INLO-PUB-01/00

MONOPOLES IN DISGUISE [a]

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We review results concerning caloron solutions of unit charge with non-trivial holonomy, revealing the constituent monopole nature of these instantons. For $SU(n)$ there are n such BPS constituents. We discuss the possible physical applications.

1 Introduction

Calorons are finite temperature instanton solutions. They are defined on $\mathbb{R}^3 \times S^1$. Due to the periodic boundary conditions in the time direction, the Polyakov loop at spatial infinity (the so-called holonomy) can take on a non-trivial value (independent of directions)

$$\mathcal{P}_\infty = \lim_{|\vec{x}|\to\infty} P(\vec{x}), \quad P(\vec{x}) = \mathrm{P}\exp\left(\int_0^\beta A_0(t,\vec{x})dt\right). \tag{1}$$

A non-trivial value reveals that the charge one $SU(n)$ caloron actually contains n constituent BPS monopoles, whose masses are determined by the eigenvalues of the Polyakov loop

$$\mathcal{P}^0_\infty \equiv \exp(2\pi i \mathrm{diag}(\mu_1,\mu_2,\cdots,\mu_n)), \quad \sum_{i=1}^n \mu_i = 0. \tag{2}$$

For defining the constituent masses it is important to note that one can choose a gauge in which $\mu_1 \le \cdots \mu_n \le \mu_{n+1} \equiv \mu_1 + 1$ (for a proof see [1]). This guarantees that the masses $8\pi^2\nu_m/\beta$, with $\nu_m \equiv \mu_{m+1} - \mu_m$, add up to $8\pi^2/\beta$ such that the action equals that of a charge one instanton.

The separation of the constituents can be chosen freely and for large separation the action density becomes static. The fermion zero-mode is localised on one of the constituents, as is discussed in sect. 2. In the core of the constituents the Polyakov loop generically will have two of its eigenvalues degenerate. For $SU(2)$ this implies that the Polyakov loop becomes $\pm I_2$ (the "zero's of the Higgs field"), as was verified explicitly [2]. In sect. 3 we also review the fact that in a suitable gauge only one of the constituents' gauge fields is non-static, in accordance to the Taubes-winding [3] required to form out of monopoles a four dimensional gauge field configuration with non-trivial topological charge.

[a] Presented at the workshop "Light-Cone QCD and Nonperturbative Hadron Physics", CSSM, Adelaide, 13-22 December, 1999.

We conclude in sect. 4 with some comments concerning the fact that not only instantons can be made from monopoles, but as we have seen also monopoles can be made out of an array of instantons. The "democratic" vacuum model described in ref. [4] naturally unifies both interpretations.

2 Densities

Using the classical scale invariance we can always arrange $\beta = 1$, as will be assumed throughout. A remarkably simple formula for the $SU(n)$ action density exists [5]

$$\mathrm{Tr}F^2_{\alpha\beta}(x) = \partial^2_\alpha\partial^2_\beta \log\psi(x), \quad \psi(x) = \tfrac{1}{2}\mathrm{tr}(\mathcal{A}_n\cdots\mathcal{A}_1) - \cos(2\pi t), \tag{3}$$
$$\mathcal{A}_m \equiv \frac{1}{r_m}\begin{pmatrix} r_m & |\vec\rho_{m+1}| \\ 0 & r_{m+1}\end{pmatrix}\begin{pmatrix}\cosh(2\pi\nu_m r_m) & \sinh(2\pi\nu_m r_m)\\ \sinh(2\pi\nu_m r_m) & \cosh(2\pi\nu_m r_m)\end{pmatrix},$$

with $r_m \equiv |\vec x - \vec y_m|$ and $\vec\rho_m \equiv \vec y_m - \vec y_{m-1}$, where $\vec y_m$ is the location of the $m^{\rm th}$ constituent monopole with a mass $8\pi^2\nu_m$. Note that the index m should be considered mod n, such that e.g. $r_{n+1} = r_1$ and $\vec y_{n+1} = \vec y_1$ (but note $\mu_{n+1} = 1+\mu_1$). It is sufficient that only one constituent location is far separated from the others, to show that one can neglect the $\cos(2\pi t)$ term in $\psi(x)$, giving rise to a static action density in this limit [5,6]. For an example see figure 1.

These generalised caloron solutions can be constructed [7,5] using the Nahm transformation [8] and the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction [9], related by a suitably defined Fourier transformation. Other methods, relying more exclusively on the Nahm transformation, were developed as well [10].

The Nahm equation for charge one calorons reduces to an abelian problem on the circle, parametrised by $z \bmod 1$,

$$\frac{d}{dz}\hat A_j(z) = 2\pi i\sum_m \rho^j_m\delta(z-\mu_m), \tag{4}$$

giving $\hat A_j(z) = 2\pi i y^j_m$, for $z\in[\mu_m,\mu_{m+1}]$. In the monopole literature $\hat A_j(z)$ is usually denoted by $T_j(z)$.

The basic ingredient in the construction of caloron solutions is a Green's function, defined on the circle $z\in[0,1]$, satisfying

$$\left(\left(\frac{1}{2\pi i}\frac{d}{dz} - t\right)^2 + r^2(x;z) + \frac{1}{2\pi}\sum_m \delta(z-\mu_m)|\vec\rho_m|\right)\hat f_x(z,z') = \delta(z-z'), \tag{5}$$

where $r^2(x;z) = r^2_m(x)$ for $z\in[\mu_m,\mu_{m+1}]$. This can be solved using a similarity with the impurity scattering problem on the circle [5,1], which we present here for the case that $\mu_m \leq z' \leq z \leq \mu_{m+1}$ (extended to $z < z'$ by $\hat{f}_x(z', z) = \hat{f}_x^*(z, z')$)

$$\hat{f}_z(z, z') = \frac{\pi e^{2\pi i t(z-z')}}{r_m \psi}\Big(e^{-2\pi i t} \sinh\left(2\pi(z-z')r_m\right) + \\ < v_m(z')|\mathcal{A}_{m-1}\cdots\mathcal{A}_1\mathcal{A}_n\cdots\mathcal{A}_m|w_m(z)>\Big), \quad (6)$$

where the spinors v_m and w_m are defined by

$$\begin{aligned} v_m^1(z) = -w_m^2(z) &= \sinh\left(2\pi(z-\mu_m)r_m\right), \\ v_m^2(z) = \quad w_m^1(z) &= \cosh\left(2\pi(z-\mu_m)r_m\right). \end{aligned} \quad (7)$$

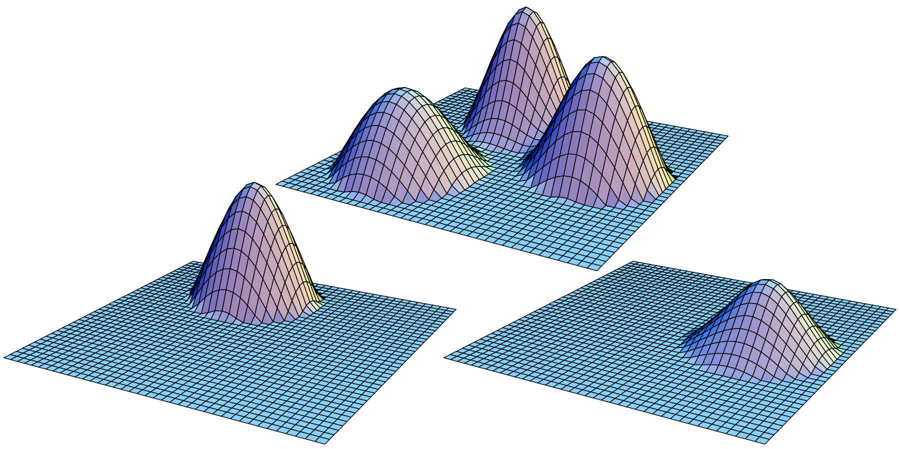

Figure 1: The action density (top) for a $SU(3)$ caloron at $t = 0$, for $\beta = 1$, on a logarithmic scale cut off at $1/(2e)$, with $(\mu_1, \mu_2, \mu_3) = (-17, -2, 19)/60$ shown in the plane defined by $\vec{y}_1 = (-2, -2, 0)$, $\vec{y}_2 = (0, 2, 0)$ and $\vec{y}_3 = (2, -1, 0)$. The masses $8\pi^2\nu_i$ are given by $(\nu_1, \nu_2, \nu_3) = (0.25, 0.35, 0.4)$. On the bottom-left is shown the zero-mode density for fermions with anti-periodic boundary conditions in time and on the bottom-right for periodic boundary conditions, at equal logarithmic scales cut off below $1/e^5$.

With the gauge field in the periodic gauge ($A_\alpha(t+1, \vec{x}) = A_\alpha(t, \vec{x})$) one can look for the fermion zero-modes satisfying $\Psi_z(t+1, \vec{x}) = \exp(2\pi i z)\Psi_z(t, \vec{x})$. To obtain the finite temperature fermion zero-mode one puts $z = \frac{1}{2}$, whereas for the fermion zero-mode with periodic boundary conditions one takes $z = 0$. For the density of these fermion zero-modes a very simple result in terms of the Green's function can be derived [11,1]

$$|\Psi_z(x)|^2 = -(2\pi)^{-2}\partial_\alpha^2 \hat{f}_x(z, z). \quad (8)$$

From this it is easily seen that in case of well separated constituents the zero-mode is localised only at $\vec{y}_m$ for which $z \in [\mu_m, \mu_{m+1}]$. To be specific, in this limit $\hat{f}_x(z,z) = \pi \tanh(\pi r_m \nu_m)/r_m$ for $SU(2)$, and more generally

$$\hat{f}_x(z,z) = \frac{2\pi \sinh[2\pi(z-\mu_m)r_m] \sinh[2\pi(\mu_{m+1}-z)r_m]}{r_m \sinh[2\pi\nu_m r_m]}. \tag{9}$$

We illustrate the localisation of the fermion zero-modes for a typical $SU(3)$ caloron in figure 1.

3 Polyakov loops and Taubes-winding

The following exact result for the $SU(2)$ Polyakov loop along the line (taken along the z-axes) connecting the two constituents exists [2]

$$\tfrac{1}{2}\mathrm{Tr}\, P(0,0,z) = -\cos(\nu_1\pi + \tfrac{1}{2}\partial_z \mathrm{acosh}[\tfrac{1}{2}\mathrm{tr}(\mathcal{A}_2\mathcal{A}_1)]). \tag{10}$$

From this it is easily seen that $P(\vec{x})$ takes on each of the values $\pm I_2$ only once, with $P(\vec{y}_2) = -I_2$ and $P(\vec{y}_1) = I_2$ for well separated constituents. These are the equivalent to the conditions for the Higgs field to vanishing, an alternative way to specify the location of the constituents. When the constituents get nearer to each other, these "zeros" shift outwards (whereas the maxima of the energy density shift inwards), see figure 2.

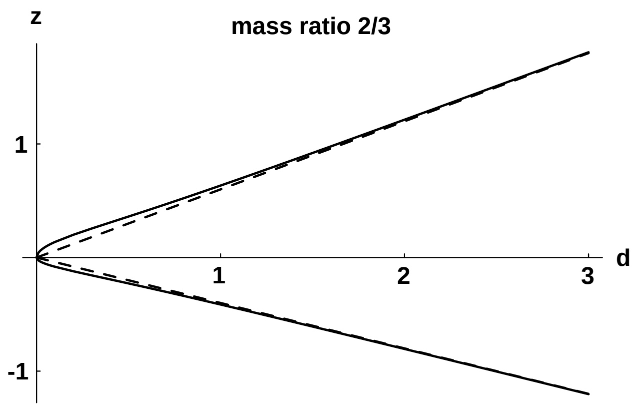

Figure 2: Shift of the locations where $P^2(\vec{x}) = I_2$ as compared to the location of the constituent monopole centers $\vec{y}_i$ for $SU(2)$. Horizontally is plotted the distance $d = \pi\rho^2$ between the constituents and vertically the position of $z_1 = \nu_2 d$ and $z_2 = -\nu_1 d$ (dashed lines), and the locations where $P(0,0,z) = I_2$ $(z > 0)$ and $P(0,0,z) = -I_2$ $(z < 0)$.

Note that the result for the Polyakov loop associated to the constituent at $\vec{y}_1$ is consistent with $A_0=0$, but for the constituent associated to $\vec{y}_2$ this only holds after applying a gauge transformation that is *anti-periodic* in the time direction (since such a gauge transformation changes the sign of the Polyakov loop). This gauge transformation has an interesting relation to the so-called Taubes-winding [3], which is most easily understood by looking at the explicit expression for the gauge field. In the *periodic gauge* one finds [7]

$$\begin{aligned} A_\alpha^{\rm per}(x) = &\frac{i}{2}\bar{\eta}^3_{\alpha\beta}\tau_3\partial_\beta\log\phi+\pi i\nu_1\tau_3\delta_{\alpha,0} \\ &+\frac{i}{2}\phi{\rm Re}\left((\bar{\eta}^1_{\alpha\beta}-i\bar{\eta}^2_{\alpha\beta})(\tau_1+i\tau_2)(\partial_\beta+2\pi i\nu_1\delta_{\beta,0})\tilde{\chi}\right), \end{aligned} \tag{11}$$

where χ has the following expansion

$$\tilde{\chi}\equiv e^{-2\pi it\nu_1}\chi=\frac{4\pi\rho^2}{(r_2+r_1+\pi\rho^2)^2}\left\{r_2e^{-2\pi r_2\nu_2}e^{-2\pi it}+r_1e^{-2\pi r_1\nu_1}\right\}, \tag{12}$$

up to relative errors $\mathcal{O}(e^{-4\pi\min(r_1\nu_1,r_2\nu_2)})$. For large ρ, $\phi(x)$ becomes time independent, confirming the static nature of the configuration for large constituent separations. The time dependence of the constituent at $\vec{y}_2$ is a full (gauge) rotation - the *Taubes-winding* - responsible for the topological charge of the otherwise time independent monopole pair [3], see figure 3. This gauge rotation is achieved by the anti-periodic gauge transformation $g(x)=\exp(-\pi it\tau_3)$, since $g(x)(\tau_1+i\tau_2)g^\dagger(x)=\exp(-2\pi it)(\tau_1+i\tau_2)$. Also note that for the spherically symmetric Bogomol'nyi-Prasad-Sommerfield (BPS) monopole [12] a gauge rotation is equivalent to an ordinary rotation.

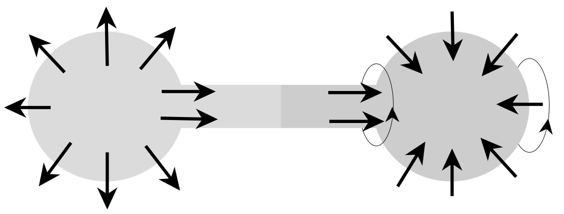

Figure 3: The gauge field with unit topological charge is constructed from two oppositely charged monopoles by rotating one of them over one full rotation, while moving over one time-period.

The information that the constituent at $\vec{y}_1$ is associated to a static BPS monopole, whereas the constituent at $\vec{y}_2$ has acquired a Taubes-winding can be extracted in a way [1] that is also valid for $SU(n)$. Provided we have ordered the μ_i as indicated below eq. (2), one finds a *static* monopole solution composed out of $n-1$ basic BPS monopoles of mass ν_m, located at $\vec{y}_m$, for $m=1,\cdots,n-1$ when sending $|\vec{y}_n|$ to infinity. The field associated with the constituent at $\vec{y}_n$ has to have a time-dependence to give rise to the topological charge of the caloron. It is this one that carries the Taubes-winding.

For $SU(n)$ we note the constituents are *basic* BPS monopoles, which are obtained by embedding $SU(2)$ in $SU(n)$. The $SU(2)$ subgroup relevant for this embedding is exactly determined by the unbroken $SU(2)$ at the core of the constituent. The restoration of the $SU(2)$ symmetry arises due to the degeneracy of two of the eigenvalues of the Higgs field, or of $P(\vec{x})$ for calorons (it implies $P(\vec{x})=\pm I_2$ as its determinant is unity). We denote by $\vec{z}_m$ the position associated to the m-th constituent where two eigenvalues of the Polyakov loop coincide. Arranging by a global gauge rotation that $\mathcal{P}_\infty=\mathcal{P}^0_\infty$ (see eq. (2)), one finds [13]

$$\begin{aligned}
P_1(\vec{z}_1) &= \mathrm{diag}(e^{\pi i(\mu_1+\mu_2)},e^{\pi i(\mu_1+\mu_2)},e^{2\pi i\mu_3},\cdots,e^{2\pi i\mu_n}),\\
P_2(\vec{z}_2) &= \mathrm{diag}(e^{2\pi i\mu_1},e^{\pi i(\mu_2+\mu_3)},e^{\pi i(\mu_2+\mu_3)},e^{2\pi i\mu_4},\cdots,e^{2\pi i\mu_n}),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
P_{n-1}(\vec{z}_{n-1}) &= \mathrm{diag}(e^{2\pi i\mu_1},\cdots,e^{2\pi i\mu_{n-2}},e^{\pi i(\mu_{n-1}+\mu_n)},e^{\pi i(\mu_{n-1}+\mu_n)}),\\
P_n(\vec{z}_n) &= \mathrm{diag}(e^{\pi i(\mu_n+\mu_{n+1})},e^{2\pi i\mu_2},\cdots,e^{2\pi i\mu_{n-1}},e^{\pi i(\mu_n+\mu_{n+1})}).
\end{aligned} \tag{13}$$

Note that (since $\mu_{n+1}=1+\mu_1$) P_n can also be written as

$$P_n(\vec{z}_n)=\mathrm{diag}(-e^{\pi i(\mu_1+\mu_n)},e^{2\pi i\mu_2},\cdots,e^{2\pi i\mu_{n-1}},-e^{\pi i(\mu_1+\mu_n)}). \tag{14}$$

The extra minus signs occurring in P_n reflect the Taubes-winding.

4 Discussion

As we have seen, an instanton has BPS monopoles as constituents, and the explicit results in eq. (11) also easily reveal the abelian limit. As long as[b] all $\nu_m\neq 0$, the field outside the core of both monopoles is indeed that of two "BPS" Dirac monopoles (i.e. dyons). For $SU(2)$ outside the cores, i.e. assuming $r_m\nu_m\gg 1$ for all m, one has $\tilde{\chi}=0$ and

$$\phi(x)=\frac{|\vec{x}-\vec{y}_1|+|\vec{x}-\vec{y}_2|+|\vec{y}_2-\vec{y}_1|}{|\vec{x}-\vec{y}_1|+|\vec{x}-\vec{y}_2|-|\vec{y}_2-\vec{y}_1|}. \tag{15}$$

[b] The Harrington-Shepard solution [14] with trivial holonomy (all $\mu_i=0$), can be reinterpreted as a bound state of a massive and a massless constituent BPS monopole. Massless monopole constituents, giving rise to so-called non-abelian clouds, may play an important role in electric-magnetic duality as the dual of gluons [15]. These massless constituents are delocalised and have no obvious abelian limit.

We note that when *neglecting* the exponential corrections, $\phi^{-1}(x)$ vanishes on the line connecting the two constituents, and $\log\phi(x)$ is harmonic outside of this line. The singularity represents the return flux of the Dirac monopole pair, described in the abelian limit by a term proportional to $\partial_j^2\log\phi(x)$ in the magnetic field

$$E_k=\frac{i}{2}\tau_3\partial_k\partial_3\log\phi,\quad B_k=\frac{i}{2}\tau_3\left(\partial_k\partial_3\log\phi-\delta_{k3}\partial_j^2\log\phi\right). \tag{16}$$

In the full theory the return flux is absent (indeed $\phi^{-1}(x)$ has only an isolated zero at $x=0$, corresponding to a gauge singularity [7]).

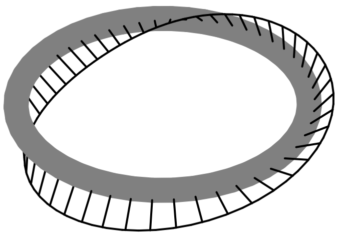

Figure 4: A closed monopole line, rotating its frame when completing the circle. The topological charge is given by the net number of windings (here one) of the frame.

As was already emphasised in ref. [4], to understand how in the abelian limit the topological charge can be recovered, one needs to preserve the Taubes-winding. This information is described by the framing of the core due to the charged components of the monopole field. Interestingly this describes a Hopf fibration [7], see figure 4. Recently the role of the Taubes-winding and Hopf fibration for retrieving topological charge was confirmed in great detail within the context of the abelian projection [16].

We have also seen that the fermion zero-mode is localised on the constituent with the Taubes-winding and this makes it likely to conjecture that this holds for general monopole loops that support non-trivial Taubes-winding, which may be relevant to understanding chiral symmetry breaking in the context of monopole degrees of freedom. The question which field configurations are more important, is however a bit like the chicken and the egg problem. After all, we can now make instantons out of monopoles *and* monopoles out of instantons. Monopoles are typically used for describing the confining large distance behaviour. Recently it has, however, been pointed out there can be a hidden large size instanton component hitherto undetected [17]. Much remains to be done here, but the calorons have provided us interesting new avenues to follow.

In the light of this we mention two interesting parallel developments. In supersymmetric theories at finite temperature, interpolating between three and four dimensions, it has been shown that one can introduce a superpotential on the space of possible holonomies, whose minimum is obtained exactly at the point where all constituents have equal mass [18]. These therefore give rise to instanton lumps of topological charge $1/N$, each responsible for two gluino zero-modes saturating the gluino condensate, resolving an old problem with the gluino condensate computations that relied on the cluster decomposition property of the ordinary instanton contribution. The other recent development aims at constructing the θ dependence of the effective action [19] for QCD. It allows for the interpretation in terms of a Coulomb gas of magnetic monopole charges, called instanton quarks, that share the properties of the constituent monopoles at the symmetric point where all masses are equal. One may wonder if the vanishing of the trace of the asymptotic Polyakov loop at this symmetric point, as would be appropriate in the confining phase, has any dynamical significance. Semiclassical arguments are no good for describing the long distance structure of QCD, and quantum fluctuations are as important as any classical large field backgrounds, it seems [4]. This does not rule out the existence of a simple low-energy description in terms of suitable degrees of freedom; it merely means it is hard to reveal this structure from the microscopic details of the theory. It is this that is still lacking.

Acknowledgements

Discussions with Eric Zhitnitsky and other participants of this workshop are greatly appreciated. I thank all my collaborators that contributed to the work reported here. I am grateful to Andreas Schreiber and Tony Williams for their invitation, for hospitality at CSSM, and for organising this workshop in wonderful Adelaide. Finally, cheers to Robert Perry for demonstrating his excellent taste, presenting me with a fine bottle of Shiraz, now maturing to be enjoyed at the right moment.

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