Viewpoints on Abelian Projections

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We discuss the relation of the Faddeev-Niemi model with $SU(2)$ gauge theory, and the identification of constituent monopoles in the $SU(3)$ caloron solutions through degenerate eigenvalues of the Polyakov loop. These have in common abelian projection and Hopf invariant.

1. Introduction

In its simplest form abelian projection [1] involves choosing an observable $X(x)$ that transforms as $g^\dagger(x)X(x)g(x)$ under gauge transformations, which can be used to diagonalise $X(x)$. The remaining gauge freedom is $U(1)^r$, where r is the rank of the gauge group. These are associated with the r neutral gauge bosons in this gauge. Singularities occur when two (or more) eigenvalues coincide, which can be shown to be magnetic monopoles, as defined with respect to the remnant abelian gauge group. Another popular choice is to minimise $\int |A_\mu^{\rm ch}(x)|^2$ along the gauge orbit [1,2]. It leads to the background gauge condition on the charged component of the gauge field, leaving the abelian subgroup unfixed.

Inspired by the abelian projection, Faddeev and Niemi attempted to identify the field $\vec{n}(x)$ (here of unit length, $|\vec{n}(x)| = 1$) in an $O(3)$ non-linear sigma model with the local colour direction for $SU(2)$ gauge theory. The hope was that the static *knotted* solutions constructed numerically [3] in the model originally introduced by Faddeev [4], were possibly related to glueballs [5]. Much work has been invested in interpreting this model as an effective low-energy representation of $SU(2)$ gauge theory [5–7]. The quality of this approximation is being investigated by inverse Monte Carlo techniques [8]. Amongst other things, these numerical results have reported the presence of unwanted (massless) Goldstone bosons, if one does not include terms that explicitly break the $O(3)$ symmetry. Also a perturbative renormalisation group study showed there are difficulties with the restricted formulation of the model, as an effective description of $SU(2)$ gauge theory [9]. This difficulty seems to be related to the fact that one expands around a background that is actually unstable, as will be discussed below. Nevertheless the $\vec{n}$ field *can* be identified with an $SU(2)$ gauge field, albeit with zero field strength. Under this identification we have $(\partial_\mu\vec{n}(x))^2 = 4|A_\mu^{\rm ch}(x)|^2$, which leads to the interpretation in terms of maximal abelian gauge fixing [10].

The static knotted solutions, as maps $\vec{n}(\vec{x})$ from S^3 (compactified R^3) to S^2, are classified by the Hopf invariant. The pre-image of a generic value of $\vec{n}$ traces out a loop in R^3 (i.e. the collection of points $\vec{x}$, where $\vec{n}(\vec{x}) = \vec{n}$). The linking number of any two such loops is equal to this Hopf invariant. It also coincides with the winding number of the gauge function $g(\vec{x})$, such that $n^a(\vec{x})\tau_a = g(\vec{x})\tau_3 g^\dagger(\vec{x})$, with τ_a the Pauli matrices. The associated $SU(2)$ gauge field with zero field strength is $A(\vec{x}) = g^\dagger(\vec{x})dg(\vec{x})$.

Such a relation between Hopf invariant and topological charge can also occur for monopoles. A monopole-antimonopole pair is created and separated to a finite distance. Kept at this separation one of them is rotated around the axis connecting the two. After this is completed they are brought together and made to annihilate. The time is considered to be euclidean and the rotation introduces a "twist" in the field that prevents the four dimensional configuration from decaying to $A = 0$. The obstruction is precisely the topological charge, whose value is equal to the net number of rotations [11]. This can just as well be described in terms of a closed monopole loop (in the same way that the Wilson loop is associated with the creation, propagation and annihilation of a heavy quark-antiquark pair), see Fig. 1. The identification with the Hopf invariant can be understood as follows: the orientation of the monopole is described by $SU(2)/U(1)\sim S^2$ at each point on the monopole loop (S^1), describing a twisted S^2 bundle over S^1, e.g. making the total space into an S^3 for one full "frame" rotation. This is the Hopf fibration, although we have interchanged fibre and base space as compared to the usual formulation.

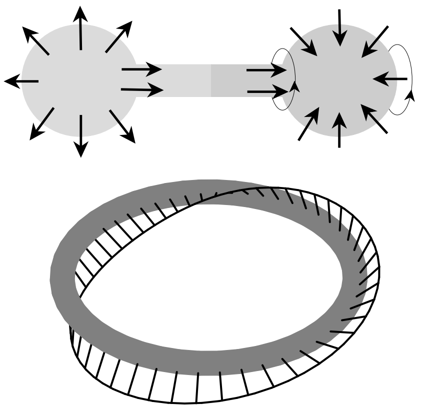

Figure 1. Topological charge constructed from oppositely charged monopoles by rotating one of them. For a closed monopole line, the embedding of the unbroken subgroup makes a full rotation.

At finite temperature A_0 plays the role of the Higgs field and the calorons (periodic instantons) provide an explicit example of this Taubes winding, when there is a non-trivial value of the Polyakov loop at infinity. This Polyakov loop is independent of the directions at infinity, because the finite action of the caloron forces the field strength to vanish at infinity. In this case the caloron splits in (n for $SU(n)$) constituent monopoles, which are the basic spherically symmetric BPS monopoles. Of these, $n-1$ are time independent, whereas the time dependence of the other exactly coincides with the rotation (at a uniform rate) [12,13] of the Taubes winding. In terms of the abelian projection and the introduction of a composite Higgs field, the relation of the Hopf invariant with twisted monopole loops has been extensively studied by Jahn [14]. We will use coinciding eigenvalues of the Polyakov loop to locate the defects. We study in how far their locations are always associated to physical lumps for the $SU(3)$ caloron, by considering the case where two of the constituents are brought close together [15].

2. The Faddeev-Niemi model

The model is defined in terms of a three-vector $\vec{n}(x)$ of fixed (here chosen unit) length. To allow for non-trivial static solutions a Skyrme-like higher-order term is added [4], through the introduction of a composite gauge field strength $F_{\mu\nu}(x) = \frac{1}{2}\vec{n}(x)\cdot(\partial_\mu\vec{n}(x)\wedge\partial_\nu\vec{n}(x))$. A useful identity is $F^2_{\mu\nu}(x) = \frac{1}{4}(\partial_\mu\vec{n}(x)\wedge\partial_\nu\vec{n}(x))^2$, which follows from the fact that $\partial_\mu\vec{n}(x)\wedge\partial_\nu\vec{n}(x) = 2F_{\mu\nu}(x)\vec{n}(x)$. Its proportionality to $\vec{n}(x)$ follows from the fact that the latter is perpendicular to $\partial_\mu\vec{n}(x)$ (since $\vec{n}(x)$ is a unit three-vector). The action is given (up to an overall factor) by

$$S=\int d^4x\left(\partial_\mu\vec{n}(x)\cdot\partial^\mu\vec{n}(x)-\frac{1}{2}F_{\mu\nu}(x)F^{\mu\nu}(x)\right), \quad (1)$$

brought to this simple form by a suitable rescaling of x. Finite energy requires $\vec{n}(\vec{x})$ to approach a constant vector at spatial infinity. In this way static configurations are classified by the topological maps from S^3 into S^2, characterised by the Hopf invariant. The two-form $F(\vec{x}) = \vec{n}(\vec{x})\cdot(d\vec{n}(\vec{x})\wedge d\vec{n}(\vec{x}))$ implicitly defines an abelian gauge field one-form $A(\vec{x})$ through $F(\vec{x}) = dA(\vec{x})$, in terms of which the Hopf invariant is given by $Q = \frac{1}{4\pi^2}\int A(\vec{x})\wedge F(\vec{x})$. Remarkably, the energy is bounded by a fractional power of this Hopf invariant [16,17].

$$E = \int d^3x\left((\partial_i\vec{n}(x))^2+\frac{1}{2}F^2_{ij}(x)\right)\geq c|Q|^{3/4}, \quad (2)$$

with $c = 16\pi^2 3^{3/8}$. This gives a rough bound, which can be improved on [18] (by roughly a factor 2). Extensive numerical studies [19,20] have gone up to $Q = 8$, with energies indeed more or less following the fractional power of Q.

2.1. Relation to $SU(2)$ gauge fields

We reformulate the non-linear sigma model in two steps, both well known [21,22]. The first involves CP_1 fields. Its main advantage is that the abelian gauge field appearing in the Hopf invariant no longer needs to be defined implicitly. To be specific, one introduces a complex two-component field $\Psi(x)$, also having unit length. A further phase is removed by the local abelian gauge invariance, obvious from the following relation to the n field:

$$n^a(x) = \Psi^\dagger(x)\tau^a\Psi(x). \tag{3}$$

The abelian gauge invariance of the CP_1 model leads to a composite gauge field

$$A_\mu(x) = -i\Psi^\dagger(x)\partial_\mu\Psi(x), \tag{4}$$

and one verifies that indeed $F(x) = dA(x)$. Useful identities for these computations are the completeness relation $\delta_{ij}\delta_{kl} + \tau^a_{ij}\tau^a_{kl} = 2\delta_{il}\delta_{jk}$ and $i\varepsilon_{abc}\tau^b_{ij}\tau^c_{kl} = \tau^a_{kj}\delta_{il} - \tau^a_{il}\delta_{jk}$. The energy, Eq. (2), becomes

$$E = \int d^3x \left(4|D_i\Psi(\vec{x})|^2 - \frac{1}{2}F^2_{ij}(\vec{x})\right), \tag{5}$$

with $D_\mu = \partial_\mu - iA_\mu(x)$ the covariant derivative.

Next we make use of the fact that any complex two-component vector of unit length is in one to one relation to an $SU(2)$ group element. Alternatively we can write $\Psi(x) = g(x)\Psi_0$. For convenience we choose $\Psi_0^\dagger = (1,0)$, such that

$$n_a(x) = \frac{1}{2}\mathrm{tr}\left(\tau_3 g^\dagger(x)\tau_a g(x)\right). \tag{6}$$

We introduce $J_\mu(x) \equiv i\tau_a J^a_\mu(x) \equiv g^\dagger(x)\partial_\mu g(x)$, which can be interpreted as the components of an $SU(2)$ gauge connection with vanishing curvature, that is $G(x) = dJ(x) + J(x)\wedge J(x) = 0$, where $J(x) \equiv J_\mu(x)dx_\mu$. A simple calculation shows that $A_\mu(x) = J^3_\mu(x)$ and $\partial_\mu\Psi^\dagger(x)\partial^\mu\Psi(x) = J^a_\mu(x)J^\mu_a(x)$. The zero non-abelian field strength leads to $F_{\mu\nu}(x) = 2(J^1_\mu(x)J^2_\nu(x) - J^1_\nu(x)J^2_\mu(x))$, or

$$F(x) = dJ^3(x) = 2J^1(x)\wedge J^2(x), \tag{7}$$

in terms of $J^a(x) \equiv J^a_\mu(x)dx_\mu$. This implies that $A(\vec{x})\wedge F(\vec{x}) = 2J^3(\vec{x})\wedge J^1(\vec{x})\wedge J^2(\vec{x})$, and substituting $J(x) = g^\dagger(x)dg(x)$ one finds that the Hopf invariant is exactly equal [19] to the winding number of the gauge function $g(\vec{x})$,

$$\frac{1}{4\pi^2}A(\vec{x})\wedge F(\vec{x}) = \frac{1}{24\pi^2}\mathrm{tr}(g^\dagger(\vec{x})dg(\vec{x}))^3. \tag{8}$$

Using that J has zero curvature, we can of course also relate the Hopf invariant to the non-abelian Chern-Simons form, giving $\frac{1}{4\pi^2}A(\vec{x})\wedge F(\vec{x}) = -\frac{1}{8\pi^2}\mathrm{tr}[J(\vec{x})\wedge\left(dJ(\vec{x}) + \frac{2}{3}J(\vec{x})\wedge J(\vec{x})\right)] \equiv CS(J)$.

2.2. The gauge fixing interpretation

The natural question to ask now, is what the interpretation of the energy functional becomes in terms of the non-abelian gauge field

$$E = \int d^3x \left(4\left(J^\alpha_i(\vec{x})\right)^2 + 2\left(\epsilon_{\alpha\beta}J^\alpha_i(\vec{x})J^\beta_j(\vec{x})\right)^2\right), \tag{9}$$

where the index α and β run only over 1 and 2. With the absence of the neutral component, $J^3_\mu(x)$, the $SU(2)/U(1)$ formulation is evident. It defines a positive definite functional on the gauge orbit, and its minimum can thus be seen as a particular (non-linear) maximal abelian gauge fixing, leaving the abelian subgroup generated by τ_3 unfixed [1,2]. As the reparametrisation is mathematically equivalent, we are entitled to interpret the minima of the energy functional in the sector with a given value of Q as gauge fixed pure gauge (i.e. curvature free, or flat) connections in a sector with gauge field winding number Q. Therefore, there is a gauge fixing in terms of which the gauge vacua with different winding number can be characterised by inequivalent knots. We thus conclude that instantons "knit", interpolating between different types of "knots".

Finding the *absolute* minimum of a gauge fixing functional is known to be a hard problem, seemingly reflected in the difficulty of finding the knots with the lowest energy for a given value of Q, as clearly illustrated in Fig. 2 of Ref. [20]

2.3. Background fields

The conventional starting point in these studies has been the decomposition [23]

$$\vec{A}_\mu(x) = \partial_\mu\vec{n}(x)\wedge\vec{n}(x) + C_\mu(x)\vec{n}(x) + \vec{W}_\mu(x), \tag{10}$$

where C_μ describes an abelian gauge field and W^a_μ are the charged components of the field, with respect to $n^a(x)\tau_a$. One attempts to integrate out both W^a_μ *and*, unlike in abelian projection, C_μ. In perturbation theory this amounts to a background field calculation. For later use we consider the one parameter family of backgrounds $\vec{A}_\mu(x)=q\partial_\mu\vec{n}(x)\wedge\vec{n}(x)$. At $q=1$ $\vec{n}(x)$ is covariantly constant, i.e. $\partial_\mu\vec{n}(x)+\vec{A}_\mu(x)\wedge\vec{n}(x)=0$, which remains true when adding $C_\mu\vec{n}$ to the background. It ensures that the abelian gauge field transforms properly under gauge transformations after abelian projection [23]. However, to address the issue of perturbative stability, we compute the field strength for general q,

$$\vec{F}_{\mu\nu}(\vec{x})=q(q-2)\partial_\mu\vec{n}(\vec{x})\wedge\partial_\nu\vec{n}(x), \qquad (11)$$

which is non-zero for $q=1$. The energy of this background field is proportional to $q^2(2-q)^2$ and actually has a local maximum at $q=1$, and hence is unstable. On the other hand, the energy of the background vanishes not only for $q=0$ (the trivial background), but also for $q=2$. The latter corresponds to the flat connection J that has topological charge equal to Q, given by the integral of the non-abelian Chern-Simons form over space. Rescaling the flat connection J with $q/2$ one easily finds its value for any q, $\int CS(qJ(\vec{x})/2)=q^2(3-q)Q/4$. For $q=1$, this relation between the Hopf invariant and the non-abelian Chern-Simons form can also be found in Ref. [6]. We note that the integral gives *half* the Hopf invariant for $q=1$, thus in some sense this background is "half-way" between two vacua.

3. The $SU(3)$ Caloron

We next discuss the finite temperature instantons (calorons). We assume the Polyakov loop, $P(\vec{x})=P\exp(\int_0^\beta A_0(\vec{x},t)dt)$ (in the periodic gauge $A_\mu(t,\vec{x})=A_\mu(t+\beta,\vec{x})$), is non-trivial at spatial infinity. The spontaneous breakdown of gauge symmetry leads to constituent monopoles [13]. Their locations can be identified through:

- Points where two eigenvalues of the Polyakov loop coincide, which is where the $U^{n-1}(1)$ symmetry is partially restored to $SU(2)\times U^{n-2}(1)$.
- The centers of mass of the (spherical) lumps.
- The Dirac monopoles (or rather dyons, due to self-duality) as the sources of the abelian field lines, extrapolated back to the cores.

If well separated and localised, all these coincide [12,24]. This is no longer the case when two constituents come close together, as shown for $SU(2)$ in Ref. [25], and for $SU(3)$ in Ref. [15].

The masses of the constituent monopoles are fixed by the eigenvalues of $\mathcal{P}_\infty\equiv\lim_{|\vec{x}|\to\infty}P(\vec{x})$. These eigenvalues can be ordered by a constant gauge transformation W_∞

$$W_\infty^\dagger\mathcal{P}_\infty W_\infty=\mathcal{P}^0_\infty=\exp[2\pi i\,\mathrm{diag}(\mu_1,\ldots,\mu_n)],$$
$$\mu_1\leq\ldots\leq\mu_n\leq\mu_{n+1}\equiv 1+\mu_1, \qquad (12)$$

with $\sum_{m=1}^n\mu_m=0$. Using the classical scale invariance to put the extent of the euclidean time direction to one, $\beta=1$, the masses of the constituent monopoles are now given by $8\pi^2\nu_i$, where $\nu_i\equiv\mu_{i+1}-\mu_i$.

Similarly we can bring $P(\vec{x})$ to the diagonal form, with eigenvalues ordered on the circle, by a *local* gauge function, $P(\vec{x})=W(\vec{x})P_0(\vec{x})W^\dagger(\vec{x})$. We note that $W(\vec{x})$ (unique up to a residual abelian gauge rotation) and $P_0(\vec{x})$ will be smooth, except where two (or more) eigenvalues coincide. The ordering shows there are n different types of singularities (called defects [26]), for each of the *neighbouring* eigenvalues to coincide. The first $n-1$ are associated with the basic monopoles (as part of the inequivalent $SU(2)$ subgroups related to the generators of the Cartan subgroup). The $n^{\rm th}$ defect arises when the first and the last eigenvalue (still neighbours on the circle) coincide. Its magnetic charge ensures charge neutrality of the caloron, and it carries the Taubes winding supporting the non-zero topological charge [12,13,27].

The topological charge can be reduced to surface integrals near the singularities with the use of $\mathrm{tr}(P^\dagger dP)^3=d\,3\mathrm{tr}((P_0^\dagger A_W P_0+2P_0^\dagger dP_0)\wedge A_W)=d\,3\mathrm{tr}(A_W\wedge(2A_W\log P_0+P_0A_WP_0^\dagger))$, where $A_W\equiv W^\dagger dW$. If one assumes *all* defects are pointlike, this can be used to show that for each of the n types the (net) number of defects has to equal the topological charge, the type being selected by the branch of the logarithm (associated with the n elements in the center) [26]. This is the generic situation, but in special cases defects may form a submanifold, as we will find for a global embedding of the $SU(2)$ caloron in $SU(3)$.

3.1. Coinciding constituents

One might expect defects to merge when constituent monopoles do. The resulting triple degeneracy of eigenvalues for $SU(3)$ implies that the Polyakov loop takes on a value in the center. Yet this can be shown *not* to occur for the $SU(3)$ caloron with *unequal* masses. We therefore seem to have (at least) one more defect than the number of constituents, when two merging constituents will manifest themselves as one constituent. To study what happens in this case we first recall the simple formula for the $SU(n)$ action density [13]

$$\mathrm{Tr}F_{\mu\nu}^2(x)=\partial_\mu^2\partial_\nu^2\log\left[\tfrac{1}{2}\mathrm{tr}(\mathcal{A}_n\cdots\mathcal{A}_1)-\cos(2\pi t)\right],$$
$$\mathcal{A}_m\equiv\frac{1}{r_m}\begin{pmatrix} r_m & |\vec{y}_m-\vec{y}_{m+1}| \\ 0 & r_{m+1}\end{pmatrix}\begin{pmatrix} c_m & s_m \\ s_m & c_m\end{pmatrix}, \quad (13)$$

with $\vec{y}_m$ the center of mass location of the $m^{\rm th}$ constituent monopole. We defined $r_m\equiv|\vec{x}-\vec{y}_m|$, $c_m\equiv\cosh(2\pi\nu_m r_m)$, $s_m\equiv\sinh(2\pi\nu_m r_m)$, as well as $\vec{y}_{n+1}\equiv\vec{y}_1$, $r_{n+1}\equiv r_1$. We are interested in the case where the problem of two coinciding constituents in $SU(n)$ is mapped to the $SU(n-1)$ caloron. For this we restrict to the case where $\vec{y}_m=\vec{y}_{m+1}$ for some m, which for $SU(3)$ is *always* the case when two constituents coincide. Since now $r_m=r_{m+1}$, one easily verifies that $\mathcal{A}_{m+1}\mathcal{A}_m=\mathcal{A}_{m+1}[\nu_{m+1}\to\nu_m+\nu_{m+1}]$, describing a *single* constituent monopole (with properly combined mass), reducing Eq. (13) to the $SU(n-1)$ action density, with $n-1$ constituents.

3.2. Defect locations

We will study in detail a generic example in $SU(3)$, with $(\mu_1,\mu_2,\mu_3)=(-17,-2,19)/60$. We denote by $\vec{z}_m$ the position associated with the $m^{\rm th}$ constituent, where two eigenvalues of the Polyakov loop coincide (the defect locations). Numerically it was established that [24], in the gauge where $\mathcal{P}_\infty=\mathcal{P}_\infty^0$ (see Eq. (12)),

$$\begin{aligned}P_1&=P(\vec{z}_1)=\mathrm{diag}(\ e^{-\pi i\mu_3},\ e^{-\pi i\mu_3},\ e^{2\pi i\mu_3}),\\ P_2&=P(\vec{z}_2)=\mathrm{diag}(\ e^{2\pi i\mu_1},\ e^{-\pi i\mu_1},\ e^{-\pi i\mu_1}),\\ P_3&=P(\vec{z}_3)=\mathrm{diag}(-e^{-\pi i\mu_2},\ e^{2\pi i\mu_2},-e^{-\pi i\mu_2}).\end{aligned}\quad (14)$$

This is for *any* choice of $\mathcal{P}_\infty^0$ and constituent locations (with the proviso they are well separated, i.e. their cores do not overlap, in which case to a good approximation $\vec{z}_m=\vec{y}_m$). Now we take $\vec{y}_1=(0,0,10+d)$, $\vec{y}_2=(0,0,10-d)$ and $\vec{y}_3=(0,0,-10)$. The limit of coinciding constituents is achieved by taking $d\to 0$.

When the cores of the two approaching constituents start to overlap, P_1 and P_2 are no longer diagonal (mixing the lower 2×2 components). At $d=0$ they are diagonal again, but P_2 will be no longer in the fundamental Weyl chamber (the "logarithm" of the Cartan subgroup). A Weyl reflection maps it back, while for $d\neq 0$ a more general gauge rotation back to the Cartan subgroup is required to do so, see Fig. 2. For $d\to 0$, *each* P_m (and $\mathcal{P}_\infty$) falls on the dashed line, defining the reduction to $SU(2)$.

To further illustrate this, we list P_m for $d\to 0$

$$\begin{aligned}\hat{P}_1&=P(\vec{z}_1)=\mathrm{diag}(\ e^{2\pi i\mu_2},\ e^{2\pi i\mu_2},\ e^{-4\pi i\mu_2}),\\ \hat{P}_2&=P(\vec{z}_2)=\mathrm{diag}(\ e^{-\pi i\mu_2},\ e^{2\pi i\mu_2},\ e^{-\pi i\mu_2}),\\ \hat{P}_3&=P(\vec{z}_3)=\mathrm{diag}(-e^{-\pi i\mu_2},\ e^{2\pi i\mu_2},-e^{-\pi i\mu_2}),\end{aligned}\quad (15)$$

(which we believe to hold for any non-degenerate choice of the μ_i). These can be factorised as $\hat{P}_m=\hat{P}_2Q_m$, where $\hat{P}_2$ describes an overall $U(1)$

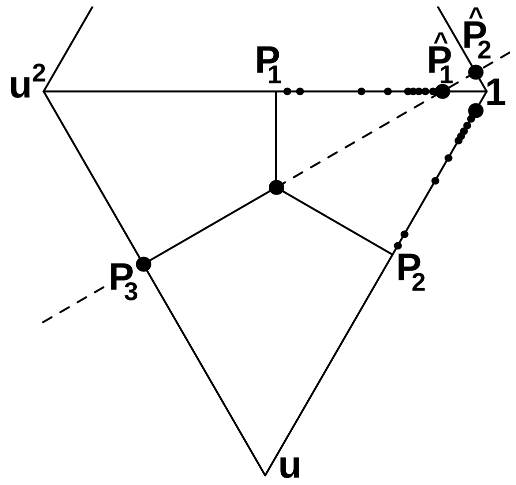

Figure 2. The fundamental Weyl chamber with the positions of P_m indicated at $d=2$, 1, .2, .1, .05, .04, .03, .02, .01, .005, .001, .0005, and (large dots) 0. The perpendiculars point to $\mathcal{P}_\infty$ (center), and emanate from the values of P_m for well separated constituents. The dashed line shows the $SU(2)$ embedding for $d=0$. ($u\equiv\exp(2\pi i/3)\,\mathbf{1}$)

factor. In terms of $Q_1 = \mathrm{diag}(e^{3\pi i\mu_2}, 1, e^{-3\pi i\mu_2})$, $Q_2 = \mathrm{diag}(1{,}1{,}1) = \mathbf{1}$ and $Q_3 = \mathrm{diag}(-1, 1, -1)$ the $SU(2)$ embedding in $SU(3)$ becomes evident. It leads for Q_2 to the trivial and for Q_3 to the non-trivial element of the center of $SU(2)$ (appropriate for the latter, carrying the Taubes winding). On the other hand, Q_1 corresponds to $\mathrm{diag}(e^{3\pi i\mu_2}, e^{-3\pi i\mu_2})$, which for the $SU(2)$ caloron is not related to coinciding eigenvalues. For $d\to 0$, Fig. 3 shows that $\vec{z}_1$ gets "stuck" at a *finite* distance (0.131419) from $\vec{z}_2$.

3.3. Spurious defects

The $SU(2)$ embedding determines the caloron solution for $d=0$, with constituent locations $\vec{y}_1{}' = \vec{y}_2$ and $\vec{y}_2{}' = \vec{y}_3$, and masses $\nu_1' = \nu_1 + \nu_2 = \mu_3 - \mu_1$ and $\nu_2' = \nu_3$. The spurious nature of the defect is obvious by calculating its location purely in terms of this $SU(2)$ caloron, demanding the $SU(2)$ Polyakov loop to equal $\mathrm{diag}(e^{3\pi i\mu_2}, e^{-3\pi i\mu_2})$. For this we can use the analytic expression [25] of the $SU(2)$ Polyakov loop along the z-axis. The location of the spurious defect, $\vec{z}_1 = (0, 0, z)$, is found by solving $3\pi\mu_2 = \pi\nu_2' - \frac{1}{2}\partial_z \mathrm{acosh}[\frac{1}{2}\mathrm{tr}(\mathcal{A}_2'\mathcal{A}_1')]$. For our example, $z = 10.131419$ indeed verifies this equation.

Fig. 4 gives the spurious location as a function of the mass difference of the two coinciding constituents. We find that only when this difference *vanishes*, the defects merge to form a triple degeneracy. Using the relation $3\mu_2 = \nu_1 - \nu_2$, the case of *equal* masses for the coinciding constituents corresponds to $\mu_2 = 0$. For *unequal* masses the defect is always spurious, but it tends to stay within reach of the non-abelian core of the coinciding constituent monopoles, $(\pi\nu_1')^{-1} \sim 0.53$, except when the mass difference approaches its extremal values $\pm\nu_1' = \pm(1-\nu_3)$. At these extremal values one of the $SU(3)$ constituents becomes massless and *delocalised*, which we excluded for $d \neq 0$.

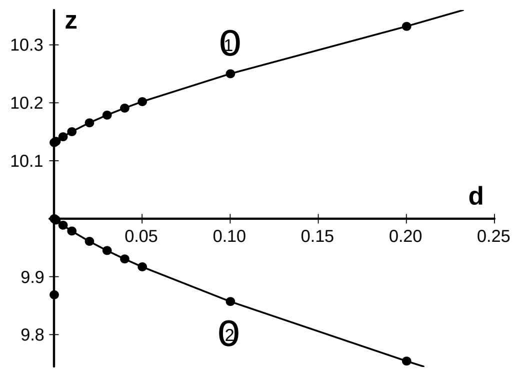

Figure 3. The defect locations $\vec{z}_1$ and $\vec{z}_2$, along the z-axis, for $M \equiv \nu_2 - \nu_1 = 0.1$ as a function of d.

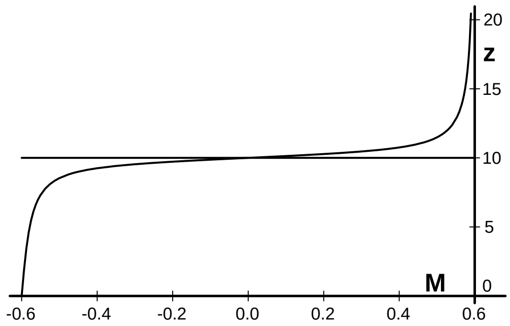

Figure 4. The defect locations $\vec{z}_1$ and $\vec{z}_2$, along the z-axis, as a function of $M \equiv \nu_2 - \nu_1$ for $d \to 0$.

Actually, the limit $d\to 0$ is singular due to the *global* decomposition into $SU(2)\times U(1)$ at $d = 0$. Gauge rotations U in the global $SU(2)$ subgroup do not affect $\hat{P}_2$, and therefore any $UQ_1U^\dagger$ gives rise to the *same* accidental degeneracy. In particular solving $-3\pi\mu_2 = \pi\nu_2' - \frac{1}{2}\partial_z \mathrm{acosh}[\frac{1}{2}\mathrm{tr}(\mathcal{A}_2'\mathcal{A}_1')]$ (corresponding to the Weyl reflection $Q_1 \to Q_1^\dagger$) yields $z = 9.868757$ for $\mu_2 = -1/30$ (isolated point in Fig. 3). Indeed, $U \in SU(2)/U(1)$ traces out a (nearly spherical) *shell* where two eigenvalues of P coincide (note that for $\mu_2 = 0$ this shell collapse to a single point, $z = 10$). A perturbation tends to remove this accidental degeneracy.

4. Conclusions

Topology is important for the non-perturbative understanding of non-abelian gauge theories, because it forces one to take serious that the configuration space is non-trivial [28]. That this has a role to play in the quantum theory can be easily understood from the difference between quantisation on the line versus the circle.

Attempts to *localise* topology by gauge fixing (and its defects), should always be treated with healthy suspicion. This is illustrated by our interpretation of the knotted solitons in the Faddeev-Niemi model in terms of a non-linear maximal abelian gauge fixing of the gauge vacuum: There is nothing *physical* to be localised in a classical vacuum. Also we have shown that the notion of abelian projected monopoles has to be considered with care. For "smooth" configurations and "reasonable" choices of projections defects may well be correlated to action lumps, but this is by no means guaranteed.

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