based on the fact that Witten's work gives explicit formulas for Donaldson and Jones polynomials, and especially in the latter case it provides many clues for generalizations.

In section 5 we will discuss Witten's construction of the topological Yang-Mills action. It was later realized [16], that there was an underlying BRST symmetry and that one could view Witten's Lagrangian as coming from the gauge fixing of an action which is given purely by the topological charge ($\frac{1}{8\pi^2}\int_M \mathrm{Tr}(F \wedge F)$) which is obviously independent of the metric. We also mention the underlying equivariant or basic cohomology [17] which can be formulated in terms of this BRST or Slavnov symmetry [18]. Finally in section 6 we will consider Witten's construction [1] of the Donaldson polynomials. We will not attempt to describe Donaldson's [2] original formulation.

These notes are mainly intended for a readership of physicists. We attempt to use as much as possible physical intuition to outline the various developments, in order to hide the author's inadequacy in achieving mathematical rigour. Nevertheless, he hopes that these lecture notes will contribute, not only to his own, but also to the reader's understanding of this new field at the borderline between physics and mathematics.

2. De Rham cohomology and Hodge theory

As the best known example of cohomology we discuss De Rham cohomology, denoted by $H^p(M,R)$, where M is an n dimensional manifold and p runs from 0 to n. (The R stand for the real numbers, but other types like integer cohomology will not concern us here.) The manifold actually needs a differential structure to define De Rham cohomology, but the invariants will turn out not to depend on this structure. We can now define the space of smooth p-forms Λ_p, where we recall that a p-form ω can locally be written as:

$$\omega = \omega_{i_1,i_2,\cdots,i_p} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p} \quad . \tag{1}$$

The indices i_j run from 1 to n, $\omega_{i_1,i_2,\cdots,i_p}$ is antisymmetric in its indices and we use Einstein's summation convention throughout this paper. The space of 0-forms are simply the set of all (differential) functions on M and the volume form for M is an example of a n-form. We can now introduce the exterior derivative $d : \Lambda_p \to \Lambda_{p+1}$ through the following local definition:

$$d\omega = \frac{\partial \omega_{i_1,i_2,\cdots,i_p}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \quad . \tag{2}$$

For later purposes we rewrite this in an operator form as:

$$d\omega = a^{*j} \frac{\partial \omega}{\partial x^j} \quad , \tag{3}$$

where the operator a^* is defined locally as:

$$a^{*j} \omega = dx^j \wedge \omega \quad . \tag{4}$$

The exterior derivative d is called a coboundary operator (for reasons which will become clear shortly) and it satisfies the important property that it squares to zero: $d^2 = 0$. This implies the following crucial relation

$$\text{im } d|_{\Lambda_{p-1}} \subset \ker\, d|_{\Lambda_p} \quad , \tag{5}$$

where im d is the image of the operator d, i.e. $\text{im } d|_{\Lambda_{p-1}} = \{\omega \in \Lambda_p | \exists \lambda \in \Lambda_{p-1},\ \omega = d\lambda\}$, and ker d is the kernel of the operator d, i.e. $\ker d|_{\Lambda_p} = \{\omega \in \Lambda_p | d\omega = 0\}$. The following definition of De Rham cohomology will therefore make sense:

$$H^p(M,R) = \ker\, d / \text{im } d\ \cap \Lambda_p \quad . \tag{6}$$

In the future, as in eq. (6), we will assume the space on which d acts implicitly defined. Thus $H^p(M,R)$ is the set of closed ($d\omega = 0$), non-exact p-forms (ω cannot be written as $d\lambda$). The Betti numbers $B_p(M)$ are equal to the dimension of $H^p(M,R)$, that is the number of independent closed, non-exact p-forms. An example for a well-known topological invariant is the Euler characteristic:

$$\chi(M) = \sum_{q=0}^{n} (-1)^q B_q(M) \quad . \tag{7}$$

The best way to see that the Betti numbers are topological invariants is to note that the De Rham cohomology $H^p(M,R)$ is dual to the real homology $H_p(M,R)$, which can be defined purely topologically (provided M has an orientation). We will be a bit sloppy in its description and refer the reader to standard mathematics textbooks for more details on singular or simplicial homology. One considers the so-called cell complex C_p of p-dimensional oriented subspaces (cells) embedded in the manifold M. They can be formally added and multiplied with real numbers. Multiplying with -1 will change the orientation. Two cells can join into one if their orientations match along the boundaries. We can now define the boundary operator $\partial : C_p \to C_{p-1}$ (taking the boundary of a set reduces its dimension) which squares again to zero ($\partial^2 = 0$), since the boundary of a boundary is empty. Completely analogous to the De Rham cohomology one can now define the homology by:

$$H_p(M,R) = \ker\, \partial / \mathrm{im}\ \partial \ \cap C_p \quad , \tag{8}$$

which is called the set of p-cycles, that is the set of closed p-dimensional cells, that are not the boundary of some (p+1)-dimensional cell. Note that in homology ∂ decreases the grading by one whereas in cohomology d increases the grading by one. This is of course a more or less artificial distinction between homology and cohomology. However, there is a more precise relation between cohomology and homology, which shows that they are actually dual to each other and that the exterior derivative is the dual of the boundary operator. This is why the exterior derivative is also called a coboundary operator. The duality is established with the following bilinear form:

$$H_p(M,R) \times H^p(M,R) : (\gamma, \omega) \to \oint_\gamma \omega \in \ R \quad . \tag{9}$$

To show that this is indeed a proper map we have to show that $\oint_\gamma \omega$ does not change if we add the boundary of a (p+1)-cell α to γ or if we add the exterior derivative of a (p-1)-form

λ to ω. This follows by Stokes' law from the fact that ω is a closed p-form ($d\omega = 0$) and γ is a closed p-cell ($\partial\gamma = 0$). To be more precise, Stokes' theorem for integration yields:

$$\oint_{\partial\alpha} \omega = \oint_{\alpha} d\omega = 0 \quad \text{and} \quad \oint_{\gamma} d\lambda = \oint_{\partial\gamma} \lambda = 0 \quad , \tag{10}$$

which is easily seen to prove the above statements. One can also show that the bilinear form is non-degenerate, which establishes the duality between homology and cohomology and shows that the Betti numbers are topological invariants. Stokes' theorem also illustrates that the boundary operator is dual to the exterior derivative with respect to the bilinear form defined in eq. (9), i.e. $\oint_\gamma d\omega = \oint_{\partial\gamma} \omega$.

Before we discuss the Hodge theory we introduce a terminology which will be of use later. Two p-cycles $\gamma_{1,2}$ are called homologous if their difference γ_1-γ_2 is the boundary of a (p+1)-dimensional cell, which we illustrate in figure 1. As we have just seen, the integral of a closed p-form over γ_1 equals its integral over γ_2. That is, the integral only depends on the homology class of the p-cycle.

To discuss the Hodge theory, which basically states that B_p is the number of harmonic p-forms, we first have to define what is meant by a harmonic p-form. For this we need a metric g_{ij} on the manifold M, which is needed to define the adjoint d^* of d. If we talk about an adjoint operator, we need an inner product on the space of p-forms. This inner product is defined with the help of the Hodge $*$ operator $* : \Lambda_p \to \Lambda_{n-p}$, which in local coordinates reads as follows:

$$*\omega = \frac{1}{(n-p)!} \omega^{i_1, i_2, \cdots, i_p} \varepsilon_{i_1, \cdots, i_p, j_1, \cdots, j_{n-p}} dx^{j_1} \wedge \cdots \wedge dx^{j_{n-p}} \quad . \tag{11}$$

Note that we used the metric to raise the indices for ω. The inner product is now given by:

$$\Lambda_p \times \Lambda_p : (\alpha, \beta) \to <\alpha, \beta> = \int_M \alpha \wedge *\beta \quad . \tag{12}$$

Since $\alpha \wedge *\beta$ is a n-form the integral is well defined and the adjoint of d follows from:

$$<\alpha, d^*\beta> = <d\alpha, \beta> \quad , \quad \alpha \in \Lambda_p \quad , \quad \beta \in \Lambda_{p+1} \quad . \tag{13}$$

We have the following useful properties, which involve the Hodge $*$ operator on p-forms:

$$d^* = -(-1)^{n(p-1)} * d * \quad \text{and} \quad *^2 = (-1)^{p(n-p)} \quad . \tag{14}$$

It is now easily verified that in local coordinates one finds the following expression for $d^*\omega$, when ω is a p-form:

$$d^*\omega = -\sum_{j=1}^{p}(-1)^j g^{\ell i_j}\frac{\partial\omega_{i_1,\cdots,i_p}}{\partial x^\ell}dx^{i_1}\wedge\cdots\wedge d\hat{x^{i_j}}\wedge\cdots\wedge dx^{i_p} \quad , \tag{15}$$

where the hat over x^{i_j} means that this differential should be eliminated from the wedge product. Again for later purposes we can rewrite this in an operator formulation:

$$d^*\omega = -a^i\frac{\partial\omega}{\partial x^i} \quad . \tag{16}$$

The a^i act on a basis element $dx^{i_1}\wedge\cdots\wedge dx^{i_p}\in\Lambda_p$ by simply leaving out dx^i from this wedge product, after anti-commuting dx^i to the left (giving zero if dx^i does not occur). The $a^i(x)$ form a basis for the tangent space T_xM and act by exterior multiplication on the cotangent space T^*_xM, for which $a^{*i}(x)$ forms the basis dual to $a^i(x)$. We leave it to the reader to verify that these operators satisfy the Dirac algebra:

$$\{a^i,a^j\}=\{a^{*i},a^{*j}\}=0 \quad \text{and} \quad \{a^i,a^*_j\}=\delta^i_j \quad . \tag{17}$$

In the next section we will indeed see that it is natural to call a^i creation and a^{*j} annihilation operators. After all this preparation we are ready to define harmonic p-forms as those p-forms ω which satisfy the Laplace equation:

$$(d\ d^* + d^*\ d)\omega = 0 \quad . \tag{18}$$

One easily verifies that if ω is a 0-form, eq. (18) indeed reduces to the normal (covariant) Laplace equation, we are all so familiar with. In particular on a flat space, where the metric is constant, using eqs. (3,16,17) one finds $d\ d^* + d^*\ d = -\partial^2/\partial {x_i}^2$.

To prove the Hodge theorem, we first note that

$$\Lambda_p = \ker\ d\oplus \text{im}\ d^* \quad , \tag{19}$$

since $d\omega=0$ iff for all $\alpha\in\Lambda_{p+1}$ we have $<d\omega,\alpha>=0$, which is equivalent to $<\omega,d^*\alpha>=0$, or $\omega\in(\text{im}\ d^*)^\perp$. Using now the definition of $H^p(M,R)$ in eq. (6) we find the following direct sum decomposition for Λ_p:

$$\Lambda_p = \text{im}\ d\oplus H^p(M,R)\oplus\text{im}\ d^* \quad . \tag{20}$$

This then implies that $H^p(M,R)$ lies in the intersection of the orthogonal complements of im d and im d^*, i.e. $H^p(M,R) = \ker\, d \cap \ker\, d^*$. Hence it remains to prove that ω is harmonic iff $d\omega = d^*\omega = 0$, but this follows from the fact that if ω is harmonic, one has $0 =< \omega, (d\ d^* + d^*\ d)\omega >=< d^*\omega, d^*\omega > + < d\omega, d\omega >$, and from the fact that $< \alpha, \alpha >$ is always strictly positive, except for $\alpha = 0$.

Thus we have the interesting result that although one had to define a metric and a differential structure on the manifold M in order to define harmonic forms, the number of harmonic forms is actually independent of the choice of metric and differential structure, they are topological invariants. Harmonic forms occur naturally in physics, as we mentioned in the introduction. There we promised to show that the solutions to the Maxwell equations are equivalent to harmonic 2-forms. The Maxwell equations are defined in terms of the curvature $F_{\mu\nu}$ on four-dimensional space-time, where $E_i = F_{0i}$ is the electric and $B_i = \varepsilon_{ijk}F_{jk}/2$ the magnetic field. These Maxwell equations are equivalent to $d^*F = 0$, where the 2-form F is given in local coordinates by $F = F_{\mu\nu}dx^\mu \wedge dx^\nu/2$. On the other hand, we know that we can write the curvature in terms of a connection 1-form A: $F = dA$, where $A = A_\mu dx^\mu$ and A_0 is called the scalar potential, whereas A_i is called the vector potential in electrodynamics. This fact is easily seen to imply that the curvature, or field strength, F satisfies the constraint of the Bianchi identities $dF = 0$. We have just seen that $dF = d^*F = 0$ indeed implies that F is harmonic.

3. Morse theory and supersymmetric quantum mechanics

In this section we will discuss Morse theory as an alternative way to study topology of a manifold. In particular we will consider Witten's [9] formulation of Morse theory to prepare us for an attempt to understand Floer homology [13]. For a more detailed discussion of the mathematical aspects of Morse theory we refer to the literature [19].

If $h : M \to R$ is a generic function with isolated critical points $P_1, P_2, \cdots, P_q$, where a critical point is defined as a zero of the gradient vector field (i.e. $\partial h(P_i)/\partial x_k = 0$ for $k = 1, 2, \cdots, n = \dim M$), then the Morse index $\mu(P_i)$ of a critical point P_i is the number of negative eigenvalues for the Hessian of h evaluated at the point P_i (the Hessian for a function h is given by the matrix $\partial^2 h(x)/\partial x_i \partial x_j$). We define M_p as the number of critical points with Morse index p. The weak Morse inequality states the following result [19]: $M_p \geq B_p$

One can prove stronger versions of the Morse inequalities [19],

$$\sum_{p=0}^{n}(M_p - B_p)t^p = (1+t)\sum_{p=0}^{n} Q_p t^p \quad , \quad Q_p \geq 0 \quad , \tag{21}$$

They will not concern us here, but as an illustrative example we give a useful corollary of these inequalities. Namely, by substituting $t = -1$ one gets an equality for the Euler characteristic (compare with eq. (7))

$$\chi(M) = \sum_{q=0}^{n}(-1)^q M_q \quad . \tag{22}$$

It is instructive to illustrate the topological nature of this formula in an example. In figure 2 we have sketched a two-dimensional manifold M with two holes, such that its Euler characteristic is $\chi(M) = 2 - 2H = -2$, where H is the number of holes (or handles). Let us choose for h the height function (the gravitational potential) with respect to the horizontal plane. Then there are 6 critical points, of which the lowest one is obviously stable and has Morse index 0, the highest one is unstable with Morse index 2 and the remaining four critical points are saddle points, with one stable and one unstable direction, hence these have Morse index 1 (see figure 2). Therefore one has the result: $\sum_p(-1)^p M_p = M_0 - M_1 + M_2 = 1 - 4 + 1 = -2$, which is indeed the Euler characteristic. We can now give a rule of thumb which shows that it is indeed a topological invariant by pressing with our thumb at the maximum. Generically, this will create one additional local minimum and one additional saddle point (see figure 3). Therefore $\delta M_0 = 1$, $\delta M_1 = 1$, $\delta M_2 = 0$, but $\delta(\sum_p(-1)^p M_p) = 0$. It could happen that we press our thumb in such a way that the maximum will become degenerate with the saddle point, i.e. the maximum is obtained along a closed curve (however, try it and you will see that this is very hard to arrange, generically this will therefore not occur). There is so-called degenerate Morse theory which is able to deal with these situations. However, typically, this degeneracy is unstable against small perturbations, and for things to make sense topologically, definitions should not depend on these perturbations. Thus the only situation where one really has to worry about degenerate critical points is when there is a symmetry (for example axial symmetry along the vertical in figure 2) and perturbations are required to respect the symmetry. In that case one talks about equivariant Morse theory, this is however all we will say about it.

As we have already hinted at when we introduced the operators a^i and a^{*i}, there is an underlying supersymmetry in the exterior algebra. After all, the exterior derivative squares to zero, as does a supercharge Q, since it is an anticommuting object. Q changes the fermion number, whereas d changes the grading (defined by the p-forms). This then implies that one can identify p with the fermion number F. To make things more precise we have the following identifications:

$$\begin{aligned} (-1)^p &\leftrightarrow (-1)^F \\ d &\leftrightarrow Q_1 \\ d^* &\leftrightarrow Q_2 = Q_1^* \\ d\ d^* + d^*\ d &\leftrightarrow 2H = \{Q_1, Q_2\} \end{aligned} \tag{23}$$

Furthermore, since B_p is precisely the number of zero-energy p-forms, and since due to the supersymmetry, non-zero energy eigenstates will always occur in pairs of fermions and bosons, the Euler characteristic is exactly equal to the Witten index:

$$\sum_p (-1)^p B_p = \mathrm{Tr}((-1)^F) \quad . \tag{24}$$

This connection is more or less the reason Witten [9] discovered his formulation of Morse theory in terms of the exterior algebra, which will occupy the rest of this section.

Motivated by removing degeneracies in a study of the Witten index for supersymmetric sigma models [8], Witten introduced the following modification of the exterior algebra:

$$\begin{aligned} d &\to d_t = e^{-th} d\ e^{th} \\ d^* &\to d_t^* = e^{th} d^*\ e^{-th} \quad . \end{aligned} \tag{25}$$

One can then define $B_p(t)$ as the number of harmonic forms for the exterior algebra d_t, i.e.

$$B_p(t) = \dim\{\ker(d_t\ d_t^* + d_t^*\ d_t) \cap \Lambda_p\} \quad . \tag{26}$$

Clearly, $B_p(t)$ will depend continuously on t, however, $B_p(t)$ is a discrete function, hence it is independent of t and one can therefore find $B_p = B_p(0)$ by studying the vacua (zero-energy states) of the Hamiltonian

$$H_t = \frac{1}{2}(d_t\ d_t^* + d_t^*\ d_t) \quad , \tag{27}$$

in the limit of $t \to \infty$. This has a tremendous advantage, since we will find the wave functions in this limit to be highly peaked around the critical points of the Morse function h, such that in the lowest non-trivial order the wave function is a p-form with a harmonic oscillator type coordinate dependence, centered at a critical point. This immediately implies that in this approximation (which will neglect tunnelling) the number of vacua, which are in Λ_p, equals precisely M_p. The higher-order analysis can only have the effect of lifting the energy of some (or all) of the states thus constructed. This gives Witten's very simple proof of the weak Morse inequality, B_p(=the number of exact zero energy states) $\leq$ M_p(=the number of approximate zero energy states).

Let us now make these considerations more precise by calculating in some detail H_t from the exterior algebra and by performing the large t asymptotic expansion for the eigenstates of this Hamiltonian. In the next section it is this part of the analysis which allows generalization to the infinite dimensional context of the Yang-Mills configuration space, connected to the Floer homology. Let us first remind ourselves of the operator expressions for the exterior algebra:

$$d\omega = a^{*i}\frac{\partial\omega}{\partial x^i} \quad , \quad d^*\omega = a^i\frac{\partial\omega}{\partial x^i} \quad , \quad d^*\omega = -a^i\frac{\partial\omega}{\partial x^i} \quad , \quad \{a^i, a^{*j}\} = g^{ij} \quad . \tag{28}$$

With the help of these we have the following results:

$$\begin{aligned} d_t &= e^{-th} d\ e^{th} = d + ta^{*i}\partial h/\partial x^i \\ d_t^* &= e^{th} d^*\ e^{-th} = d^* + ta^i\partial h/\partial x^i \quad , \end{aligned} \tag{29}$$

which, when substituted in the expression for H_t, eq. (27) gives:

$$2H_t\ \omega = (d\ d^* + d^*\ d)\omega + t^2 g^{ij}\frac{\partial h}{\partial x^i}\frac{\partial h}{\partial x^j}\ \omega + t[a^{*i}, a^j]D_i D_j h\ \omega \quad . \tag{30}$$

To obtain this result we used:

$$\begin{aligned} d(a^i\frac{\partial h}{\partial x^i}\omega) &= a^{*j}\frac{\partial}{\partial x^j}(a^i\frac{\partial h}{\partial x^i})\omega - a^i\frac{\partial h}{\partial x^i}d\omega \\ d^*(a^{*j}\frac{\partial h}{\partial x^j}\omega) &= a^i\frac{\partial}{\partial x^i}(a^{*j}\frac{\partial h}{\partial x^j})\omega - a^{*j}\frac{\partial h}{\partial x^j}d\omega \end{aligned} \tag{31}$$

Finally D_i stands for the covariant derivative. By definition one has $D_i h = \partial h/\partial x^i$ and $\frac{\partial}{\partial x^j}(a^i D_i h) = a^i D_j D_i h$, which completes the derivation of eq. (30). The Hamiltonian has now acquired a potential term proportional to the square of the gradient of h. Its minima are therefore the critical points of the Morse function and for large t these minima become increasingly localized. We can expand around a critical point, and choose locally flat coordinates, such that $g_{ij}(x) = \delta_{ij} + \mathcal{O}(x^2)$ (which means that the connection Γ^i_{jk} vanishes to $\mathcal{O}(x)$) and $h(x) = h(0) + \lambda_i x_i^2/2 + \mathcal{O}(x^3)$. Note that the number of negative λ_i is precisely the Morse index of the critical point we are expanding about. Thus we get the following expansion for the Hamiltonian around the critical points:

$$2H_t = \sum_i \{-\frac{\partial^2}{\partial x_i^2} + t^2\lambda_i^2 x_i^2 + t\lambda_i[a_i^*, a_i]\} \quad , \tag{32}$$

which everybody will recognize as the n-dimensional harmonic oscillator (in diagonal form), plus something that commutes with that, which is easily seen from the following properties (no summation over i):

$$[a_i^*, a_i]dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p} = \pm dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p} \quad , \tag{33}$$

with the eigenvalue +1 if $i \in \{i_1, i_2, \cdots, i_p\}$ and -1 otherwise. From this one immediately obtains the spectrum for the Hamiltonian:

$$E_t = \frac{1}{2}t\sum_i (|\lambda_i|(1+2N_i) + \lambda_i n_i) + \mathcal{O}(t^0) \quad , \quad n_i = \pm 1 \quad , \quad N_i \in N \quad . \tag{34}$$

This could only give zero energy if $N_i = 0$ for all i and if $n_i = -\text{sign}\ \lambda_i$. Since the number of negative eigenvalues λ_i is precisely the Morse index p for the critical point we are expanding about, we have p indices i for which $n_i = +1$ and the eigenfunction is therefore a p-form. In the approximation we are currently working with, each critical point gives a suitable groundstate wave function whose energy vanishes to order t, and as asserted before there are M_p such wave functions in Λ_p.

One might think that higher order perturbation theory will remove the degeneracy among the zero-energy states, however, supersymmetry will actually guarantee that the energy will vanish to all orders in perturbation theory, and only tunnelling effects will be able to remove some of the degeneracies. The number of harmonic forms is determined by topology and this is consistent with the fact that perturbation theory is a local expansion, which is blind to the topology of the manifold. Tunnelling will involve paths that do probe large portions of the manifold and should be able to distinguish which critical points are removable. These considerations therefore motivate that one can refine the weak Morse inequalities by studying tunnelling for the Hamiltonian H_t. This is what we will consider next.

Before we discuss the tunnelling analysis, it is instructive to give the supersymmetric non-linear sigma-model. Its action is given by [8]:

$$\begin{aligned} S =& \frac{1}{2}\int d^2x\{g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi_j + ig_{ij}(\phi)\bar{\psi}^i\gamma^\mu D_\mu\psi^j + \frac{1}{12}R_{ik\ell j}(\phi)\bar{\psi}^i\psi^\ell\bar{\psi}^k\psi^j \\ & - g^{ij}(\phi)\frac{\partial h}{\partial\phi^i}\frac{\partial h}{\partial\phi^j} - D_iD_jh(\phi)\bar{\psi}^i\psi^j\} \quad , \end{aligned} \tag{35}$$

where the covariant derivative of the spinor fields is given by:

$$D_\mu\psi^i = \partial_\mu\psi^i + \Gamma^i_{jk}(\phi)\partial_\mu\phi^j\psi^k \quad . \tag{36}$$

For the Witten-index computation all non-zero energy levels have boson-fermion degeneracy, so that one can restrict oneself to the zero-momentum sector, which gives a supersymmetric quantum mechanics equivalent to the exterior algebra which we considered before. To be more precise, since ψ is a Majorana fermion, in the representation of the Dirac-matrices where $\gamma_0 = \mathrm{diag}(1,-1)$, ψ^i has two components, which are each others conjugate. If we call the upper component a^i, then Q and Q^* in this supersymmetric quantum mechanics are exactly equal to d and d^*. Note that ϕ^i are the coordinates on the target manifold M, which are to be identified with x^i in our earlier discussion of the exterior algebra. Originally Witten introduced the Morse function h to get rid of a degeneracy in the classical potential (which is zero for all ϕ^i if $h=0$). In the superfield formulation of the supersymmetric sigma model h appears simply as a magnetization:

$$S = \frac{1}{2}\int d^2xd^2\theta\{g_{ij}(\Phi)\bar{D}\Phi^iD\Phi^j + h(\Phi)\} \quad , \tag{37}$$

where the superfield and superderivative are given by:

$$\Phi^i = \phi^i + \bar{\theta}\psi^i + \bar{\theta}\theta F^i/2 \quad , \quad D_\alpha = \partial/\partial\bar{\theta}_\alpha - i(\bar{\theta}\gamma^\mu)_\alpha\partial_\mu \quad . \tag{38}$$

We will now discuss the tunnelling calculation. For $|P_i>$ the perturbative vacua we constructed before, we can compute the matrix elements $<P_i|d_t|P_j>$, which will allow us to calculate the Hamiltonian with respect to this basis. This matrix element can be calculated approximately by an instanton calculation, but will in general be zero due to the fermionic zero-modes. $< P_i|d_t|P_j >$ will only be non-zero if the instantons relevant for the tunnelling from P_i to P_j have exactly one fermionic zero-mode, which will be absorbed by d_t. Due to the supersymmetry, the number of fermionic zero-modes equals the number of bosonic zero-modes. There is always at least one bosonic zero-mode, which is related to the invariance of the instanton solution under time-translation. We cannot have more, otherwise the matrix element of interest would vanish. Therefore, the only instantons relevant for computing $< P_i|d_t|P_j >$ correspond to tunnelling paths that are isolated. They form a discrete moduli space, see figure 4 for an example.

We can now make use of a version of the index theorem [20] relevant to the present tunnelling analysis. It states that the number of fermionic zero-modes is equal to the spectral flow of the Hessian $D_iD_jh(x)$ along the tunnelling path from P_i to P_j; in more mundane terminology, this spectral flow is the number of eigenvalues that change sign, when following the Hessian along the tunnelling path from one critical point to the other. Since the number of negative eigenvalues at a critical point equals its Morse index, the only tunnelling paths we need to consider are those between critical points for which the respective Morse indices differ by 1. To be even more precise the instanton path has to go from P_i, with Morse index $p+1$, *down* the gradient lines of h to P_j, with Morse index p. One easily verifies that the instantons are determined by the equations:

$$\frac{dx^i(\tau)}{d\tau} = -g^{ij}(x(\tau))\frac{\partial h(x(\tau))}{\partial x^j} \quad , \tag{39}$$

with the boundary conditions $x(-\infty) = P_i$ and $x(\infty) = P_j$. The total action is found to be $S = t(h(P_i) - h(P_j)) > 0$.

The steepest descent approximation for the instanton calculation [8,9,21] gives us:

$$< P_i|d_t|P_j > = \sum_r K(r) \exp[-t(h(P_i) - h(P_j))] \quad , \tag{40}$$

where r runs over the discrete set of instantons and K(r) is the prefactor, which one obtains by calculating the determinant of the quadratic piece in the action, when expanding around the instantons (taking into account the transverse fluctuations) and a factor coming from the zero-modes. One can now show that the supersymmetry will cancel the following two contributions to K(r) up to a possible overall minus sign, namely those coming from the fermionic and the bosonic transverse fluctuations and the contribution coming from the fermionic and bosonic zero-mode (a subtle factor may remain, affecting the normalization of $|P_k>$). Hence for each tunnelling path, $K(r) = \pm 1$, so that one has the following result:

$$< P_i|d_t|P_j >= n(P_j, P_i) \exp[-t(h(P_i) - h(P_j))] \quad , \tag{41}$$

where $n(P_j, P_i)$ is an integer. To determine the sign that each instanton will contribute requires a more careful analysis of the instanton calculation and involves the parallel transport of a certain tangent frame along the tunnelling path, which allows one to compare the orientation of the frame at P_i, to the frame at P_j. These notions all occur quite naturally in the WKB analysis involved in the tunnelling calculation [22], but we will only state its result (see figure 4). Let P_i be the critical point with index $p+1$, and V_i the (p+1)-dimensional subspace of the tangent space at P_i, spanned by the eigenvectors of the Hessian with a negative eigenvalue (V_i is the tangent space of the unstable manifold associated with P_i). Let v be the tangent vector to the tunnelling path at P_i, then the tangent frame we wish to transport along the tunnelling path is $V_i \cap v^{\perp}$, which is p-dimensional and can therefore be compared with the p-dimensional space V_j (the tangent space to the unstable manifold, associated to the critical point P_j). The orientations of the spaces V_i and V_j are the ones induced respectively by the (p+1)-form and the p-form, which arise as the eigenfunctions in the perturbative analysis. This prescription gives us a unique way of determining $n(P_j, P_i)$ and Witten used these to define a new cohomology (called twisted cohomology [9]):

$$\delta|Q >= \sum_{P \in W_{p+1}} n(Q, P)|P > \quad , \tag{42}$$

where $Q \in W_p$ and the W_p form the so-called Witten complex:

$$W_p = \{|P > |\mu(P) = p\} \quad . \tag{43}$$

Thus $\delta : W_p \to W_{p+1}$ and the matrix elements of δ are precisely those of d_t, with respect to the set of perturbative vacua $= \cup_p W_p$. Consequently $\delta^2 = 0$ and we can form a cohomology. The instanton calculation proves that if $(\delta\ \delta^* + \delta^*\ \delta)|\chi >= \lambda|\chi >$, with $\lambda \neq 0$, then $|\chi >$

has non-zero energy. It has required some work to rigorously prove that the converse is also true, that is when $\lambda = 0$ the corresponding eigenstate (which is in the twisted cohomology) does indeed have zero energy (Witten [9] only supplies some intuitive arguments). This then shows that $B_p = \dim((\ker\, \delta/\mathrm{im}\, \delta) \cap W_p)$, establishing that the Betti numbers can be determined from Morse theory. For the strong Morse inequalities, see ref. [23].

4.Floer homology

We consider now Y to be a three-manifold over which there is defined a Yang-Mills gauge theory, that is one associates to Y a fibre bundle, with fibres in a compact gauge group, for which we will choose SU(2). The analogue of the manifold M is now played by the configuration space, $\mathcal{C} = \mathcal{A}/\mathcal{G}$ which is the space $\mathcal{A}$ of connections A on the manifold Y, modulo the SU(2) gauge transformations $\mathcal{G}$ (maps g of Y into SU(2) where g acts as follows: ${}^gA = gAg^\dagger + gdg^\dagger$). The infinite-dimensional configuration space $\mathcal{C}$ is a manifold away from the so-called reducible connections (A is reducible if there is a gauge function not in the centre of the gauge group, which leaves the connection invariant: $\exists g \notin Z_2, {}^gA = A$). The reducible connections, of which 0 is an obvious example, give rise to conic (orbifold-type) singularities. However, $\mathcal{C}$ is smooth enough to allow for a formulation of Morse theory. We will sketch here some of the results without going into too much detail and we refer to Floer's original paper [13] (of which the introduction is quite readable for a physicist), or to Braam's review [14]. Atiyah's paper [15] provides the grand scheme of how everything is to fit together.

Floer takes as a Morse function the Chern-Simons functional ($A = A^a_\mu \frac{\sigma_a}{2i} dx^\mu$):

$$h(A) = \int_Y \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) = \int_Y d_3 x \varepsilon^{ijk} \mathrm{Tr}(A_i \partial_j A_k + \frac{2}{3} A_i A_j A_k) \quad . \tag{44}$$

We note that h is not quite an appropriate function on $\mathcal{C}$, since it is not exactly invariant under gauge transformations,

$$h({}^gA) = h(A) - \frac{1}{3} \int_Y \mathrm{Tr}((g^\dagger dg)^3) = h(A) - 8\pi^2 \deg(g : Y \to SU(2)) \quad . \tag{45}$$

Therefore one rather considers h as a function on $\mathcal{A}/\mathcal{G}^0$, where $\mathcal{G}^0$ is the connected component of the space of gauge transformations. The critical points of h are easily found to be the curvature free configurations, called flat connections in the mathematical terminology. They are the classical minima of the Yang-Mills potential. Indeed one easily verifies:

$$\frac{\partial h(A)}{\partial A_i^a(x)} = -\varepsilon^{ijk} F_{jk}^a(x)/2 = -B_i^a(x) \quad , \tag{46}$$

where $F_{ij}^a(x) = \partial_i A_j^a(x) - \partial_j A_i^a(x) + \varepsilon_{abc} A_i^b(x) A_j^c(x)$ is the curvature (as a 2-form it is given by $F = dA + A \wedge A = F_{\mu\nu}^a \frac{\sigma_a}{4i} dx^\mu \wedge dx^\nu$). Similarly it is very easy to determine the gradient flow of this Morse function, whose solutions will describe the tunnelling in this infinite-dimensional analogue of the exterior algebra (furtheron we will explicitly give the relevant supersymmetric Hamiltonian). We find:

$$\frac{\partial A_i^a(x;\tau)}{\partial \tau} = B_i^a(x;\tau) \quad . \tag{47}$$

These are exactly the anti-self-duality equations if we extend the connection A over Y, to a connection over $M = Y \times R$, with of course τ being identified with the fourth extra coordinate. One easily verifies that eq. (47) is equivalent to $F = -*F$. The Hodge $*$ is defined with respect to the metric of M, which is in an obvious way obtained from the metric of Y, by defining $g_{44} = 1,\ g_{i4} = 0$ for $i = 1, 2, 3$. Thus we see that the Yang-Mills instantons occur in a natural way in Floer's analysis.

Similar to the Witten complex, one can now consider what we will call the Floer complex. It is simply the set of gauge equivalence classes of the flat connections $\mathcal{V}$. A flat connection is classified by its holonomy along a loop, the well-known Wilson loop $\mathrm{P}\exp(\oint_\gamma A)$, and one can show that this holonomy only depends on the homotopy of the loop γ. The argument is that under a small deformation of the loop, the Wilson loop changes proportional to the curvature (this is obvious for an abelian gauge group, but can be suitably extended to non-Abelian groups [24]) and is hence zero. Since the homotopy does not change under continuous deformation, the holonomy stays constant under these changes. The Floer complex is therefore in one-to-one relation with the set of SU(2) representations of the fundamental group $\pi_1(Y)$. In general, however, the space of equivalence classes of flat connections is not discrete. My favourite example is of course the case that Y is the three-torus T^3, where it can be shown that $\mathcal{V}$ is the orbifold T^3/Z_2 [25]. Also in general, the Floer complex will contain reducible connections (since 0 is always a flat connection). For these reasons one imposes some constraints on the manifolds Y to be considered. One requires them to be so-called homology three spheres, which means that their (integer) first homology $H_1(Y, Z)$ is zero. This is sufficient to show that $\pi_1(Y)$ is finite and to guarantee that the only reducible connection in the now finite Floer complex is the 0 connection.

There are, however, two (related) technical problems one has to deal with. Firstly, for h to be an appropriate Morse function we were actually working on $\mathcal{A}/\mathcal{G}^0$, whereas for the Floer complex, we considered dividing out *all* gauge transformations. Thus, we have to convolute the Floer complex with $\mathcal{G}/\mathcal{G}^0 \sim \pi_3(SU(2))$. The second problem is that the Hessian of the Chern-Simons functional is no longer an elliptic operator, it is actually the covariant derivative operator, which is of Dirac-type and is unbounded from below. Therefore, it is impossible to define the Morse index, since there will always be an infinite number of negative eigenvalues. One can, however, define a relative Morse index, by declaring the Morse index related to a preferred flat connection equal to 0. The most obvious choice is the 0 connection, but it is reducible and needs special care [13,14]. In the physical terminology, together with the fact that the Hessian is a Dirac-type operator, this choice corresponds to a choice of vacuum, or Dirac-sea. The relative Morse index follows now directly from the index theorem [20] for the Hessian, similar to what we saw in the previous section.

From instanton calculations on $M = Y \times S^1$, we know that the spectral flow for tunnelling from a flat connection A to a flat connection ${}^gA$ is equal to $8k$, when k is the winding number of the gauge transformation g. It is essential that M is a compact manifold, which is achieved because we tunnel to *gauge equivalent* configurations. This implies that one can divide out the action of $\mathcal{G}/\mathcal{G}^0$, provided the grading due to the relative Morse index is defined modulo 8. This brings us back to a finite Floer complex, with a grading defined modulo 8 and, as for Witten's formulation of Morse theory for a finite dimensional manifold, Floer was able to define a boundary operator ∂, which squares to zero and defines a homology, where the eight homology groups

$$HF^i(Y) = (\ker \partial / \mathrm{im}\ \partial) \cap F^i \quad . \tag{48}$$

are called the Floer groups, with F^i the Floer complex of the flat connections which have a Morse index i (mod 8) with respect to the 0 connection. Floer has proved that these homology groups lead to invariants for the manifold Y (i.e. do not dependent on the choice of metric on Y)

We end this section with the promised construction of the supersymmetric Hamiltonian, whose zero-energy ground states should correspond to the Floer groups (or to $\ker(\partial^* \ \partial + \partial \ \partial^*)$). As in the previous section this Hamiltonian should be given by the Laplacian on $\mathcal{A}$ in terms of the twisted exterior algebra on $\mathcal{A}$. The standard exterior algebra on $\mathcal{A}$ is given by (compare eqs. (3,16)):

$$\mathbf{d} = \int_Y d^3x \psi_i^a(x) \frac{\delta}{\delta A_i^a(x)} \quad , \quad \mathbf{d}^* = -\int_Y d^3x \bar{\psi}_i^a(x) \frac{\delta}{\delta A_i^a(x)} \quad , \tag{49}$$

where $\psi_i^a(x)$ form a basis for the exterior algebra of $\mathcal{A}$ at A ($T_A^*\mathcal{A}$) and $\bar{\psi}_i^a(x)$ form the dual basis (in $T_A\mathcal{A}$). They are anticommuting $spin-one$ fields and satisfy a Dirac algebra:

$$\{\psi_i^a(x), \psi_j^b(y)\} = \{\bar{\psi}_i^a(x), \bar{\psi}_j^b(y)\} = 0 \quad , \quad \{\psi_i^a(x), \bar{\psi}_j^b(y)\} = g_{ij}(x)\delta^{ab}\delta_3(x-y) \quad . \tag{50}$$

The twisted exterior algebra is now given, in terms of the Chern-Simons functional which Floer chose as his Morse function, by:

$$\mathbf{d}_{e^{-2}} = \exp(-h(A)/e^2)\mathbf{d}\exp(h(A)/e^2) \quad , \quad \mathbf{d}^*_{e^{-2}} = \exp(h(A)/e^2)\mathbf{d}^*\exp(-h(A)/e^2) \quad . \tag{51}$$

For reasons which will become obvious, we have chosen e^{-2} instead of t and we are now interested in the limit $e \to 0$. The Hamiltonian to consider is given in terms of the twisted Laplacian on $\mathcal{A}$ by:

$$2e^{-2}H = \mathbf{d}_{e^{-2}} \ \mathbf{d}^*_{e^{-2}} + \mathbf{d}^*_{e^{-2}} \ \mathbf{d}_{e^{-2}} \quad . \tag{52}$$

A straightforward computation, exactly analogous to eq. (30), gives the supersymmetric Hamiltonian:

$$H = \int_Y d^3x \mathrm{Tr}(e^2\vec{\Pi}^2 + \frac{1}{e^2}\vec{B}^2 + \varepsilon^{ijk}\psi_i D_j \bar{\psi}_k) \quad , \tag{53}$$

where $D_j = \partial_j + \mathrm{ad}A_j$ is the covariant derivative in the adjoint representation and $\Pi_i^a(x) = -i\frac{\delta}{\delta A_i^a(x)}$ are the canonical momenta. Restriction to the gauge invariant configuration space $\mathcal{C}$ is easily achieved by imposing Gauss' law, in the same way as this is done in the Hamiltonian theory for ordinary Yang-Mills theory in the $A_0 = 0$ gauge. We now see that e plays the role of the coupling constant and that the large t asymptotic analysis of the previous section corresponds to the weak coupling expansion. Finally we note, as mentioned in the introduction, that the bosonic part of the Hamiltonian is exactly the ordinary Yang-Mills Hamiltonian.

5. The Lagrangian for topological Yang-Mills theory.

As we mentioned in the introduction, there is a relation of the Floer groups to the Donaldson invariants, which arise when one "cuts" a closed four manifold M into two pieces along a three manifold. That is, each half has the same three manifold Y as a boundary. For the interested reader we refer to Atiyah's paper [15] for more details. Here we only remark, that due to this relation one anticipates that there should be a Lorentz-invariant formulation in four dimensions, which reduces to the Floer theory on a manifold of type $M = Y \times R$ in the Hamiltonian formulation, but which on a closed four manifold would be intimately connected to the Donaldson invariants. In physical terminology this means that one should look for a Lagrangian formulation of the Hamiltonian in eq. (53). This is the challenge Atiyah [15] put to the physics community and we will discuss how Witten addressed the question.

He introduced a U-quantum number, which corresponds to the Floer groups (and is conserved mod 8), with the following assignment:

$$U(A) = 0 \quad , \quad U(\psi) = 1 \quad , \quad U(\bar{\psi}) = -1 \quad . \tag{54}$$

One now likes to fit $(A, \psi, \bar{\psi})$ into a Lorentz multiplet, which will somehow have to play a role in the instanton calculation on the four manifold M. There is an obvious difficulty with this, which is related to the fact that ψ and $\bar{\psi}$ are anticommuting spin-one fields, which already indicates that the supersymmetry, alluded to in the previous section is not quite standard (as we will see, it is more natural to see ψ as a ghost field and to talk about a Slavnov or BRST symmetry).

If we consider deformations δA along a given instanton moduli space $\mathcal{M}_k$, then in order for $A + \delta A$ to still be a solution it has to satisfy the deformation equations:

$$D_\alpha \delta A_\beta - D_\beta \delta A_\alpha - \varepsilon_{\alpha\beta\mu\nu} D^\mu \delta A^\nu = 0 \quad , \quad D_\alpha \delta A^\alpha = 0 \quad . \tag{55}$$

The first equation is simply the deformation of the self-duality equations, the second is a gauge condition. The δA form a tangent vector to the moduli space $\mathcal{M}_k$, which in the physical terminology is called a zero-mode. Usually $\mathcal{M}_k$ is considered the space of all anti-self-dual solutions (or anti-instantons). A simple change of orientation of M together with some field redefinitions will give Witten's [1] results, but here we follow the notations of ref. [18]. We want every bosonic zero-mode to be cancelled by an anti-commuting zero-mode, which will be established by the following term in the Lagrangian:

$$4\bar{\psi}^{\mu\nu}D_\mu\psi_\nu + \beta D_\mu\psi^\mu \quad . \tag{56}$$

Here $\bar{\psi}_{\mu\nu}$ is an anti-commuting anti-self-dual tensor field, related to the fields in the previous section by $\bar{\psi}_i = \varepsilon_{ijk}\bar{\psi}^{jk}/2$. We therefore see that we had to introduce β and ψ_0 as new fields. One should compensate those by commuting ghost of ghost fields $\bar{\phi}$ and ϕ to cancel these added degrees of freedom.

Witten [1] conjectured the following Lagrangian that could be related to the Hamiltonian of eq. (53):

$$\begin{aligned}\mathcal{L} =&\mathrm{Tr}\{-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + 4\bar{\psi}^{\mu\nu}D_\mu\psi_\nu + \beta D_\mu\psi^\mu + \bar{\phi}D_\mu^2\phi + \bar{\phi}[\psi_\mu,\psi^\mu] + \phi[\bar{\psi}_{\mu,\nu},\bar{\psi}^{\mu\nu}]\\ &+(D_\mu^{bg}A^\mu)^2 + \bar{c}D_\mu^{bg}D^\mu c + \cdots\} \quad ,\end{aligned} \tag{57}$$

where the dots stand for terms higher order in the fields, D_μ^{bg} is the background covariant derivative for the background gauge, where the last two terms describe the standard Faddeev-Popov gauge fixing, with $c,\bar{c}$ the normal ghost fields. The main motivation for this expression is largely that in a naive instanton calculation (ignoring for a moment the zero-modes, by assuming that the moduli space is a single point) the partition function is given by

$$Z = \frac{\mathrm{Pf}(\mathbf{D})}{\sqrt{\det(\mathbf{\Delta})}} \quad . \tag{58}$$

In this equation $\mathrm{Pf}(\mathbf{D})$ is the Pfaffian of the antisymmetric operator $\mathbf{D}$, appearing in eq. (56). The Pfaffian comes from the Grassmann integration over the anti-commuting fields and up to a factor ± 1 is equal to the square root of the determinant of the operator. The factor in the denominator is of course coming from the Gaussian integration over the commuting fields. The operator $\mathbf{\Delta}$ is the one occurring in eq. (55), which is, by construction, related to the operator $\mathbf{D}$ through the anti-commuting symmetry $\delta A_\mu = \epsilon\psi_\mu$, where ϵ is in this case a *scalar* Grassmann variable. In the BRST language this defines an anti-commuting operation s, such that $sA_\mu = \psi_\mu$. This means that in eq. (58) the denominator will cancel against the nominator, up to a factor ± 1. Note that the Faddeev-Popov determinant cancels exactly against the determinant coming from the integration over the ghost of ghost fields $\bar{\phi}$, ϕ, which are assumed to be complex fields. Requiring the quadratic part of the action to be invariant fixes the symmetry for the other fields and the U quantum numbers for ϕ and $\bar{\phi}$ to be $U(\phi) = 2$ and $U(\bar{\phi}) = -2$. It also almost fixes the form of the higher order part of the Lagrangian (Witten showed that there is a choice which corresponds exactly to a so-called "twisted" version of $N = 2$ super Yang-Mills theory). The action in eq. (57) presents potential pitfalls. It is not obviously positive definite and it might be plagued by Gribov ambiguities. As we will see, most of the fields are ghost or ghost of ghost fields, and the action is actually constant. Thus, we could phrase the question as whether the physicist's description of how to deal with the Lagrangian in eq. (57) leads to a normalizable integration measure on the configuration space (in a fixed topological sector).

Let us now come back to the result of the partition function in eq. (58), which more or less by construction is equal to ± 1 in the case that $\mathcal{M}_k$ exists of one point. Similarly, when $\mathcal{M}_k$ is discrete, one finds $Z = \sum_{\mathcal{M}_k}(-1)^{n_i}$ and Witten argues [1] that this is precisely one of the Donaldson invariants (by studying how the sign of the Pfaffian is determined). The strength of Witten's analysis [1] is the way one shows that the partition function is an invariant, i.e. does not depend on the metric. This property is at the heart of what topological field theories are about. In Witten's original analysis this invariance seems to come out of the blue, however, furtheron we will analyse the deeper reasons for this. At this point we will only specify the properties, which will guarantee the topological nature of the theory. Firstly the anti-commuting symmetry s is a BRST or Slavnov symmetry, whose square is zero, $s^2 = 0$. Witten [1] did not take ordinary gauge fixing into account, implicitly working on the space $\mathcal{A}/\mathcal{G}$, rather than on $\mathcal{A}$. In that case s^2 is only zero on gauge invariant states, which is all one needs anyhow. The advantage is furthermore that in this way one does not have to address the issue of Gribov ambiguities in the usual gauge fixing. The second property we already alluded to before, is that the action is actually constant. This is expressed by the fact that one can write the action $S = \int_M \sqrt{g}\mathcal{L}$ as ($F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$):

$$S = 2\int_M \mathrm{Tr}(F \wedge F) + sS_{gf} \quad , \tag{59}$$

where S_{gf} is an integral over a local gauge invariant polynomial of the fields. Finally and crucially, one can show [1] that the energy-momentum tensor $T_{\mu\nu}$, which is obtained by varying the action with respect to the metric g, is BRST-trivial: $T_{\mu\nu} = s\lambda_{\mu\nu}$, where again λ is a local gauge invariant polynomial in the fields (this actually follows from eq.(59)). Following Witten [1], the fact that $s^2 = 0$ and $T = s\lambda$ will lead to the fact that the partition function is independent of the metric:

$$\delta_g Z = -\frac{1}{e^2}\int \mathcal{D}A\delta_g S \exp(-\frac{1}{e^2}S) = -\frac{1}{2e^2}Z < s\int_M \sqrt{g}\delta g_{\mu\nu}\lambda^{\mu\nu} >= 0 \quad , \tag{60}$$

It is crucial to observe here that one has to assume that there is no anomaly in the energy-momentum tensor, i.e. one has to make sure that quantum corrections do not spoil the BRST-triviality of the energy-momentum tensor. This has successfully been shown to one-loop in ref. [26], but it is expected that the symmetries are strong enough to allow one to derive Ward identities that will establish the BRST-triviality to all orders. Similarly one can now ask which operators $\mathcal{O}$ will have invariant expectation values, i.e. satisfy the equation:

$$\delta_g < O >=< \delta_g \mathcal{O} - \frac{1}{2e^2}\mathcal{O}s\int_M \sqrt{g}\delta g_{\mu\nu}\lambda^{\mu\nu} >= 0 \quad . \tag{61}$$

We see that a sufficient condition is that $\delta_g \mathcal{O} = s\rho$ for some gauge invariant operator ρ (we will, however, only consider operators that do not depend on the metric) and that $\mathcal{O}$ is BRST-trivial, i.e. $s\mathcal{O} = 0$. When, however, $\mathcal{O}$ itself is the s of something ($\mathcal{O} = s\rho$, for some *gauge invariant* operator ρ) then its expectation value is zero and does not lead to anything useful. In conclusion, the interesting observables, which have invariant expectation values are those with a non-trivial equivariant (or basic) cohomology [17]:

$$\mathcal{O} \in \text{ker } \hat{s}/\text{im } \hat{s} \quad , \tag{62}$$

where $\hat{s}$ restricts s to the *gauge invariant* set of operators (which are polynomials in the fields). However, it is not known whether *all* non-trivial observables are of this type.

Until now, we only considered the case where the moduli space is discrete, but in the examples mostly known to physicists like for S^4, this is not the case. Then in general the partition function will be zero, due to the presence of anti-commuting zero-modes. To have a non-zero result one needs to consider operators which will cancel all these zero-modes and this is what the quantum number U is useful for. One can show [1], that the operator in question should be a polynomial in the fields, with a net value for U (which is additive), equal to the number of zero-modes. This is all quite similar to 't Hooft's construction of the effective action for the breaking of chiral $U_A(1)$ through instantons [27]. Again, by the index theorem the number of zero-modes is equal to the dimension of the moduli space:

$$U(\mathcal{O}) = \dim(\mathcal{M}_k) = 8k - \frac{3}{2}(\chi(M) + \sigma(M)) \quad , \tag{63}$$

where $\chi(M)$ is the Euler characteristic and $\sigma(M)$ is the so-called signature of M, e.g. $\chi(S^4) = 2$, $\sigma(S^4) = 0$. In the instanton calculation the expectation value of the operator $\mathcal{O}$ is calculated by first integrating out the non-zero modes, including the ϕ and $\bar{\phi}$ field, after which $\mathcal{O}$ takes the form

$$\mathcal{O} = \Phi_{i_1,i_2,\cdots,i_n}(a_k)\theta^{i_1}\theta^{i_2}\cdots\theta^{i_n} \quad , \quad U(\mathcal{O}) = n \quad , \tag{64}$$

where a_k are parameters describing the instantons (moduli parameters) and θ^k the zero-modes for ψ. The expectation value of $\mathcal{O}$ then reduces to an integration over moduli space, with the canonical measure $d\mu$ defined on $\mathcal{M}_k$ [1]. Actually, since at least for a physicist, it can be easily shown that the expectation values of the operators are independent of the coupling constant (if $\partial\mathcal{O}/\partial e = 0$), one can work in the weak-coupling limit, and the only integration over non-zero modes that survives in the limit $e \to 0$, is the integration over the fields ϕ and $\bar{\phi}$. From eq. (57) we see that $[\psi_\mu, \psi^\mu]$ acts as a source ϕ. Thus [1], one replaces in $\mathcal{O}$ ϕ by

$$\phi(x) = -\int_M \sqrt{g}G(x,y)[\psi_\mu(y), \psi^\mu(y)] \quad , \tag{65}$$

ψ by their zero-modes and A by the self-dual connections, where $G(x,y)$ is the Greens function for $D_\mu D^\mu$. This defines therefore explicit formulas for the invariants in terms of integrals over the moduli space, which we will discuss in the next section.

We will end this section by a discussion of the BRST formulation of topological Yang-Mills theory [16,18,28,29,30]. The strategy is to start with an action which is independent of the metric (to guarantee the topological properties). This will in general have large symmetries. In the case of Yang-Mills theory, one starts [16] with an action that is proportional to the topological charge:

$$S_0 = 2\int_M \mathrm{Tr}(F \wedge F) \quad . \tag{66}$$

This action is clearly invariant under an arbitrary variation of the gauge field (provided we stay in the same topological sector). This symmetry needs to be gauge fixed: $\mathcal{L} = \mathcal{L}_0 + sV_{gf}$.

The normal gauge fixing (dividing out the action of the gauge group $\mathcal{G}$) can and should be done separately [28]

$$\mathcal{L} = \mathcal{L}_0 + sV_{gf} + s_g V_{FP} \quad , \tag{67}$$

see also [30]. Both s and s_g generate Slavnov or BRST symmetries and satisfy $s^2 = 0$, $s_g^2 = 0$ and $\{s, s_g\} = 0$.

To be more precise, the invariance is given by $\delta A_\mu = \psi_\mu$, which we fix by the gauge condition $F^- = (F - *F) = 0$. Note that this does not completely fix the gauge, but this is on purpose, since we wish to be left with the (finite set of) degrees of freedom that describe the instantons, which form exactly the kernel of this gauge condition. Thus, as is familiar in the BRST formalism, ψ_μ becomes a ghost. We also have a Lagrange multiplier field $b_{\mu\nu}$, which will enforce the constraint $F^- = 0$ and is therefore an anti-self-dual tensor field. Finally one completes the set of fields by the anti-ghost $\bar{\psi}_{\mu\nu}$. However, the variations of A in the direction of the gauge orbit for which $\psi_\mu = D_\mu \phi$, with ϕ some function in the adjoint representation of the gauge group, is a redundant symmetry (because it will be described by dividing out the action of the gauge group $\mathcal{G}$) and needs therefore to be removed too. This symmetry is then fixed by $D_\mu \psi^\mu = 0$. The field ϕ will hence be the ghost of a ghost (and is a commuting field). We will call the Lagrange multiplier, used to enforce this gauge condition β and the anti-ghost of the ghost $\bar{\phi}$. This completes the description of the field content and explains the origin of all the fields in Witten's [1] original formulation. It is easy to find the appropriate gauge fixing Lagrangian $\mathcal{L}_{gf} = sV_{gf}$ of eq. (67)

$$\mathcal{L}_{gf} = s\mathrm{Tr}(\bar{\phi} D_\mu \psi^\mu + \bar{\psi}^{\mu\nu}(b_{\mu\nu} - F^-_{\mu\nu})) + s\mathrm{Tr}(\beta[\phi, \bar{\phi}]) \quad , \tag{68}$$

where the last term can be freely added; however, with it the Lagrangian can be seen as a "twisted" version of $N = 2$ supersymmetric Yang-Mills [1]. The Slavnov or BRST symmetry s is given by the following formula [18] (for ease of notation we suppressed the indices):

$$\begin{aligned}
sA = \psi' \quad &, \quad s\psi' = 0 \quad , \quad \psi' = \psi - D\omega \\
s\bar{\psi} = b' \quad &, \quad sb' = 0 \quad , \quad b' = b - [\omega, \bar{\psi}] \\
s\bar{\phi} = \beta' \quad &, \quad s\beta' = 0 \quad , \quad \beta' = \beta - [\omega, \bar{\phi}] \\
s\omega = \phi' \quad &, \quad s\phi' = 0 \quad , \quad \phi' = \phi - [\omega, \omega]/2
\end{aligned} \tag{69}$$

We have written the action of s in terms of shifted fields to demonstrate the fact that the ordinary local s-cohomology is trivial. Any operator which is in the kernel of s, can also be written as the s of some other operator. That would not leave any interesting observables. As we saw, it is not the local, but the equivariant cohomology that determines the interesting observables. Related to this is that in eq.(69) the field ω appears, which is a ghost field, with values in the adjoint representation of the gauge group. It plays the role of an infinitesimal gauge transformation on which the action does not depend. This is crucial for defining equivariant cohomology, which goes back to Cartan [17], who called it basic cohomology. For this cohomology one restricts s to the gauge invariant and ω independent operators. It is this cohomology which is non-trivial, leading to the interesting observables that will be discussed in the next section. To complete the discussion, the ordinary gauge fixing does not interfere with the above construction and is as usual formulated as follows:

$$\mathcal{L}_{FP} = s_g V_{FP} = s_g \ \bar{c}(b - D_\mu^{bg} A^\mu) \quad , \tag{70}$$

$$s_g A_\mu = D_\mu c \quad , \quad s_g c = -[c,c]/2 \quad , \quad s_g \bar{c} = b \quad , \quad s_g b = 0 \quad , \tag{71}$$

Let us emphasize again that Gribov ambiguities will in general be present, but to the point of dividing out the gauge group, this seems to be only a technical handicap. As we said before, and which is the attitude taken by Witten [1], one can entirely work on $\mathcal{A}/\mathcal{G}$, where this issue need not be addressed. It is, however, possible that Gribov will still take revenge through the gauge condition $D_\mu \psi^\mu = 0$, but we have nothing sensible to say on this right now.

6. Donaldson polynomials

In this section we will finally be able to construct the Donaldson polynomials. As observed before, we need operators with a net U charge equal to the dimension of the moduli space and we will build these operators as a product of operators $\mathcal{O}_i$, which are non-trivial elements of the equivariant s-cohomology with charge U_i. Thus $\mathcal{O} = \Pi \mathcal{O}_i$ with $\sum U_i = \dim(\mathcal{M}_k)$. The simplest such non-trivial element of the equivariant s-cohomology is:

$$W_0 = \frac{1}{2}\mathrm{Tr}(\phi^2(x)) \quad , \quad U(W_0) = 4 \quad . \tag{72}$$

Note that $W_0 = sW = \frac{1}{2}s\mathrm{Tr}(\phi\omega - \frac{1}{3}\omega^3)$, so indeed $sW_0 = 0$, but nevertheless it is a non-trivial element of the equivariant s-cohomology, due to the ω dependence of W. We now need to verify explicitly that W_0 is indeed an invariant, by demonstrating that it does not depend on the coordinate $x \in M$. For this it is sufficient to show that dW_0 is s-trivial, i.e. there exists a 1-form W_1 such that $dW_0 = sW_1$. This is easily checked:

$$dW_0 = \mathrm{Tr}(\phi D\phi) = -s\mathrm{Tr}(\phi\psi) = sW_1 \quad , \quad W_1 = -\mathrm{Tr}(\phi\psi) \quad . \tag{73}$$

In this equation D is the covariant differential and ψ is the ghost 1-form $\psi_\mu dx^\mu$. From eq. (73) we see that W_1 itself is not s-trivial, however, since it is a 1-form on M, we can integrate it over a 1-cycle γ_1, which by the use of Stokes' law, will show that $\oint_{\gamma_1} W_1$ is s-trivial:

$$s\oint_{\gamma_1} W_1 = \oint_{\gamma_1} dW_0 = \oint_{\partial\gamma_1} W_0 = 0 \quad . \tag{74}$$

However, in order for $\oint_{\gamma_1} W_1$ to have a chance to be an invariant, it should not depend on the particular choice of the (closed) 1-cycle γ_1, but only on its homology class. For this it is sufficient to show that the integral of sW_1 over the boundary of a 2-cell α_2, is zero. Again by Stokes' law, this is equivalent to demanding dW_1 to be s-trivial. One easily finds this to be the case:

$$dW_1 = -\mathrm{Tr}(\phi D\psi + D\phi\wedge\psi) = s\mathrm{Tr}(\frac{1}{2}\psi\wedge\psi - \phi F) = sW_2 \quad , \quad W_2 = \mathrm{Tr}(\frac{1}{2}\psi\wedge\psi - \phi F) \quad . \tag{75}$$

Now W_2 is a 2-form on M and its integral over a 2-cycle γ_2 (i.e. a closed two-dimensional cell) should be s-trivial, we leave this to the reader to verify, by using Stokes' law. Again demanding that $\oint_{\gamma_2} W_2$ will depend only on the homology class of the 2-cycle γ_2, leads to the requirement that dW_2 is s-trivial, and sure enough things work out beautifully:

$$dW_2 = \mathrm{Tr}(D\psi\wedge\psi - D\phi\wedge F) = s\mathrm{Tr}(\psi\wedge F) = sW_3 \quad , \quad W_3 = \mathrm{Tr}(\psi\wedge F) \quad . \tag{76}$$

It starts to get boring, but hang on, we are almost at the end. Again W_3 itself would not lead to an observable but it is a 3-form on M and we can integrate over a 3-cycle γ_3, which in order to lead to an invariant should only depend on the homology of γ_3, leading to the condition that dW_3 is s-trivial. So for the last time:

$$dW_3 = \mathrm{Tr}(D\psi\wedge F) = -\frac{1}{2}s\mathrm{Tr}(F\wedge F) = sW_4 \quad , \quad W_4 = \mathrm{Tr}(F\wedge F) \quad . \tag{77}$$

We recognize that the 4-form is precisely proportional to the Pontryagin class and the integral over the only 4-cycle, which is the manifold M, is thus proportional to the topological charge and is clearly an invariant. And if one really wants to go to the bottom of it, the Bianchi identity $DF = 0$, will show that $s \int_M W_4 = 0$.

We have thus constructed a chain of observables $\mathcal{O}_i = \oint_{\gamma_i} W_i$, with $U_i = 4 - i$, where γ_0 is a point x and $\gamma_4 = M$. They form a map, which we will call the Donaldson map, from the homology $H_k(M, R)$ of the base manifold M into the equivariant s-cohomology of the field theory. As we observed in the previous section around eq. (65), these observables descent to objects on the moduli. Those objects are polynomials in the zero-modes θ_j (with coefficients that are functions on the moduli space) of a degree which is exactly the U quantum number of the operator in question. The θ_j can be naturally interpreted as differential forms on the moduli space $\mathcal{M}_k$ and therefore $\mathcal{O}_i$ will become a $(4-i)$-form on $\mathcal{M}_k$ after substituting in $\mathcal{O}_i$ ϕ by eq. (65), ψ by their zero-modes and A by the self-dual connections. This also means that s, after this restriction to the moduli space, should play the role of the exterior derivative on $\mathcal{M}_k$. Although this seems obvious, the author believes that this has not yet been established in a sufficiently clear way in the literature up to now, but let us here assume it to be the case. Then the Donaldson map is a map from $H_i(M, R)$ into $H^{4-i}(\mathcal{M}_k, R)$ and a Donaldson polynomial is simply the wedge product of these elements in the cohomology of the moduli space, such that the total degree is the dimension of the moduli space. The invariant is obtained by integrating over moduli space. If Φ^{γ_k} is the image of the Donaldson map for $\mathcal{O}_k = \oint_{\gamma_k} W_k$, then the Donaldson invariant is given by:

$$\int_{\mathcal{M}_k} \Phi^{\gamma_{k_1}} \wedge \Phi^{\gamma_{k_2}} \wedge \cdots \wedge \Phi^{\gamma_{k_n}} \quad , \quad \sum_i (4 - k_i) = \dim(\mathcal{M}_k) \quad . \tag{78}$$

This gives very explicit formulas for the Donaldson invariants, which is the strength of Witten's [1] construction. Not surprisingly, there exists a topological version of quantum mechanics, which will give the Euler characteristic as an invariant [31]. It provides an interesting framework in which many of the above manipulations can be defined more rigorously.

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Figure Captions

Figure 1: We illustrate here that the 1-cycles γ_1 and γ_2 are homologous. This is because cutting the figure along both of these curves we get two 2-cells. For each of these it is true that the boundary of the 2-cell is $\pm(\gamma_1 - \gamma_2)$ (the simplest 2-cell with this property is the tube). However, γ_1 and γ_3 are clearly not homologous.

Figure 2: Here we illustrate Morse theory, where h is the height function. We indicate the critical points and their Morse indices μ.

Figure 3: The same as figure 2, but now deformed by pressing with our thumb at the maximum. This illustrates that the Euler characteristic, calculated by using Morse theory will not change under this deformation.

Figure 4: Here we illustrate the tunnelling paths between two critical points, which differ in Morse index by one. In this example we see that there are two instantons. We also indicate the tangent space $V_P \cap v^{\perp}$, which is transported along the tunnelling path, so that we can compare its orientation with V_Q. In this example both instantons contribute with the same sign.