CHAPTER 1

NON-PERTURBATIVE ASPECTS OF GAUGE FIXING

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We review in this contribution to celebrate fifty years of Yang-Mills theory the intricate non-perturbative aspects of gauge fixing and its relation to the topologically non-trivial gauge invariant configuration space.

1. Introduction

The introduction of non-abelian gauge theories in the Yang-Mills paper,[1] now fifty years ago, has revolutionized particle physics. Its remarkable achievements are of course largely due to it contributing so successfully to our understanding of the subatomic world around us, through the electro-weak theory[2] and it is most pure form through Quantum Chromodynamics.[3] It was after establishing renormalizability of non-abelian gauge theories[4] that progress accelerated. What is equally remarkable is that it was not only successful in describing experimental data, but also formed the source of many beautiful results in mathematics. Pure non-abelian gauge theory is believed to be defined in a mathematically rigorous way, and finding a genuine proof for the formation of a mass gap has become one of the seven millennium prize problems of the Clay Institute for mathematics.[5]

Much of the foundations for the applications of the Yang-Mills theory were already in place when I started my research in the early eighties of the previous century. It is therefore with humble feelings that I contribute to this volume to celebrate fifty years of Yang-Mills theory. I dedicate this review to my heros and mentors in this field, in particular to Chen Ning Yang, who not only made the first steps but contributed to so many other deep results,[6] as well as to Tini Veltman and of course to Gerard 't Hooft, who brought me under the spell of the gauge principle.[7] I also dedicate this review to the late Vladimir Gribov. His ideas on gauge fixing will play an important role in this review.

In this review we will describe how one can describe the non-trivial nature of the physical configuration space, defined through the space of gauge equivalence classes of Yang-Mills gauge fields. We will argue that this has important dynamical consequences for the low-energy dynamics of the theory, to be described in the absence of dynamical fermions. This restriction, as well as technical limitations in taking the infinite volume limit has prevented us to test these ideas against experiment. Nevertheless, comparison with numerical results obtained with lattice gauge theory,[8] in particular in the finite volume domain, has provided important support.

This review is organized as follows. In Sec. 2 we discuss why it is useful to reconsider hamiltonian methods, in particular in the context of gauge fixing through minimalization, presented in Sec. 3. We apply these methods to computing the low-lying spectrum in pure gauge theory in a finite volume in Sec. 4, and where applicable compare with lattice Monte Carlo data to demonstrate this approach is essential to give the proper non-perturbative description of the theory. In Sec. 5 we discuss some recent application to the issue of correctly defining the expectation value of A^2, relevant to the possible existence of a dimension-two condensate. We end with some conclusions.

2. The hamiltonian setting

Field theory was formulated in the early days within the hamiltonian framework, but these methods have long since been abandoned, or so it seems. After all, why should one follow a formalism that is so much more complicated than the covariant path integral approach of Feynman. Yet, when it comes to non-perturbative aspects there are instances where perhaps one should reconsider this point of view, as the use of wave functionals on configuration space has some clear advantages in this respect. In the light of the success of incorporating instantons in the path integral,[9] this may sound heretic. But these non-perturbative path integral calculations are particularly successful for quantities that have no perturbative contribution and can be addressed through a saddle point approximation. When observables are involved that will receive perturbative contributions as well, as happens in particular for low-lying states, the path integral becomes more difficult to deal with, in a large part due to the geometrically non-trivial physical configuration space.

For Yang-Mills theory the hamiltonian is typically formulated in the $A_0 = 0$ and the Coulomb gauge.[10] Putting A_0 to zero as a gauge condition is somewhat misleading, since A_0 plays the role of a Lagrange multiplier and as such in the path integral generates the time-dependent gauge transformations. The hamiltonian still "suffers" from the invariance under time-independent gauge transformations. One could insist wave functionals to be invariant under gauge transformations, like in quantum mechanics when formulating wave functions that are invariant under rotations. In this case the hamiltonian takes a simple form. We absorb the bare coupling constant in the gauge field (in the quantum theory Ward identities show that this gives a field definition without anomalous scaling). In these conventions, with $A_i(\vec{x}) = A_i^a(\vec{x})T^a$ (T^a the anti-hermitian generators, normalized according to $\mathrm{tr}\,(T^aT^b) = -\frac{1}{2}\delta_{ab}$), the field strength is given by

$$F_{ij}(\vec{x}) = F_{ij}^a(\vec{x})T^a = \partial_i A_j(\vec{x}) - \partial_j A_i(\vec{x}) + [A_i(\vec{x}), A_j(\vec{x})] \tag{1}$$

and the hamiltonian density reads

$$\mathcal{H}(\vec{x}) = -\tfrac{1}{2}g^2\partial^2/\partial A_j^a(\vec{x})^2 + \tfrac{1}{2}g^{-2}\left(B_j^a(\vec{x})\right)^2. \tag{2}$$

To separate the variables in gauge group (angular) and gauge orbit (radial) parameters is not an easy task, even though in 2+1 dimensions some success has been achieved along this line.[11] Alternatively one introduces a gauge condition. The Coulomb gauge condition ($\partial_i A_i(\vec{x}) = 0$) is one particular choice that can be made, which unlike axial gauge conditions, is well-defined in a finite physical volume. Such a finite volume one might want to use to regulate the infrared divergencies.

The aim of fixing the gauge is to give a unique parametrization of the set of gauge orbits $\mathcal{A}/\mathcal{G}$, where $\mathcal{A}$ is the collection of gauge fields (i.e. connections) and $\mathcal{G}$ is the group of local gauge transformations. Due to the non-abelian nature of the theory this gauge invariant configuration space has a non-trivial geometry.[12] Factoring out an infinite dimensional symmetry group is no small matter, but Faddeev and Popov have shown how to deal with this.[13] However, as was discovered by Gribov,[14] the Coulomb gauge suffers from what has become known as Gribov ambiguities. Due to the non-linearities there will be a (large) number of solutions to the gauge condition on the same gauge orbit[a]. This remains true, under certain conditions, for any gauge choice linear in the gauge field.[16] Thus, further conditions are required to restrict to a unique representative, which gives the so-called fundamental (modular) domain[17] on which the wave functionals are to be defined.

The advantage of using the Coulomb gauge is that it reduces to the perturbative formulation for the high energy modes, which can be well-approximated by harmonic oscillator contributions to the wave functional. In the direction of these field modes the potential energy rises steeply. Their contribution, including regularizing the ultraviolet behavior, gives in particular rise to the running of the coupling. A finite volume allows us to have a well-defined mode expansion, and if the volume is taken small, asymptotic freedom[18] ensures a small coupling constant for all field modes. With the volume set by the length scale L, rescaling $A_i(\vec{x})$ with L the hamiltonian can be formulated in terms of dimensionless fields. Apart from the overall scaling dimension of the hamiltonian, only the running coupling introduces a non-trivial volume dependence. Arranging the low-energy field modes (and all those modes not affected by the cutoff) to be scale invariant, the spreading of the wave functional is entirely caused by an increasing coupling. This is what leads to non-perturbative effects, once wave functionals start to probe the non-trivial geometry of the configuration space. In a periodic geometry, the perturbative analysis in a small volume was pioneered by Bjørken[19] and further developed by Lüscher.[20]. Addressing non-perturbative effects in this formalism is reduced to the proper formulation of boundary conditions in *field space*, at the boundary of the fundamental domain, which as we will see are directly related to its subtle gauge redundancies.

[a]The fact that the gauge orbit through $A = 0$ has many distinct configurations that satisfy the Coulomb gauge condition was observed before in Ref. 15.

3. Gauge fixing

To fix to Coulomb gauge it has turned out to be advantageous to formulate this in terms of minimizing the L^2 norm of the vector potential along the gauge orbit[17,21,22]

$$G_A(h) \equiv \|[h]A\|^2 = -\int_M \mathrm{tr}\,\left(\left(h^{-1}(\vec{x})A_i(\vec{x})h(\vec{x}) + h^{-1}(\vec{x})\partial_i h(\vec{x})\right)^2\right). \quad (3)$$

The integral over the finite spatial volume M is with the appropriate volume form and $[h]A$ indicates the gauge transformation. Expanding around the minimum of Eq. (3), writing $h(\vec{x}) = \exp(X(\vec{x}))$ one easily finds

$$\begin{aligned}\|\,[h]A\|^2 &= \|A\|^2 + 2\int_M \mathrm{tr}\,(X\partial_i A_i) + \int_M \mathrm{tr}\,(X^\dagger FP(A)X) \quad (4)\\ &+\frac{1}{3}\int_M \mathrm{tr}\,\left(X\,[[A_i,X],\partial_i X]\right) + \frac{1}{12}\int_M \mathrm{tr}\,\left([\mathcal{D}_i X, X][\partial_i X, X]\right) + \mathcal{O}\left(X^5\right),\end{aligned}$$

where $FP(A)$ is the Faddeev-Popov operator

$$FP(A)X = -\partial_i(\partial_i X + [A_i, X]). \quad (5)$$

Stationarity of the norm functional with respect to X precisely gives the Coulomb gauge condition, $\partial_i A_i = 0$, therefore making the vector potential transverse. At a local minimum $FP(A)$ is in addition a positive operator. The set of all these vector potentials associated to local minima of the norm functional defines the so-called Gribov region Ω. Since $FP(A)$ is linear in A, Ω is seen to be a convex subspace of the set of transverse connections Γ. Its boundary $\partial\Omega$ is called the Gribov horizon. At the Gribov horizon, the *lowest* non-trivial eigenvalue of the Faddeev-Popov operator vanishes, and points on $\partial\Omega$ are associated with coordinate singularities. Any point on $\partial\Omega$ can be seen to have a finite distance to the origin of field space and in some cases even uniform bounds can be derived.[23]

The Gribov region as the set of *local* minima of the norm functional needs to be further restricted to the *absolute* minima to form a fundamental domain,[17] which will be denoted by Λ. The fundamental domain is clearly contained within the Gribov region. To show that also Λ is convex, we note that

$$\begin{aligned}\|\,[h]A\|^2 - \|A\|^2 &= \int_M \mathrm{tr}\,\left(A_i^2\right) - \int_M \mathrm{tr}\,\left(\left(h^{-1}A_i h + h^{-1}\partial_i h\right)^2\right) \quad (6)\\ &= \int_M \mathrm{tr}\,\left(h^\dagger FP_f(A)\,h\right) \equiv \langle h, FP_f(A)\,h\rangle\,, \quad FP_f(A) \equiv -\partial_i(\partial_i + A_i),\end{aligned}$$

where $FP_f(A)$ acts on the fundamental representation and is similar to the Faddeev-Popov operator. We can define Λ in terms of the absolute minima over $h \in \mathcal{G}$ of $\langle h, FP_f(A)\,h\rangle$

$$\Lambda = \{A \in \Gamma |\, \min_{h\in\mathcal{G}} \langle h, FP_f(A)\,h\rangle = 0\}. \quad (7)$$

Using that $FP_f(A)$ is linear in A, the convexity of Λ is automatic: A line connecting two points in Λ lies within Λ.

A convex space is contractible and one might now wonder how the fundamental region could ever reproduce the non-trivial topology of the configuration space. But we should not forget that Λ is contained in Ω, which implies Λ is bounded and must have a boundary.[24] Making use of the fact that obviously $A = 0$ is in the interior of Λ, we can consider a ray extending out from the origin into a given direction, where it will have to cross the boundary of Λ (and of Ω). For any point along this ray in Λ, the norm functional is at its absolute minimum as a function of the gauge orbit. However, for points in Ω that are not also in Λ, the norm functional is necessarily at a relative minimum. The absolute minimum for this orbit is an element of Λ, but in general not along the ray. Continuity therefore tells us that at some point along the ray, this absolute minimum has to pass the local minimum. At the point where they are exactly degenerate, there are two gauge equivalent vector potentials with the same norm, both at the absolute minimum. Since by construction the norm functional has a unique minimum in the interior of Λ, these two degenerate configurations have to *both* lie on the boundary of Λ.

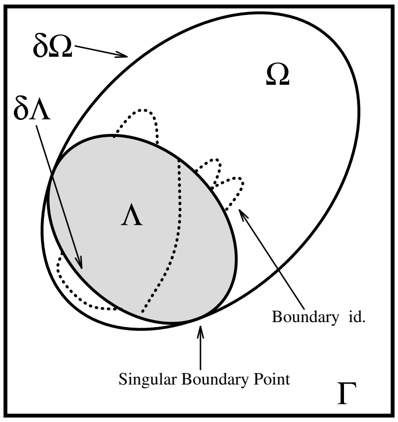
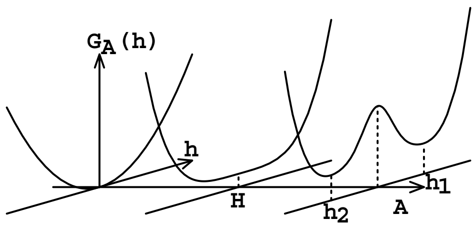


Fig. 1. Sketch of the fundamental (shaded) and Gribov regions. The dotted lines indicate the boundary identifications.

It may come as a surprise, but the shapes of the Gribov and fundamental regions are scale independent, as is clear when expressing the gauge field in terms of the dimensionless combination LA. Note that the norm functional is still degenerate along the constant gauge transformations. We deal with this by insisting the wave functional is in the singlet representation under the constant gauge transformations (which now is *finite* dimensional) such that $\Lambda/G = \mathcal{A}/\mathcal{G}$, where Λ is assumed to include the boundary identifications that restore the non-trivial topology of $\mathcal{A}/\mathcal{G}$. If a degeneracy at the boundary is continuous, other than by constant gauge transformations, one necessarily has at least one non-trivial zero eigenvalue for $FP(A)$ and the Gribov horizon will touch the boundary of the fundamental domain at these so-called singular boundary points, as indicated in Fig. 1. In principle, by choosing a different gauge fixing in the neighborhood of these points one could resolve the singularity. Since the boundary identifications are by gauge transformations the boundary condition on the wave functionals is simply that they are identical, possibly up to a phase in case the gauge transformation is homotopically non-trivial.



(a)

Fig. 2. Sketch of a singular boundary point due to a bifurcation of the norm functional, where the boundary of the fundamental domain touches the Gribov horizon (H).

It is interesting to note that generically at singular boundary points the norm functional undergoes a bifurcation moving from inside to outside the fundamental (and Gribov) region.[24] The absolute minimum turns into a saddle point and two local minima appear, as indicated in Fig. 2. These are necessarily gauge copies of each other. This gauge transformation is homotopically trivial as it reduces to the identity at the bifurcation point, evolving continuously from there on. Also Gribov's original arguments for the existence of gauge copies[14] (showing that points just outside the horizon are gauge copies of points just inside) can be easily understood from the perspective of bifurcations in the norm functional. It describes the generic case where the zero-mode of the Faddeev-Popov operator arises because of the coalescence of a local minimum with a saddle point with only one unstable direction. At the Gribov horizon the norm functional locally behaves in that case as X^3, with X the relevant zero eigenfunction of the Faddeev-Popov operator.

The necessity to restrict to the fundamental domain, a subset of the transverse gauge fields, introduces a non-local procedure in configuration space. This cannot be avoided since it reflects the non-trivial geometry of this space, a geometry which we stress again is scale independent when rescaling the gauge field with L. Homotopically non-trivial gauge transformations are in one to one correspondence with non-contractible loops in configuration space, which give rise to conserved quantum numbers. The quantum numbers are like the Bloch momenta in a periodic potential and label representations of the homotopy group of gauge transformations. On the fundamental domain the non-contractible loops arise through identifications of boundary points. The fundamental domain will therefore contain all the information relevant for the topological quantum numbers. Sufficient knowledge of the boundary identifications will allow for an efficient and natural projection on the various superselection sectors. Typically we integrate out the high-energy modes, being left with the low-energy modes whose dynamics is determined by an effective hamiltonian defined on the fundamental domain (restricted to these low-energy modes). In this it is assumed that the contributions of the high-energy modes can be dealt with perturbatively, generating the running coupling and the effective interactions of the low-energy modes.

4. Low-lying spectrum

We will now illustrate the relevance of the fundamental domain through a study of gauge fields on the torus, since it allows for a comparison with results obtained by Monte Carlo studies in lattice gauge theory. The most accurate results have been obtained for the gauge group SU(2), but we will mention at the end of this section some other results as well. First consider the spatially constant gauge fields, $A_j = i\frac{C_j}{2L}\tau_3$ (L is the size of the torus, τ_a are the Pauli matrices). These form the set of gauge fields on which the classical potential vanishes. It is called the vacuum valley (also referred to as toron valley, or moduli space of flat connections) and one can perform a Born-Oppenheimer-like approximation for deriving an effective hamiltonian in terms of these "slow" degrees of freedom. To find the Gribov horizon within the vacuum valley we note that the $\vec{C}$ dependent eigenvalues of $FP(A)$ are given by $\lambda^{gh}_{\vec{n}}(\vec{C}) = 2\pi\vec{n}\cdot(2\pi\vec{n}\pm\vec{C})/L^2$, with $\vec{n}\neq\vec{0}$ an integer vector. The lowest eigenvalue vanishes if $C_j = \pm 2\pi$, such that the Gribov region is given by a cube with sides of length 4π, centered at the origin, see Fig. 3.

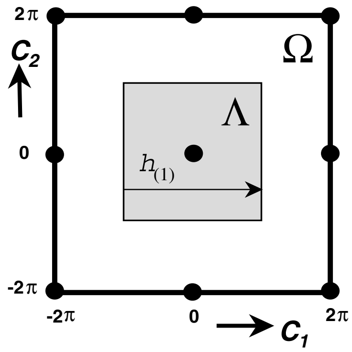


Fig. 3. A two dimensional cross-section of the vacuum valley with the fat square indicating the Gribov horizon and the grey square the fundamental domain. The dots at the Gribov horizon are Gribov copies of the origin.

The gauge transformation $h_{(j)} = \exp(\pi i x_j \tau_3/L)$ shifts C_j to $C_j + 2\pi$, leaving the other components $C_{i\neq j}$ unchanged. As $h_{(j)}$ is anti-periodic it is homotopically non-trivial, they are 't Hooft's twisted gauge transformations.[25] Gauge copies are thus explicitly seen to occur inside Ω. Furthermore the naive vacuum $A = 0$ has (many) gauge copies that lie on the Gribov horizon. Actually, in the Coulomb gauge for *any* three-manifold, a Gribov copy by a homotopically non-trivial gauge transformation of $A = 0$ will necessarily have a non-trivial zero eigenvalue for the Faddeev-Popov operator.[24] Taking the symmetry under homotopically non-trivial gauge transformations properly into account is crucial for describing the non-perturbative dynamics and one sees that the singularity of the hamiltonian at Gribov copies of $A = 0$ could form a severe obstacle in obtaining reliable results.

The boundary of the fundamental domain restricted to the vacuum valley occurs where $|C_j| = \pi$, well inside the Gribov region, see Fig. 3. Indeed, one verifies that the gauge copies $\vec{C} = (\pi, C_2, C_3)$ and $\vec{C} = (-\pi, C_2, C_3)$ have equal norm. Boundary identifications are now by the homotopically non-trivial gauge transformations $h_{(j)}$. This has the topology of a torus. Taking the invariance under constant gauge transformations into account, which on the vacuum valley acts as $\vec{C}$ to $-\vec{C}$ (associated to the Weyl group), we find for Λ/G restricted to vacuum valley the orbifold T^3/Z_2.

Formulating the hamiltonian on Λ, with the boundary identifications implied by the gauge transformations $h_{(j)}$, avoids the singularities at the Gribov copies of $A = 0$. "Bloch momenta" associated with the 2π shift, implemented by the non-trivial homotopy of $h_{(j)}$, label 't Hooft's electric flux quantum numbers[25] and leads to wave functions that should satisfy $\Psi(C_j = -\pi) = \exp(\pi i e_j)\Psi(C_j = \pi)$. The phase factor is not arbitrary, but ± 1, because $h^2_{(j)}$ is homotopically trivial. To see that a loop that winds around the vacuum valley twice is contractible in Λ requires one to include modes transverse to the vacuum valley, which will make the Z_2 nature of the relevant homotopy group evident.[26] These considerations are valid in the absence of fields in the fundamental representation (quarks), as assumed to be the case here.

In weak coupling Lüscher showed that the wave functionals are localized at $A = 0$ and that the spectrum is discrete,[20] and in this limit insensitive to the boundary identifications. The resulting degeneracy in the topological quantum numbers implies a vanishing electric flux energy, defined by the difference in energy of a state with $|\vec{e}| = 1$ and the vacuum state with $\vec{e} = \vec{0}$. Although there is no classical potential barrier to achieve this suppression, it comes about by a quantum induced barrier, in strength down by two powers of the coupling constant.

$$V(\vec{C}) = 2L^{-1} \sum_{\vec{p} \in \mathbb{Z}^3} \left(|2\pi\vec{p} + \vec{C}| - |2\pi\vec{p}|\right) = \frac{4}{L\pi^2} \sum_{\vec{n} \neq \vec{0}} \frac{\sin^2(\frac{1}{2}\vec{n}\cdot\vec{C})}{(\vec{n}\cdot\vec{n})^2}. \tag{8}$$

The second identity is obtained by Poisson resummation,[20] and the periodicity reflects the gauge symmetry discussed above. This quantum induced barrier leads to a suppression[27] with a factor $\exp(-S/g)$ instead of the usual factor of $\exp(-8\pi^2/g^2)$ for instantons.[9] At stronger coupling the wave functional spreads out over the vacuum valley and drastically changes the spectrum.[28] At this point the energy of electric flux suddenly switches on.

There was a difficult problem to be solved due to the fact that the potential is *quartic* in the zero-momentum gauge fields (their field strength is quadratic in the field) when expanding around $A = 0$. This was solved by Lüscher using Bloch degenerate perturbation theory[29] to derive an effective hamiltonian in terms of the constant modes.[20] Away from $A = 0$ one could further reduce the dynamics to one along the vacuum valley, but near the origin the adiabatic approximation breaks down. This gives rise to a conic singularity, $V(\vec{C}) = 2|\vec{C}|/L + \mathcal{O}\left(C^2\right)$, due to fluctuations of the non-abelian zero-momentum modes (contributing the $\vec{p} = \vec{0}$ term in Eq. (8)). This is a manifestation of the so-called orbifold singularities of the vacuum valley, which arise as fixed points under the Weyl reflections.

The effective hamiltonian is expressed in terms of the coordinates c_j^a, where $j = \{1, 2, 3\}$ is the spatial index ($c_0^a = 0$) and $a = \{1, 2, 3\}$ the gauge index. Apart from an overall factor $1/L$, the effective hamiltonian depends on L only through the renormalized coupling constant g at the scale $\mu = 1/L$, or to one loop order $g^2 = 12\pi^2/[-11\ln(\Lambda_{MS}L)]$,

$$\begin{aligned} LH_{\rm eff}(c) = &-\frac{g^2}{2}(1+\alpha_1 g^2)\sum_{i,a}\frac{\partial^2}{(\partial c_i^a)^2} + \frac{1}{4}(\frac{1}{g^2}+\alpha_2)\sum_{ij,a}(f_{ij}^a)^2 + \gamma_1\sum_i r_i^2 \\ &+\gamma_2\sum_i r_i^4 + \gamma_3\sum_{i>j} r_i^2 r_j^2 + \gamma_4\sum_i r_i^6 + \gamma_5\sum_{i\neq j} r_i^2 r_j^4 + \gamma_6\prod_i r_i^2 \\ &+\alpha_3\sum_{ijk,a} r_i^2 (f_{jk}^a)^2 + \alpha_4\sum_{ij,a} r_i^2 (f_{ij}^a)^2 + \alpha_5 {\rm det}^2 c, \end{aligned} \tag{9}$$

where $f_{jk}^a \equiv -\epsilon_{abd}c_j^b c_k^d$ and $r_j^2 \equiv \sum_a c_j^a c_j^a$, such that the zero-momentum gauge field and field strength read $A_j(\vec{x}) = ic_j^a\frac{\tau_a}{2L}$ and $F_{jk}(\vec{x}) = if_{jk}^a\frac{\tau_a}{2L^2}$. One expresses the masses and the size of the finite volume in dimensionless quantities, like mass-ratios and the parameter $z = mL$. In this way, the dependence of g on L becomes irrelevant. We have organized the terms according to the importance of their contributions. The first two give (when ignoring $\alpha_{1,2}$) the lowest order effective hamiltonian,[19] whose energy eigenvalues are $\mathcal{O}\left(g^{2/3}\right)$, as can be seen by rescaling c with $g^{2/3}$. We will not provide here the numerical values of the parameters that appear in the effective hamiltonian, which were determined to two loop order.[30]

It is interesting to note that the energy ΔE for one unit of electric flux is determined by tunneling through a *quantum induced* potential barrier, with the following asymptotic expansion[31,32]

$$\Delta E = 2L^{-1}\lambda B^2 g^{5/3}\exp\left[-S/g + \varepsilon_1 T/g^{2/3}\right](1+\mathcal{O}(g)), \tag{10}$$

where $S = 12.4637$ is the tunneling action and $T = 3.9186$ is related to the tunneling time, and comes from the "classical" turning points of the quantum induced potential $V(\vec{C})$, which also determines $\lambda = 0.6997$ through its transverse fluctuations along the tunneling path. The constant $\varepsilon_1 = 4.116719735$ also gives to lowest order the mass of the scalar glueball,[33] $m_0 = L^{-1}\varepsilon_1 g^{2/3}$. Finally, the quantity $B = 0.206$ is taken from the asymptotic expansion of the lowest order perturbative wave function in the direction of the vacuum valley.

Of more practical importance is, however, to go to the domain where the typical energies are above those of the quantum induced potential barrier, when the wave functional has spread out over the full fundamental domain, see Fig. 3. In this case one will become sensitive to the boundary identifications on the fundamental domain. The influence of the boundary conditions on the low-lying glueball states is felt as soon as $z = m_0 L > 0.9$. When this happens imposing proper boundary conditions on the wave function becomes mandatory. They depend on the parity $p_i = \pm 1$ under the coordinate reflection $c_i^a \to -c_i^a$ (for all a) and the electric flux quantum numbers $q_i \equiv \exp(i\pi e_j)$ of the wave function $\Psi_{\vec{q},\vec{p}}(c)$, and are given by[26,34]

$$\begin{aligned} \Psi_{\vec{q},\vec{p}}(c)|_{r_i=\pi} &= 0 \quad , \quad \text{if} \quad p_i q_i = -1 \\ \frac{\partial}{\partial r_i}(r_i \Psi_{\vec{q},\vec{p}}(c))|_{r_i=\pi} &= 0 \quad , \quad \text{if} \quad p_i q_i = +1 \end{aligned} \tag{11}$$

It should be noted that this does not apply to the negative parity states ($\prod_i p_i = -1$) since parity restricted to the vacuum valley is equivalent to a Weyl reflection.[34] Using the fact that for fixed i each c_i^a behaves as a vector under SU(2) (gauge) rotations, a complete basis of wave functions can be constructed in terms of spherical harmonics for each of these, whose angular momenta l_i have to add to zero to ensure gauge invariance. The "angular" momentum quantum numbers l_i are restricted to be even or odd, for resp. $p_i = 1$ or $p_i = -1$, and the radial wave functions which depend on r_i either vanish, or have vanishing derivative at $r = \pi$ (for resp. $l_i + e_i$ even or odd, cmp. Eq. (11)). It is then straightforward to compute the matrix elements of the effective hamiltonian (Eq. (9)) and solve for the low-lying spectrum by Rayleigh-Ritz. The first two lines in Eq. (9) are already sufficient to obtain the mass-ratios to an accuracy of better than 5%.

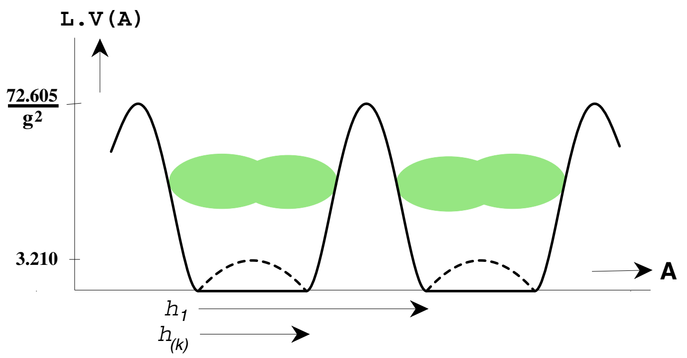


Fig. 4. Sketch for the potential on the torus. Shown are two vacuum valleys, related to each other by a gauge transformation h_1 with winding number $\nu(h_1) = 1$. The induced one-loop effective potential, of height $3.210/L$, has degenerate minima related to each other by the anti-periodic gauge transformations $h_{(k)}$. The classical barrier, separating the two valleys, has a height $72.605/Lg^2$.

For the self-consistently of this approach it is important to identify the range of validity for the effective hamiltonian. Naturally, at larger volumes extra degrees of freedom will behave non-perturbatively. We know from the existence of gauge transformations with non-trivial winding number, $\nu(h) = \frac{1}{24\pi^2}\int_M \mathrm{tr}\,((h^{-1}dh)^3)$, there are gauge equivalent vacuum valleys that can be reached only by crossing a saddle point called the finite volume sphaleron, see Fig. 4. Typically this saddle point lies on a tunneling path (instanton). The degrees of freedom along this tunneling path go outside of the space of zero-momentum gauge fields and if the energy of a state flows over the barrier, its wave functional will no longer be exponentially suppressed below the barrier and will in particular be non-negligible at the boundary of the fundamental domain. Boundary identifications in this direction of field space now become dynamically important as well.

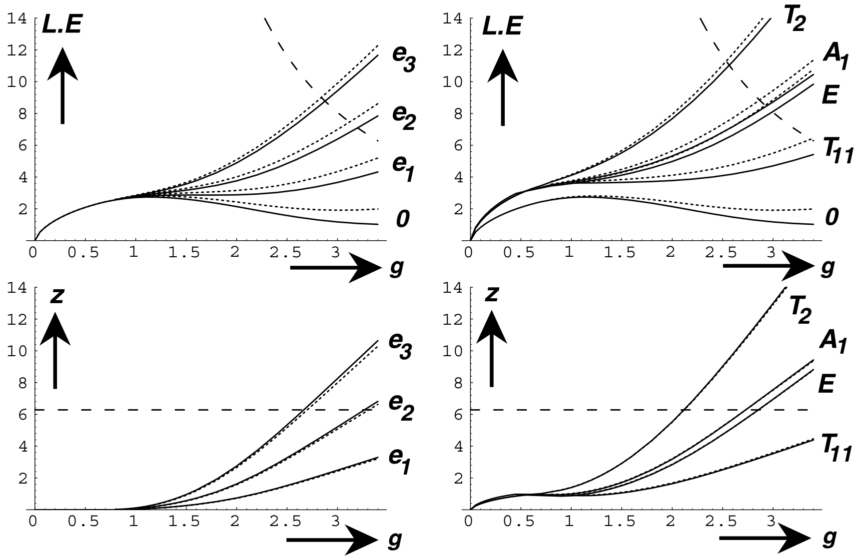


Fig. 5. Top row shows $LE(g)$ for the relevant positive parity representations; dotted lines are without the two loop correction included. The dashed curve denotes the barrier height $LE_{sph}(g)$. Bottom row shows $z(g) = Lm(g) = L(E(g) - E_0(g))$, and the dashed line is at $z = 2\pi$.

The height of this barrier is given by[35] $E_{sph}(g) = 72.605/Lg^2$, and as long as the states under consideration have energies below this value, the barrier transitions can be neglected (or treated semi-classically if there is no perturbative contribution) and the zero-momentum effective hamiltonian provides an accurate description. To determine for which volume our approximations break down we inspect the energy levels as a function of g in Fig. 5. In the top two figures we have plotted $LE(g)$ for the various representations of the cubic group O(3,$\mathbb{Z}$) (0 stands for the vacuum state, which is in the A_1^+ representation). The dashed curve shows the height of the instanton barrier. In the lower two figures we plot $z = Lm \equiv L(E - E_0)$. Here the dashed line is at $z = 2\pi$, associated with the energy scale $2\pi/L$ of the non-zero momentum field modes which were integrated out. Note that the sensitivity to two loop corrections (included in the dotted curves) almost entirely drops out for the energy differences at the scale of this figure. We read off that instantons only become important for $z_0 = 5$ to 6, i.e. for L roughly 5 to 6 times the correlation length set by the scalar glueball mass.

On the three-torus we have therefore achieved a self-contained picture of the low-lying glueball spectrum in intermediate volumes from first principles with *no free parameters*, apart from the overall scale. For very small volumes the energy of electric flux vanishes and there is an accidental rotational symmetry, only split by the $\mathcal{O}\left(g^{8/3}\right)$ terms in Eq. (9). The 2^+ tensor glueball splits in a nearly degenerate doublet E^+ and a triplet T_2^+. Both are lighter than the scalar 0^+, also denoted by A_1^+ as the scalar singlet representation of the cubic group. Going to larger volumes, $L > 0.1$ fermi, the energy of electric flux per unit length, which in an infinite volume would be the string tension K, is surprisingly constant in intermediate volumes, whereas the splitting of the tensor states becomes quite large. In intermediate volumes the doublet E^+ and triplet T_2^+ have respectively masses of roughly 0.9 and 1.7 times the scalar mass. This doublet E^+, *lighter* than the scalar, was first observed in the lattice studies of Berg and Billoire,[36] and caused some stir at the time.

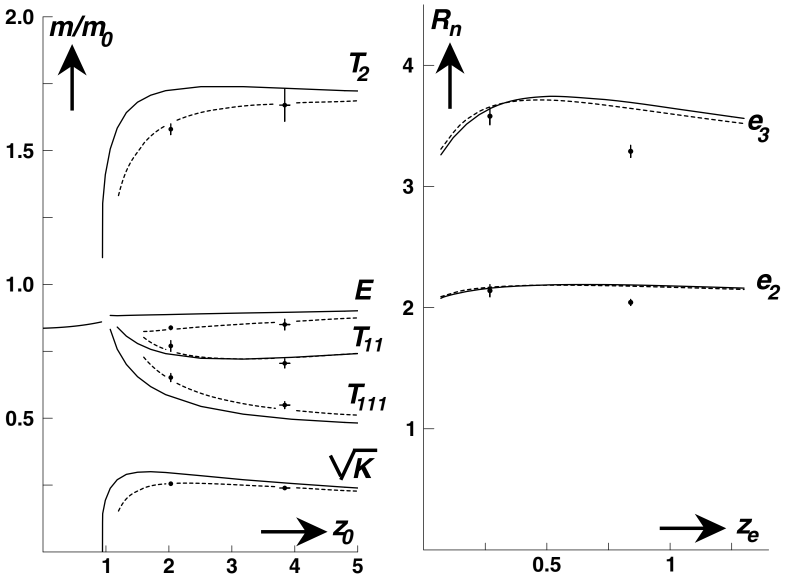


Fig. 6. Left: mass ratios m/m_0 as a function of $z = m_0(L)L$, with m_0 the scalar glueball mass. Full (continuum) and dashed (4^3 lattice) curves give the hamiltonian results. Shown are string tension $\sqrt{K(L)}$, E^+ and T_2^+ tensor masses and the exotic states T_{11} with two, and $T_{111} = T_2(111)$ with three units of electric flux. Right: electric flux ratios R_n as a function of z_e. Monte Carlo data[41] obtained with Wilson action on $4^3 \times N_t$ lattice ($N_t = 32, \beta = 2.4$ and $N_t = 99, \beta = 3.0$).

Electric flux energies (for the trivial representation) are labeled by $e = e_1$ for $\vec{e} = (1, 0, 0)$, e_2 for $(1, 1, 0)$ and e_3 for $(1, 1, 1)$. For e_n we speak of n units of electric flux, sometimes called “torelon” energies. As was argued by ’t Hooft,[25] if a confining string would have formed, such that $E_e = KL$ (or $z_e = LE_e = KL^2$), it would become energetically favorable to run along the direction of $\vec{e}$, giving $R_n \equiv E(e_n)/E(e_1) = \sqrt{n}$. This in contrast with separate strings, each winding in the direction for which $e_j = 1$, which would give $R_n = n$. In intermediate volumes it is the latter behavior that we found,[28] confirmed by the Monte Carlo results.[38] One has to go to quite large volumes to start to observe the expected $\sqrt{n}$ behavior.[39]

After the early work of Berg and Billoire (their collected data can be found in Ref. 40), considerable progress was achieved by the so-called variational method, which allowed much more accurate results.[37] In Fig. 6 we present a comparison with the Monte Carlo results of Michael,[41] obtained for a lattice of spatial size 4^3. The hamiltonian results below $z = 0.95$ are due to Lüscher and Münster,[33] which is where the spectrum is insensitive to *any* identifications at the boundary of Λ. The electric flux energy ratios R_2 and R_3 are shown on the right.

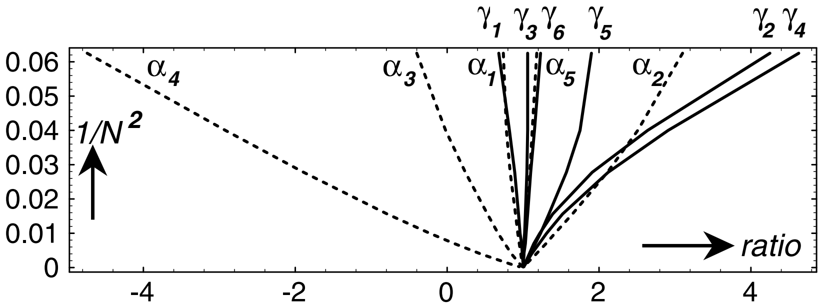


Fig. 7. Flow of the lattice hamiltonian coefficients to their continuum ($N \to \infty$) values. Shown are $\alpha_i(N)/\alpha_i$ (dashed lines) and $\gamma_i(N)/\gamma_i$ (full lines) as a function of the square of the lattice spacing $(a/L)^2 = 1/N^2$.

We note that the solid curves in Fig. 6 representing the continuum results, deviate significantly from the lattice data. Even though one should not expect a 4^3 lattice to be an accurate approximation for the continuum, it was cause for some concern. To settle this issue we redid the complete derivation of the effective hamiltonian starting from the lattice theory, *without taking the continuum limit*.[30,43] The hamiltonian is basically of the same form as in Eq. (9), except that its coefficients depend on the lattice spacing. One can follow the renormalization group flow of the hamiltonian to its continuum fixed point in this formalism in all detail, see Fig. 7. Using the same analysis as in the continuum leads for a finite lattice to the dashed curves in Fig. 6. The lattice data now agree perfectly, up to a volume of about 0.75 fermi, the regime for which we have shown that the effective hamiltonian in the zero-momentum modes should provide a good approximation. The deviations for R_2 and in particular R_3 are in accordance with the fact that these are of relatively high energy, and therefore expected to be sensitive to other non-perturbative effects that invalidate integrating out all the non-zero momentum modes, see Fig. 5.

The question of extending the previous results to SU(3) is a natural one in the light of QCD. The perturbative expansion,[20,44] vacuum-valley effective potential,[20,32] and the semi-classical evaluation of the energy of electric flux[45] (due to the tunneling through the quantum induced vacuum-valley effective potential) are more or less straightforward. Figure 8 sketches for SU(3) the properties of the vacuum valley and the one-loop effective potential. The non-trivial problem of formulating the appropriate boundary conditions on the boundary of the fundamental domain for SU(3) was solved by Vohwinkel and qualitative agreement with the lattice data was found.[42,46] With the results for SU(3) in hand generalization to SU(N) was achieved.[47] Like for SU(2) the approximations break down for L larger than 5 to 6 times the correlation length of the scalar glueball.

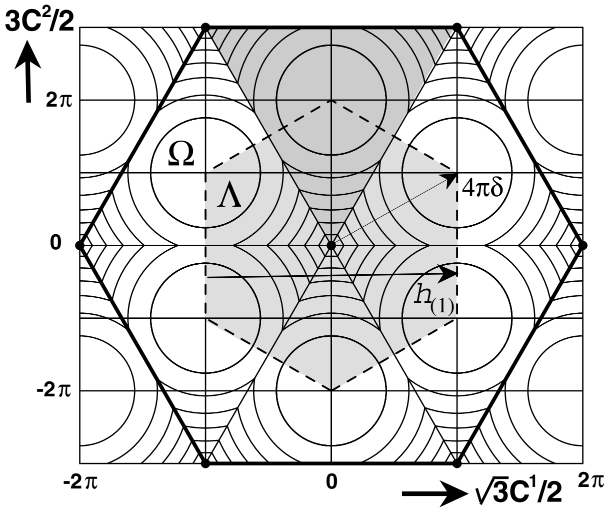


Fig. 8. Two dimensional cross-section of the SU(3) vacuum valley for $C_1^b T_b = C_1^1\lambda_8/2 + C_1^2\lambda_3/2$ and $C_{2,3}^b = 0$, with λ_a the Gell-Mann matrices. Circles indicate the equipotential lines for the quantum induced effective potential. The light shaded area represents the fundamental region, bounded by the dashed hexagon. The fat hexagon represents the Gribov horizon, with gauge copies of $A = 0$ indicated by the dots. The darker shaded area is the fundamental Weyl chamber.

A similar analysis for calculating the low-lying glueball spectrum has been performed for SU(2) on a spherical geometry. In this case there is no continuous vacuum degeneracy, and the wave functional first starts to feel the boundary of the fundamental domain when reaching the finite volume sphaleron configuration (associated to a homogeneous color magnetic field on the sphere), which can be shown to lie on the boundary of the fundamental domain.[48] One can formulate an effective hamiltonian in terms of an 18 dimensional space of gauge fields whose zero-point energies are degenerate to lowest order, and which contains the sphaleron degree of freedom. From a conceptual point of view finding an effective hamiltonian for this space and implementing the appropriate boundary conditions was considerably more complicated.[49] The results for the spectrum are shown in Fig. 9 as a function of $f \equiv g^2/2\pi^2$, where g is now the coupling constant as running with the radius of the sphere.

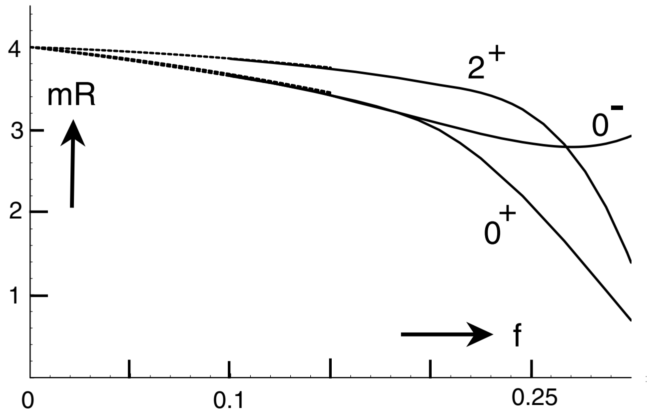


Fig. 9. The full one loop results for the masses of the scalar and tensor glueball and of the oddball on the trhee-sphere as a function of $f = g^2/2\pi^2$. The dashed lines gives the perturbative result.

In lowest order in perturbation theory the 0^+, 0^- and 2^+ glueball states are all degenerate, lifted in the next order in perturbation theory, but unexpectedly making the 0^- "oddball" slightly *lighter* than the 0^+ glueball state. It is an important test to verify that non-perturbative effects that set in when going to larger volumes are able to correct for this "unwanted" behavior. Indeed, when including the effects of the boundary of the fundamental domain the $0^-/0^+$ mass ratio rapidly increases. Also the $2^+/0^+$ mass ratio reaches a value more in accordance with what is expected in a large volume, where the results on the torus and sphere should agree.

Beyond $f \sim 0.28$ it can be shown that the wave functionals start to feel parts of the boundary of the fundamental domain which the present calculation is not representing properly. This value corresponds to a circumference of roughly 1.3 fm, when setting the scale as for the torus, assuming the scalar glueball mass in both geometries at this intermediate volume to coincide. Like for SU(3) we refer to Ref. 50 for a more extensive review.

5. Dimension-two condensate

Condensates are believed to be connected to the non-perturbative structure of the theory and it has been speculated that a dimension-two condensate in QCD or in pure gauge theories should be present.[51] In a theory without manifest dimensional parameters the only evident operator for such a condensate is in terms of the vector potential itself $V^{-1}\,\|A\|^2$. Minimizing over the gauge orbit deals with the problem that this would otherwise be gauge dependent and it is crucial to note that the quantity $\|A\|^2_{\rm min}$ is properly defined on the fundamental domain. The only issue here is to verify that on the boundary, where points are related by a gauge transformation, $\|A\|^2_{\rm min}$ is independent of the representative on the gauge orbit, but this is precisely implied by the way we have identified the boundary of the fundamental domain. Therefore,

$$\langle 0|\,\|A\|^2_{\rm min}\,|0\rangle \equiv \int_\Lambda \mu(A)\mathcal{D}A\;\Psi_0^*(A)\,\|A\|^2_{\rm min}\,\Psi_0(A) \tag{12}$$

is in principle well-defined, and could form the basis for establishing the existence of a non-perturbative dimension-two condensate. Here $\Psi_0(A)$ is the groundstate wave functional, and $\mu(A)$ is the appropriate measure on field space[10,12]. The finite volume cutoff allows us to define the contribution coming from the low-lying modes for which the wave functional is affected by the boundary of the fundamental domain. By excluding the perturbative contributions coming from the modes that are integrated out, in as far as they do *not interact* with the low-lying modes kept in the effective hamiltonian, this gives by construction a finite result. In a small volume with periodic boundary conditions, to lowest order the wave functional is simply a product of harmonic oscillators for each of the field modes. The dynamics of the zero-momentum modes are, however, governed by Eq. 9. In computing $\langle 0|\,\|A\|^2_{\rm min}\,|0\rangle$ to lowest order, we therefore use for these zero-momentum modes the lowest order groundstate wave function (with all coefficients put to zero in Eq. 9. A simple rescaling of the fields with a factor $g^{2/3}$ shows that the zero-momentum contribution to $\langle A^2\rangle \equiv L^{-3}\langle 0|\,\|A\|^2_{\rm min}\,|0\rangle$ is proportional to $g^{4/3}/L^2$.

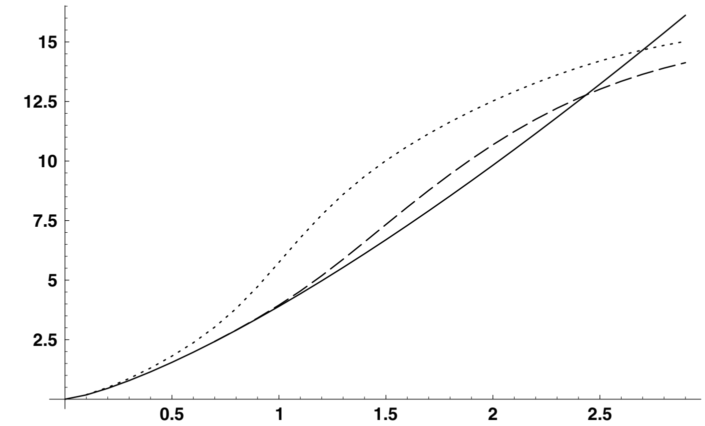


Fig. 10. The non-perturbative finite volume contribution to $L^2\langle A^2\rangle$ for $SU(2)$ as a function of g. The full lines give the lowest order result, $3.89775g^{4/3}$. The dashed line include the effect of the boundary conditions for the lowest order effective hamiltonian. The dotted line includes higher order contributions to the effective hamiltonian as discussed in the text.

An efficient way to compute this expectation value is by adding $\lambda\left(A_j^a(\vec{x})\right)^2$, with $A_j^a(\vec{x})$ assumed to be transverse and to lie within the fundamental domain, to the hamiltonian density of Eq. (2). We consider volumes where the restriction to the fundamental domain is felt for the zero-momentum modes only. Integrating out the non-zero momentum modes perturbatively now introduces a λ dependent term in the effective hamiltonian Eq. 9. The term independent of the zero-momentum gauge field will be dropped, as its value and its derivative with respect to λ, at $\lambda = 0$, can be interpreted as the perturbative contribution to the vacuum energy E_0, resp. $\langle 0| \, \|A\|^2_{\rm min} \, |0\rangle$. Keeping only the terms in the effective hamiltonian that depend on the zero-momentum gauge field we may calculate the groundstate energy as a function of λ. Its derivative at $\lambda = 0$ gives the *non-perturbative* finite volume contribution to $\langle 0| \, \|A\|^2_{\rm min} \, |0\rangle$. In Fig. 10 we illustrate for $SU(2)$ the lowest order result, with and without incorporating the boundary conditions.[53] In this figure the result can be trusted up to $g = 2.6$, corresponding to $L = 0.75$ fermi. To indicate the magnitude of higher order contributions the dotted curves gives the result based on the expectation value of the λ dependent effective hamiltonian evaluated with the groundstate wave function for $\lambda = 0$. The significance of these results is mainly to illustrate there is a way one can make sense of $\langle A^2 \rangle$. For it to be relevant to the issue of a dimension-two condensate, one would have to take the infinite volume limit, which is out of reach with the current methods.

6. Epilogue

We have seen that the non-abelian configuration space has a complicated geometry, which can nevertheless be probed successfully by restricting to a finite physical volume. Perhaps it feels somewhat strange not to address the problems directly in the infinite volume, but we have to remember that non-abelian gauge theories in the confined phase suffer from severe infrared problems, and it remains unclear how on the one hand to explain the success of a parton description at high energies and at the other hand the complete absence of these degrees of freedoms at low energies. This is not the place to list the many uncontrolled attempts that were made in an infinite volume. Fortunately lattice gauge theory promises a controlled approach, but does not always shed light on the underlying dynamics. Light-cone methods still suffer from infrared divergencies, and the unfulfilled dream to keep the coupling constant in check. The latter also is a problem in the many variational methods, which have difficulty in giving reasons why to trust one ansatz over the other. Much remains to be done.

We are in no way pretending here to have given the answer to these questions. The finite volume is too small for comfort to make contact with the real world. Nevertheless, the methods presented control its errors, as is evident from the comparison with the lattice Monte Carlo data. But in this approach it remains open how the infinite volume limit will be independent of the geometry of the finite volume, when scaled to infinity.[50] It is here, of course, where lies the hidden treasure of understanding the formation of a mass gap. The finite range of the physical degrees of freedom should make the geometry irrelevant, once there is enough room to move around. After fifty years of Yang-Mills theory, the insisting question of Pauli to Yang,[6] "what is the mass of this field" remains unanswered, but now carries a reward of a million dollars,[5] which I am sure makes no difference to *why* we want to know.

Acknowledgments

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