Higher charge calorons with non-trivial holonomy

Falk Bruckmann, Dániel Nógrádi and **Pierre van Baal**

Instituut-Lorentz for Theoretical Physics, University of Leiden,
P.O.Box 9506, NL-2300 RA Leiden, The Netherlands

Abstract

The full ADHM-Nahm formalism is employed to find exact higher charge caloron solutions with non-trivial holonomy, extended beyond the axially symmetric solutions found earlier. Particularly interesting is the case where the constituent monopoles, that make up these solutions, are not necessarily well-separated. This is worked out in detail for charge 2. We resolve the structure of the extended core, which was previously localized only through the singularity structure of the zero-mode density in the far field limit. We also show that this singularity structure agrees exactly with the abelian charge distribution as seen through the abelian component of the gauge field. As a by-product zero-mode densities for charge 2 magnetic monopoles are found.

1 Introduction

Calorons are instantons at finite temperature. For a long time the influence of a background Polyakov loop on the properties of these topological excitations has been neglected. Solutions were constructed long ago [1] and were studied in detail in the semi-classical approximation [2]. In all these studies the Polyakov loop at spatial infinity (also called the holonomy) was trivial, i.e. an element of the center of the gauge group. That the influence of the background Polyakov loop on the topological excitations can be dramatic is particularly clear in the confined phase, where on average its trace vanishes. Caloron solutions in such backgrounds were constructed only relatively recently [3, 4] and can be seen as composed of massive monopole constituents with their magnetic charges adding to zero.

It was observed that the one-loop correction to the action for configurations with a non-trivial asymptotic value of the Polyakov loop gives rise to an infinite action barrier, which were therefore considered irrelevant [2]. However, the infinity simply arises due to the integration over the finite energy *density* induced by the perturbative fluctuations in the background of a non-trivial Polyakov loop [5]. The proper setting would therefore rather be to calculate the non-perturbative contribution of calorons (with a given asymptotic value of the Polyakov loop) to this energy density, as was first successfully implemented in supersymmetric theories [6], where the perturbative contribution vanishes. It has a minimum where the trace of the Polyakov loop vanishes, i.e. at maximal non-trivial holonomy.

In a recent study at high temperatures, where one presumably can trust the semi-classical approximation, the non-perturbative contribution of these monopole constituents (also called dyons) was computed [7]. When added to the perturbative contribution [5] with its minima at center elements, a local minimum develops where the trace of the Polyakov loop vanishes, deepening further for decreasing temperature. This gives support for a phase in which the center symmetry, broken in the high temperature phase, is restored and provides an indication that the monopole constituents are the relevant degrees of freedom for the confined phase.

Also lattice studies, both using cooling [8] and chiral fermion zero-modes [9] as filters, have now confirmed that monopole constituents do dynamically occur in the confined phase. A charge 1 caloron is seen for SU(n) to consist of n constituent monopoles. In the deconfined phase, due to the fact that the average Polyakov loop becomes a center element, the caloron returns to the form known as the Harrington-Shepard solution [1]. The latter can also be interpreted as consisting of constituent monopoles, however, with $n-1$ of them being massless.

To be precise, for self-dual configurations in the background of non-trivial holonomy the masses of constituent monopoles are given by $8\pi^2\nu_j/\beta$, with $\nu_j \equiv \mu_{j+1}-\mu_j$. The μ_i are related to the eigenvalues of the Polyakov loop at spatial infinity,

$$\mathcal{P}_\infty = \lim_{|\vec{x}|\to\infty} \mathrm{Pexp}(\int_0^\beta A_0(t,\vec{x})dt) = g^\dagger \exp(2\pi i \mathrm{diag}(\mu_1,\mu_2,\ldots,\mu_n))g, \tag{1}$$

(this expression assumes the periodic gauge $A_\mu(t,\vec{x}) = A_\mu(t+\beta,\vec{x})$) where g is the gauge rotation used to diagonalize $\mathcal{P}_\infty$ and β is the period in the imaginary time direction, related to the inverse temperature. The eigenvalues $\exp(2\pi i\mu_j)$ are to be ordered on the circle such that $\mu_1 \le \mu_2 \le \ldots \le \mu_n \le \mu_{n+1}$, with $\mu_{n+j} \equiv 1+\mu_j$ and $\sum_{i=1}^n \mu_i = 0$, which guarantees that the masses add up to $8\pi^2/\beta$, the instanton action per unit (imaginary) time. At higher topological charge k, the parameter space of widely separated constituent monopoles is described by kn monopole constituents, k of each of the n types of abelian charges (with overall charge neutrality).

We established in an earlier paper [10] that well-separated constituents act as point sources for the so-called far field (that is far removed from any of the cores). When constituents of opposite charge (n constituents of different type) come together, the action density no longer deviates significantly from that of a standard instanton. Its scale parameter ρ is related to the constituent separation d through $\rho^2/\beta \approx d$. Yet, the gauge field is vastly different, as is seen from the fact that within the confines of the peak there are n locations where two of the eigenvalues of the Polyakov loop coincide [11, 12], thus in some sense varying over the maximal range available (e.g. for SU(2) from $\mathbb{1}_2$ to $-\mathbb{1}_2$), whereas for trivial holonomy only one such location is found.

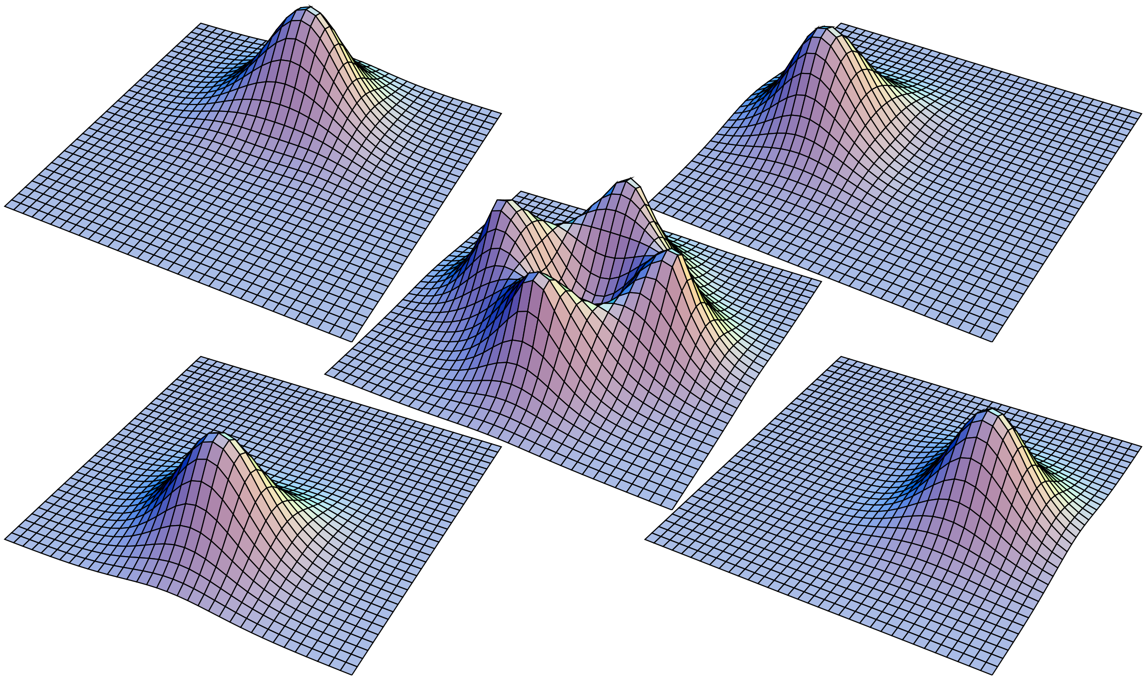

Figure 1: In the middle is shown the action density in the plane of the constituents at $t = 0$ for an SU(2) charge 2 caloron with $\mathrm{tr}\,\mathcal{P}_\infty = 0$, in the regime where constituents are not well-separated. On a scale enhanced by a factor $10\pi^2$ are shown the densitities for the two zero-modes, using either periodic (left) or anti-periodic (right) boundary conditions in the time direction. This solution is for the so-called "crossed" configuration with $\mathbf{k} = 0.997$ and $D = 8.753$, see Sect. 4 for more details.

On the other hand, when constituents of equal charge come together typically an extended core structure is found. This was deduced, in particular for the case of charge 2 calorons, from our ability to analytically determine the zero-mode density (summed over the two zero-modes implied by the index theorem) in the far field limit, neglecting exponential contributions[1]. In this limit it forms a singular distribution on a disc bounded by an ellipse, but approaches two delta functions for well-separated like-charge constituents. This zero-mode density only sees constituents of a given charge, depending on the boundary condition for the fermions in the imaginary time direction, which can be chosen to be a U(1) phase (containing the physically relevant choice of anti-periodic boundary conditions for thermal field theory). We will show for

[1]This is in some sense equivalent to the high temperature limit, with constituent masses given by $8\pi^2\nu_m/\beta$, such that the range of the exponential contributions shrinks inversely proportional with the temperature.

SU(2) that their difference for periodic and anti-periodic boundary conditions coincides *exactly* with the (abelian) charge distribution extracted from the gauge field in the same limit, making contact with an old result due to Hurtubise [13] for the asymptotic behaviour of the monopole Higgs field.

We found two particular parametrizations within the SU(2) charge 2 moduli space that exhibited these extended charge distributions. The first includes as a limit arbitrary charge 2 monopoles. The second of these parametrizations contains as a limiting case the axially symmetric configurations constructed for arbitrary charge in Ref. [14]. Deforming away from the axial configuration the two discs overlap. Describing the intricate behaviour for the non-abelian core of these configurations in this region of the parameter space requires one to find exact solutions, which are presented here. We rely on early work of Nahm [15] and Panagopoulos [16] for charge 2 magnetic monopoles, which is simplified to some extent by our formalism that uses the relation between the Fourier transformation of the ADHM construction (as relevant for the finite temperature case) and the Nahm transformation, a crucial ingredient for our success to find explicit solutions [3]. Fig. 1 gives a particular example for the action and zero-mode densities of a charge 2 caloron solution. The two dimensional zero-mode basis is chosen such that each zero-mode only localizes on one of the constituents of a given charge, showing both the zero-modes with periodic and anti-periodic boundary conditions in the imaginary time direction.

This paper is organized as follows. In Sect. 2 we will outline the construction, introduce the Green's function that is computed through the analogy of an impurity scattering problem, and summarize the various limits that can be formulated before explicitly solving for the Green's function. In Sect. 3 we present the method that allows one to find the exact solution for the Green's function, first for the general case and then applied in more detail to that of topological charge 2 calorons. Readers only interested in the results could skip Sects. 2.2 and 3. In Sect. 4 we discuss the two classes of configurations in the moduli space of the charge 2 calorons and provide plots of the various quantities to illustrate the properties of the exact results. In Sect. 5 we discuss the relation between the algebraic tail of the gauge field and the zero-mode density. We end with some discussions. An Appendix presents a new result for the limiting behaviour of the action density.

2 Outline of the construction

There are two steps in the construction of charge k caloron solutions. The first step involves finding a $U(k)$ gauge field $\hat{A}_\mu(z)$, which satisfies the self-duality equation, i.e. the Nahm equation [15], on a circle parametrized by z, with z introduced through replacing the original SU(n) gauge field $A_\mu(x)$ by $A_\mu(x) - 2\pi i z \delta_{0\mu} \mathbb{1}_n$. Although not affecting the field strength, this changes the holonomy to $\exp(-2\pi iz\beta)\mathcal{P}_\infty$, revealing that z has period β^{-1}. The index theorem guarantees the existence of k zero-modes $\Psi(x;z)$ which satisfy the Dirac equation, or in the two-component Weyl form

$$\bar{D}_z\Psi(x;z) \equiv \bar{\sigma}_\mu D_z^\mu \Psi(x;z) \equiv \bar{\sigma}_\mu(\partial_\mu + A_\mu(x) - 2\pi iz\delta_{0\mu}\mathbb{1}_n)\Psi(x;z) = 0. \tag{2}$$

with $\bar{\sigma}_\mu = \sigma_\mu^\dagger = (\mathbb{1}_2, -i\vec{\tau})$ (τ_i are the usual Pauli matrices). We may remove z from the gauge field $A_\mu(x)$ by transforming the zero-mode to $\hat{\Psi}_z(x) \equiv \exp(-2\pi itz)\Psi(x;z)$, which is at the expense of making the zero-mode only periodic up to a phase factor, $\hat{\Psi}_z(t+\beta,\vec{x}) = \exp(-2\pi iz\beta)\hat{\Psi}_z(t,\vec{x})$. In a similar way we could introduce $\vec{z}$ through $\Psi(x;z,\vec{z}) \equiv \exp(2\pi i\vec{z}\cdot\vec{x})\Psi(x;z)$, which replaces in Eq. (2) $A_\mu(x)$ by $A_\mu(x) - 2\pi iz_\mu\mathbb{1}_n$, where $z_0 \equiv z$. Assuming the k zero-modes $\Psi^{(a)}(x;z,\vec{z})$ to be orthonormal one has

$$\hat{A}_\mu^{ab}(z) = \int \Psi^{(a)}(x;z,\vec{z})^\dagger \frac{\partial}{\partial z_\mu}\Psi^{(b)}(x;z,\vec{z})d^4x, \tag{3}$$

or equivalently (demonstrating as well that $\hat{A}$ does not depend on $\vec{z}$)

$$\hat{A}_0^{ab}(z) = \int \Psi^{(a)}(x;z)^\dagger \frac{\partial}{\partial z}\Psi^{(b)}(x;z)d^4x, \quad \hat{A}_k^{ab}(z) = 2\pi i\int \Psi^{(a)}(x;z)^\dagger x_k \Psi^{(b)}(x;z)d^4x. \tag{4}$$

We have shown how $\hat{A}_\mu(z)$ can alternatively be related to a Fourier transformation of the ADHM construction of instantons [17], that periodically repeat (up to a gauge rotation with $\mathcal{P}_\infty$) in the imaginary time direction, so as to turn an infinite charge instanton in R^4 to one of finite charge and finite temperature. The derivation of this relation will not be repeated here, see Ref. [3, 14] for the details.

The connection to the ADHM construction has been useful to simplify the second step in the construction of the caloron solutions, namely how to reconstruct the original gauge field when given a solution to the Nahm equation [15] (which is equivalent to the quadratic ADHM constraint),

$$\frac{d}{dz}\hat{A}_j(z) + [\hat{A}_0(z), \hat{A}_j(z)] + \tfrac{1}{2}\varepsilon_{jk\ell}[\hat{A}_k(z), \hat{A}_\ell(z)] = 2\pi i\sum_m \delta(z-\mu_m)\rho_m^j, \tag{5}$$

For convenience of notation we will henceforth use the classical scale invariance to set $\beta = 1$. The singularities in the Nahm equation appear precisely for those values where $e^{-2\pi iz}\mathcal{P}_\infty$ has one of its eigenvalues equal to 1, at $z = \mu_i$. This is where some fermion field components become massless, i.e. the zero-mode becomes delocalized, whereas for generic values of z it is exponentially localized, which has turned out to be a useful tool to pinpoint the constituent monopoles.

One could apply the Nahm transformation again, introducing $\hat{A}_\mu(z) - 2\pi ix_\mu\mathbb{1}_k$ (with $x_0 \equiv t$) and find the n chiral fermion zero-modes in the background of this gauge field. The construction is somewhat complicated due to the presence of the singularities, whose structure is determined from the matrices $\vec{\rho}_m$ which appear in the Nahm equation. Not all $\vec{\rho}_m$ are independent, e.g. integrating and tracing the Nahm equation yields the conditions, $\sum_{m=1}^n \mathrm{Tr}\,\vec{\rho}_m = \vec{0}$. Further constraints are implied [14] by the fact that one may introduce (as is most easily seen in relation to the ADHM construction) k two-component spinors $\zeta_a^\dagger$ in the $\bar{n}$ representation of SU(n), such that

$$2\pi\zeta_a^\dagger P_m \zeta_b = \sigma_0 \hat{S}_m^{ab} - \vec{\tau}\cdot\vec{\rho}_m^{\,ab}, \tag{6}$$

where P_m are projections defined through $\mathcal{P}_\infty = \sum_{m=1}^n e^{2\pi i\mu_m} P_m$ ($\hat{S}_m$ will appear in Eq. (8)).

2.1 The Green's function

However, with reference to the ADHM construction, there is great benefit in first finding the solution for the Green's function, $\hat{f}_x(z,z') \equiv \hat{g}^\dagger(z) f_x(z,z')\hat{g}(z')$, where

$$\left\{-\frac{d^2}{dz^2} + V(z;\vec{x})\right\} f_x(z,z') = 4\pi^2 \mathbb{1}_k \delta(z-z'), \tag{7}$$

with

$$\begin{aligned} V(z;\vec{x}) &\equiv 4\pi^2 \vec{R}^2(z;\vec{x}) + 2\pi\sum_m \delta(z-\mu_m)S_m, \quad S_m \equiv \hat{g}(\mu_m)\hat{S}_m\hat{g}^\dagger(\mu_m), \\ R_j(z;\vec{x}) &\equiv x_j - (2\pi i)^{-1}\hat{g}(z)\hat{A}_j(z)\hat{g}^\dagger(z), \end{aligned} \tag{8}$$

and S_m playing the role of "impurities". This is formulated in the gauge where first we transform $\hat{A}_0(z)$ to a constant (diagonal) matrix, $2\pi i\xi_0$, as is always possible in one dimension, and then use

$$\hat{g}(z) \equiv \exp(2\pi i(\xi_0 - x_0\mathbb{1}_k)z), \quad \mathrm{Tr}\,\xi_0 = 0 \tag{9}$$

(when $\mathrm{Tr}\,\xi_0 \neq 0$ it is absorbed in a shift of x_0) to transform $\hat{A}_0 - 2\pi i x_0 \mathbb{1}_k$ to zero. This is at the expense of introducing periodicity up to a gauge transformation; although $\hat{f}_x(z,z')$ is periodic in z and z' with period 1 (for $\beta = 1$), $f_x(z,z')$ no longer is[2].

Given a solution for the Green's function, there are straightforward expressions for the gauge field [14] (only involving the Green's function evaluated at the "impurity" locations) and the fermion zero-modes [10, 18]. For the zero-mode density this gives

$$\hat{\Psi}_z^{(a)}(x)^\dagger \hat{\Psi}_z^{(b)}(x) = -(2\pi)^{-2}\partial_\mu^2 \hat{f}_x^{ab}(z,z). \tag{10}$$

[2] It is in this respect interesting to note that $\hat{g}(1)$ plays the role of the holonomy associated to the dual Nahm gauge field $\hat{A}_\mu(z)$.

In this paper we will only have need for the Green's function at $z' = z$, which formally can be expressed as

$$f_x(z,z) = -4\pi^2 \left((\mathbb{1}_{2k} - \mathcal{F}_z)^{-1}\right)_{12}, \quad \mathcal{F}_z \equiv \hat{g}^\dagger(1)\mathrm{Pexp}\int_z^{z+1} \begin{pmatrix} 0 & \mathbb{1}_k \\ V(w;\vec{x}) & 0 \end{pmatrix} dw, \quad (11)$$

where the $(1,2)$ component on the right-hand side of the first identity is with respect to the 2×2 block matrix structure. In particular this leads to a compact expression for the action density [3, 14]

$$\mathcal{S}(x) \equiv -\tfrac{1}{2}\mathrm{tr}F_{\mu\nu}^2(x) = -\tfrac{1}{2}\partial_\mu^2\partial_\nu^2 \log\det\left(ie^{-\pi i x_0}(\mathbb{1}_{2k} - \mathcal{F}_z)\right), \quad (12)$$

which can be shown to be independent of the choice of z.

The formal expression for $\mathcal{F}_z$ can be made explicit by a decomposition into the "impurity" contributions T_m at $z = \mu_m$ and the "propagation" $H_m \equiv W_m(\mu_{m+1}, \mu_m)$ between μ_m and μ_{m+1}. For $z \in (\mu_m, \mu_{m+1})$ this gives

$$\mathcal{F}_z = W_m(z,\mu_m)T_m H_{m-1}\cdots T_2 H_1 T_1 \hat{g}^\dagger(1) H_n T_n H_{n-1}\cdots T_{m+1} H_m W_m(\mu_m, z), \quad (13)$$

with

$$T_m \equiv \begin{pmatrix} \mathbb{1}_k & 0 \\ 2\pi S_m & \mathbb{1}_k \end{pmatrix}, \quad W_m(z,z') \equiv \begin{pmatrix} f_m^+(z) & f_m^-(z) \\ \frac{d}{dz}f_m^+(z) & \frac{d}{dz}f_m^-(z) \end{pmatrix} \begin{pmatrix} f_m^+(z') & f_m^-(z') \\ \frac{d}{dz}f_m^+(z') & \frac{d}{dz}f_m^-(z') \end{pmatrix}^{-1}. \quad (14)$$

The columns of the two $k\times k$ matrices $f_m^\pm(z)$, defined for $z \in (\mu_m, \mu_{m+1})$, form the $2k$ solutions of the homogeneous Green's function equation,

$$\left(\frac{d^2}{dz^2} - 4\pi^2 \vec{R}^2(z;\vec{x})\right)\hat{v}(z) = 0, \quad (15)$$

of which those in $f_m^+(z)$ are exponentially rising and those in $f_m^-(z)$ are exponentially falling. This implies [14] $f_m^\pm(z) \to \exp\left(\pm 2\pi|\vec{x}|(z-\mu_m)\mathbb{1}_k\right) C_m^\pm$ for $|\vec{x}| \to \infty$, in which $C_m^\pm \equiv f_m^\pm(\mu_m)$ can be arbitrary non-singular (to ensure a complete set of solutions) matrices. In Ref. [10] we put $C_m^\pm = \mathbb{1}_k$, but here we find it convenient to leave this choice open. With the "impurity" scattering problem solved, constructing the exact solutions of the homogeneous Green's function equation is the last step in finding analytic expressions for the higher charge calorons.

2.2 Limiting cases

Nevertheless, approximate solutions can be derived, either assuming $\vec{x}$ is far removed from any core such that the gauge field has become abelian (which we called the far field limit, denoted by a subscript "ff"), or assuming $\vec{x}$ and the constituents of type m

(belonging to the m-th interval) are well separated from all others, but not necessarily from each other (which was called the zero-mode limit, denoted by a subscript "zm"). This is because we found [10] that for $z \in (\mu_m, \mu_{m+1})$ the k zero-modes $\hat{\Psi}_z^{(a)}(x)$ only "see" the constituents of type m. For $\mu_m \leq z \leq \mu_{m+1}$ we have

$$\begin{aligned} f_x^{\rm zm}(z,z) &= \pi\left(f_m^-(z)f_m^-(\mu_{m+1})^{-1} - f_m^+(z)f_m^+(\mu_{m+1})^{-1}Z_{m+1}^-\right)\times \\ &\quad \left(f_m^-(\mu_m)f_m^-(\mu_{m+1})^{-1} - Z_m^+f_m^+(\mu_m)f_m^+(\mu_{m+1})^{-1}Z_{m+1}^-\right)^{-1}\times \\ &\quad \left(f_m^-(\mu_m)f_m^-(z)^{-1} - Z_m^+f_m^+(\mu_m)f_m^+(z)^{-1}\right)R_m^{-1}(z), \end{aligned} \tag{16}$$

up to *exponential* corrections in the distance to the constituents of type $m' \neq m$, with

$$\begin{aligned} Z_m^- &\equiv \mathbb{1}_k - 2\Sigma_m^{-1}R_{m-1}(\mu_m), \qquad Z_m^+ \equiv \mathbb{1}_k - 2\Sigma_m^{-1}R_m(\mu_m), \\ R_m(z) &\equiv \tfrac{1}{2}(R_m^+(z)+R_m^-(z)), \qquad \Sigma_m \equiv R_m^-(\mu_m)+R_{m-1}^+(\mu_m)+S_m. \end{aligned} \tag{17}$$

and

$$2\pi R_m^\pm(z) \equiv \pm\left(\frac{d}{dz}f_m^\pm(z)\right)f_m^\pm(z)^{-1}. \tag{18}$$

In the zero-mode limit Z_m^+ and Z_{m+1}^- approach $\mathbb{1}_k$ up to *algebraic* corrections. Neglecting these contributions as well, e.g. by sending the constituents of type $m' \neq m$ to infinity, leads to the so-called monopole limit (denoted by a subscript "mon"), further simplifying the expression for the Green's function to

$$f_x^{\rm mon}(z,z) = -\pi U(z,\mu_{m+1})U_m^{-1}(\mu_m,\mu_{m+1})U_m(\mu_m,z)R_m^{-1}(z), \tag{19}$$

with[3]

$$U_m(z,z') \equiv f_m^+(z)f_m^+(z')^{-1} - f_m^-(z)f_m^-(z')^{-1}. \tag{20}$$

In turn, from Eq. (16) one may derive the far field limit, giving up to *exponential* corrections in the distance to *all* constituents

$$f_x^{\rm ff}(\mu_m,\mu_m) = 2\pi\Sigma_m^{-1}, \tag{21}$$

as is relevant for the expression of the gauge field in this limit [14]. For the zero-mode density (Eq. (10)) we may use for $\mu_m < z < \mu_{m+1}$ (z strictly different from μ_m, μ_{m+1})

$$f_x^{\rm ff}(z,z) = \pi R_m^{-1}(z). \tag{22}$$

The discontinuity in this limit, $\pi R_m(\mu_m) \neq 2\pi\Sigma_m^{-1} \neq \pi R_{m-1}(\mu_m)$, arises due to the zero-mode developing a massless component when z approaches μ_m. It might seem that

[3]Note that $U_m(z,z')$ satisfies the Green's function equation with boundary conditions $U_m(z,z)=0$ and $\frac{d}{dz}U_m(z,z') = -\frac{d}{dz'}U_m(z,z') = 2\pi R_m(z)$, for $z' \to z$.

Eq. (22), combined with Eq. (10), is inconsistent with an exponential decay. However, it turns out that [10]

$$\mathcal{V}_m(\vec{x}) \equiv (4\pi)^{-1}\mathrm{Tr}\left(R_m^{-1}(z)\right) \tag{23}$$

is independent of z in the m-th interval and harmonic everywhere (hence giving vanishing zero-mode density) except for some singularities in the cores of the constituents of type m. It is this feature, and our ability to compute $\mathcal{V}_m(\vec{x})$ exactly for SU(2) charge 2 calorons, that allowed us to make statements about the localization of the cores, without solving the Green's function exactly.

In Appendix A we derive the following new result for the monopole limit of the action density (Eq. (12)). In the limit where $\vec{x}$ and the constituent locations of type m are well separated from all other constituents, for which both the action and zero-mode densities are static, we find (with U_m defined in Eq. (20))

$$\mathcal{S}^{\mathrm{mon}}(\vec{x}) = -\tfrac{1}{2}\partial_i^2\partial_j^2 \log\det\left[U_m(\mu_{m+1},\mu_m)R_m^{-1}(\mu_m)\right], \tag{24}$$

up to *algebraic* corrections. This is a direct generalization for the action density of a single BPS monopole, $\mathcal{S}(\vec{x}) = -\frac{1}{2}\partial_i^2\partial_j^2\log\left[\sinh(2\pi\nu_m|\vec{x}|)/|\vec{x}|\right]$ (located at the origin and with mass $8\pi^2\nu_m$). We emphasize that this result can be used irrespective of the distance between the constituents of the same type m. Eq. (24) therefore gives in terms of $f_m^{\pm}(z)$ a closed form expression for the multi-monopole energy density. The same holds, when using Eq. (19), for the zero-mode density (Eq. (10)). For the caloron it involves taking the limit where all constituents of type $m' \neq m$ are sent to infinity, which is why this is called the monopole limit. We recall, that in the far field one should use [14] (see also App. A)

$$\mathcal{S}^{\mathrm{ff}}(\vec{x}) = -\tfrac{1}{2}\partial_i^2\partial_j^2\sum_{m=1}^{n}\log\det\left(f_m^+(\mu_{m+1})f_m^+(\mu_m)^{-1}R_m^{-1}(\mu_m)\Sigma_m\right). \tag{25}$$

3 Exact results

Finding the exact homogeneous solutions of the Green's function equation, Eq. (15), closely follows Nahm's method [15] to construct the dual chiral zero-modes. The main advantage of our approach is that we need not worry about boundary conditions, as this is solved by the "impurity" scattering formalism [14]. In the following we restrict ourselves to a given interval $z \in (\mu_m, \mu_{m+1})$ and work in the gauge where $\hat{A}_0(z) - 2\pi i x_0 \mathbb{1}_k = 0$.

Using that $\hat{A}_\mu(z)$ is self-dual, a consequence of the Nahm equation, one easily shows that $\hat{D}_x^\dagger\hat{D}_x = -\frac{d^2}{dz^2} + 4\pi^2\vec{R}^2(z;\vec{x})$, such that it is natural to consider the equation

$$\hat{D}_x\hat{\psi}(z) = \sigma_\mu\hat{D}_x^\mu\hat{\psi}(z) = \left(\frac{d}{dz} - 2\pi\vec{\tau}\cdot\vec{R}(z;\vec{x})\right)\hat{\psi}(z) = 0. \tag{26}$$

It follows that $\hat{\psi}(z)$ would be a homogeneous solution of the Green's function equation, albeit that $\hat{\psi}(z)$ is a spinor (with a chirality opposite to that for the zero-modes involved in the Nahm transformation, cmp. Eq. (2)). We follow Nahm [15] and use the ansatz $\hat{\psi}(z) = (\mathbb{1}_2 + \vec{u}(\vec{x}) \cdot \vec{\tau})|s\rangle \otimes \hat{v}(z)$, where $\hat{v}(z)$ is a k dimensional complex vector, $\vec{u}(\vec{x})$ is a unit vector that does not depend on z and $|s\rangle$ is an arbitrary normalized constant spinor (as long as it is not annihilated by $\mathbb{1}_2 + \vec{u}(\vec{x}) \cdot \vec{\tau}$). It then follows that $\hat{v}(z) = \langle s|\,(\mathbb{1}_2 - \vec{u}(\vec{x}) \cdot \vec{\tau})\,\hat{\psi}(z)$ satisfies Eq. (15).

The unit vector $\vec{u}(\vec{x})$ is found from a complex vector $\vec{y}(x)$ which squares to 0, $\vec{y}(\vec{x}) \cdot \vec{y}(\vec{x}) = 0$, implying its real and imaginary parts are perpendicular and of equal length ($\neq 0$ as long as $\vec{y}(\vec{x}) \neq \vec{0}$), such that (for ease of notation the $\vec{x}$ dependence of $\vec{y}$ and $\vec{u}$ will henceforth be left implicit)

$$\vec{u} = i\vec{y} \times \vec{y}^*/\Big(\vec{y} \cdot \vec{y}^*\Big) \tag{27}$$

is well defined and $\vec{u} \times \vec{y} = -i\vec{y}$, i.e. $\mathrm{Re}(\vec{y})$, $\mathrm{Im}(\vec{y})$ and $\vec{u}$ form an orthogonal set of vectors. Using the ansatz for $\hat{\psi}(z)$ and introducing

$$\hat{Y}(z) \equiv -\vec{y} \cdot \vec{R}(z;\vec{x}), \quad \hat{U}(z) \equiv -2\pi\vec{u} \cdot \vec{R}(z;\vec{x}), \tag{28}$$

leads to the equations

$$\hat{Y}(z)\hat{v}(z) = 0, \quad \frac{d}{dz}\hat{v}(z) = -\hat{U}(z)\hat{v}(z) \tag{29}$$

for which the first one can only have a solution provided $\det \hat{Y}(z) = 0$.

It is the great beauty of Nahm's formalism that $\det \hat{Y}(z)$ is a conserved quantity. That is, $\frac{d}{dz}\hat{A}_j(z) = -\frac{1}{2}\varepsilon_{jk\ell}[\hat{A}_k(z), \hat{A}_\ell(z)]$ (the Nahm equation Eq. (5) restricted to an interval and in the gauge where $\hat{A}_0 = 0$) implies $\frac{d}{dz}\det \hat{Y}(z) = 0$ for any choice of $\vec{x}$ and $\vec{y}$ on a null-cone in C^3 (or rather in CP^2 since we may rescale $\vec{y}$ with a non-zero complex factor without changing the equations). The conserved quantities are generated by the symmetric traceless monomials, $M_{i_1 i_2 \cdots i_\ell}$, built from $\mathrm{Tr}\left(\hat{A}_{i_1}(z)\hat{A}_{i_2}(z)\cdots\hat{A}_{i_\ell}(z)\right)$ with ℓ arbitrary, as one readily verifies. For example, $\mathrm{Tr}\,\hat{A}_i(z)$ is constant and defines the center of mass. A natural way to project on the traceless symmetric monomials is precisely through introducing $\vec{y} \in CP^2$ on a null-cone, forming $y_{i_1}y_{i_2}\cdots y_{i_\ell}M_{i_1 i_2 \cdots i_\ell} = y_{i_1}y_{i_2}\cdots y_{i_\ell}\mathrm{Tr}\left(\hat{A}_{i_1}(z)\hat{A}_{i_2}(z)\cdots\hat{A}_{i_\ell}(z)\right)$. It is interesting to note that $x_{i_1}x_{i_2}\cdots x_{i_\ell}M_{i_1 i_2 \cdots i_\ell}|\vec{x}|^{-\ell}$ is always a spherical harmonic of order ℓ, used in Ref. [10] to show through the multipole expansion of $\mathrm{Tr}\,(R_m^{-1}(z))$ that it is conserved and harmonic, except for singularities in the core of the constituents.

Once it is established that $\det \hat{Y}(z)$ is independent of z for any choice of $\vec{x}$, we can look for its zeros. Using the null-cone parametrization $\vec{y} = (\frac{1}{2}(1-\zeta^2), -\frac{i}{2}(1+\zeta^2), \zeta)$, and the fact that the matrix $\hat{Y}(z)$ is k dimensional, we obtain a polynomial equation in

ζ of order $2k$ and hence there are for generic $\vec{x}$ exactly $2k$ solutions. It is useful to note that these solutions come in complex conjugate pairs, where the symmetry $\vec{y} \to \vec{y}^*$ implies $\vec{u} \to -\vec{u}$ and $\zeta \to -1/\zeta^*$ (giving $\vec{y}^*$ up to a multiple, equivalent to $\vec{y}^*$ in CP^2).

Given a particular zero $\vec{y}$, we may conveniently write a vector in the kernel of $\hat{Y}(z)$ as [19] $\hat{v}_a(z) = \hat{\phi}(z)(\mathrm{adj}\,\hat{Y}(z))_{ac}$ for a fixed choice of c, where $\mathrm{adj}\,\hat{Y}(z)$ is the matrix formed by the minors of $\hat{Y}(z)$. The Nahm equation is easily seen to imply $\frac{d}{dz}\hat{Y}(z) = [\hat{U}(z), \hat{Y}(z)]$ for *any* choice of $\vec{y}$ on the null-cone. From this one derives that[4] $\frac{d}{dz}\mathrm{adj}\,\hat{Y}(z) = [\hat{U}(z), \mathrm{adj}\,\hat{Y}(z)]$. Substituting $\hat{v}_a(z) = \hat{\phi}(z)(\mathrm{adj}\,\hat{Y}(z))_{ac}$ into Eq. (29) gives

$$\frac{d\hat{\phi}(z)}{dz}(\mathrm{adj}\,\hat{Y}(z))_{ac} = \hat{\phi}(z)\Big(\mathrm{adj}\,\hat{Y}(z)\hat{U}(z)\Big)_{ac}. \tag{30}$$

Using the fact that $\hat{U}(z) = -iu_j\hat{g}(z)\hat{A}_j(z)\hat{g}^\dagger(z) - 2\pi u_j x_j$, we get

$$\hat{\phi}(z) = \frac{\exp\Big(\hat{\mu}(z) - 2\pi z\vec{u}\cdot\vec{x}\Big)}{\sqrt{(\mathrm{adj}\,\hat{Y}(z))_{ac}}}, \quad \frac{d\hat{\mu}(z)}{dz} = -\frac{i\Big\{\mathrm{adj}\,\hat{Y}(z), u_j\hat{g}(z)\hat{A}_j(z)\hat{g}^\dagger(z)\Big\}_{ac}}{2(\mathrm{adj}\,\hat{Y}(z))_{ac}}, \tag{31}$$

where the equation for $\hat{\mu}(z)$ is the same for any value of a (it may depend on the value of c). This is useful for studying the asymptotic behaviour of the solution. For large $|\vec{x}|$, $\det\hat{Y}(z) = 0$ implies that $\vec{y}\cdot\vec{x} \to 0$, such that (cmp. Eq. (27)) $\vec{u} \to \pm\vec{x}/|\vec{x}|$. We may use the symmetry $\vec{u} \to -\vec{u}$ to guarantee that there are k zeros $\vec{y}^{(b)}$ with the sign of $\vec{u}^{(b)}\cdot\vec{x}$ negative, leading to solutions that rise as $\exp(2\pi z|\vec{x}|)$. It then follows that the k zeros $\vec{y}^{(b+k)} = \vec{y}^{(b)*}$ give $\vec{u}^{(b+k)} = -\vec{u}^{(b)}$, leading to the solutions that decay as $\exp(-2\pi z|\vec{x}|)$ for large $|\vec{x}|$. Hence we may put $f^+_{ab}(z) = \hat{v}^{(b)}_a(z)$ and $f^-_{ab}(z) = \hat{v}^{(b+k)}_a(z)$. Defining

$$\hat{m}^+_{ab}(z) = -\Big(\mathrm{adj}\,\vec{y}^{(b)}\cdot\vec{R}(z;\vec{x})\Big)_{ac}, \quad \hat{m}^-_{ab}(z) = -\Big(\mathrm{adj}\,\vec{y}^{(b)*}\cdot\vec{R}(z;\vec{x})\Big)_{ac}, \tag{32}$$

which are *algebraic* in $\hat{A}_j(z)$ and x_j, we find

$$f^+_{ab}(z) = \hat{m}^+_{ab}(z)\hat{\phi}^{(b)}(z), \quad f^-_{ab}(z) = \hat{m}^-_{ab}(z)\hat{\phi}^{(b+k)}(z), \tag{33}$$

where $\hat{\phi}(z)$ contains the exponential dependencies, and thus seemingly all the information about the cores of the constituents. To make this more precise we compute $R^\pm(z)$, see Eq. (18), using that Eq. (29) implies (with $\vec{u}^{(b+k)} = -\vec{u}^{(b)}$) $\frac{d}{dz}f^\pm_{ab}(z) = \mp 2\pi\sum_{d=1}^k \vec{u}^{(b)}\cdot\vec{R}_{ad}(z;\vec{x})f^\pm_{db}(z)$, finding that the factors $\hat{\phi}^{(b)}(z)$ *drop out*

$$R^\pm_{ad}(z) = -\sum_{b,e=1}^{k}\vec{u}^{(b)}\cdot\vec{R}_{ae}(z;\vec{x})\hat{m}^\pm_{eb}(z)(\hat{m}^\pm(z)^{-1})_{bd}. \tag{34}$$

[4] Assume first that $\vec{y}$ is such that $\det\hat{Y} \neq 0$, in which case $\mathrm{adj}\,\hat{Y} = \hat{Y}^{-1}\det\hat{Y}$ and therefore $\frac{d}{dz}\mathrm{adj}\,\hat{Y} = -\hat{Y}^{-1}[\hat{U},\hat{Y}]\hat{Y}^{-1}\det\hat{Y} + \hat{Y}^{-1}\frac{d}{dz}\det\hat{Y} = [\hat{U},\mathrm{adj}\,\hat{Y}] + \hat{Y}^{-1}\mathrm{Tr}(\hat{Y}^{-1}\frac{d}{dz}\hat{Y})\det\hat{Y} = [\hat{U},\mathrm{adj}\,\hat{Y}]$. Observe that $\mathrm{adj}\,\hat{Y}$ is analytic in $\vec{y}$, such that the result is valid also when $\vec{y}$ leads to $\det\hat{Y} = 0$.

This proves that $R_m^{\pm}(z)$ is purely algebraic in $\hat{A}_j(z)$ and x_j, as are Σ_m and $R_m(z)$, which determine the far field limit for the zero-mode density and the gauge field.

One might wonder how, given that the $\hat{\psi}^{(b)}(z)$ are of the "wrong" chirality in the context of the Nahm transformation, one could use these results to reconstruct the gauge field for magnetic monopoles where the relation to the ADHM construction is not readily available. For this one observes that the columns of $\hat{\psi}^{(b)}(z)$ can be used to form a $2k\times 2k$ matrix $w(z)$. Using that $\hat{D}_x w(z)=0$, one finds $\hat{D}_x^{\dagger}\left(w^{\dagger}(z)^{-1}\right)=0$. Thus the columns of $w^{\dagger}(z)^{-1}$ give $2k$ independent solutions for each interval, from which n normalizable solutions $\hat{\Psi}^{(p)}(z;\vec{x})$ should remain after imposing the appropriate boundary (cq. matching) conditions. These are then used in Nahm's original construction to compute the gauge field (cmp. Eq. (3))

$$A_\mu^{pq}(x)=\int \hat{\Psi}^{(p)}(z;x_0,\vec{x})^{\dagger}\frac{\partial}{\partial x_\mu}\hat{\Psi}^{(q)}(z;x_0,\vec{x})dz, \tag{35}$$

where $\hat{\Psi}^{(p)}(z;x_0,\vec{x})\equiv \hat{g}^{\dagger}(z)\hat{\Psi}^{(p)}(z;\vec{x})$.

There seems to be considerable advantage in using the Green's function (Fourier transformed ADHM) method, since it can solve the matching conditions without relying on the availability of exact solutions for the normalizable dual zero-modes. To go beyond the approximations discussed in Sect. 2.2 and resolve the constituent cores we need to solve for $\hat{\mu}(z)$. This can not always be done in closed form, but it is given by an explicit integral which can be performed numerically when required. For charge 2 monopoles Panagopoulos [16] was, however, able to find the exact integral. We can use the same ingredients for the caloron case and explicitly solve for the Green's function in the case of charge 2 calorons.

3.1 Analytic expressions for charge 2

For charge 2 the number of invariants associated to the conserved quantities of the Nahm equation is 8, of which $\mathrm{Tr}\,\hat{A}_j(z)\equiv 4\pi i a_j$ are related to the center of mass for the constituents of given magnetic charge, coming from the interval under consideration. Assuming now that $\hat{A}_j(z)$ is traceless, 5 invariants remain, given in terms of the symmetric traceless tensor

$$M_{ij}\equiv -\tfrac{1}{2}\left(\mathrm{Tr}(\hat{A}_i(z)\hat{A}_j(z))-\tfrac{1}{3}\delta_{ij}\mathrm{Tr}(\hat{A}_k^2(z))\right). \tag{36}$$

Three of its parameters are associated to the rotation $\mathcal{R}$ which diagonalizes the 3×3 matrix, $M=\mathcal{R}\mathrm{diag}(c_1,c_2,c_3)\mathcal{R}^t$, where $\mathcal{R}$ is fixed by requiring $c_2\le c_1\le c_3$. The c_i add to zero and can be expressed in terms of the so-called scale (D) and shape ($\mathbf{k}$) parameters,

$$c_1=D^2\frac{1-2\mathbf{k}^2}{12},\quad c_2=D^2\frac{\mathbf{k}^2-2}{12},\quad c_3=D^2\frac{1+\mathbf{k}^2}{12}. \tag{37}$$

The Nahm equation for the case of charge 2 can be solved completely in terms of Jacobi elliptic functions [20, 21], which was summarized in Ref. [10]

$$\hat{g}(z)\hat{A}_j(z;\vec{a},\mathcal{R},h,D,\mathbf{k})\hat{g}^\dagger(z) \equiv 2\pi i a_j \mathbb{1}_2 + \tfrac{1}{2} iD\mathcal{R}_{jb} f_b\left(D(z-z_0)\right) h^\dagger \tau_b h, \tag{38}$$

where h is a global gauge parameter and

$$f_1(z) \equiv \frac{\mathbf{k}'}{\mathrm{cn}_\mathbf{k}(z)}, \quad f_2(z) \equiv \frac{\mathbf{k}'\mathrm{sn}_\mathbf{k}(z)}{\mathrm{cn}_\mathbf{k}(z)}, \quad f_3(z) \equiv \frac{\mathrm{dn}_\mathbf{k}(z)}{\mathrm{cn}_\mathbf{k}(z)}, \quad \mathbf{k}' \equiv \sqrt{1-\mathbf{k}^2}. \tag{39}$$

with[5] $\mathrm{sn}_\mathbf{k}(z) = \sin(\varphi(z))$, $\mathrm{cn}_\mathbf{k}(z) = \cos(\varphi(z))$ and $\mathrm{dn}_\mathbf{k}(z) = \sqrt{1-\mathbf{k}^2\mathrm{sn}^2_\mathbf{k}(z)}$ the standard Jacobi elliptic functions. This does not yet address the matching of $\hat{A}_j(z)$ on the different intervals, where some difference between the monopole and caloron application appears. For the caloron, apart from the axially symmetric solutions constructed in Ref. [14], we found two sets of non-trivial solutions that interpolate between overlapping and well-separated constituents. It is for these classes of solutions that we will resolve the cores, when constituents overlap and the non-linearity plays an important role.

The next step in the construction is finding the zeros $\vec{y}$ of $\det\hat{Y}(z)$. One has

$$\det\hat{Y}(z) = y_i y_j (x_i x_j - \tfrac{1}{3}\vec{x}^{\,2}\delta_{ij} - (2\pi)^{-2}M_{ij}), \tag{40}$$

where we used $\vec{y}^{\,2}=0$. Substituting a parametrization for this null-cone in CP^2, e.g. $\vec{y} = (\frac{1}{2}(1-\zeta^2), -\frac{i}{2}(1+\zeta^2), \zeta)$, gives a 4th order polynomial. However, for finding the 4 solutions we find it in this case more convenient to first diagonalize the matrix $x_i x_j - \frac{1}{3}\vec{x}^{\,2}\delta_{ij} - (2\pi)^{-2}M_{ij} = \mathcal{O}_{ik}\lambda_k\mathcal{O}_{jk}$. Introducing $\vec{y}\,' = \mathcal{O}^t\vec{y}$, the equation for the zeros reduces to $(y_1')^2\lambda_1 + (y_2')^2\lambda_2 + (y_3')^2\lambda_3 = 0$, which in addition to the null-cone condition, $(y_1')^2 + (y_2')^2 + (y_3')^2 = 0$, is now easily solved by

$$\vec{y}^{\,(a)\,\prime} = \left(\sqrt{\lambda_2-\lambda_3}, (-1)^{a+1}\sqrt{\lambda_3-\lambda_1}, i\sqrt{\lambda_2-\lambda_1}\right), \quad \vec{y}^{\,(a+2)\,\prime} = (\vec{y}^{\,(a)\,\prime})^*,\ a=1,2, \tag{41}$$

where we fixed (for generic $\vec{x}$) the diagonalizing rotation $\mathcal{O}$ by ordering $\lambda_1 \le \lambda_3 \le \lambda_2$. Using Eq. (27) we find for $\vec{u}\,' \equiv \mathcal{O}^t\vec{u}$

$$\vec{u}^{\,(a)\,\prime} = -\left((-1)^a\frac{\sqrt{\lambda_3-\lambda_1}}{\sqrt{\lambda_2-\lambda_1}}, \frac{\sqrt{\lambda_2-\lambda_3}}{\sqrt{\lambda_2-\lambda_1}}, 0\right), \quad \vec{u}^{\,(a+2)\,\prime} = -\vec{u}^{\,(a)\,\prime},\ a=1,2. \tag{42}$$

One easily checks that $\vec{u}^{\,(1,2)}$ and $\vec{u}^{\,(3,4)}$ give rise to respectively the exponentially rising and falling solutions.

It is also instructive to give the explicit expressions for the matrices $\hat{m}^\pm(z)$ in Eq. (32) (choosing $c=2$). We note that for a 2×2 matrix $\mathrm{adj}\,\hat{Y} = (\mathrm{Tr}\,\hat{Y})\mathbb{1}_2 - \hat{Y}$,

[5] $\varphi(z)$ implicitly defined by the elliptic integral of the first kind $z = \int_0^{\varphi(z)} 1/\sqrt{1-\mathbf{k}^2\sin^2\theta}\, d\theta$.

and without loss in generality[6] we take $z_0 = 0$, $\vec{a} = \vec{0}$, $\mathcal{R} = \mathbb{1}_3$, $D = 1$ and $h = \mathbb{1}_2$ in Eq. (38), such that

$$\hat{m}^+(z) = -\frac{1}{4\pi}\begin{pmatrix} y_1^{(1)} f_1(z) - iy_2^{(1)} f_2(z) & y_1^{(2)} f_1(z) - iy_2^{(2)} f_2(z) \\ 4\pi\vec{x}\cdot\vec{y}^{(1)} - y_3^{(1)} f_3(z) & 4\pi\vec{x}\cdot\vec{y}^{(2)} - y_3^{(2)} f_3(z) \end{pmatrix},$$

$$\hat{m}^-(z) = -\frac{1}{4\pi}\begin{pmatrix} y_1^{(3)} f_1(z) - iy_2^{(3)} f_2(z) & y_1^{(4)} f_1(z) - iy_2^{(4)} f_2(z) \\ 4\pi\vec{x}\cdot\vec{y}^{(3)} - y_3^{(3)} f_3(z) & 4\pi\vec{x}\cdot\vec{y}^{(4)} - y_3^{(4)} f_3(z) \end{pmatrix}. \tag{43}$$

We checked that Eq. (23), $\mathcal{V}(\vec{x}) = (2\pi)^{-1}\mathrm{Tr}(R^+(z)+R^-(z))^{-1}$, evaluated using Eq. (34) is independent of z and agrees with the result derived in Ref. [10].

We next address solving Eq. (31), which for charge 2 can be written as[7]

$$\frac{d\hat{\mu}(z)}{dz} = \frac{M_{ij}u_i y_j \delta^{ac} - 2\pi i(\vec{x}\cdot\vec{y})u_i \hat{A}_i^{ac}(z)}{2\pi(\vec{x}\cdot\vec{y})\delta^{ac} - iy_i\hat{A}_i^{ac}(z)}. \tag{44}$$

For the same choice of parameters in Eq. (38) as above, $z_0 = 0$, $\vec{a} = \vec{0}$, $\mathcal{R} = \mathbb{1}_3$, $D = 1$ and $h = \mathbb{1}_2$, this gives (with $a = 1$ and $c = 2$)

$$\frac{d\hat{\mu}(z)}{dz} = 2\pi(\vec{x}\cdot\vec{y})\frac{u_1 f_1(z) - iu_2 f_2(z)}{y_1 f_1(z) - iy_2 f_2(z)}. \tag{45}$$

Although the dependence on $\vec{x}$ is complicated, the integral over z turns out to be manageable (as was observed before in the context of charge 2 monopoles, although we here choose not to express the solution in terms of theta functions [16]). To solve the equation we first rewrite the right-hand side of Eq. (45) using the fact that $\vec{u}\times\vec{y} = -i\vec{y}$ (cmp. Eq. (27)),

$$\begin{aligned}\frac{u_1 f_1(z) - iu_2 f_2(z)}{y_1 f_1(z) - iy_2 f_2(z)} &= \frac{(u_3 y_1 + iy_2)f_1(z) - i(u_3 y_2 - iy_1)f_2(z)}{y_3\,(y_1 f_1(z) - iy_2 f_2(z))} \\ &= \frac{f_1(z)f_2(z)y_3^2 + 4iy_1y_2(\mathbf{k}')^2}{y_3\,(16\pi^2(\vec{x}\cdot\vec{y})^2 - y_3^2 f_3^2(z))} + \frac{u_3}{y_3}.\end{aligned}$$

In the last identity we used that $\vec{y}^2 = 0$, $f_1^2(z) - f_2^2(z) = 1 - \mathbf{k}^2 = (\mathbf{k}')^2$ and the fact that $\det\hat{Y}(z) = 0$ implies $(y_1 f_1(z) - iy_2 f_2(z))(y_1 f_1(z) + iy_2 f_2(z)) = 16\pi^2(\vec{x}\cdot\vec{y})^2 - y_3^2 f_3^2(z)$. With $\frac{d}{dz}f_3(z) = f_1(z)f_2(z)$ we can now integrate Eq. (45),

$$\begin{aligned}\hat{\mu}(z) &= 2\pi z(\vec{x}\cdot\vec{y})\frac{u_3}{y_3} + \tfrac{1}{4}\log\left(\frac{4\pi\vec{x}\cdot\vec{y} + f_3(z)y_3}{4\pi\vec{x}\cdot\vec{y} - f_3(z)y_3}\right) + i\frac{\mathrm{sign}(z)(\mathbf{k}')^2}{2\pi(\vec{x}\cdot\vec{y})}\frac{y_1y_2}{4y_3}I(z), \\ I(z) &\equiv \Pi_{\mathbf{k}}(f_3^{-1}(z), n) - \Pi_{\mathbf{k}}(1, n) + |z|, \quad n \equiv \frac{(4\pi\vec{x}\cdot\vec{y})^2}{y_3^2},\end{aligned} \tag{46}$$

[6] We may change z_0, $\vec{a}$, h, $\mathcal{R}$ and D by resp. translations, (gauge) rotations, and suitable rescalings.

[7] Using that $\frac{1}{2}\{\hat{A}_i, \hat{A}_j\}y_i u_j = \frac{1}{2}\mathbb{1}_2\mathrm{Tr}(\hat{A}_i\hat{A}_j)y_i u_j = -M_{ij}y_i u_j\mathbb{1}_2$, since $\vec{u}\cdot\vec{y} = 0$.

up to an irrelevant constant, where $\Pi_{\mathbf{k}}(s,n)$ is the elliptic integral of the third kind[8],

$$\Pi_{\mathbf{k}}(s,n)\equiv\int_0^s\frac{dt}{(1-nt^2)\sqrt{(1-\mathbf{k}^2t^2)(1-t^2)}}.\tag{47}$$

We now combine these ingredients to give in terms of $\vec{y}$ the exact form for the homogeneous solution of the Green's function equation,

$$\hat{v}_a(z)=\hat{\phi}(z)(\mathrm{adj}\,\hat{Y}(z))_{a2}=\exp\left(\hat{\mu}(z)-2\pi z\vec{u}\cdot\vec{x}\right)\frac{(\mathrm{adj}\,\hat{Y}(z))_{a2}}{\sqrt{(\mathrm{adj}\,\hat{Y}(z))_{12}}}\tag{48}$$

or putting in all the relevant expressions

$$\begin{aligned}\hat{v}_a(z)\;&=\;\exp\left(i\frac{z}{y_3}\left[2\pi(\vec{y}\times\vec{x})_3+\frac{y_1y_2(\mathbf{k}')^2}{8\pi(\vec{x}\cdot\vec{y})|z|}\Big(|z|+\Pi_{\mathbf{k}}(f_3^{-1}(z),n)-\Pi_k(1,n)\Big)\right]\right)\times\\&\quad(4\pi)^{-\frac{1}{2}}\Big(-y_1f_1(z)-(-1)^a iy_2f_2(z)\Big)^{\frac{1}{2}}\left(\frac{4\pi\vec{x}\cdot\vec{y}-(-1)^a y_3f_3(z)}{4\pi\vec{x}\cdot\vec{y}+(-1)^a y_3f_3(z)}\right)^{\frac{1}{4}}.\end{aligned}\tag{49}$$

Substituting $\vec{y}=\vec{y}^{(b)}=\mathcal{O}^t\vec{y}^{(b)\prime}$, with $\vec{y}^{(b)\prime}$ as defined in Eq. (41) gives $f^+_{ab}(z)=\hat{v}^{(b)}_a(z)$ and $f^-_{ab}(z)=\hat{v}^{(b+2)}_a(z)$, and thereby the Green's function, once we specify the parameters involved in the solutions to the Nahm equation.

4 Action and zero-mode density plots

The discontinuities in $\hat{A}_j(z)$ at $z=\mu_1$ and $z=\mu_2$ implied by the Nahm equation, Eq. (5), impose constraints which are (like the quadratic ADHM constraint) in general difficult to solve. Work is in progress to describe the full parameter space for SU(2) and charge 2, but in Ref. [10] we did find two non-trivial parametrizations for which we illustrate here in a number of figures how the full caloron solutions look like, using the exact Green's function as constructed in the previous section. Taking advantage of the possibility to work with arbitrary arithmetic precision the programme Maple has been used for these calculations. The configurations are formulated in terms of the two intervals $z\in[\mu_1,\mu_2]$ and $z\in[\mu_2,1+\mu_1]$, each associated with two constituent monopoles of equal magnetic charge, but opposite in sign from one interval to the next. Apart from a shift and (gauge) orientation, the configuration is described by a

[8]More commonly the elliptic integral of the third kind is defined as $\Pi(n;\varphi,\mathbf{k})=\Pi_{\mathbf{k}}(\sin\varphi,n)$. Note that $I(z)$ can alternatively be written as $\int_1^{f_3(z)}(1-t^2/n)^{-1}(t^2-\mathbf{k}^2)^{-\frac{1}{2}}(t^2-1)^{-\frac{1}{2}}dt$, from which it follows that $\frac{d}{dz}I(z)=\big((1-nf_3^2(z))|f_1(z)f_2(z)|\big)^{-1}\frac{d}{dz}f_3(z)=\mathrm{sign}(z)(1-nf_3^2(z))^{-1}$, using $f_3^2(z)-1=f_2^2(z)$, $f_3^2(z)-\mathbf{k}^2=f_1^2(z)$ and $\frac{d}{dz}f_3(z)=f_1(z)f_2(z)$.

shape ($\mathbf{k}$) and scale (D) parameter (see Sect. 3.1), for simplicity assumed to be the same on both intervals. We also take all constituents to be of equal mass, $\nu_{1,2} = \frac{1}{2}$ ($\mu_2 = -\mu_1 = \frac{1}{4}$), most relevant for the confined phase with $\mathrm{tr}\,\mathcal{P}_\infty = 0$.

The two periodic and two anti-periodic chiral fermion zero-modes each have support on oppositely charged constituents (see Fig. 1) and in the far field limit it was found that the zero-mode density (summed over the two zero-modes implied by the index theorem) is described by a disc singularity, bounded by an ellipse with semi-major axis $D/4\pi$ and eccentricity $(1-\mathbf{k}')/(1+\mathbf{k}')$ (where $\mathbf{k}' = \sqrt{1-\mathbf{k}^2}$). This revealed that the core of a cluster of like-charged constituents is in general extended, unless the individual constituents are well separated. The far field only describes the (algebraic) abelian component of the gauge field and to resolve the structure of the core we need to determine the full non-abelian structure.

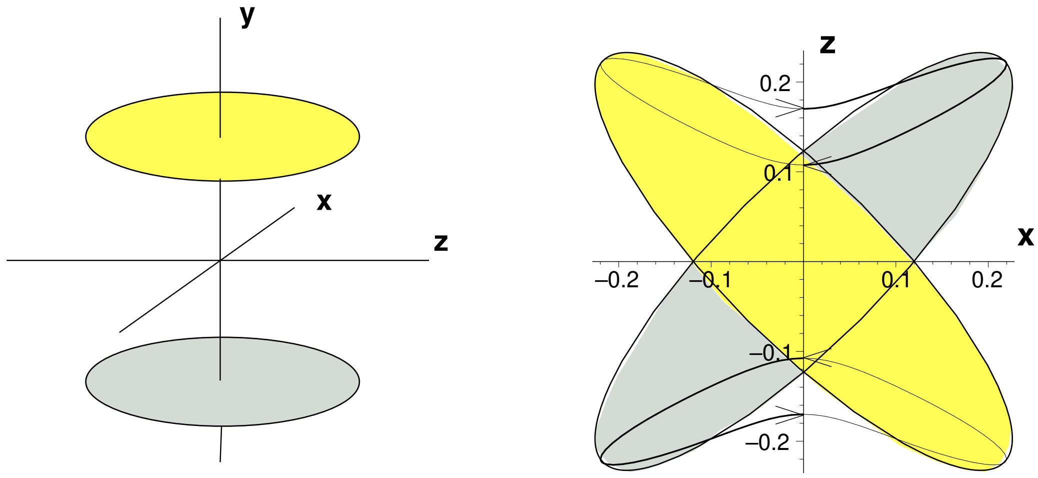

Figure 2: An example to illustrate the location of the disc singularities (light and dark shaded according to magnetic charge) for a rectangular (left) and crossed (right) configuration. The latter is shown at $\alpha = -\pi/2$, for $d = \pi/32$ and $\theta = \pi/4$ ($\mathbf{k} = 0.962$, $D = 3.894$ and $\varphi = -\pi/4$). The curves indicate the would-be constituent locations at fixed d and θ, varying α from $-\pi$ (to which the arrows point) to 0. For $\alpha = -\pi$ and 0 the discs collapse to lines ($\mathbf{k} = 1$) with no singularities remaining, except at the endpoints.

For the first parametrization (called "rectangular") the two discs are parallel and separated in height by a distance d. The configuration is charaterized by the two extremal points along the major axis of the disc, also called *would-be* constituent locations[9],

$$\vec{y}_m^{\,(j)} = \Big(0\,,\, \tfrac{1}{2}(-1)^m d\,,\, (-1)^j (4\pi)^{-1} D\Big)\,, \quad 2\pi d = D f_2(\tfrac{1}{4}D), \tag{50}$$

[9]In their immediate neighbourhood the action density is maximal; they are the constituent locations in the point-like limit, $\mathbf{k} \to 1$.

up to an overall shift and orientation (the definition of $f_2(z)$, which involves $\mathbf{k}$, can be found in Eq. (39)). A typical example is shown in Fig. 2 (left). Apart from $\mathbf{k}$ and D, the parameters that enter Eq. (38) for the m-th interval are $\vec{a} = (0, \frac{1}{2}(-1)^m d, 0)$, $\mathcal{R} = \mathbb{1}_3$, $h = \mathbb{1}_2$ and $z_0 = \frac{1}{4}(1 + (-1)^m)$. In all cases discussed here we have also put $\xi_0 = 0$. One verifies that the discontinuities of $\hat{A}_j(z)$ at $z = \mu_m$ are given by $2\pi i \rho^j_m$ with appropriately chosen ζ_a (cmp. Eq. (6)), as discussed in detail in Ref. [10].

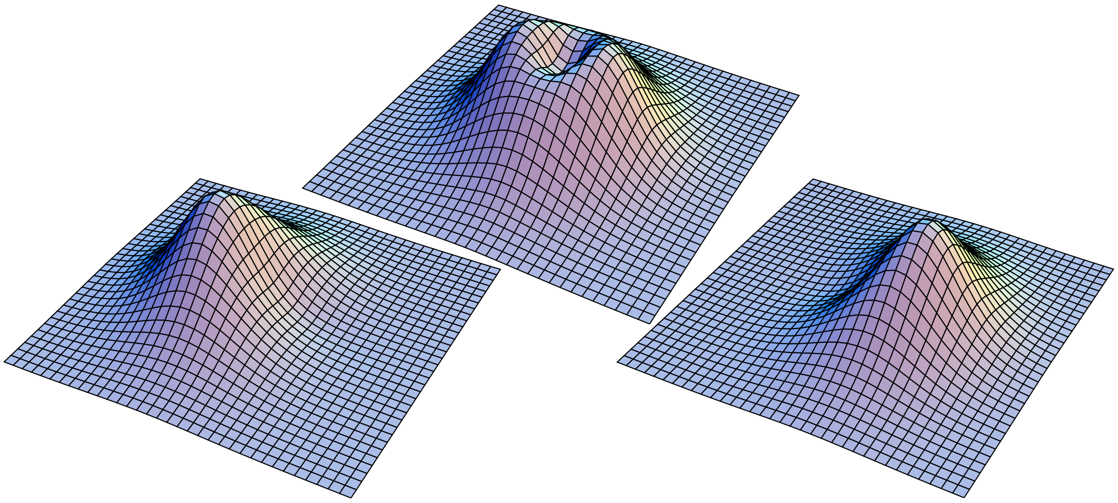

Figure 3: Shown is the energy density (middle) for $\mu_2 = 0.25$, $\mathbf{k} = 0.570$ and $D = 6.915$, in the monopole limit $d \to \infty$ and in the x-z plane through one of the discs for the rectangular configuration, see Fig. 2 (left). On a scale enhanced by a factor $4\pi^2$ are shown the densities for the two monopole zero-modes (left and right).

For the second parametrization (called "crossed") the two discs are coplanar and intersect, see Fig. 2 (right) for a typical example. Their relative orientations can vary between perpendicular and coinciding (for which $\mathbf{k}$ is forced to 1). Here we choose for the parameters in Eq. (38) h and $\mathcal{R}$ to be non-trivial (isospin) rotations around the y-axis with angles $(-1)^m\theta$, resp. $(-1)^m\varphi$, and $\vec{a} = (0, 0, -\frac{1}{2}(-1)^m d\cos\alpha)$, whereas z_0 and ξ_0 are as in the rectangular case. The would-be constituent locations are now given (up to an overall shift and orientation) by

$$\vec{y}^{\,(j)}_m = \Big((-1)^j(4\pi)^{-1}D\sin\varphi\,,\,0\,,\,(-1)^{m+j}(4\pi)^{-1}D\cos\varphi - \tfrac{1}{2}(-1)^m d\cos\alpha\Big)\,, \qquad (51)$$

where $\pm\varphi$ conveniently gives the orientation of each of the two discs with respect to the z-axis. The angle α originates from the definition of ζ_a, which through Eq. (6) determines the discontinuity of $\hat{A}_j(z)$. To ensure the proper matching, the following three equations need to be satisfied [10]

$$D\sin(\theta\pm\varphi)\,[f_3(\tfrac{1}{4}D) \pm f_1(\tfrac{1}{4}D)] = 8\pi d(1\pm\sin\alpha), \quad Df_2(\tfrac{1}{4}D) + 8\pi d\sin\alpha = 0. \qquad (52)$$

which determine φ, $\mathbf{k}$ and D for given α, θ and d. Fig. 2 illustrates that for these crossed configurations the two discs always overlap, unlike for the rectangular case. This formed an important motivation for the present study, so as to determine in how far the overlapping discs would affect the behaviour in the core.

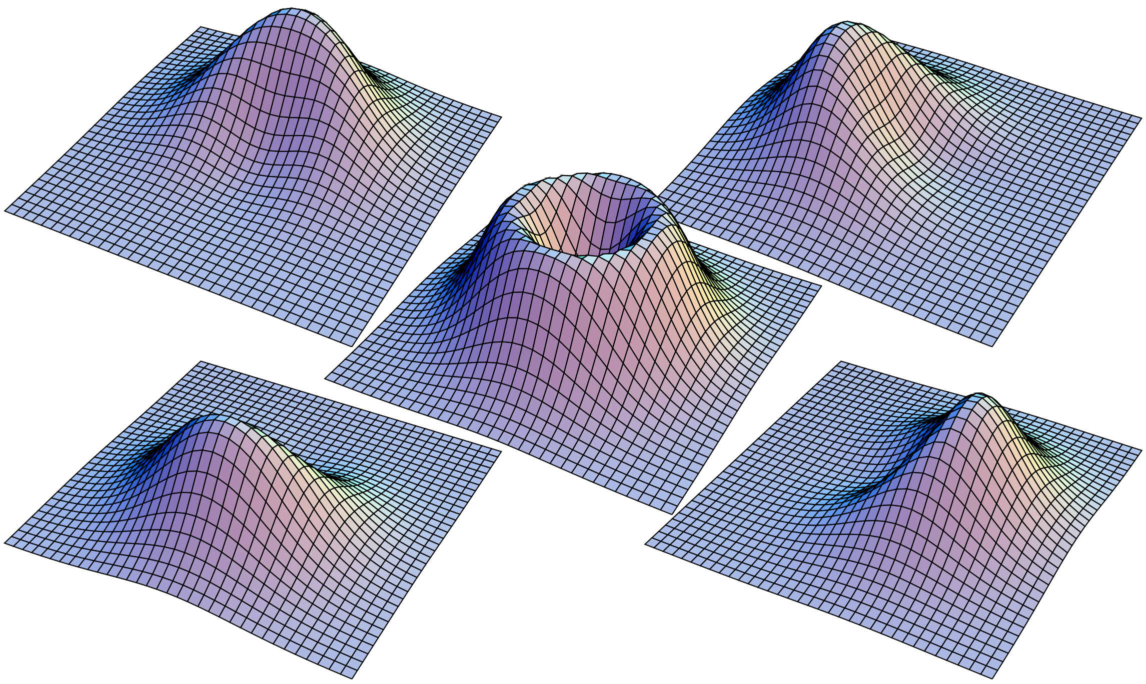

Figure 4: In the middle is shown the action density in the plane of the constituents at $t = 0$ for an SU(2) charge 2 caloron with $\mathrm{tr}\,\mathcal{P}_\infty = 0$ in the crossed configuration of Fig. 2, hence with $\mathbf{k} = 0.962$ and $D = 3.894$. On a scale enhanced by a factor $16\pi^2$ are shown the densities for the two zero-modes, using either periodic (left) or anti-periodic (right) boundary conditions in the time direction.

We will illustrate, using the exact formalism, how the action and zero-mode densities for these solutions behave. For the rectangular case we are mostly interested in the monopole limit, $d \to \infty$. For finite d, the density separates in two contributions, where each in the limit $d \to \infty$ is exactly the density for a charge 2 monopole with the same values of $\mathbf{k}$ and D. Note that in the limit $d \to \infty$ the matching conditions for $\hat{A}_j(z)$ "decouple" the intervals, turning into pole conditions at $z = \mu_{1,2}$, as is appropriate for multi-monopoles [20]. We do recover from this the known results for the charge 2 monopoles [22], like the doughnut structure for $\mathbf{k} = 0$, corresponding to two superimposed monopoles. The interest in the monopole limit comes from the fact that the zero-mode densities for multi-monopoles had not been studied in detail before [23]. In Fig. 3 we give the densities for $\mathbf{k} = 0.570$ and $D = 6.915$, which is intermediate between the doughnut and well-separated monopole configurations. On the other hand, for d small (compared to $\beta = 1$) the configuration will look like two non-dissociated calorons, and in particular is no longer static. When D remains much bigger than d, forcing $\mathbf{k} \to 1$, these behave as two well-separated charge 1 instantons. Otherwise, when D is comparable to d, one finds overlapping instantons [24].

An example for the crossed configuration with $\mathbf{k} = 0.997$ and $D = 8.753$ was already shown in Fig. 1. In this case D is large enough for the like-charge constituents to be separated, as is particularly clear from the zero-modes, which are essentially no longer overlapping. But two nearest neighbour (oppositely) charged constituents still show appreciable overlap. The distance between these nearest neighbours is $\frac{1}{4}D\sqrt{2}/\pi = 0.985$. As this is comparable to $\beta = 1$ we would expect the configuration to depend on time. Indeed, at the maxima of the action density its value of $1.18 \times 16\pi^2$ at $t = 0$ is reduced by almost 50% at $t = 0.5$. At the center of mass, where the action density is much lower, there is still a time dependence. However, far from all cores the field becomes static[10]. Increasing D further (which will push $\mathbf{k}$ closer to 1) the configuration quickly turns into well separated spherically symmetric static BPS monopoles.

More interesting is to consider the case with smaller D, like $D = 3.894$ and $\mathbf{k} = 0.962$, for which the disc singularities were illustrated in Fig. 2. The corresponding densities are shown in Fig. 4. We see that the constituents are now so close that they form a doughnut, but we stress this is different from the *static* monopole doughnut, which has $\mathbf{k} = 0$. Since the oppositely charged constituents now are as close as 0.438, which is considerably smaller than the time extent, the solution will have a strong time dependence. When D is decreased even further, it will turn into a charge 2 instanton localized in both space and time.

5 Higgs field asymptotics

In this section we make some comments on the far field limit of the gauge field. As we discussed before, the gauge field far removed from any core becomes abelian (as well as static). The abelian subgroup is the one that leaves the holonomy invariant, in the periodic gauge equivalent to leaving the constant asymptotic value of the adjoint Higgs field A_0 invariant. For definiteness, let us consider the case of SU(2) with $k = 2$, $\beta = 1$ and $\mathcal{P}_\infty = \exp(2\pi i\vec{\omega} \cdot \vec{\tau})$ (i.e. $\mu_2 = -\mu_1 = |\vec{\omega}|$). Up to exponential corrections we have

$$A_0^{\rm ff}(\vec{x}) = 2\pi i\vec{\omega} \cdot \vec{\tau} - \tfrac{1}{2}i\hat{\omega} \cdot \vec{\tau}\Phi(\vec{x}), \tag{53}$$

where we can express $\Phi(\vec{x})$ in terms of the far field limit of the Green's function at the impurities [14]

$$\Phi(\vec{x}) = \pi^{-1}\left[1 - \pi^{-1}{\rm Tr}\left(\hat{f}_x^{\rm ff}(\mu_2,\mu_2)\hat{S}_2\right)\right]^{-1}\partial_i{\rm Tr}\left(\hat{f}_x^{\rm ff}(\mu_2,\mu_2)\rho_2^i\right). \tag{54}$$

[10]For any SU(n) and topological charge k the far field limit is static, *provided* all constituent monopoles have a non-vanishing mass

Using the twistor description of magnetic monopoles Hurtubise [13] was able to explicitly compute the asymptotic Higgs field for the SU(2) magnetic monopole long ago. The function he found for this algebraic tail amazingly agrees exactly with $\mathcal{V}_m(\vec{x})$, Eq. (23), which was introduced to describe the caloron zero-mode density (m denotes the interval and hence the type of constituents to which the corresponding zero-modes would localize [10]). As mentioned before, from our multi-caloron results one can recover the multi-monopole results by sending the constituent monopoles with the "unwanted" magnetic charge to infinity, cmp. Eq. (24).

Although this tends to be cumbersome to show, $\Phi(\vec{x})$ in the far field can be written as $\Phi_1(\vec{x})-\Phi_2(\vec{x})$, where $\Phi_m(\vec{x})$ is the contribution coming from the type m constituent monopoles and the difference in sign is due to the sign change in the magnetic charge. This is simply because the field is abelian in the far field and linear superposition preserves the self-duality. Hence, $\Phi_m(\vec{x})=2\pi\mathcal{V}_m(\vec{x})$, such that for the SU(2) caloron

$$\Phi(\vec{x})=2\pi\mathcal{V}_1(\vec{x})-2\pi\mathcal{V}_2(\vec{x}). \tag{55}$$

We checked that this result indeed holds for the solutions discussed in Sect. 4, even when the two types of monopole structures are not well-separated. This relation trivially holds for the axially symmetric solutions that were introduced in Ref. [14], where $\Phi(\vec{x})$ was explicitly shown to factorize in a sum of point charge contributions, compatible with what was found in Ref. [10] for the zero-mode densities. Therefore, in the far field limit (i.e. for the algebraic part) the singularity structure in the zero-mode density agrees *exactly* with the abelian charge distribution, as given by $\partial_i^2\Phi(\vec{x})$. Such a relation is at the heart of using chiral fermion zero-modes as a filter to isolate the underlying topological lumps from rough lattice Monte Carlo configurations [9].

6 Discussions

In this paper we have analyzed the higher charge caloron solutions and showed how to obtain exact results by suitably combining techniques developed in the context of the Nahm transformation and the ADHM formalism. The aim of these studies has been to establish that SU(n) caloron solutions of charge k can be described in terms of kn monopole constituents, and that these can be viewed as independent constituents. A natural way to get an ensemble would be to consider approximate superpositions of k charge 1 calorons, but this would lead to an unwanted memory effect, with constituents remembering from which caloron they originated [14]. Our studies, within the context of self-dual configurations, have shown nevertheless that the constituents have an independent identity, with the only requirement that the net magnetic and electric charge of the configuration vanishes (each of the n types of constituents should occur with the same number). A recent lattice study [25], using the technique of over-improvement [26], fully confirms this picture.

It is therefore reasonable to consider the constituents as the independent building blocks for constructing an ensemble of monopole constituents, something that was not questioned in Ref. [7], but like for the instanton liquid [27] forms an essential assumption in a semi-classical study. Clearly the expectation is that semi-classical methods no longer work in the confined regime, at least for the part of the parameter space that corresponds to well-separated constituents, that is typically associated to instantons with a large scale parameter. It is not unlikely that the density of these constituents at low temperatures is so high that they form a coherent background and as such will no longer easily be recognized as lumps. With high quark densities leading to deconfinement, it may perhaps be that a high constituent monopole density will lead to confinement [28], although for now we have to leave this as a speculation.

Instantons that overlap get deformed and depending on the relative gauge orientation tend to "repel", i.e. inspecting the action density distribution they do not get closer than a certain distance [24]. When deconstructing instantons in monopole constituents, interestingly only like-charge constituents will show this effect, manifesting itself through the extended core structure. For unlike charges, from the point of view of the abelian field, the configuration behaves as with linear superposition. If as a consequence of this all abelian charge is annihilated, it disappears through forming a small instanton (localized in space and time), which in the limit of zero size describes the boundary of the moduli-space. The interaction between constituents of opposite duality is more complicated [8, 29].

In conclusion, calorons with non-trivial holonomy have revealed a rich structure, incorporating traditional instanton physics, but allowing for gauge fields that inherit some essential features associated to a confining background not present in the traditional formulations. The fact that the underlying constituents are monopoles opens the way to describe the confining aspects of the theory in terms of these degrees of freedom. Much work remains to be done when it comes to understanding the dynamics, but we hope to have convinced the reader that a consistent picture is developing that holds considerable promise for the future.

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Appendix A

In this appendix we derive the zero-mode limit for the action density, which assumes that the distance of $\vec{x}$ and the constituents of type m to all constituents of type $m' \neq m$ is large, but where $\vec{x}$ and the constituent locations of type m may otherwise be arbitrary.

As in Ref. [10] we take $z = \mu_m + 0$ for computing $\mathcal{F}_z$ and use that we can write $\det(\mathbb{1}_{2k} - \mathcal{F}_{\mu_m}) = \det(\mathbb{1}_{2k} - LK)$, where $K \equiv F_{m-1}\Theta_{m-1} \cdots \Theta_1 \hat{g}^\dagger(1) F_n \Theta_n \cdots \Theta_{m+2} F_{m+1}$ and $L \equiv \Theta_{m+1} F_m \Theta_m$, with

$$\Theta_m \equiv \begin{pmatrix} \mathbb{1}_k & \mathbb{1}_k \\ 2\pi R_m^+(\mu_m) & -2\pi R_m^-(\mu_m) \end{pmatrix}^{-1} T_m \begin{pmatrix} \mathbb{1}_k & \mathbb{1}_k \\ 2\pi R_{m-1}^+(\mu_m) & -2\pi R_{m-1}^-(\mu_m) \end{pmatrix},$$

$$F_m \equiv \begin{pmatrix} f_m^+(\mu_{m+1}) f_m^+(\mu_m)^{-1} & 0 \\ 0 & f_m^-(\mu_{m+1}) f_m^-(\mu_m)^{-1} \end{pmatrix}. \tag{56}$$

We note the K has *no* remaining dependence on the constituent locations of type m. Writing $LK \equiv \hat{L}\hat{K} + \tilde{L}\tilde{K}$, with

$$\hat{K} \equiv \begin{pmatrix} K_{++} & K_{+-} \\ 0 & \mathbb{1}_k \end{pmatrix}, \quad \tilde{K} \equiv \begin{pmatrix} 0 & 0 \\ K_{-+} & K_{--} \end{pmatrix}, \quad \hat{L} \equiv \begin{pmatrix} L_{++} & 0 \\ L_{-+} & 0 \end{pmatrix}, \quad \tilde{L} \equiv \begin{pmatrix} 0 & L_{+-} \\ 0 & L_{--} \end{pmatrix}. \tag{57}$$

we find $\det(\mathbb{1}_{2k} - LK) = \det(\hat{K}) \det(\hat{K}^{-1} - \hat{L} - \tilde{L}\tilde{K}\hat{K}^{-1})$.

We next use

$$\hat{K}^{-1} = \begin{pmatrix} K_{++}^{-1} & -K_{++}^{-1} K_{+-} \\ 0 & \mathbb{1}_k \end{pmatrix}, \quad \tilde{K}\hat{K}^{-1} = \begin{pmatrix} 0 & 0 \\ K_{-+} K_{++}^{-1} & (K^{-1})_{--} \end{pmatrix} \tag{58}$$

and note that in the zero-mode limit K_{++}^{-1}, $K_{++}^{-1} K_{+-}$, $K_{-+} K_{++}^{-1}$ and $(K^{-1})_{--}$ are exponentially small (cmp. Ref. [10], App. A), such that

$$\det\left(ie^{-\pi i x_0}(\mathbb{1}_{2k} - LK)\right) = \det(e^{-2\pi i x_0} K_{++}) \det(L_{++}). \tag{59}$$

With the definition of L we now find

$$L_{++} = \tfrac{1}{4} R_{m+1}^{-1}(\mu_{m+1}) \left(R_{m+1}^-(\mu_{m+1}) + S_{m+1}\right) \tilde{U}_m \left(R_{m-1}^+(\mu_m) + S_m\right), \tag{60}$$

where

$$\begin{aligned} \tilde{U}_m &= \mathcal{Z}_{m+1}^+ f_m^+(\mu_{m+1}) f_m^+(\mu_m)^{-1} R_m^{-1}(\mu_m) \tilde{\mathcal{Z}}_m^+ - \mathcal{Z}_{m+1}^- f_m^-(\mu_{m+1}) f_m^-(\mu_m)^{-1} R_m^{-1}(\mu_m) \tilde{\mathcal{Z}}_m^-, \\ \mathcal{Z}_m^\pm &= \mathbb{1}_k \pm (R_m^-(\mu_m) + S_m)^{-1} R_{m-1}^\pm(\mu_m), \;\; \tilde{\mathcal{Z}}_m^\pm = \mathbb{1}_k \pm R_m^\mp(\mu_m)(R_{m-1}^+(\mu_m) + S_m)^{-1}. \end{aligned} \tag{61}$$

and $\tilde{U}_m$ contains *all* contributions due to the constituent locations of type m, up to *exponential* corrections in the distance of these, *and* of $\vec{x}$, to the other constituents. Hence $\log\det\left(ie^{-\pi i x_0}(\mathbb{1}_{2k} - LK)\right)$ splits into the sum of two contributions, $\log\det(\tilde{U}_m)$

and $\log\det[\frac{1}{4}(R^+_{m-1}(\mu_m)+S_m)e^{2\pi ix_0}K_{++}R^{-1}_{m+1}(\mu_{m+1})(R^-_{m+1}(\mu_{m+1})+S_{m+1})]$, where the last term only depends on the constituent locations of type $m'\neq m$ whose contribution will decay inversely proportional to the fourth power of their distance. Allowing for *algebraic* decay (or in the monopole limit, sending all constituents of type $m'\neq m$ to infinity) such that in addition $\mathcal{Z}^\pm_{m+1}=\tilde{\mathcal{Z}}^\pm_m=\mathbb{1}_k$, one thus finds Eq. (24).

A simple way to derive the result for the far field limit in Eq. (25) is by noting that in this case *all* $f^-_m(\mu_{m+1})f^-_m(\mu_m)^{-1}$ are exponentially small and F_m can be approximated by $\mathrm{diag}(F^{++}_m,0)$, with $F^{++}_m=f^+_m(\mu_{m+1})f^+_m(\mu_m)^{-1}$. This therefore acts as a projection on the ++ component and is thus seen to lead to $\det(ie^{-\pi ix_0}(\mathbb{1}_{2k}-LK))=\det(e^{-2\pi ix_0}\hat{g}^\dagger(1)F^{++}_n\Theta^{++}_n\cdots F^{++}_1\Theta^{++}_1)$. Using the fact that [10] $\Theta^{++}_m=\frac{1}{2}R^{-1}_m(\mu_m)\Sigma_m$ gives the required result.

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