# NEW TOPOLOGICAL STRUCTURES IN QCD*

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We review the recent progress made in understanding instantons at finite temperature (calorons) with non-trivial holonomy, and their monopole constituents as relevant degrees of freedom for the confined phase.

## 1. Introduction

There has been a revived interest in studying instantons at finite temperature $T$, so-called calorons.[1,2] The main reason is that new explicit solutions could be obtained in the case where the Polyakov loop at spatial infinity (the so-called holonomy) is non-trivial, necessary to reveal more clearly the monopole constituent nature of these calorons.[3,4] Trivial holonomy, i.e. with values in the center of the gauge group, is typical for the deconfined phase. Non-trivial holonomy is therefore expected to play a role in the confined phase (i.e. for $T < T_c$) where the trace of the Polyakov loop fluctuates around small values. The properties of instantons are therefore directly coupled to the order parameter for the deconfining phase transition.

At finite temperature $A_0$ plays in some sense the role of a Higgs field in the adjoint representation, which explains why magnetic monopoles (in the BPS limit)[5,6] occur as constituents of calorons. Since $A_0$ is not necessarily static it is better to consider the Polyakov loop as the analog of the Higgs field,

$$P(t,\vec{x}) = \mathrm{Pexp}\left(\int_0^\beta A_0(t+s,\vec{x})ds\right), \tag{1}$$

which, like an adjoint Higgs field, transforms under a periodic gauge transformation $g(x)$ to $g(x)P(x)g^{-1}(x)$. Here $\beta = 1/kT$ is the period in the imaginary time direction, under which the gauge field is assumed to be

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periodic. Finite action requires the Polyakov loop at spatial infinity to be constant. For SU($n$) gauge theory this gives

$$\mathcal{P}_\infty = \lim_{|\vec{x}|\to\infty} P(0,\vec{x}) = g^\dagger \exp(2\pi i \mathrm{diag}(\mu_1,\mu_2,\ldots,\mu_n))g, \tag{2}$$

where $g$ is chosen to bring $\mathcal{P}_\infty$ to its diagonal form, with the $n$ eigenvalues being ordered according to

$$\textstyle\sum_{i=1}^n \mu_i = 0, \quad \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \leq \mu_{n+1} \equiv 1+\mu_1. \tag{3}$$

In the so-called algebraic gauge, where $A_0(x)$ is transformed to zero at spatial infinity, the gauge fields satisfy the boundary condition

$$A_\mu(t+\beta,\vec{x}) = \mathcal{P}_\infty A_\mu(t,\vec{x})\mathcal{P}_\infty^{-1}. \tag{4}$$

Caloron solutions are such that the total magnetic charge vanishes. The "force" stability of these solutions in terms of its constituent monopoles is based, as for exact BPS multi-monopole solutions, on balancing the electromagnetic with the scalar (Higgs) force,[7,8] except that for calorons repulsive and attractive forces are interchanged as compared to multi-monopoles. A single caloron with topological charge one contains[9] $n-1$ monopoles with a unit magnetic charge in the $i$-th U(1) subgroup, which are compensated by the $n$-th monopole of so-called type $(1,1,\ldots,1)$, having a magnetic charge in each of these subgroups. This special monopole is also called a Kaluza-Klein (KK) monopole.[10] At topological charge $k$ there are $kn$ constituents, $k$ monopoles of each of the $n$ types. The monopole of type $j$ has a mass $8\pi^2\nu_j/\beta$, with $\nu_j \equiv \mu_{j+1}-\mu_j$. The sum rule $\sum_{j=1}^n \nu_j = 1$ guarantees the correct action, $8\pi^2 k$, for calorons with topological charge $k$.

Prior to their explicit construction, calorons with non-trivial holonomy were considered irrelevant,[2] because the one-loop correction gives rise to an infinite action barrier. However, the infinity simply arises due to the integration over the finite energy density induced by the perturbative fluctuations in the background of a non-trivial Polyakov loop.[11] The non-perturbative contribution of calorons (with a given asymptotic value of the Polyakov loop) to this energy density as the relevant quantity to be considered, was first calculated in supersymmetric theories,[12] where the perturbative contribution vanishes. It has a minimum where the trace of the Polyakov loop vanishes, i.e. at maximal non-trivial holonomy. Recently the calculation of the non-perturbative contribution was performed in ordinary gauge theory at high temperatures.[13] When added to the perturbative contribution with its minima at center elements,[11] these minima turn unstable for decreasing temperature right around the expected value of $T_c$. This lends some support to monopole constituents being the relevant degrees of freedom which drive the transition from a phase in which the center symmetry is broken at high temperatures to one in which the center symmetry is restored at low temperatures. Lattice studies, both using cooling[14] and chiral fermion zero-modes[15] as filters, have also conclusively confirmed that monopole constituents do dynamically occur in the confined phase.

## 2. Properties of caloron solutions

Well-separated constituents can be shown to act as point sources for the so-called far field, that is far removed from any of the cores, where the gauge field is abelian.[16] When constituents of opposite charge ($n$ constituents of different type) come together, the action density no longer deviates significantly from that of a standard instanton. Its scale parameter $\rho$ is related to the constituent separation $d$ through $\pi\rho^2/\beta = d$. A typical example for a charge one SU(2) caloron with far and nearby constituents is shown in Fig. 1 (left). When $\rho \ll \beta$ no difference would be seen with the Harrington-Shepard solution,[1] the gauge field is nevertheless vastly different, as follows from the fact that within the confines of the peak there are $n$ locations where two of the eigenvalues of the Polyakov loop coincide.[17,18] This difference is also seen to play an important role in the supersymmetric case.[19] When, on the other hand, constituents of equal charge come together (which requires $|k| > 1$) an extended core structure appears.[16] For two coinciding

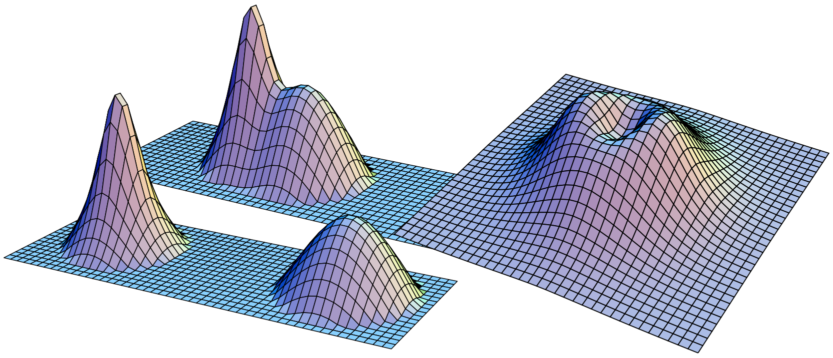

Figure 1. On the left are shown two charge one SU(2) caloron profiles at $t = 0$ with $\beta = 1$ and $\mu_2 = -\mu_1 = 0.125$, for $\rho = 1.6$ (bottom) and 0.8 (top) on equal logarithmic scales, cutoff below an action density of $1/(2e^2)$. On the right we show the action density (on a linear scale) of a typical SU(2) caloron with topological charge 2 and $\mu_2 = -\mu_1 = 0.25$ for which two constituents of equal magnetic charge are closer than their individual sizes (but not exactly on top). When the other two constituents are far away, as for the case shown here, this becomes a charge two monopole solution.

constituents this gives rise to the typical doughnut structure also observed for monopoles,[20] see Fig. 1 (right).

### 2.1. *Fermion zero-modes*

An essential property of calorons is that the chiral fermion zero-modes, whose existence is guaranteed by the index theorem,[22] are localized to constituents of a certain charge only. The latter depends on the choice of boundary condition for the fermions in the imaginary time direction (allowing for an arbitray U(1) phase $\exp(2\pi i z)$).[21] This result can be understood in terms of the Callias index theorem,[23] and provides an important signature for the dynamical lattice studies, using chiral fermion zero-modes as a filter.[15] To be precise, the zero-modes are localized to the monopoles of type $m$ provided $\mu_m < z < \mu_{m+1}$ (most of the time we use the classical scale invariance to set $\beta = 1$). Denoting the zero-modes by $\hat{\Psi}_z^a(x)$, where $a = 1, \ldots, k$ for a caloron of topological charge $k$, we can write

$$\hat{\Psi}_z^a(x)^\dagger \hat{\Psi}_z^b(x) = -(2\pi)^{-2}\partial_\mu^2 \hat{f}_x^{ab}(z,z). \tag{5}$$

where $\hat{f}^{ab}(z,z')$ is a Green's function that appears in the construction to be discussed below. The trace, i.e. the sum over the zero-mode densities, has a remarkably simple form in the far field limit (denoted by ff and defined by neglecting terms that decay exponentially with the distance to any of the constituent cores)

$$\mathrm{Tr}\hat{f}_x^{\mathrm{ff}}(z,z) = 4\pi^2 \mathcal{V}_m(\vec{x}), \quad \text{for} \quad \mu_m < z < \mu_{m+1}. \tag{6}$$

As is implicit in the notation, $\mathcal{V}_m$ is static and independent of $z$ within its interval of definition. It has been calculated explicitly for $k = 1$ and 2. In addition $\mathcal{V}_m$ has to be harmonic (up to singularities), because the zero-modes decay exponentially as long as $z \neq \mu_j$ (for any $j$), and therefore do not survive in the far field limit. For $k = 1$ and $\vec{y}_m$ the constituent location one simply has $\mathcal{V}_m(\vec{x}) = 1/(4\pi|\vec{x} - \vec{y}_m|)$, whereas for $k = 2$ we found[16]

$$\mathcal{V}_m(\vec{x}) = \frac{1}{2\pi|\vec{x}|} + \frac{\mathcal{D}}{4\pi^2}\int_{r<\mathcal{D}} dr d\varphi \, \frac{\partial_r |\vec{x} - r\vec{y}(\varphi)|^{-1}}{\sqrt{\mathcal{D}^2 - r^2}}, \tag{7}$$

where $\vec{y}(\varphi) = (\sqrt{1-\mathbf{k}^2}\cos\varphi, 0, \sin\varphi)$, up to an arbitrary coordinate shift and rotation. Here $\mathcal{D}$ is a scale and $\mathbf{k}$ a shape parameter to characterize *arbitrary* SU(2) charge 2 solutions. In this representation it is clear that $\mathcal{V}_m(\vec{x})$ is harmonic everywhere except on a disk bounded by an ellipse with minor axes $2\mathcal{D}\sqrt{1-\mathbf{k}^2}$ and major axes $2\mathcal{D}$. Although not directly obvious, when $\mathbf{k} \to 1$ the support of the singularity structure is on two points only, separated by a distance $2\mathcal{D}$. Taking an arbitrary test function $f(\vec{x})$ one can prove that[16]

$$-\lim_{\mathbf{k}\to 1} \int f(\vec{x}) \partial_i^2 \mathcal{V}_m(\vec{x}) d^3x = f(0,0,\mathcal{D}) + f(0,0,-\mathcal{D}). \tag{8}$$

Monopoles of different charges have to adjust to each other to form an exact caloron solution, such that $\mathbf{k}$ and $\mathcal{D}$ are in general not independent. So far we constructed two classes of solutions illustrated in Fig. 2 for both of which a large value of $\mathcal{D}$ implies that $\mathbf{k}$ approaches 1 exponentially. Hence we find point-like constituents, a necessary requirement to describe the field configurations at larger distances in terms of these objects. When all constituents of other types are sent to infinity, we recover the exact multi-monopole solutions of a given type of magnetic charge, and our results therefore also provide explicit solutions for the monopole zero-modes, which were not known before for the multi-monopole configurations.

Surprisingly, the charge distribution that gives rise to the abelian field far from any of the constituent cores (even when extended due to overlap) can be calculated exactly from $\mathcal{V}_m(\vec{x})$. Hurtubise[24] found earlier for the asymptotic behavior of the abelian field exactly the same result through his study of the algebraic tail of the Higgs field, using twistor methods. Let us first consider SU(2), for which we can parametrize the holonomy by $\mathcal{P}_\infty = \exp(2\pi i \vec{\omega}\cdot\vec{\tau})$ ($\mu_2 = -\mu_1 = |\vec{\omega}|$), where $\tau_a$ are the usual Pauli matrices.

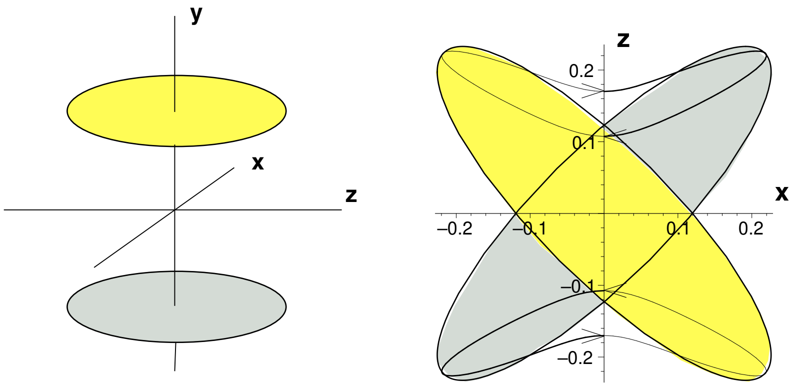


Figure 2. Illustration of the location of the disk singularities (light and dark shaded according to magnetic charge) for a so-called “rectangular” (left) and “crossed” (right) configuration, as used for the $k = 2$ solutions shown in resp. Fig. 1 and Fig. 3). The curves for the “crossed” case represent a one parameter family of solutions interpolating between two axially symmetric solutions for which $\mathbf{k} = 1$ independent of $\mathcal{D}$, giving point-like constituents without the need to take $\mathcal{D}$ large. See Ref. 16 for more details.

The field asymptotically becomes abelian, and is necessarily proportional to $\hat{\omega}\cdot\vec{\tau}$. One can therefore write $A_0^{\rm ff}(\vec{x}) = 2\pi i\vec{\omega}\cdot\vec{\tau} - \frac{1}{2}i\hat{\omega}\cdot\vec{\tau}\Phi(\vec{x})$, where the constant term (absent in the algebraic gauge) shows again how $A_0$ plays the role of a Higgs field. We found for $k=1$ and 2 that $\Phi(\vec{x}) = 2\pi(\mathcal{V}_1(\vec{x}) - \mathcal{V}_2(\vec{x}))$. Therefore, the singularity structure in the zero-mode density agrees *exactly* with the abelian charge distribution, as given by $\partial_i^2\Phi(\vec{x})$. Clearly our expectation is that this result will hold for arbitrary $k$ and will generalize to SU($n$) with each $\mathcal{V}_m(\vec{x})$ simply associated to the U(1) generator with respect to which the magnetic charge of the monopole of type $m$ is defined. Since the electric field is given in terms of the gradient of $A_0$, which due to the self-duality is equal to the magnetic field, a particularly simple formula results for the action density $\mathcal{S}$. Using in addition that outside the cores there is no source for the abelian field, implying $A_0$ to be harmonic in the far field (as is also clear from the relation to $\mathcal{V}_m$), we find

$$\mathcal{S}^{\rm ff}(\vec{x}) = -2{\rm tr}(\partial_i A_0(\vec{x}))^2 = 2\partial_i^2{\rm tr}A_0^2(\vec{x}) = -\partial_i^2\Phi^2(\vec{x}). \tag{9}$$

Therefore, the algebraic tail of the action density is known to all orders in $1/|\vec{x}|$, which is of course equivalent to the fact that the charge distribution, $\partial_i^2\Phi(\vec{x})$, giving rise to this asymptotic field is also exactly known, even though the formula for the action density can of course only be used outside any of the constituent cores.

## 3. The construction – in brief

Our construction of caloron solutions is in outline simple, and in its practical implementation relies heavily on the Atiyah-Drinfeld-Hitchin-Manin construction[25] of multi-instantons and on the closely related Nahm transformation.[26] For the $k=1$ Harrington-Shepard solution[1] one simply takes a periodic array of $\mathbb{R}^4$ instanton solutions, which can be easily constructed through the 't Hooft ansatz, provided the relative color orientation between subsequent periods is trivial. This necessarily gives trivial holonomy. Here the algebraic gauge discussed earlier is useful, as it shows that with non-trivial holonomy, shifting over each period the gauge field rotates in color space by an amount exactly given by the holonomy. It necessitates the use of the full ADHM formalism. The crucial observation has been that the "twisted" shift symmetry lends itself very well to Fourier transformation and makes contact with the Nahm transformation for calorons.[26] The variable $z$ we introduced in formulating the generalized boundary conditions for the chiral fermions is precisely the dual of $t$ under this transformation.

### 3.1. *From ADHM to Nahm*

The ADHM construction for charge $k$ instantons[25] starts with a $2k \times 2k$ complex matrix $B = \sigma_\mu \otimes B_\mu$ (each $B_\mu$ is a hermitian $k \times k$ matrix, whereas $\sigma_\mu = (\mathbb{1}_2, i\vec{\tau})$) and a $k$ dimensional vector $\lambda = (\lambda_1, \ldots, \lambda_k)$, where $\lambda_i^\dagger$ is a two-component spinor in the $\bar{n}$ representation of SU($n$) (i.e. $\lambda$ is a $n \times 2k$ complex matrix). These are combined to form a $(n+2k) \times 2k$ dimensional matrix $\Delta(x)$, which has $n$ normalized eigenvectors with vanishing eigenvalue, $\Delta^\dagger(x)v(x)=0$. The matrix $v(x)$ is of size $(n+2k)\times n$ and one has

$$\Delta(x) = \begin{pmatrix} \lambda \\ B(x) \end{pmatrix}, \quad B(x) = B - x \otimes \mathbb{1}_k, \quad v(x)=\begin{pmatrix} -\mathbb{1}_n \\ B^\dagger(x)^{-1}\lambda^\dagger \end{pmatrix} \phi^{-1/2}, \tag{10}$$

where the quaternion $x = x_\mu \sigma_\mu$ (a $2 \times 2$ matrix with spinor indices) defines the position. The hermitian positive definite $n \times n$ matrix $\phi(x)$,

$$\phi(x)=\mathbb{1}_n+\lambda\left(B^\dagger(x)B(x)\right)^{-1}\lambda^\dagger, \tag{11}$$

is required to normalize $v(x)$.

The gauge field $A_\mu(x) = v^\dagger(x)\partial_\mu v(x)$ is self-dual provided

$$(\Delta^\dagger(x)\Delta(x))^{-1} = \mathbb{1}_2 \otimes f_x, \tag{12}$$

where $f_x$ is a hermitian $k \times k$ matrix. This condition states that $\Delta^\dagger(x)\Delta(x)$ is invertible and commutes with the quaternions, and is known as the quadratic ADHM constraint, turning a set of non-linear partial differential equations into a quadratic matrix equation.

To construct a charge $k$ caloron with non-trivial holonomy we place $k$ instantons in the time interval $[0,\beta[$, performing a color rotation with $\mathcal{P}_\infty$ for each shift of $t$ over $\beta$, cmp. Eq. (4). This is implemented by requiring (suppressing color and spinor indices and scaling $\beta$ to 1)

$$\lambda_{pk+k+a} = \mathcal{P}_\infty \lambda_{pk+a}, \quad B_{pk+a,qk+b} = B_{pk-k+a,qk-k+b} + \sigma_0\delta_{pq}\delta_{ab}. \tag{13}$$

Solutions to these equations are parametrized by $\zeta_a$ and $\hat{A}_p^{ab}$,

$$\lambda_{pk+a} = \mathcal{P}_\infty^p \zeta_a, \quad B_{pk+a,qk+b} = p\sigma_0\delta_{pq}\delta_{ab} + \hat{A}^{ab}_{p-q}, \tag{14}$$

with $\hat{A}$ to be determined so as to satisfy the quadratic ADHM constraint. To perform the Fourier transformation we introduce the $n$ projectors $P_m$, $\mathcal{P}_\infty = \sum_m e^{2\pi i \mu_m} P_m$, such that $\lambda_{pk+a} = \sum_m e^{2\pi i p \mu_m} P_m \zeta_a$. This gives

$$2\pi i \sum_p e^{2\pi i p z} \hat{A}_p^{ab} = \hat{A}^{ab}(z), \quad \sum_p e^{-2\pi i p z} \lambda_{pk+a} = \sum_m \delta(z-\mu_m) P_m \zeta_a. \tag{15}$$

Here $\zeta_a^\dagger$ is again a two-component spinor in the $\bar{n}$ representation of SU$(n)$ and $\hat{A}^{ab}(z) = \sigma^\mu \hat{A}_\mu^{ab}(z)$, with $\hat{A}_\mu(z)$ an anti-hermitian $k \times k$ matrix. In terms of the latter

$$\sum_{p,q} B_{pk+a,qk+b}(x) e^{2\pi i(pz-qz')} = \frac{\delta(z-z')}{2\pi i} \hat{D}_x^{ab}(z'), \tag{16}$$
$$\hat{D}_x^{ab}(z) \equiv \hat{D}^{ab}(z) - 2\pi i x \delta^{ab} = \sigma_0 \delta^{ab} \frac{d}{dz} + \hat{A}^{ab}(z) - 2\pi i x \delta^{ab},$$

which is the positive chirality Dirac (i.e. Weyl) operator for the U$(k)$ gauge field $\hat{A}_\mu(z) - 2\pi i x_\mu \mathbb{1}_k$ defined on the circle, $z \in [0,1]$, i.e. with periodicity 1 ($\beta^{-1}$ in case $\beta \neq 1$). Introducing $\hat{S}_m^{ab}$ and $\vec{\rho}_m^{\,ab}$, through

$$2\pi \zeta_a^\dagger P_m \zeta_b \equiv \mathbb{1}_2 \hat{S}_m^{ab} - \vec{\tau} \cdot \vec{\rho}_m^{\,ab}, \tag{17}$$

the quadratic ADHM constraint gives precisely Nahm's equation,[26]

$$\frac{d}{dz}\hat{A}_j(z) + [\hat{A}_0(z), \hat{A}_j(z)] + \tfrac{1}{2}\varepsilon_{jk\ell}[\hat{A}_k(z), \hat{A}_\ell(z)] = 2\pi i \sum_m \delta(z-\mu_m)\rho_m^j. \tag{18}$$

### 3.2. *From Green's function to solution*

For topological charge 1 the dual gauge field is abelian, and the commutator terms in Eq. (18) vanish, such that $\hat{A}_\mu(z)$ is simply piecewise constant, jumping at the singularities. On the other hand the power of the ADHM formalism lies in the "magic" formulae[27,28] that express the gauge field, fermion zero-mode, its density, and the action density in terms of $f_x$ as defined in Eq. (12). Fourier transformation turns the matrix $f_x$ into a Green's function $\hat{f}_x(z,z') \equiv \hat{g}^\dagger(z) f_x(z,z') \hat{g}(z')$, where $f_x(z,z')$ satisfies

$$\left\{ -\frac{d^2}{dz^2} + V(z;\vec{x}) \right\} f_x(z,z') = 4\pi^2 \mathbb{1}_k \delta(z-z'), \tag{19}$$

with

$$V(z;\vec{x}) \equiv 4\pi^2 \vec{R}^2(z;\vec{x}) + 2\pi \sum_m \delta(z-\mu_m) S_m, \quad S_m \equiv \hat{g}(\mu_m) \hat{S}_m \hat{g}^\dagger(\mu_m),$$
$$R_j(z;\vec{x}) \equiv x_j - (2\pi i)^{-1} \hat{g}(z) \hat{A}_j(z) \hat{g}^\dagger(z). \tag{20}$$

Note that the $S_m$ play the role of "impurities" and that

$$\hat{g}(z) \equiv \exp(2\pi i(\xi_0 - x_0 \mathbb{1}_k) z), \tag{21}$$

defined in terms of the dual holonomy $\exp(2\pi i \xi_0) \equiv \mathrm{Pexp}(\int_0^1 \hat{A}_0(z) dz)$, can be used to transform $\hat{A}_0 - 2\pi i x_0 \mathbb{1}_k$ to zero, in order to simplify as much as possible the Green's function equation. This is at the expense of introducing periodicity up to a gauge transformation. Although $\hat{f}_x(z,z')$ is periodic in $z$ and $z'$ with period 1 (for $\beta=1$), $f_x(z,z')$ no longer is.

Given a solution for the Green's function, there are straightforward expressions for the gauge field[29] only involving the Green's function evaluated at the "impurity" locations and for the fermion zero-modes.[16] For the zero-mode density see Eq. (5). As an example we give the Green's function at $z'=z$, which formally can be expressed as ($x$ dependence of $\mathcal{F}_z$ is implicit)

$$f_x(z,z) = -4\pi^2\left((\mathbb{1}_{2k}-\mathcal{F}_z)^{-1}\right)_{12},\ \mathcal{F}_z \equiv \hat{g}^\dagger(1)\mathrm{Pexp}\int_z^{z+1} dw \begin{pmatrix} 0 & \mathbb{1}_k \\ V(w;\vec{x}) & 0 \end{pmatrix}, \tag{22}$$

where the $(1,2)$ component on the right-hand side of the first identity is with respect to the $2\times 2$ block matrix structure. This has allowed us to find a particularly compact expression for the action density,[29]

$$\mathcal{S}(x) \equiv -\tfrac{1}{2}\mathrm{tr}F_{\mu\nu}^2(x) = -\tfrac{1}{2}\partial_\mu^2\partial_\nu^2 \log\det\left(ie^{-\pi i x_0}(\mathbb{1}_{2k}-\mathcal{F}_z)\right), \tag{23}$$

which can be shown to be independent of $z$.

The formal expression for $\mathcal{F}_z$ can be made more explicit by a decomposition into the "impurity" contributions $T_m$ at $z=\mu_m$ and the "propagation" $H_m \equiv W_m(\mu_{m+1},\mu_m)$ between $\mu_m$ and $\mu_{m+1}$, with for $z,z' \in (\mu_m,\mu_{m+1})$

$$T_m \equiv \begin{pmatrix} \mathbb{1}_k & 0 \\ 2\pi S_m & \mathbb{1}_k \end{pmatrix},\ W_m(z,z') \equiv \begin{pmatrix} f_m^+(z) & f_m^-(z) \\ \frac{d}{dz}f_m^+(z) & \frac{d}{dz}f_m^-(z) \end{pmatrix}\begin{pmatrix} f_m^+(z') & f_m^-(z') \\ \frac{d}{dz}f_m^+(z') & \frac{d}{dz}f_m^-(z') \end{pmatrix}^{-1},$$

$$\mathcal{F}_z = W_m(z,\mu_m)T_m H_{m-1}\cdots H_1 T_1 \hat{g}^\dagger(1) H_n T_n \cdots T_{m+1} H_m W_m(\mu_m,z). \tag{24}$$

The $k\times k$ matrices $f_m^\pm(z)$ are defined for $z \in (\mu_m,\mu_{m+1})$ and behave as $f_m^\pm(z) \to \exp\left(\pm 2\pi|\vec{x}|(z-\mu_m)\mathbb{1}_k\right) f_m^\pm(\mu_m)$ for $|\vec{x}|\to\infty$, where $f_m^\pm(\mu_m)$ can be arbitrary non-singular matrices. Together these form the $2k$ solutions of the homogeneous Green's function equation,

$$\left(\frac{d^2}{dz^2} - 4\pi^2\vec{R}^2(z;\vec{x})\right)\hat{v}(z;\vec{x}) = 0. \tag{25}$$

We have been able to use this "mix" of the ADHM and Nahm formalism both in making powerful approximations, like in the far field limit (based on our ability to identify the exponentially rising and falling terms), and for finding exact solutions through solving the homogeneous Green's function equation for $k=2$.

What makes this case tractable is the fact that the Nahm equations on each interval can be solved in terms of elliptic functions,[30]

$$\hat{g}(z)\hat{A}_j(z)\hat{g}^\dagger(z) \equiv 2\pi i\left[a_j\mathbb{1}_2 + \mathcal{D}\mathcal{R}_{jb}f_b\left(4\pi\mathcal{D}(z-z_0)\right)h^\dagger\tau_b h\right], \tag{26}$$

where $h, \mathcal{R}$ is a (gauge) rotation, $\vec{a}$ is a coordinate shift and

$$f_1(z) \equiv \frac{\mathbf{k}'}{\mathrm{cn}_{\mathbf{k}}(z)},\ f_2(z) \equiv \frac{\mathbf{k}'\mathrm{sn}_{\mathbf{k}}(z)}{\mathrm{cn}_{\mathbf{k}}(z)},\ f_3(z) \equiv \frac{\mathrm{dn}_{\mathbf{k}}(z)}{\mathrm{cn}_{\mathbf{k}}(z)},\ \mathbf{k}' \equiv \sqrt{1-\mathbf{k}^2}. \quad (27)$$

with $\mathrm{sn}_{\mathbf{k}}$, $\mathrm{cn}_{\mathbf{k}}$ and $\mathrm{dn}_{\mathbf{k}}(z) = \sqrt{1-\mathbf{k}^2\mathrm{sn}^2_{\mathbf{k}}}$ the standard Jacobi elliptic functions. This does not yet address matching of $\hat{A}_j(z)$ on different intervals, where some difference between the monopole and caloron application appears. For the caloron, apart from the axially symmetric solutions constructed in Ref. 29, we found two sets of non-trivial solutions that interpolate between overlapping and well-separated constituents, see Fig. 2.

To resolve the full structure of the cores we had to find the exact solutions to Eq. (25). For this task we could make convenient use of an existing analytic results for charge 2 monopoles,[31] adapting it to the case calorons. Essential is that once the solutions $f_m^{\pm}(z)$ are known, everything else can be easily determined in terms of these, cmp. Eq. (24). We conveniently expressed the explicit solutions for Eq. (25) in terms of the elliptic integral of the third kind, but refer to Ref. 16 for further details. A sample of the results can be found in Fig. 1 (right) and Fig. 3.

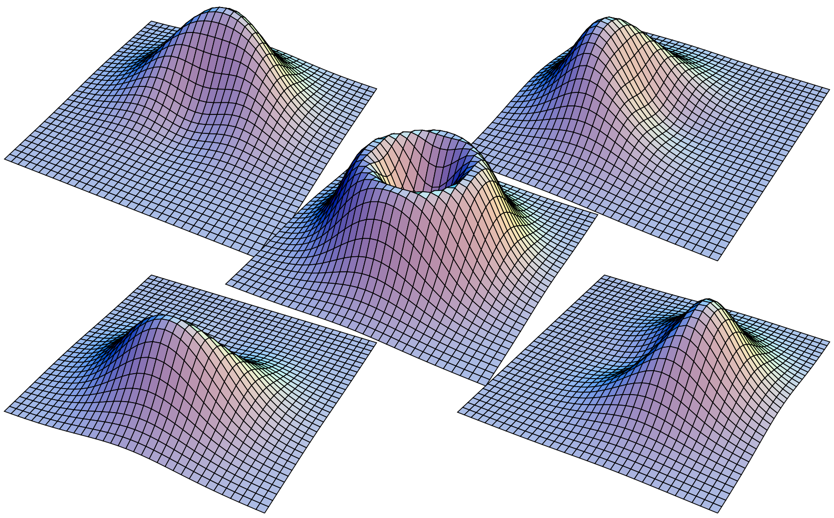

Figure 3. In the middle is shown the action density in the plane of the constituents at $t = 0$ for an SU(2) charge 2 caloron with $\mathrm{tr}\,\mathcal{P}_\infty = 0$ in the "crossed" configuration of Fig. 2, where all constituents strongly overlap. On a scale enhanced by a factor $16\pi^2$ are shown the densities for the two zero-modes, using either periodic (left) or anti-periodic (right) boundary conditions in the time direction. Note that these still are able to identify four individual, albeit deformed, constituents.

## 4. Conclusions

We have seen that instantons at finite temperature are composed of constituent monopoles. Of course, the hope is to develop a calculational scheme to address questions like monopole condensation in the popular scenario of the dual supercondutor.[32] However, this is very much complicated by the fact that at low temperature the coupling constant is large, and instantons form a dense ensemble. This does imply that the monopoles form a dense ensemble as well. One may then have some hope that the confining electric phase could be characterized by a dual deconfining magnetic phase, where the dual deconfinement is due to the large monopole density, in a similar spirit to high density induced quark deconfinement.

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