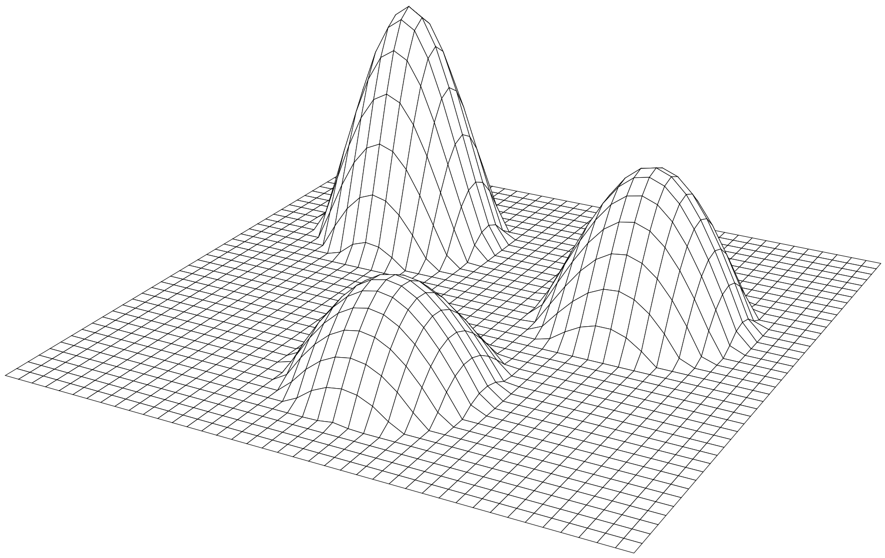
Periodic Instantons and Monopoles



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1 Introduction

1.1 Perspective

It has been appreciated for a long time that the interactions between elementary particles are described by gauge theories. In these theories matter particles, quanta of the matter fields, have an internal space on which a symmetry group acts. The gauge field is quantised in bosonic gauge particles. When gauge particles hit a matter particle, the effect is a rotation in its internal space. In this way matter particles can influence each other by exchanging a gauge particle, which affects the internal spaces of both particles. The simplest example of a gauge theory is electromagnetism, where the internal space is a circle and the gauge group is the abelian group $U(1)$, affecting the phases of the particles. For more general, non-abelian gauge groups, the order of successive group actions is of relevance, which is reflected in self-interactions of the gauge field. The gauge principle states that the physics is invariant under local symmetry group transformations of the particles' internal spaces.

The matter particles, which are fermionic, are the leptons and quarks. Electrons, muons and neutrinos are examples of leptons. Quarks are the constituents of hadrons, of which there are two types, baryons and mesons. Baryons are particles like the proton and neutron, in which there are (on average) three quarks. Mesons consist of a quark and an anti-quark.

The electro-weak theory describes the weak interaction, responsible for radioactive decay, and the electromagnetic interaction. It is a spontaneously broken gauge theory where the $SU(2)\times U(1)$ symmetry group is broken down to $U(1)$. The field describing the Higgs particle is responsible for the symmetry breaking, which gives the matter particles and some of the gauge bosons a mass. The gauge particles in the electro-weak theory are the photon, a massless particle mediating the electromagnetic interaction, and the massive $W^{\pm}$ and Z^0 particles, mediating the weak interaction.

The strong interaction that holds protons and neutrons together within the nucleus and quarks within hadrons is described by an $SU(3)$ gauge theory, quantum chromodynamics (QCD). The three directions in the corresponding internal space are labelled by three colours: red, green and blue. The gauge particles are called gluons, which are massless and of which there are eight different types. When hitting a quark, they transform it from one colour to another.

The electroweak theory and QCD together form the Standard Model for elementary particles. The Standard Model is renormalisable, i.e. infinities due to quantum fluctuations can be made to cancel in the calculation of physical quantities. This makes the theory well defined and scattering processes as studied in experiments with particle accelerators can be predicted with high accuracy within the Standard Model. In these calculations one uses perturbative expansions around the vacuum, which is a reliable approximation for high momenta or equivalently distance scales smaller than the size of a proton.

At larger scales, or lower energies, QCD should explain how the quarks together with the gluons build up the hadrons. From experiments it followed that quarks are never observed as single particles: they always occur in a bound state in baryons and mesons. This phenomenon is called quark confinement. Perturbative techniques that worked so well in capturing the short-range interactions are insufficient to explain the long-range forces involved. Though progress has been made in this direction, a full understanding and an analytic proof of quark confinement is still lacking. The non-abelian nature of the gauge group and thereby the self-interactions of the gauge particles are held responsible for the confinement process. Indeed, configurations consisting purely of gauge particles ("glueballs") exist in the theory and one can envisage a hadron as a system of quarks bound together by this glue. For these calculations, one needs the full properties of the configuration space of the gauge theory, going beyond perturbation theory.

Part of these so-called non-perturbative effects can be studied by considering other stationary points of the gauge theory action than the vacuum state. These are the classical solutions. Instantons and monopoles, studied in this thesis and defined below, are examples of classical solutions. They feature at various places in the analysis of non-perturbative dynamics. Instantons are solutions existing in the pure gauge theory, whereas monopoles are solitons in the Yang-Mills-Higgs system. This thesis deals with periodic instantons, which interpolate between ordinary instantons and monopoles.

It is expected that at sufficently high energies, or sufficiently small scales -much smaller than the size of a proton- the various interactions become of the same strength. In the quest for a unified description of all interactions, including gravity, string theory is the most promising candidate. In string theory, one considers extended objects, rather than pointlike particles. For strings, the extension in dimension is one. Different vibration modes of these strings then represent the various elementary particles, including the graviton that mediates gravity. The interaction vertices in field theory Feynman diagrams, i.e. the points where the world-lines of interacting particles meet, in string theory get replaced by topologically nontrivial two-surfaces (world sheets) in which the interaction point is smeared out. This makes string theory diagrams finite, curing the divergencies that plagued earlier attempts toward a quantum gravity. To make string theory consistent, the dimension of the space-time in which it is defined has to be larger than four. The way the extra dimensions are wrapped up to give back four-dimensional space-time, influences the effective low energy theory. There is not yet a compelling reasoning which singles out a particular compactification scheme leading to the Standard Model.

A recent development in string theory is the study of higher dimensional extended objects, generalising the particle and the string, for which the number of world-volume dimensions can be higher than one (a particle) or two (a string). Examples are D-p-branes, extended objects with a $p+1$-dimensional world volume, on which ordinary strings end, satisfying certain boundary conditions. Stacks of D-p-branes can represent gauge theories and other configurations in string theory have properties in common with monopoles and instantons. Certain D-brane configurations can be shown to be mathematically equivalent to monopoles. The non-trivial part of H-monopoles, a configuration in a low energy effective theory of heterotic string theory, is formed by periodic instantons. Classical solutions, featuring so prominently in gauge theories, also have a part to play in string theory.

Mathematically, gauge fields are the connections on principal fibre bundles. A principal fibre bundle consists of a base manifold, i.e. a set of points described by one or more overlapping charts (coordinate patches), for which each point has a local copy of the gauge group. The coordinates on the overlap of two patches can be given in terms of coordinates on either patch, being related by smooth functions (coordinate diffeomorphisms). Going from one point to another, the corresponding local gauge groups may be relatively rotated. This is probed by a matter particle, i.e. a vector acted upon by the gauge group, as a rotation of its internal space. How strongly the internal space varies from one point to another is measured by the gauge connection. The effect that the internal vector space may be rotated after going a round-trip, starting and ending in the same point, is measured by the gauge curvature: the field strength of the gauge field, generalising the electric and magnetic fields of electromagnetism. The internal space may vary in such a non-trivial way that more than one coordinate patch is needed to characterise the gauge connection. The gauge transformation interpolating between the gauge choices (the transition functions or cocycles) on the overlapping coordinate patches can have a "winding", i.e. can be topologically nontrivial. Instantons and monopoles have gauge connections for which this is the case. The solution space of instantons and monopoles is infinite dimensional as each new gauge choice gives a new solution. Dividing out this gauge ambiguity gives the space of gauge invariant parameters or moduli, which is finite dimensional for a specific type of winding. This so-called moduli space has a differential geometry of its own. Moduli spaces for instantons and monopoles are complex manifolds, i.e. the coordinate diffeomorphisms are analytic, guaranteeing the existence of a complex structure. Actually, they have even three complex structures, all three compatible with the metric on the moduli space and satisfying the algebra of the quaternions. This beautiful fact, which makes moduli spaces of instantons and monopoles so interesting from a differential geometric point of view, is called the hyperKähler property. It will play an important role in the analysis of metrics on moduli spaces.

The remainder of this introductory chapter is used to give some essential background material and ends with an outline of this thesis.

1.2 Gauge theories

In this thesis we consider exact classical solutions in gauge theories defined on euclidean four-manifolds M. The euclidean space-time described by M is obtained from ordinary space time by going to imaginary time, ($x_0 = it$). Imaginary time is used in the analytic continuation of the field theory partition function to make it well defined. As M will have a flat metric, there will be no difference between upper and lower indices. A pure gauge theory for gauge group G ($SU(n)$ or $U(n)$ throughout this thesis) in euclidean space-time is defined by the Yang-Mills action functional

$$S = -\int_M d_4x \frac{1}{2} \mathrm{Tr} F_{\mu\nu} F_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3. \tag{1.2.1}$$

Here repeated indices imply summation and $F_{\mu\nu}$ is the field strength or curvature associated with the gauge field or connection A_μ, taking values in the Lie algebra $\mathfrak{g}$ (usually $\mathfrak{su}(n)$ or $\mathfrak{u}(n)$) of G,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{1.2.2}$$

Finally, M denotes the euclidean space-time manifold of interest. In this thesis, M will be the euclidean flat space $\mathbb{R}^4/H$, where H is the subgroup of translational symmetries under which the physics is invariant. We sometimes will adopt form notation, $A = A_\mu dx_\mu$, $F = \frac{1}{2} F_{\mu\nu} dx_\mu \wedge dx_\nu$, in which $F = dA + A \wedge A$. As $F_{\mu\nu} = [D^{\mathrm{ad}}_\mu, D^{\mathrm{ad}}_\nu]$, $\quad D^{\mathrm{ad}}_\mu = \partial_\mu + [A_\mu, \cdot]$ denoting the covariant derivative, the curvature satisfies the Bianchi identity

$$\varepsilon_{\mu\nu\rho\sigma} D^{\mathrm{ad}}_\nu F_{\rho\sigma} = 0, \quad \varepsilon_{0123} = 1. \tag{1.2.3}$$

The equations of motion derived from this action, the Yang-Mills equations, read

$$D^{\mathrm{ad}}_\mu F_{\mu\nu} = 0. \tag{1.2.4}$$

Under the gauge action $g : M \to G$, the gauge potential and curvature transform as follows:

$$\begin{aligned} A_\mu(x) &\to {}^{[g]}A_\mu = g(x)(\partial_\mu + A_\mu(x))g^{-1}(x), \\ F_{\mu\nu}(x) &\to {}^{[g]}F_{\mu\nu}(x) = g(x) F_{\mu\nu}(x) g^{-1}(x). \end{aligned} \tag{1.2.5}$$

The action is gauge invariant. Therefore, the configuration space containing the physically relevant degrees of freedom, and on which computations should be performed, is the quotient space $\mathcal{A}/\mathcal{G}$, $\mathcal{A}$ being the space of all gauge connections $A_\mu(x)$ and $\mathcal{G}$ denoting the space of all gauge transformations $g(x)$.

This space is highly non-trivial, not only because of the infinite number of dimensions but also because of the existence of non-contractable loops. Following the action around such a loop, one may encounter a cycle vacuum V-energy barrier-vacuum V'. V and V' are gauge equivalent vacua, connected via a topologically non-trivial gauge transformation. It is the energy barrier, the fact that V and V' are vacua and the nontriviality of the gauge transformation that makes the loop non-contractable. For small excitations of the field, the wave functional in a quantum-mechanical calculation will be localised around V whereas with increasing energy the wave functional will start to spread out and make excursions taking tunneling paths to vacua $V', V'', \ldots$. These tunneling paths are the instantons, to be discussed in the next section.

1.3 Yang-Mills instantons and calorons

The Yang-Mills action can be rewritten as

$$S = -\int_M d_4x \tfrac{1}{4}\mathrm{Tr}(F_{\mu\nu} \pm {}^*F_{\mu\nu})^2 \mp 8\pi^2 k, \tag{1.3.1}$$

where ${}^*F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$ denotes the dual of F. The topological charge is defined by

$$\begin{aligned} k &= -\frac{1}{16\pi^2}\int d_4x \mathrm{Tr} F_{\mu\nu}\, {}^*F_{\mu\nu} \\ &= -\frac{1}{8\pi^2}\int d_4x \partial_\mu \varepsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left(A_\nu \partial_\rho A_\sigma + \frac{2}{3}A_\nu A_\rho A_\sigma\right) \end{aligned} \tag{1.3.2}$$

and is indeed topological as it reduces to a boundary term. Eq. (1.3.1) shows that within a class of gauge potentials with fixed k the action is minimal if

$$F = \pm\, {}^*F, \tag{1.3.3}$$

in which case the Yang-Mills equations (1.2.4) are satisfied automatically. These selfdual ($F = {}^*F$) and anti-selfdual ($F = -\,{}^*F$) solutions are called instantons. For the action to be finite, the instanton should be localised in both space and time, i.e. the action density is concentrated around an instant. This is what the name instantons is derived from. On $\mathbb{R}^4$, these classical solutions have a clear interpretation as tunneling paths between vacua, as both for $x_0 \to -\infty$ and for $x_0 \to \infty$, the configuration approaches a vacuum, $F_{\mu\nu} \to 0$, for all $\vec{x}$. It is because of this relation with tunneling paths that instantons, finite action solutions in the euclidean version of the theory, are important for the quantum theory in ordinary Minkowski space-time.

The standard $SU(2)$ instanton [9, 50, 51, 87] on $\mathbb{R}^4$ is given by the $SU(2)$ gauge potential

$$A_\mu(x) = -\frac{\rho^2\bar{\eta}_{\mu\nu}(x_\mu - y_\nu)}{|x-y|^2(|x-y|^2+\rho^2)} = \tfrac{1}{2}\bar{\eta}_{\mu\nu}\partial_\nu \log(1 + \frac{\rho^2}{|x-y|^2}). \tag{1.3.4}$$

Here, $\bar{\eta}_{\mu\nu}$ is the anti-selfdual quaternionic 't Hooft tensor, defined as $\bar{\eta}_{\mu\nu} \equiv \frac{1}{2}(\bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\mu\sigma_\nu) \equiv \bar{\eta}^a_{\mu\nu}\sigma_a$ in terms of the basic quaternions. The basic quaternions are defined as $\sigma_\mu = (1_2, -i\vec{\tau}) = (1, \mathbf{i}, \mathbf{j}, \mathbf{k})$ and $\bar{\sigma}_\mu = (1_2, i\vec{\tau})$, with

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \tag{1.3.5}$$

The τ_i are the Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.3.6}$$

The tensor $\eta_{\mu\nu} \equiv \frac{1}{2}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu) \equiv \eta^a_{\mu\nu}\sigma_a$ is selfdual. Using these definitions, the 't Hooft tensors take values

$$\begin{aligned} \eta^a_{0\nu} &= -\delta^a_\nu, \quad \eta^a_{ij} = -\varepsilon_{ija}, \\ \bar{\eta}^a_{0\nu} &= \delta^a_\nu, \quad \bar{\eta}^a_{ij} = -\varepsilon_{ija}. \end{aligned} \tag{1.3.7}$$

These values differ slightly from those in [50], as in the conventions chosen here time is labelled by x_0 rather than x_4. The unit quaternions are quaternions q for which $|q|^2 \equiv q_\mu^2 = 1$. These form the group $SU(2)$.

At infinity, the solution behaves like $\mathcal{O}(1/|x|^3)$ and hence it has a finite action. In $x_\mu = y_\mu$ there is a gauge singularity. It is removed by a gauge transformation $g_{(1)}(x) = (x-y)/|x-y| \in SU(2)$, where $x = x_\mu \sigma_\mu$ is a quaternionic notation of the position coordinate on $\mathbb{R}^4$. After this gauge transformation

$$
\begin{aligned}
A'_\mu &= {}^{[g_{(1)}]}A_\mu \qquad (1.3.8)\\
&= \frac{\eta_{\mu\nu}(x-y)_\nu}{|x-y|^2+\rho^2} = \frac{|x-y|^2}{|x-y|^2+\rho^2} g^\dagger_{(1)}\partial_\mu (g^\dagger_{(1)})^{-1} \to g^\dagger_{(1)}\partial_\mu (g^\dagger_{(1)})^{-1}, \quad |x| \to \infty,
\end{aligned}
$$

and the potential manifestly approaches a vacuum at infinity indeed. The corresponding curvature

$$
F'_{\mu\nu} = {}^{[g_{(1)}]}F_{\mu\nu} = \frac{2\rho^2 \eta_{\mu\nu}}{(|x-y|^2+\rho^2)^2} \qquad (1.3.9)
$$

is clearly selfdual due to the selfduality of $\eta_{\mu\nu}$. The action density,

$$
-\tfrac{1}{2}\mathrm{Tr} F_{\mu\nu}F_{\mu\nu} = \frac{48\rho^4}{(\rho^2+|x-y|^2)^4}, \qquad (1.3.10)
$$

is calculated using $\eta^a_{\mu\nu}\eta^b_{\mu\nu} = 4\delta_{ab}$, and reveals that the instanton is localised around y_μ which therefore denotes the position of the instanton. The scale of the solution is set by ρ. Thus ρ and y are the parameters or *moduli* of the instanton. Integrating the action density reveals a value $k = 1$ for the topological charge. This is precisely the winding number of the gauge transformation that connects the behaviour near the origin to that at infinity: one quickly convinces oneself that $g_{(1)}(x) = (x-y)/|x-y| \in SU(2)$, as map from the unit three-sphere at infinity to $SU(2)$, sweeps once over the three-sphere, now considered as the group space of $SU(2)$. Thus the third homotopy class of the gauge group, the classification $\pi_3(SU(2)) = \mathbb{Z}$ of maps of the unit three sphere to $SU(2)$, gives the topological characterisation of instantons on $\mathbb{R}^4$. The topologically non-trivial nature of the solution is also read off from the fact that two coordinate patches rather than one, with an interpolating gauge transformation $g_{(1)}$, are needed to describe the instanton, when compactified to $S^4 = \mathbb{R}^4 \cup \infty$. This illustrates the notion of a principal fibre bundle. It should be noted that it is the combination of base manifold and gauge group that gives rise to possibly topologically non-trivial windings of the transition functions.

A generalisation of eq. (1.3.4) with charge k is given by the 't Hooft Ansatz

$$
A_\mu = \tfrac{1}{2}\bar{\eta}_{\mu\nu}\partial_\nu \log\phi(x), \quad \phi(x) = 1 + \sum_{p=1}^{k} \frac{\rho_p^2}{|x-y_p|^2}, \qquad (1.3.11)
$$

for which the action density reads

$$
-\mathrm{Tr} F_{\mu\nu}^2 = -\partial_\mu^2 \partial_\nu^2 \log\phi(x), \qquad (1.3.12)
$$

outside the points y_p. A profile now shows k lumps with scales ρ_p located at y_p, representing k instantons of charge one, all of which having the same gauge orientation. This results in $5k$ moduli for the 't Hooft Ansatz. As will be explained later, there are $8k-3$ parameters needed to describe all gauge inequivalent instantons on $\mathbb{R}^4$. The $3k-3$ parameters missing in the 't Hooft Ansatz precisely correspond to the various relative gauge orientations. A complete construction of general $SU(n)$ instantons on $\mathbb{R}^4$ of arbitrary charge k was given by Atiyah, Drinfeld, Hitchin and Manin (ADHM), see chapter 3.

Using the 't Hooft Ansatz, it is possible to construct examples of periodic instantons. For periodic instantons the base manifold $M = \mathbb{R}^3 \times S^1 = \mathbb{R}^4/H, \quad H = \mathbb{Z}$. The period is the circumference of the circle S^1 which will be denoted $\mathcal{T}$. These solutions first appeared in the context of finite temperature field theory, where there is a natural period $\mathcal{T} = \beta = (\text{temperature})^{-1}$, and are called *calorons* because of this origin.

The simplest caloron is built from an infinite array of identical instantons with scale ρ located at $y_p = y + p\mathcal{T}$, $p \in \mathbb{Z}$. This caloron due to Harrington and Shepard [42] is therefore given in terms of the 't Hooft potential

$$\begin{aligned}\phi(x) &= 1 + \sum_{p\in\mathbb{Z}} \frac{\rho^2}{|\vec{x}-\vec{y}|^2 + (x_0 - y_0 - p\mathcal{T})^2} \\ &= 1 + \frac{\pi\rho^2}{\mathcal{T}|\vec{x}-\vec{y}|} \frac{\sinh\frac{2\pi}{\mathcal{T}}|\vec{x}-\vec{y}|}{\cosh\frac{2\pi}{\mathcal{T}}|\vec{x}-\vec{y}| - \cos\frac{2\pi}{\mathcal{T}}(x_0-y_0)}.\end{aligned} \tag{1.3.13}$$

The solution is characterised by a position y and a scale ρ. The limit $\mathcal{T} \to \infty$, corresponding to an infinite compactification circumference, gives back the standard instanton on $\mathbb{R}^4$.

For finite scale ρ, the asymptotics of the gauge potential and curvature is

$$A \sim \frac{1}{|\vec{x}|^2}, \quad F \sim \frac{1}{|\vec{x}|^3}, \quad |\vec{x}| \to \infty. \tag{1.3.14}$$

A dramatic transition occurs in the limit $\mathcal{T} \to 0$, which corresponds to the circle S^1 shrinking to a point, and therefore to the solution turning static. One can show the action density to indeed become time-independent whereas gauge potential and curvature become

$$A \sim \frac{1}{|\vec{x}|}, \quad F \sim \frac{1}{|\vec{x}|^2}, \quad |\vec{x}| \to \infty, \quad \mathcal{T} \to 0, \tag{1.3.15}$$

which is the behaviour of a monopole. The $\mathcal{T} \to 0$ limit is equivalent to the $\rho \to \infty$ limit, using classical scale invariance. Unlike the $U(1)$ Dirac monopole, this static solution is not pointlike. It is an extended object, regular near the origin. Actually, it can be shown to be gauge equivalent to the simplest monopole in the Bogomol'nyi-Prasad-Sommerfield limit [88], to be discussed in the next section. This limit reveals an important aspect of instantons on a compactified manifold: there are monopoles inside.

From the instanton and monopole limits of the Harrington-Shepard caloron we learn that calorons interpolate between instantons and monopoles. This will be an important aspect of calorons, which can be deduced from the topology of selfdual connections on $\mathbb{R}^3 \times S^1$ and which will be encountered at various places in this thesis.

1.4 Bogomol'nyi-Prasad-Sommerfield monopoles

The action of the Yang-Mills-Higgs theory, a gauge theory with a Higgs field $\Phi(x)$, is given by

$$S = \int d_4x \left(\frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} - \mathrm{Tr}(D^{\mathrm{ad}}_\mu\Phi)(D^{\mathrm{ad}\,\mu}\Phi) - \frac{\lambda}{2}(\frac{1}{2}\mathrm{Tr}\Phi^\dagger\Phi - c^2)^2\right). \qquad (1.4.1)$$

Here, and in the remainder of this section, we use real time ($x_0 = t$) and there is a difference between upper and lower indices, which are transformed into each other by the metric diag$(1,-1,-1,-1)$. The Higgs field transforms in the adjoint representation of the gauge group, ${}^{[g]}\Phi(x) = g(x)\Phi(x)g^{-1}(x)$. The third term on the rhs. spontaneously breaks the gauge symmetry due to the Higgs field acquiring an expectation value Φ_∞ minimising the potential term in the action. When the Higgs self-coupling goes to zero ($\lambda \to 0$, the Bogomol'nyi-Prasad-Sommerfield (BPS) limit), the only vestige of the symmetry breaking is a boundary condition for the asymptotic behaviour Φ_∞ of Φ. In that case the potential energy of a static configuration is given by

$$E^{\mathrm{BPS}} = -\int d_3x \left(\mathrm{Tr}B_i^2 + \mathrm{Tr}D^{\mathrm{ad}}_i\Phi D^{\mathrm{ad}}_i\Phi\right), \qquad (1.4.2)$$

in the gauge where $A_0 = 0$. Here the magnetic field is defined as $B_i = -\frac{1}{2}\varepsilon_{ijk}F^{jk}$. (The electric field is defined as $E_i = -F^{0i}$). The energy may be rewritten as

$$\begin{aligned} E^{\mathrm{BPS}} &= -\int d_3x \mathrm{Tr}\left((B_i \pm D^{\mathrm{ad}}_i\Phi)^2\right) \mp 8\pi\mathcal{Q}\|\Phi_\infty\|, \quad \|\Phi\| = \sqrt{-\tfrac{1}{2}\mathrm{Tr}\Phi^2}, \\ \mathcal{Q} &= -\frac{1}{4\pi}\int d\Omega|x|x_i\mathrm{Tr}\left(B_i\hat{\Phi}\right), \quad \hat{\Phi} = \|\Phi_\infty\|^{-1}\Phi, \end{aligned} \qquad (1.4.3)$$

using the Yang-Mills equation for B_i. $\mathcal{Q}$ is a topological index, related to the winding numbers of the gauge transformation g describing the Higgs field at infinity in terms of the symmetry breaking Higgs vacuum expectation value Φ_∞,

$$\begin{aligned} \Phi(x) &\to {}^{[g]}\Phi_\infty, \quad |\vec{x}| \to \infty, \\ g : S^2 &\to G/H_\infty. \end{aligned} \qquad (1.4.4)$$

Here H_∞ is the isotropy subgroup of G leaving the vacuum expectation value Φ_∞ invariant. For gauge group $SU(2)$, $\mathcal{Q}$ is an integer: the monopole charge. The winding numbers and the related magnetic charges are thus classified according to the second homotopy group $\pi_2(G/H_\infty) \simeq \pi_1(H_\infty)$. The topology will be discussed in detail in chapter 2. Within a class of configurations having the same topology the minimal energy is assumed by configurations satisfying the Bogomol'nyi equation

$$B_i = \pm D_i^{\mathrm{ad}}\Phi. \tag{1.4.5}$$

These static solutions are called *monopoles*, as the projection of B_i on the Higgs field asymptotically approaches an abelian magnetic monopole configuration. It is now observed [69] that with A_0 replaced by Φ, the selfduality equation eq. (1.3.3) for instantons becomes the Bogomol'nyi equation eq. (1.4.5) for monopoles. Therefore, the static BPS monopoles may be considered $\mathbb{R}$ invariant instantons (or S^1 invariant calorons). The large scale limit of the Harrington-Shepard caloron in eq. (1.3.15) illustrates this point of view [88] . Henceforth, we usually denote the monopole Higgs field by A_0.

The simplest BPS monopole is the $\mathcal{Q} = 1$ $SU(2)$ monopole due to Bogomol'nyi, Prasad and Sommerfield [11, 86], given by

$$\begin{aligned} A_0 &= \frac{i}{2}\hat{x}\cdot\vec{\tau}\left(2\pi\nu\coth 2\pi\nu|\vec{x}| - \frac{1}{|\vec{x}|}\right), \\ A_i &= -\frac{i}{2}\varepsilon_{ijk}\frac{x_j\tau_k}{|\vec{x}|^2}\left(1 - \frac{2\pi\nu|\vec{x}|}{\sinh 2\pi\nu|\vec{x}|}\right). \end{aligned} \tag{1.4.6}$$

For this configuration, the asymptotic value of the Higgs field is

$$A_0^\infty = 2\pi i \mathrm{diag}(\nu/2, -\nu/2), \tag{1.4.7}$$

up to an $\hat{x}$ dependent gauge rotation. The energy (or mass) of the monopole is proportional to this asymptotic value and amounts to $8\pi^2\nu$.

The force between two monopoles of the same charge vanishes when they are far apart, because the electromagnetic repulsion is exactly balanced by the attractive interaction of the long-range Higgs fields [68]. Therefore, a system of well-separated monopoles is stable and a so-called multi-monopole can exist. As there is no interaction energy involved, the energy of the multimonopole does not depend on the moduli. It is proportional to the integer monopole charge, cf. eq. (1.4.3), i.e. the number of constituent monopoles. This is similar to the case of instantons on $\mathbb{R}^4$, where the action is proportional to the winding number k which counts the number of constituent single instantons. It can be shown that there are $4\mathcal{Q} - 1$ gauge invariant parameters or moduli describing a charge $\mathcal{Q}$ $SU(2)$ monopole [97]. These parameters may be interpreted as positions and relative phases of the $\mathcal{Q}$ constituent monopoles.

1.5 The geometry of the moduli spaces of selfdual connections

Part of the mathematical interest in selfdual connections, instantons and BPS monopoles, lies in the fact that moduli spaces can be endowed with hyperKähler metrics.

In this section we discuss some basic notions of hyperKähler geometry and the geometry of moduli spaces. The moduli space is the space of solutions divided by the group of gauge transformations. It is the space to be integrated over when the contribution of instantons to the partition function is determined. In this calculation the metric on the moduli space, defined below, enters naturally as the norms of the gauge zero-modes in the instanton background. For monopoles, the low energy scattering can be described as geodesic motion on the monopole moduli space. As explained in section 1.5.3, the metric then enters in the kinetic part of the effective Lagrangian, having the moduli and their time derivatives as coordinates. This is called the moduli space approximation for monopoles, cf. [70, 92].

1.5.1 HyperKähler manifolds

Manifolds with metric g are hyperKähler if they have three independent complex structures I, J, K that satisfy the quaternion algebra, $IJ = -JI = K$ and cyclic, whose associated Kähler forms $\omega^I(\cdot,\cdot) = g(\cdot, I\cdot)$, $\omega^J(\cdot,\cdot) = g(\cdot, J\cdot)$, $\omega^K(\cdot,\cdot) = g(\cdot, K\cdot)$ are closed. As will be outlined later, the moduli spaces of selfdual connections inherit their hyperKähler property from the hyperKähler structure of the base space manifold $M = \mathbb{R}^4/H$, where $H = \emptyset, \mathbb{Z}, \mathbb{R}$ for instantons, calorons and monopoles respectively. The position coordinate on $\mathbb{R}^4$ will again be denoted as a quaternion, $x = x_\mu\sigma_\mu$ and $\bar{x} = x_\mu\bar{\sigma}_\mu$. Identifying the tangent space to the quaternions $\mathbb{H} \simeq \mathbb{R}^4$ with the vector space itself, the complex structures act on x as right multiplication with $-\mathbf{i}, -\mathbf{j}, -\mathbf{k}$, such that $(I, J, K)_{\mu\nu} = \bar{\eta}^{1,2,3}_{\mu\nu}$. It is sometimes convenient to combine the metric and Kähler forms into one quaternion,

$$(g, \vec{\omega}) = g\sigma_0 + \vec{\omega}\cdot\vec{\sigma}. \tag{1.5.1}$$

This implies for $\mathbb{R}^4$,

$$\begin{aligned} (g, \vec{\omega}) &= d\bar{x} \otimes dx, \\ g &= ds^2 = (dx_\mu)^2, \\ \vec{\omega}\cdot\vec{\sigma} &= d\bar{x}\wedge dx = \bar{\eta}_{\mu\nu} dx_\mu \wedge dx_\nu = (2dx_0 \wedge d\vec{x} - d\vec{x}\wedge d\vec{x})\cdot\vec{\sigma} \quad . \end{aligned} \tag{1.5.2}$$

Here, $(d\vec{a}\wedge d\vec{b})^i = \varepsilon_{ijk} da^j \wedge db^k$. The 't Hooft matrices give a convenient splitting of the two-forms into selfdual and anti-selfdual combinations,

$$\eta^i_{\mu\nu} dx_\mu \wedge dx_\nu, \quad \bar{\eta}^i_{\mu\nu} dx_\mu \wedge dx_\nu. \tag{1.5.3}$$

The first occur in the description of the selfdual instanton curvatures whereas the second span the Kähler forms.

Many examples of hyperKähler manifolds emerge as hyperKähler quotients [47]. Consider a hyperKähler manifold $\mathcal{M}$ acted upon freely by a group G (with algebra $\mathfrak{g}$) of isometries. The metric g then satisfies

$$(L_X g)_{\mu\nu} \equiv X^\lambda \partial_\lambda g_{\mu\nu} + (\partial_\mu X^\lambda) g_{\lambda\nu} + (\partial_\nu X^\lambda) g_{\mu\lambda} = 0, \tag{1.5.4}$$

L denoting the Lie derivative and $X \in \mathfrak{g}$. When G preserves the complex structures,

$$(L_X\vec{\omega})_{\mu\nu} \equiv X^\lambda \partial_\lambda \vec{\omega}_{\mu\nu} + (\partial_\mu X^\lambda)\vec{\omega}_{\lambda\nu} + (\partial_\nu X^\lambda)\vec{\omega}_{\mu\lambda} = 0, \tag{1.5.5}$$

the isometries are called triholomorphic. The moment map $\vec{\mu} : \mathcal{M} \to \mathfrak{g}^* \otimes \mathbb{R}^3$ can be defined as $X_\alpha \vec{\omega}_{\alpha\nu} = \partial_\nu \vec{\mu}^X$.

The moment map originates from symplectic geometry, used in classical mechanics. There, ω denotes the symplectic form defined on the tangent bundle to $\mathbb{R}^3$, and $\omega = dx_i \wedge dp_i$. The moment map for translations in $\mathbb{R}^3$, defined by the vector field $X = X_i\partial_i$, is then given by the momentum, $\vec{\mu} = \vec{p}$. For rotations, the moment map is given by the angular momentum, $\vec{\mu} = \vec{x} \times \vec{p}$.

The manifold $\vec{\mu}^{-1}(\vec{\mathfrak{c}})/G$, with $\vec{\mathfrak{c}} \in \mathbb{R}^3 \otimes Z_\mathfrak{g}$ ($Z_\mathfrak{g}$ the centre of $\mathfrak{g}^*$) obtained by taking the quotient of the level set $\vec{\mu}^{-1}(\vec{\mathfrak{c}})$ by G is then hyperKähler itself [47]. Isometries commuting with G descend to the quotient. When they are also triholomorphic, this property is preserved.

In this thesis, most examples start with a manifold parameterised by a coordinate $y \in \mathbb{H}^N$, with flat metric and Kähler forms $dy^\dagger \otimes dy$, where $y^\dagger = \bar{y}^t$. The group acting triholomorphically usually consists of a product of $U(1)$s, related to the monopole phases or it is identical to some gauge group. The relevant example is provided by the moduli space of ADHM data in the construction of charge k instantons on $\mathbb{R}^4$ for gauge group $SU(n)$, to be discussed in chapter 3. The calorons will be constructed using an infinite-dimensional version of the ADHM construction and its moduli space can be obtained using an infinite dimensional hyperKähler quotient. In another example, the hyperKähler quotient will act on the caloron moduli space, which gives a monopole moduli space.

1.5.2 Metrics on moduli spaces

The metric on the moduli space $\mathcal{M}$ of selfdual connections on the manifold $M = \mathbb{R}^4/H$ is computed as the L_2 norm of its tangent vectors. These are gauge orthogonal variations of the connections with respect to their moduli. Specifically, Z_μ is tangent to the moduli space when it is a solution of the deformation equation (i.e. a variation of the selfduality equation) and the so-called gauge orthogonality condition requiring it to be a zero mode of the covariant derivative D_μ^{ad},

$$D^{\mathrm{ad}}_{[\mu} Z_{\nu]} = \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta} D^{\mathrm{ad}}_{[\alpha} Z_{\beta]}, \quad D^{\mathrm{ad}}_\mu(A) Z_\mu = 0. \tag{1.5.6}$$

Written in terms of quaternions, these equations are concisely expressed as

$$D^{\mathrm{ad}\dagger} Z = 0, \tag{1.5.7}$$

where $D^{\mathrm{ad}} = \sigma_\mu D^{\mathrm{ad}}_\mu$ and $Z = Z_\mu \sigma_\mu$. One then reads off the tangent space to admit three almost complex structures I, J, K acting as $-\mathbf{i}, -\mathbf{j}, -\mathbf{k}$ on the right. Metric and Kähler forms read

$$(g, \vec{\omega})_\mathcal{M}(Z, Z') = \frac{1}{4\pi^2} \int_M d_4x \mathrm{Tr}\left(Z^\dagger(x) Z'(x)\right), \tag{1.5.8}$$

where Z, Z' are any two tangent vectors. Geometrically the gauge orthogonality assures that the deformation has a vanishing projection along the gauge fibre, i.e. it is orthogonal with respect to the metric eq. (1.5.8) to all infinitesimal gauge transformations $\Phi^{\rm inf}$ of the vacuum, $\int_M d_4x Z_\mu D_\mu^{\rm ad}\Phi^{\rm inf} = 0$. This guarantees the deformation to be horizontal, i.e. it has only a component along the gauge independent moduli, cf. [8]. See also the discussion in section 1.5.3, where the gauge orthogonality condition is derived from Gauss's law. Gauge orthogonality of a variation $\delta_r A_\mu$ of the selfdual connection with respect to the $r^{\rm th}$ modulus can be achieved by applying an infinitesimal gauge transformation Φ^r,

$$Z_\mu^r = \delta_r A_\mu + D_\mu^{\rm ad}\Phi^r, \quad (D_\nu^{\rm ad})^2\Phi^r = -D_\mu^{\rm ad}\delta_r A_\mu, \tag{1.5.9}$$

implying for the metric

$$g = -\frac{1}{4\pi^2}\int_M d_4x \mathrm{Tr}(\delta A_\mu - D_\mu^{\rm ad}(D_\nu^{\rm ad})^{-2}D_\rho^{\rm ad}\delta A_\rho)^2. \tag{1.5.10}$$

The hyperKähler property of the moduli space follows formally from considering it as the infinite dimensional hyperKähler quotient of the space of general connections $\mathcal{A}$ by the triholomorphic action of the group of gauge transformations $\mathcal{G}$[3, 26]. The moment map is $\vec{\mu}_{\mathcal{G}}\cdot\vec{\sigma} = \bar{\eta}_{\mu\nu}F_{\mu\nu}/8\pi^2$, i.e. the projection of the connection on the anti-selfdual forms. Hence the zero set is formed by the space of selfdual solutions, which quotiented by $\mathcal{G}$ gives the moduli space. That this quotient is well defined follows from the invariance of the Kähler forms

$$\vec{\omega}_{rs}\cdot\vec{\sigma} = -\frac{1}{4\pi^2}\int_M d_4x\bar{\eta}_{\mu\nu}\mathrm{Tr}(\delta_r A_\mu \delta_s A_\nu), \tag{1.5.11}$$

under infinitesimal gauge transformations, which is seen by adding arbitrary $D_\mu^{\rm ad}\Phi^r$ to the deformations, using $D_\mu^{\rm ad}\delta_s A_\nu\bar{\eta}_{\mu\nu} = \delta_s(\bar{\eta}_{\mu\nu}F_{\mu\nu}) = 0$. For calorons the boundary condition is that their connections are periodic modulo a gauge transformation. This is consistent with complex structures acting as $\bar{\eta}_{\mu\nu}$. One therefore expects caloron moduli spaces to be hyperKähler. This hyperKähler property of instanton moduli spaces applies to BPS monopoles as well, as follows immediately when considering them S^1 invariant calorons.

1.5.3 The moduli space approximation for monopoles

The moduli space approximation for monopoles assumes that the motion of a system of many monopoles follows initially the geodesics on the multi-monopole moduli space [70], rigorously proven in [92] for the gauge group $SU(2)$. We follow [70, 98]. In this subsection we do not identify A_0 as the Higgs field, but keep A_0 and Φ separate. Like in section 1.4, we consider real time.

One of the Yang-Mills-Higgs equations, derived by varying A_0 in the action, is Gauss's law. It reads in the $A_0 = 0$ gauge

$$D_i\dot{A}_i + [\Phi, \dot{\Phi}] = 0, \tag{1.5.12}$$

the dot denoting a derivative with respect to time. It constrains the dynamics of the fields. Assuming that during the slow motion of the multi-monopole system the fields still form a multi-monopole configuration, we can write down

$$A_i(\vec{x},t) = {}^{[g(\vec{x},\gamma(t))]}A_i^{\rm BPS}(\vec{x},\gamma(t)), \quad \Phi(\vec{x},t) = {}^{[g(\vec{x},\gamma(t))]}\Phi^{\rm BPS}(\vec{x},\gamma(t)), \tag{1.5.13}$$

where $\gamma(t)$ denote the moduli of the monopole configuration $(\vec{A}^{\rm BPS}(\vec{x},\gamma), \Phi^{\rm BPS}(\vec{x},\gamma))$. The time derivatives then read

$$\begin{aligned}\dot{A}_i(\vec{x},t) &= \dot{\gamma}_r(t)(D_i^{\rm ad}\Phi^r + {}^{[g(\vec{x},\gamma(t))]}\frac{\partial}{\partial\gamma_r}A_i^{\rm BPS}(\vec{x},\gamma(t))) \equiv \dot{\gamma}_r(t)Z_i^r(\vec{x},\gamma(t)),\\ \dot{\Phi}(\vec{x},t) &= \dot{\gamma}_r(t)([\Phi,\Phi^r] + {}^{[g(\vec{x},\gamma(t))]}\frac{\partial}{\partial\gamma_r}\Phi^{\rm BPS}(\vec{x},\gamma(t))) \equiv \dot{\gamma}_r(t)Z_\Phi^r(\vec{x},\gamma(t)).\end{aligned} \tag{1.5.14}$$

Here, $\Phi^r = g(\vec{x},\gamma(t))\frac{\partial}{\partial\gamma_r}g^{-1}(\vec{x},\gamma(t))$ and $(Z_\Phi^r, \vec{Z}^r)$ forms a tangent vector to the monopole moduli space corresponding to the variation of the $r^{\rm th}$ modulus. Gauss's law implies,

$$D_i^{\rm ad}Z_i^r + [\Phi, Z_\Phi^r] = 0, \tag{1.5.15}$$

which is equivalent to the gauge orthogonality condition in eq. (1.5.6) when identifying A_0 with Φ. Inserting the time derivatives in the Yang-Mills-Higgs action in the gauge where $A_0 = 0$,

$$S = \int_M d_4x{\rm Tr}\left(-\dot{A}_i^2 - \dot{\Phi}^2 + B_i^2 + (D^{\rm ad}\Phi)^2\right), \tag{1.5.16}$$

gives an action describing the dynamics on the moduli space,

$$S = 4\pi^2\int dx_0\, g_{rs}\dot{\gamma}_r\dot{\gamma}_s - E^{\rm BPS}. \tag{1.5.17}$$

Here,

$$g_{rs} = \frac{1}{4\pi^2}\int_M d_3x{\rm Tr}\left(Z_i^{r\dagger}(\vec{x})Z_i^s(\vec{x}) + Z_\Phi^{r\dagger}(\vec{x})Z_\Phi^s(\vec{x})\right) \tag{1.5.18}$$

is the metric on the monopole moduli space, cf. eq. (1.5.8). The energy $E^{\rm BPS}$ is defined in eq. (1.4.2). Clearly, the metric on the monopole moduli space has entered the effective Lagrangian in its kinetic part. The second term in the rhs. of eq. (1.5.17) forms the potential part. It is constant as it is proportional to the energy of the static monopole system, which does not depend on the moduli. We finally note that the approximation becomes exact when $\dot{\gamma} \to 0$.

1.5.4 Relative moduli spaces

Moduli spaces of selfdual connections can usually be written as a product of the base space M, describing the centre of mass and the non-trivial relative moduli space $\mathcal{M}_{\rm rel}$,

$$\mathcal{M} = M \times \mathcal{M}_{\rm rel}. \tag{1.5.19}$$

In the metric this corresponds to a part describing the flat metric on the base space M and one for the relative or centered metric on $\mathcal{M}_{\mathrm{rel}}$, containing the nontrivial part. In the moduli space approximation, the motion on M describes the motion of the centre of mass, whereas the motion on $\mathcal{M}_{\mathrm{rel}}$ describes the internal motion of the multi-monopole system. Sometimes however, when we want to take limits, it will be preferable to work with the full metric on $\mathcal{M}$. In particular it will be worthwhile to consider the boundaries of caloron moduli spaces as monopole moduli spaces. This is realised by removing constituent monopoles from the caloron. An example of this procedure is given by the infinite scale monopole limit of the caloron with trivial holonomy, which can be seen as a massless monopole constituent being moved to infinity. When the constituent is removed from the caloron, what remains is the elementary BPS monopole.

1.6 This thesis

In this thesis we will be concerned with calorons. There are many reasons to study them. There is the obvious link with finite-temperature field theory. Moreover, they provide examples of selfdual connections over partially compactified four-manifolds. Most importantly, adjusting the size of the circle allows for a smooth interpolation between instantons and monopoles. This last aspect makes it obvious that the study of calorons adds to the understanding of both objects and the formalism to investigate them. They were used for this purpose in ref. [23]. The moduli spaces are interesting because of the hyperKähler property of the metric. The metrics of the calorons we will encounter also feature in the study of D-branes.

The main part of this thesis is devoted to the construction of caloron solutions, their physical characteristics, such as action densities and fermion zero-modes, and the geometry of their moduli spaces.

There are various methods to study instantons and monopoles. In this thesis we will be primarily concerned with the Nahm transformation. The Nahm transformation considers the selfdual connection from the point of view of Weyl fermions in its background. The Nahm transformation is thus like an inverse scattering method for solitons. It maps selfdual connections to selfdual connections on a dual manifold. From the mathematical point of view, the Nahm transformed connection is a connection on a vector bundle over the moduli space of flat connections. The ADHM construction may be considered a special case of the Nahm transformation. It will turn out that it is possible to extend the range of the ADHM construction to calorons, by relating the two approaches by Fourier transformation. Thus we will profit from the advantages of both methods.

Calorons were studied before. Explicit solutions, generalising the Harrington-Shepard solution to higher charge and gauge group were found within the 't Hooft Ansatz [42, 17]. These solutions do not exhaust the topological richness of connections on $\mathbb{R}^3 \times S^1$. As will be described in more detail in section 2.2, the topological characterisation of calorons is not only in terms of their instanton charge. Another topological label is given by the holonomy around S^1, measured at spatial infinity,

$$\mathcal{P}(\vec{x}) = P\,\exp(\int_0^{\mathcal{T}} A_0(\vec{x}, x_0)dx_0) \to \mathcal{P}_\infty, \quad \text{if } |\vec{x}| \to \infty. \tag{1.6.1}$$

Here, P stands for path-ordering and we used the periodic gauge, $A_\mu(x+\mathcal{T}) = A_\mu(x)$. The solutions within the 't Hooft Ansatz all have trivial holonomy ($\mathcal{P}_\infty = 1$). In this thesis we study calorons with arbitrary holonomy. It is this extension to non-trivial holonomy for which the full machinery of the ADHM and Nahm formalisms is necessary and which gives the new and unexpected results. The most intriguing result is the fact that a charge one $SU(n)$ caloron consists of n elementary BPS monopoles. This is demonstrated in the picture on the cover of this thesis, where we see three monopoles together forming the $SU(3)$ caloron. (The figure on page 114 at the end of this thesis shows a *genuine* $SU(73)$ solution.)

The outline of this thesis is as follows. In chapter 2 we will discuss the Nahm transformation. In chapter 3 we summarise the ADHM construction and the multi-instanton calculus within this formalism, used to obtain physical relevant quantities. This calculus will be used to derive the properties of the caloron. The charge one caloron $SU(2)$ with arbitrary holonomy is presented in chapter 4. We use the ADHM construction with special periodicity constraints to rederive the Nahm formalism and to construct the caloron solution. Crucial is that we treat a caloron as a multi-instanton of infinite charge. It is made as an infinite array of single instantons. Going from one instanton to another then gets accompanied by a gauge rotation, thus giving rise to non-trivial holonomy. This generalises the Harrington-Shepard caloron which has trivial holonomy. Moreover, we study its moduli space and use the metric properties of the ADHM construction to compute the caloron metric. The action density of the charge one $SU(n)$ caloron and the energy density of a related $SU(n)$ monopole obtained as its limit are computed in chapter 5, where the $SU(2)$ techniques are generalised. Chapter 6 is devoted to the computation of the hyper-Kähler metric for the moduli space of these objects. Again using the adaptation of the ADHM calculus to periodic objects, we compute in chapter 7 the Weyl fermion zero-mode density in the background of the $SU(n)$ charge one caloron. This will be used to perform the explicit Nahm transformation for the $SU(n)$ charge one caloron. Chapter 8 contains a summary and a discussion of the results.

2 The Nahm transformation

In this chapter we discuss the Nahm transformation for selfdual connections. Initially, it arose as a modification by Nahm of the ADHM construction [1] for instantons on $\mathbb{R}^4$ to study BPS monopoles [75, 76]. Later developments culminated in the Nahm duality transformation for selfdual connections on generalised tori. It is also known as the Mukai transformation [73] between holomorphic vector bundles and maps selfdual fields on a four manifold to selfdual fields on dual four manifolds. It therefore forms a powerful tool for studying these solutions. Also calorons can and will be treated along these lines [77].

2.1 General

Consider a $U(n)$ bundle E with selfdual gauge connection A_μ on a four manifold $M = \mathbb{R}^4/H$, with instanton number k. Here H is a subgroup of translation symmetries under which the physics is invariant. When H is a four dimensional lattice, M will be the four torus [14]. Other four manifolds are obtained by taking appropriate limits [6]. We demand the gauge potential to be invariant modulo gauge transformations under the action of H.

The essential ingredient in Nahm's construction [76] is to add a curvature free abelian connection, $-2\pi i z_\mu dx_\mu$, to the gauge field and to study the Weyl operator

$$D_z(A) = \sigma_\mu D_z^\mu(A), \quad D_z^\dagger(A) = -\bar{\sigma}_\mu D_z^\mu(A), \quad D_z^\mu(A) = \partial_\mu + A_\mu - 2\pi i z_\mu. \qquad (2.1.1)$$

As compared to usual conventions [14, 6], we replaced z by $-z$ to facilitate matching with the ADHM construction. When A is without flat factors (meaning that the vector bundle E does not split in $E' \oplus L$ for any flat line bundle L), then $D_z(A)$ will have a trivial kernel [26]. For such gauge fields $G_z(x,y) = (D_z^\dagger(A)D_z(A))^{-1}$ is well-defined. The index theorem [4, 16, 13] shows that there are k normalisable zero modes of the Weyl operator $D_z^\dagger(A)$, for each value of $z \in \hat{M} = \mathbb{R}^4/\hat{H}$, $\hat{H} = \{z \in \mathbb{R}^4 | z \cdot y \in \mathbb{Z}, \forall y \in H\}$, cf. ref. [6]. We can therefore define a $U(k)$ connection on the space $\hat{M}$,

$$\hat{A}_\mu^{\hat{m}\hat{m}'}(z) = \int_M dx\ \Psi_z^{\hat{m}\dagger}(x) \frac{\partial}{\partial z_\mu} \Psi_z^{\hat{m}'}(x), \qquad (2.1.2)$$

where $\Psi_z^{\hat{m}}(x)$, $\hat{m} = 1, \ldots, k$ form an orthonormal basis for the Nahm bundle $\hat{E}$ of fermionic zero modes. This is called the Nahm transformed connection.

The Weitzenböck identity [26] states

$$D_z^\dagger(A)D_z(A) = -(D_z^\mu(A)D_z^\mu(A) + \tfrac{1}{2}\bar{\eta}_{\mu\nu}F^{\mu\nu}(x)), \qquad (2.1.3)$$

where $\bar{\eta}_{\mu\nu}$ is the anti-selfdual quaternionic 't Hooft tensor defined in section 1.3. As F is selfdual, the second term will vanish, and hence $D_z^\dagger(A)D_z(A)$ and $G_z(x,y)$

commute with the quaternions. This has profound consequences for the curvature associated to the Nahm connection. One finds [6]

$$\hat{F}_{\mu\nu}(z) = 8\pi^2 \int_{M\times M} dxdy\, \Psi_z^\dagger(x) G_z(x,y)\eta_{\mu\nu}\Psi_z(y) \tag{2.1.4}$$
$$-4\pi i \int_{\partial M\times M} dS_\lambda(x)\, dy\, \frac{\partial}{\partial z_{[\mu}}\Psi_z^\dagger(x)\sigma_\lambda\bar{\sigma}_{\nu]} G_z(x,y)\Psi_z(y),$$

where Ψ_z denotes the matrix with the zero modes $\Psi_z^{\hat{m}}$ as columns. Here we used that G_z commutes with the quaternions. The first term is clearly selfdual due to the self-duality of $\eta_{\mu\nu}$. The boundary term shows possible deviations from selfduality, which occur at the points z for which the zero modes do not decay exponentially in the non-compact directions. In these directions the connection necessarily approaches a vacuum for the action to be finite. These vacua are labelled by the eigenvalues of the Polyakov loops $\mathcal{P}_i = P\exp\int_{C_i} A_\mu dx_\mu$ along the circles C_i corresponding to the compact directions. In the case that $e^{2\pi i z_i}$ becomes equal to one of these eigenvalues, the component of $A_\mu - 2\pi i z_\mu$ along C_i in eq. (2.1.1) will develop a zero eigenvalue when approaching infinity. This gives rise to a surviving boundary term in eq. (2.1.4) and as a result a deviation from selfduality, precisely for these specific points. As the deviations occur in single points, they will be expressible in terms of delta functions. Hence, $\hat{F}_{\mu\nu}$ is selfdual almost everywhere. For the non-compact directions μ, the z_μ dependence of $\Psi_z(x)$ is a plane wave factor, and hence $\hat{A}$ is z_μ independent. Note that the $U(k)$ symmetry in the space of zero modes associated to A is mapped onto a gauge symmetry for $\hat{A}$. On the other hand, gauge transformations of A leave $\hat{A}$ unchanged.

For the four torus T^4 the boundary terms are absent and instantons are mapped onto instantons. It can be shown, using the family index theorem, that under this Nahm transformation a $U(n)$ connection with topological charge k is mapped onto a $U(k)$ connection with topological charge n. The Nahm transformation on T^4 squares to the identity [14]. More explicitly, the dual Weyl operator $\hat{D}_x^\dagger(\hat{A}) = -\bar{\sigma}_\mu(\hat{\partial}_\mu + \hat{A}_\mu - 2\pi i x_\mu)$ has n zero modes

$$\hat{D}_x^\dagger(\hat{A})\hat{\Psi}_x^m(z) = 0, \quad m = 1,\ldots,n, \tag{2.1.5}$$

in terms of which the original connection $A_\mu(x)$ is reconstructed

$$A_\mu^{mm'}(x) = \int_{\hat{M}} dz\ \hat{\Psi}_x^{\dagger m}(z)\frac{\partial}{\partial x_\mu}\hat{\Psi}_x^{m'}(z). \tag{2.1.6}$$

This suggests to use the Nahm transformation in the construction of selfdual connections on modified tori, in situations when one can explicitly find the dual connection $\hat{A}$. The Nahm transformed connection and boundary terms are then referred to as Nahm data. Generally one expects such a construction to be feasible when the Nahm transformed bundle is simpler than the original, in particular when $\hat{M}$ is of lower dimension than M. In these cases one sometimes can solve $\hat{A}_\mu$ from the selfduality equation for $\hat{F}_{\mu\nu}$ in the presence of boundary terms. Another simplification arises for the case of topological charge $k = 1$, since in that case the Nahm connection $\hat{A}$ is abelian. Because of the boundary terms, the second Nahm transformation will have to be modified by properly handling the singularities. As will be discussed in detail in section 2.3.1, the extreme case is $M = \mathbb{R}^4$, $H = 0$. There the dual manifold $\hat{M}$ is just a point and the pair formed by the dual Weyl operator and singularities reduce to matrices which precisely give the ADHM data [22, 77, 26, 6]. The Nahm transformation on $\mathbb{R}^4/H$ thus encompasses the ADHM construction as discussed in section 2.3.1 and in more detail in chapter 3. The Nahm transformation for calorons and monopoles is discussed in the next section, whereas the explicit Nahm construction of the elementary BPS monopole is given in section 2.3.2.

There are two further properties of the Nahm transformation on T^4, [14]. The first is a relation between the zero-modes of D and those of $\hat{D}$ via the Green's function G_x. The second, related to the first, is a connection between the gauge zero-modes in the background of $\hat{A}_\mu$ and those of A_μ which can be used to prove the preservation of the metric and hyperKähler structure under the Nahm transformation. This isometric property states that the metric on the moduli space $\mathcal{M}$ of selfdual connections on M is identical to that on the moduli space $\hat{\mathcal{M}}$ of selfdual connections on the dual space $\hat{M}$. The calculus to prove these statements is not unlike that needed in chapter 3 where the fermion zero-modes and moduli space metric for instantons on $\mathbb{R}^4$ are calculated within the ADHM formalism.

2.2 The Nahm transformation for calorons and monopoles

We will now consider the Nahm transformation for calorons (using the classical scale invariance of the selfduality equations, we can choose the period $\mathcal{T} = 1$ such that $H = \mathbb{Z}$) and monopoles in the BPS limit ($H = \mathbb{R}$). For the latter, A_0 is interpreted as the Higgs field. Thus we can unify these two cases by considering them as connections on $\mathbb{R}^3 \times S^1$. The connections for $\mathbb{R}^3 \times S^1$ are topologically classified according to their behaviour at the boundary, $S^2 \times S^1$. We give a short summary of the classification presented in ref. [41].

For the action to be finite, it is necessary that the connections go to a vacuum at spatial infinity. Generally, gauge vacua are labelled by the conjugacy classes of representations of maps of the first homotopy group to the gauge group [26]. For $S^2 \times S^1$, we can characterise each vacuum by a gauge equivalence class of an element of the gauge group. Using a gauge transformation, this element can be chosen diagonal. The vacuum at infinity is related to the holonomy along the S^1 (or Polyakov loop),

$$\mathcal{P}(\vec{x}) = P\,\exp\Big(\int_0^{\mathcal{T}} A_0(\vec{x}, x_0)dx_0\Big), \tag{2.2.1}$$

in the periodic gauge. The difference of a closed Wilson loop evaluated along two curves C and C' is related with the flux through the surface swept out by the curves interpolating between C and C'. Hence, at spatial infinity where the curvature vanishes, a small deformation of the path C around which the holonomy is measured does not influence $\mathcal{P}(\vec{x})$. Only the homotopy of C is important, and the eigenvalues of the holonomy at spatial infinity are topological invariants. Therefore, the gauge holonomy at infinity is diagonal up to an $\hat{x}$ dependent gauge transformation V

$$\lim_{|\vec{x}|\to\infty} \mathcal{P}(\vec{x}) = \mathcal{P}_\infty = V\mathcal{P}^0_\infty V^{-1}, \quad \mathcal{P}^0_\infty = \exp[2\pi i \mathrm{diag}(\mu_1,\ldots,\mu_n)]. \tag{2.2.2}$$

The eigenvalues can be ordered such that

$$\mu_1 < \ldots < \mu_n < \mu_{n+1} \equiv \mu_1 + 1, \quad \sum_{m=1}^{n} \mu_m = 0, \tag{2.2.3}$$

using the gauge symmetry and assuming maximal symmetry breaking for the moment. For later use, we define $\nu_m = \mu_{m+1} - \mu_m$, related to the mass of the m^{th} constituent monopole. Asymptotically,

$$A_0 = 2\pi i\, \mathrm{diag}(\mu_1,\cdots,\mu_n) - i\,\mathrm{diag}(k_1,\cdots,k_n)/(2r) + \mathcal{O}(r^{-2}), \quad \sum_i k_i = 0, \tag{2.2.4}$$

up to the gauge transformation $V(\hat{x})$ that induces a map from S^2 to the factor group $SU(n)/H_\infty$, with H_∞ the isotropy group of $\exp[2\pi i\,\mathrm{diag}(\mu_1,\cdots,\mu_n)]$. For $SU(n)$ these maps $V(\hat{x}) \to SU(n)/H_\infty$ are classified according to the fundamental group of H_∞. Generically, H_∞ consists of several $U(1)$ and $SU(N)$, $N > 1$ subgroups. Each $U(1)$ gives rise to a monopole winding number, related to the integers k_i. The enhanced residual gauge symmetry described by the $SU(N)$ subgroups arises when there is non-maximal symmetry breaking, $\nu_m = \mu_{m+1} - \mu_m = 0$ for some value(s) of m, giving rise to massless constituent monopoles. The non-trivial value of $\mathcal{P}_\infty$ breaks the gauge symmetry. This makes calorons very similar to BPS monopoles, [75, 76] which fit in the above classification as S^1 invariant selfdual connections, classified according to the magnetic charges $(m_1,\ldots,m_{n-1})$ [53], where $m_i = k_1 + \ldots + k_i$.

The other topological quantum number is related to the homotopy class of the map $\partial M = S^2 \times S^1 \to SU(n)$ which occurs in the gauge transformation connecting the behaviour near the origin to that at infinity, which is classified by the instanton number $k \in \pi_3(SU(n)) = \mathbb{Z}$. Gauge connections on $\mathbb{R}^3 \times S^1$ are therefore classified by μ_i, m_i and k. The $SU(n)$ calorons studied in this thesis have no net magnetic charges, $m_i = 0$, and their only nontrivial topological labels are the instanton number k and the eigenvalues μ_m of the holonomy.

In the Nahm transformation for monopoles and calorons the $\vec{z}$ dependence of the fermionic zero modes is that of a plane wave, $\Psi_{z_0,\vec{z}}(x) = e^{2\pi i\vec{x}\cdot\vec{z}}\Psi_{z_0,\vec{0}}(x)$ and one finds $\hat{A}$ to be $\vec{z}$ independent. Therefore, the dual manifold is one dimensional with coordinate $z_0 \equiv z$. The rank of the Nahm transformed connection is given by the appropriate index theorems. For periodic instanton with no net magnetic charges the fermionic zero modes are associated to the instanton winding number k in the usual way. Hence, the rank of the Nahm transformed gauge potential is k. For non-zero m_i there are additional zero-modes associated with the monopole content. The situation for the extreme case, $k = 0$, a static BPS monopole, is that the dimension of the space of fermionic zero modes depends on z: for $z \in (\mu_i, \mu_{i+1})$ the dimension is given by m_i and for z outside $[\mu_1, \mu_n]$ the dimensions is zero according to the Callias-Bott-Seeley theorem [16, 13]. Jumps in the rank of the Nahm transformed connection occur exactly where $z = \mu_i$, again according to the Callias-Bott-Seeley theorem [16, 13] (the situation of non-maximal symmetry breaking, where two or more μ_i coincide, is more involved).

Therefore, for calorons and monopoles, the dual space $\hat{M}$ is an interval on the real line. For monopoles, this interval is $[\mu_1, \mu_n]$, when we order $\mu_i \leq \mu_{i+1}$. So, for monopoles, $\hat{M}$ is not the entire real line as one might expect. For calorons z is the coordinate on the dual circle and $\hat{A}$ is periodic, $z = \mu_1$ and $z = \mu_1 + 1$ are identified. The Nahm transformed curvature reduces to

$$\hat{F}_{0i}(z) = \frac{d}{dz}\hat{A}_i + [\hat{A}_0, \hat{A}_i], \quad \hat{F}_{ij}(z) = [\hat{A}_i, \hat{A}_j]. \tag{2.2.5}$$

Using the selfduality of the first term in eq. (2.1.4), and the fact that the second term of this equation is zero almost everywhere, except for possible delta function singularities at $z = \mu_i$ corresponding to the holonomy, one finds

$$\frac{d}{dz}\hat{A}_i + [\hat{A}_0, \hat{A}_i] - \tfrac{1}{2}\varepsilon_{ijk}[\hat{A}_j, \hat{A}_k] = \sum_p \beta_i^p \delta(z - \mu_p). \tag{2.2.6}$$

These are the celebrated Nahm equations, extended with source terms β_i^p describing the matching conditions. There are additional boundary conditions at $z = \mu_i$ if the dimensionality of the matrices changes [54]. For monopoles, there are no matching data at the endpoints $z = \mu_1, \mu_n$. The field construction in the Nahm formalism from the Nahm data to the actual gauge potentials for monopoles and calorons can be found in [76, 54, 32]. A special case for monopoles is considered in the appendix to chapter 6. Any $\hat{A}_\mu(z)$ obeying the Nahm equation of the right dimensionalities and singularity structure gives rise to a BPS monopole or caloron. For monopoles this is generally proven using twistor methods [76, 46], but for $SU(2)$ monopoles there exist direct proofs of the equivalence, without an intermediate twistor step [78]. For calorons the construction was formulated in ref. [77] and the twistor method for these periodic instantons was given in ref. [32], but a relation with existence theorems or a full circle reciprocity proof as it exists for monopoles and for instantons on $\mathbb{R}^4$ and T^4 seems not to be present.

For the $k = 1$ $SU(n)$ caloron, $\hat{A}_\mu$ is a $U(1)$ connection on the circle with n singularities corresponding to the holonomy. For $SU(2)$ this holonomy is given by a unit quaternion, $e^{2\pi i\vec{\omega}\cdot\vec{\tau}}$, and there are two singularities μ_1, μ_2 with $\mu_2 = -\mu_1 = \omega$ and $\omega \equiv |\vec{\omega}| \in [0, \frac{1}{2}]$. For the charge one caloron, the magnetic components of $\hat{F}$ vanish and hence non-zero values and singularities are only assumed by the electric components $\hat{F}_{0i}$. By the Nahm equations $\hat{A}$ is forced to be piecewise constant. The explicit ADHM construction for these calorons will be given in chapters 4 and 5 and we will see among other things that all aspects suggested in ref. [77] arise automatically.

2.3 Examples

In this section we consider two famous examples. The first is a presentation of the ADHM construction of multi-instantons from the perspective of the Nahm transformation. The second is Nahm's original formalism for the BPS monopole.

2.3.1 The ADHM construction

In this section we review [22, 77, 6] how the Nahm transformation encompasses the ADHM construction for instantons on $\mathbb{R}^4$. For these solutions, all z_μ dependence of the zero-modes of the Weyl operator is carried by a plane wave factor (all directions are noncompact)

$$\Psi_z(x) = e^{2\pi i z_\mu x_\mu}\Psi(x). \tag{2.3.1}$$

These plane wave factors drop out in the expressions for the connection (2.1.2) and curvature (2.1.4) and therefore, the dual manifold $\hat{M}$ is just a point. As a result, the dual Weyl operator $\hat{D}_x = \sigma_\mu(\hat{A}_\mu - 2\pi i x_\mu)$ is algebraic. Here, the $\hat{A}_\mu$ are skewHermitean $k\times k$ matrices.

Asymptotically, the instanton gauge potential approaches a pure gauge transformation at infinity, $|x|\to\infty$,

$$A_\mu \to g(x)\partial_\mu g^{-1}(x), \tag{2.3.2}$$

where g has winding number k. The k fermionic zero-modes in the instanton background are then given by

$$\Psi^{\hat{m}}_{m,I}(x) = \frac{x_\mu\sigma^\mu_{IJ}}{\pi|x|^4}g_{mm'}(x)\alpha^{\hat{m}}_{m'J} + \mathcal{O}(|x|^{-4}),\quad I=1,2,\quad m=1,\ldots n,\quad \hat{m}=1,\ldots,k. \tag{2.3.3}$$

I denotes the spinor index. That the zero-modes indeed assume this form at infinity follows from $D_\mu\cdot = g\partial_\mu(g^{-1}\cdot)$ asymptotically and the identity $\sigma^\dagger_\mu\partial_\mu(x|x|^{-4}) = 0,\quad |x|\neq 0$. The matrix α describes the various orientations in spin and colour space. π is a normalisation factor. Using $G^{-1}(x)\cdot = -D^2_\mu\cdot = -g\partial^2_\mu(g\cdot)$ at infinity and $\partial^2_\mu(x|x|^{-2}) = -4x|x|^{-4}$, we also find

$$(G\Psi)^{\hat{m}}_{m,I}(x) = \frac{x_\mu\sigma^\mu_{IJ}}{4\pi|x|^2}g_{mm'}(x)\alpha^{\hat{m}}_{m'J} + \mathcal{O}(|x|^{-3}). \tag{2.3.4}$$

The boundary term in eq. (2.1.4) is now evaluated using these asymptotics, in combination with quaternion relations like $\sigma_\alpha\sigma_\mu\sigma_\alpha = -2\sigma^\dagger_\mu$ and the fact that

$$\oint d\Omega_3|x|^{-4}x_\lambda x_\alpha x_\beta x_\mu = \frac{\pi^2}{12}(\delta_{\alpha\mu}\delta_{\beta\lambda}+\delta_{\alpha\lambda}\delta_{\beta\mu}+\delta_{\alpha\beta}\delta_{\lambda\mu})$$

for the three-sphere. The Nahm transformed curvature then reads

$$\hat{F}_{\mu\nu} = 8\pi^2 \langle \Psi_z | G_z \eta_{\mu\nu} | \Psi_z \rangle + \pi^2 \alpha^\dagger \bar{\eta}_{\mu\nu} \alpha. \tag{2.3.5}$$

Due to the second term, this curvature is no longer selfdual. As a consequence, in the -now algebraic- second Nahm transformation $\hat{D}_x^\dagger \hat{D}_x$ no longer commutes with the quaternions, nor will the resulting connection be selfdual. To repair this, we extend the Weyl operator to a $(2k+n) \times 2k$ matrix

$$\hat{D}_x^{\text{ext}} = \begin{pmatrix} 2\pi\tilde{\alpha} \\ \hat{D}_x \end{pmatrix}. \tag{2.3.6}$$

The tilde denotes transposing with respect to the spinor index. We then obtain, using the Weitzenböck identity (2.1.3),

$$\begin{aligned} \hat{D}_x^{\text{ext}\,\dagger} \hat{D}_x^{\text{ext}} &= 4\pi^2 \tilde{\alpha}^\dagger \tilde{\alpha} - (\hat{A}_\mu - 2\pi i x_\mu)^2 - \tfrac{1}{2} \bar{\eta}_{\mu\nu} \hat{F}_{\mu\nu} \\ &= 4\pi^2 \alpha^\dagger \otimes \alpha - (\hat{A}_\mu - 2\pi i x_\mu)^2 + \tfrac{1}{2} \pi^2 \bar{\eta}^a_{\mu\nu} \bar{\eta}^b_{\mu\nu} \tau^a \alpha^\dagger \tau^b \alpha \\ &= 4\pi^2 \alpha^\dagger \otimes \alpha - (\hat{A}_\mu - 2\pi i x_\mu)^2 - 2\pi^2 \tau^a \alpha^\dagger \tau^a \alpha, \end{aligned} \tag{2.3.7}$$

indicating by $\otimes$ that a tensor product rather than an inner product is taken with respect to the spinor indices. The last term is rewritten by applying the completeness property of the basic quaternions $\tau^a_{IJ} \tau^a_{KL} + \delta_{IJ} \delta_{KL} = 2\delta_{IL} \delta_{JK}$, yielding

$$\tau^a \alpha^\dagger \tau^a \alpha = -\text{tr}_2(\alpha^\dagger \otimes \alpha) + 2\alpha^\dagger \otimes \alpha. \tag{2.3.8}$$

Here tr_2 denotes the trace corresponding to quaternions or spinor indices. That the extension indeed repairs the commutation of $\hat{D}_x^\dagger \hat{D}_x$ with the quaternions now follows immediately: the Green's function in the extended second Nahm transformation commutes with the quaternions,

$$(\hat{G}_x^{\text{ext}})^{-1} = -(\hat{A}_\mu - 2\pi i x_\mu)^2 + 2\pi^2 \text{tr}_2 \alpha^\dagger \otimes \alpha. \tag{2.3.9}$$

The matrix $\hat{D}_x^{\text{ext}}$ will be central to the ADHM construction, to be described at length in chapter 3. There the explicit field construction will be explained, in which the extension α features as a source term

$$\hat{D}_x^{\text{ext}\,\dagger} \hat{\psi}_x^{\text{ext}} = 0, \quad \hat{\psi}_x^{\text{ext}} = \begin{pmatrix} s_x \\ \hat{\psi}_x \end{pmatrix} \in \mathbb{C}^{n+2k,n}, \quad \hat{D}_x^\dagger \hat{\psi}_x = -2\pi \tilde{\alpha}^\dagger s_x, \quad \hat{\psi}_x^{\text{ext}\,\dagger} \hat{\psi}_x^{\text{ext}} = 1_n \tag{2.3.10}$$

and the gauge field is given by

$$A_\mu(x) = \hat{\psi}_x^{\text{ext}\dagger} \partial_\mu \hat{\psi}_x^{\text{ext}}. \tag{2.3.11}$$

2.3.2 The elementary BPS monopole

For the elementary BPS monopole, the Nahm transformation is easy to perform. As the gauge connection for the elementary BPS monopole is rotational invariant up to a compensating gauge transformation, the same holds true for the Weyl fermion zero-modes in its background. Therefore, choosing the origin as the centre of symmetry, the spatial components $\hat{A}_i$ of the Nahm transformed connection vanish, cf. eq. (2.1.2). Choosing a particular linear combinations for these zero-modes translates into a particular gauge choice for the Nahm transformed connection. Thus one also can achieve $\hat{A}_0 = 0$. We will now prove that indeed, the elementary BPS monopole can be reconstructed from $\hat{A}_\mu = 0$ using the second Nahm transformation. The dual zero-modes are contained in the 2×2 matrix $\hat{\psi}$ which is the solution of

$$(\frac{d}{dz} - 2\pi i x_\mu \sigma_\mu^\dagger)\hat{\psi}_x(z) = 0. \tag{2.3.12}$$

Therefore

$$\hat{\psi}_x(z) \propto e^{2\pi i x_0 z - 2\pi \vec{x}\cdot\vec{\tau} z} = e^{2\pi i x_0 z}(\cosh 2\pi|\vec{x}|z - \hat{x}\cdot\vec{\tau}\sinh 2\pi|\vec{x}|z) \tag{2.3.13}$$

By the Callias index theorem, the interval on which the Nahm construction is performed is restricted to $[-\nu/2, \nu/2]$, ν being related to the Higgs vacuum expectation value (cf. section 1.4). The zero-modes of the dual Weyl operator are therefore given by

$$\sqrt{\frac{2\pi|\vec{x}|}{\sinh 2\pi\nu|\vec{x}|}} e^{2\pi i x_0 z}(\cosh 2\pi|\vec{x}|z - \hat{x}\cdot\vec{\tau}\sinh 2\pi|\vec{x}|z), \tag{2.3.14}$$

whose norm squared integrated over $[-\nu/2, \nu/2]$ yields one. The monopole gauge potential and Higgs field is now retrieved from eq. (2.1.6),

$$A_\mu(x) = \int_{[-\nu/2,\nu/2]} dz \hat{\psi}_x^\dagger(z) \frac{\partial}{\partial x_\mu} \hat{\psi}_x(z). \tag{2.3.15}$$

This results in the monopole connection

$$\begin{aligned} A_0 &= -\frac{i}{2}\hat{x}\cdot\vec{\tau}\left(2\pi\nu\coth 2\pi\nu|\vec{x}| - \frac{1}{|\vec{x}|}\right), \\ A_i &= -\frac{i}{2}\varepsilon_{ijk}\frac{x_j\tau_k}{|\vec{x}|^2}\left(1 - \frac{2\pi\nu|\vec{x}|}{\sinh 2\pi\nu|\vec{x}|}\right), \end{aligned} \tag{2.3.16}$$

which is of the same form as eq. (1.4.6), up to the sign in A_0. This is related to the fact that the selfduality equation for monopoles with $A_0 = \Phi$ translates into $B_i = -D_i\Phi$, i.e. the Nahm formalism actually gives an antimonopole in the conventions chosen.

It follows immediately that the Nahm data for a monopole located at $\vec{x} = \vec{y}$ is given by $\hat{A}_i = 2\pi i y_i$, cf. eq. (2.1.2).

3 The ADHM construction

In this section we review the ADHM calculus for charge k instantons on $\mathbb{R}^4$ with gauge group $SU(n)$. It allows one to compute many quantities of physical relevance in explicit form.

3.1 Yang-Mills instantons on $\mathbb{R}^4$

The ADHM formalism for $SU(n)$ charge k instantons [1, 22] employs a k dimensional vector $\lambda = (\lambda_1, \ldots, \lambda_k)$, where $\lambda_i^\dagger$ is a two-component spinor in the $\bar{n}$ representation of $SU(n)$. Alternatively, λ can be seen as an $n \times 2k$ complex matrix. In addition one has four complex hermitian $k \times k$ matrices B_μ, combined into a $2k \times 2k$ complex matrix $B = \sigma_\mu \otimes B_\mu$. With abuse of notation, we often write $B = B_\mu \sigma_\mu$. Together λ and B constitute the $(n + 2k) \times 2k$ dimensional matrices

$$\Delta = \begin{pmatrix} \lambda \\ B \end{pmatrix} \tag{3.1.1}$$

and $\Delta(x)$, to which is associated a complex $(n+2k) \times n$ dimensional normalised zero mode vector $v(x)$,

$$\Delta(x) = \begin{pmatrix} \lambda \\ B(x) \end{pmatrix}, \quad B(x) = B - x, \quad \Delta^\dagger(x) v(x) = 0, \quad v^\dagger(x) v(x) = 1_n. \tag{3.1.2}$$

Here the quaternion $x = x_\mu \sigma_\mu$ denotes the position (a $k \times k$ unit matrix is implicit). Picking a particular gauge, we can solve $v(x)$ explicitly in terms of the ADHM data by

$$v(x) = \begin{pmatrix} -1_n \\ u(x) \end{pmatrix} \phi^{-\frac{1}{2}}, \quad u(x) = (B^\dagger - x^\dagger)^{-1} \lambda^\dagger, \quad \phi(x) = 1_n + u^\dagger(x) u(x). \tag{3.1.3}$$

As $\phi(x)$ is an $n \times n$ positive hermitian matrix, its square root $\phi^{\frac{1}{2}}(x)$ is well-defined. The gauge field is given by

$$A_\mu(x) = v^\dagger(x) \partial_\mu v(x) = \phi^{-\frac{1}{2}}(x)(u^\dagger(x) \partial_\mu u(x)) \phi^{-\frac{1}{2}}(x) + \phi^{\frac{1}{2}}(x) \partial_\mu \phi^{-\frac{1}{2}}(x). \tag{3.1.4}$$

Local gauge transformations arise from the $U(n)$ symmetry $v(x) \to v(x) g(x)$ in the solution space of eq. (3.1.2).

For $A_\mu(x)$ to be a selfdual connection, $\Delta(x)$ has to satisfy the quadratic ADHM constraint, which states that $\Delta^\dagger(x)\Delta(x) = B^\dagger(x) B(x) + \lambda^\dagger \lambda$ (considered as $k \times k$ complex quaternionic matrix) has to commute with the quaternions, or equivalently

$$\Delta^\dagger(x) \Delta(x) = \sigma_0 \otimes f_x^{-1}, \tag{3.1.5}$$

where f_x is defined as a hermitian $k \times k$ Green's function. It is sufficient for this to hold at $x = 0$, i.e., $\Delta^\dagger\Delta = B^\dagger B + \lambda^\dagger\lambda$ must be real quaternionic and invertible.

The second constraint is that $(B - x)$ should have a trivial kernel, except for k values of x, where $v(x)$ and, as a consequence, $A_\mu(x)$ are singular. These points can be shown to be gauge singularities. This implements the non-triviality of the bundle and reflects the topological nature of these solutions.

That the connection is indeed selfdual is best seen from using $F = dA + A \wedge A$. With $A = v^\dagger(x)dv(x)$ one finds

$$\begin{aligned} F &= dv^\dagger(x) \wedge dv(x) - dv^\dagger(x)v(x) \wedge v^\dagger(x)dv(x) \\ &= dv^\dagger(x) \wedge (1 - v(x) \otimes v^\dagger(x))dv(x). \end{aligned} \tag{3.1.6}$$

As $1 - v(x) \otimes v^\dagger(x) \equiv 1 - P$ is the projection on the orthogonal complement of the kernel of $\Delta^\dagger(x)$ (since $\Delta^\dagger(x)v(x) = 0$), we can use that $1 - v(x) \otimes v^\dagger(x) = \Delta(x)f_x\Delta^\dagger(x)$. Substituting this in the expression for F and using that $\Delta^\dagger(x)dv(x) = -dx_\mu \frac{\partial\Delta^\dagger(x)}{\partial x_\mu}v(x) = dx^\dagger(b^\dagger v(x))$ ($dx \equiv dx_\mu\sigma_\mu$, $b^\dagger = (0, 1_k)$, $b^\dagger v(x) \equiv \phi^{-\frac{1}{2}}(x)u(x)$), we find [2]

$$F = (v^\dagger(x)b)dx \wedge f_x dx^\dagger(b^\dagger v(x)). \tag{3.1.7}$$

The crucial observation is now that the quadratic ADHM constraint implies that f_x commutes with the quaternions and that $dx \wedge dx^\dagger = \eta_{\mu\nu}dx_\mu \wedge dx_\nu$, much like in the Nahm transformation where the Weitzenböck identity, eq. (2.1.3), guarantees that $D_z^\dagger(A)D_z(A)$ commutes with the quaternions. We thus find [22]

$$F_{\mu\nu}(x) = 2\phi^{-\frac{1}{2}}(x)u^\dagger(x)\eta_{\mu\nu}f_x u(x)\phi^{-\frac{1}{2}}(x), \tag{3.1.8}$$

which is selfdual due to the selfduality of $\eta_{\mu\nu}$.

The quadratic constraint can be reformulated as $\Im(\Delta^\dagger(x)\Delta(x)) = 0$, where $\Im W = \Im\sigma_\mu \otimes W_\mu = \sigma_i \otimes W_i$ ($\Re W = \Re\sigma_\mu \otimes W_\mu = W_0 = \frac{1}{2}\mathrm{tr}_2 W$), and one obtains

$$\bar\eta_{\mu\nu} \otimes B_\mu B_\nu + \tfrac{1}{2}\tau_a \otimes \mathrm{tr}_2(\tau_a\lambda^\dagger\lambda) = 0. \tag{3.1.9}$$

Note that this implies that $\mathrm{tr}_2(\tau_a\lambda^\dagger\lambda)$ is traceless for $a = 1, 2, 3$. The ADHM constraint is left untouched under the transformations

$$B \to TBT^\dagger, \quad \lambda \to \lambda T^\dagger, \quad T \in U(k), \tag{3.1.10}$$

$$\lambda \to g\lambda, \quad g \in U(n). \tag{3.1.11}$$

The first leaves the gauge field invariant, whereas the second induces global gauge transformations.

There is a geometrical interpretation of the non-linear ADHM constraint. Before presenting it, we need a natural metric on the space of ADHM matrices which is induced from the metric on the instanton moduli space.

As explained in the introduction, this metric on the moduli space of instantons on $\mathbb{R}^4$ is computed as the L_2 norm of its tangent vectors. These are solutions to the equations (1.5.6)

$$D^{\rm ad}_{[\mu}Z_{\nu]} = \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}D^{\rm ad}_{[\alpha}Z_{\beta]}, \quad D^{\rm ad}_\mu(A)Z_\mu = 0.$$

The tangent vectors can be computed within the ADHM formalism. A zero-mode is given by [81]

$$Z_\mu(C) = -v^\dagger(x) C \bar\sigma_\mu f_x b^\dagger v(x) + v^\dagger(x) b f_x \sigma_\mu C^\dagger v(x), \tag{3.1.12}$$

where C is a tangent vector to the moduli space of ADHM data,

$$C = \begin{pmatrix} c \\ Y \end{pmatrix}, \tag{3.1.13}$$

provided C satisfies

$$\mathrm{tr}_2(\Delta^\dagger(x) C \sigma_i^\dagger) = -\mathrm{tr}_2(C^\dagger \Delta(x) \sigma_i^\dagger), \quad \mathrm{tr}_2(\Delta^\dagger(x) C) = \mathrm{tr}_2(C^\dagger \Delta(x)). \tag{3.1.14}$$

These conditions on C follow from considering

$$D_\mu^{\mathrm{ad}} Z_\nu = (D_\mu a_\nu) C^\dagger v - v^\dagger C (D_\mu a_\nu)^\dagger + v^\dagger b f_x K_{\nu\mu} f_x b v, \tag{3.1.15}$$

where we defined the object $a_\mu = v^\dagger b f_x \sigma_\mu$ which can be shown to have the properties

$$D_\mu a_\mu = 0, \quad D_{[\mu} a_{\nu]} = -v^\dagger b f_x (B_\alpha - x_\alpha) f_x \eta_{\mu\nu} \sigma_\alpha \tag{3.1.16}$$

and the object $K_{\mu\nu} = \sigma_\mu (C^\dagger \Delta(x)) \sigma_\nu^\dagger - \sigma_\mu (\Delta^\dagger(x) C) \sigma_\nu^\dagger$. Then the deformation equation is equivalent to the requirement of $K_{\mu\nu}$ to be selfdual, implying the first identity in eq. (3.1.14), whereas the gauge orthogonality is equivalent to $K_{\mu\mu} = 0$, resulting in the second. Verifying eqs. (3.1.15, 3.1.16) requires manipulations similar to those leading to eq. (3.1.7).

Using an infinitesimal $U(k)$ transformation (3.1.10) $T = \exp(-i\delta X)$, where $\delta X = \delta X^\dagger$, the tangent vectors can be constructed as

$$C = \delta\Delta + \delta_X \Delta = \begin{pmatrix} \delta\lambda + i\lambda\delta X \\ \delta B + i[B, \delta X] \end{pmatrix}, \tag{3.1.17}$$

which automatically satisfy the deformation equation. Gauge orthogonality imposes

$$\mathrm{tr}_2 \left(B^\dagger [B, i\delta X] - [B^\dagger, i\delta X] B + 2i\delta X \lambda^\dagger \lambda + \lambda^\dagger \delta\lambda - \delta\lambda^\dagger \lambda + B^\dagger \delta B - \delta B^\dagger B \right) = 0. \tag{3.1.18}$$

The complex structures acting on tangent vectors Z extend to C in a natural way, $Z(C)\bar\sigma_i = Z(C\bar\sigma_i)$, as is seen from eq. (3.1.12) and $\sigma_\mu \bar\sigma_i = -\bar\eta^i_{\mu\nu} \sigma_\nu$.

There is a remarkable identity for the product of the gauge zero-modes due to Corrigan [20] [1]

$$\mathrm{Tr} Z^r_\mu Z^s_\mu = \tfrac{1}{2} \partial^2_\mu \mathrm{Tr}\, \mathrm{tr}_2 \left(C_r^\dagger P C_s f_x + C_s^\dagger C_r f_x \right). \tag{3.1.19}$$

This is symmetric in (r, s) because eq. (3.1.14) implies symmetry of

$$\mathrm{Tr}\, \mathrm{tr}_2 (C_r^\dagger \Delta(x) f_x \Delta^\dagger(x) C_s).$$

[1] The expression given in [81] is incorrect for gauge group $SU(n)$ and should be replaced by the one given here in eq. (3.1.19).

A proof of this identity is not given by the authors of [20]. For gauge group $SU(2)$, one is found in [27]. An adaptation of that proof to $SU(n)$ is sketched here. Using manipulations as above and the fact that $\sigma_\mu x \sigma_\mu = -2\bar{x}$ and $\sigma_\mu x \bar{\sigma}_\mu = 2\mathrm{tr}_2 x$, one can derive the following relations

$$\begin{aligned}
&\partial_\mu f_x = f_x(b^\dagger \bar{\sigma}_\mu \Delta(x) + \Delta^\dagger(x) b \sigma_\mu) f_x, \quad \partial_\mu^2 f_x = -4 f_x b^\dagger \mathrm{tr}_2(P) b f_x, \\
&\partial_\mu P = \Delta(x) f_x b^\dagger \bar{\sigma}_\mu P + P b \sigma_\mu f_x \Delta^\dagger(x), \\
&\partial_\mu^2 P = -4\{P, b f_x b^\dagger\} + 4\Delta(x) f_x b^\dagger \mathrm{tr}_2(P) b f_x \Delta^\dagger(x).
\end{aligned} \tag{3.1.20}$$

Direct substitution of these results gives

$$\begin{aligned}
&\tfrac{1}{2}\partial_\mu^2 \mathrm{Tr}\,\mathrm{tr}_2\left(C_r^\dagger(P+1)C_s f_x\right) = \\
&\quad -2\mathrm{Tr}\,\mathrm{tr}_2\left(C_r^\dagger\{P, b f_x b^\dagger\} C_s f_x\right) + 2\mathrm{Tr}\,\mathrm{tr}_2\left(C_r^\dagger \Delta(x) f_x b^\dagger \mathrm{tr}_2(P) b f_x \Delta^\dagger(x) C_s f_x\right) \\
&\quad +\mathrm{Tr}\,\mathrm{tr}_2\left(C_r^\dagger(\Delta(x) f_x b^\dagger \bar{\sigma}_\mu P + P b \sigma_\mu f_x \Delta^\dagger(x)) C_s f_x (b^\dagger \bar{\sigma}_\mu \Delta(x) + \Delta^\dagger(x) b \sigma_\mu) f_x\right) \\
&\qquad\qquad -2\mathrm{Tr}\,\mathrm{tr}_2\left(C_r^\dagger(P+1)C_s f_x b^\dagger \mathrm{tr}_2(P) b f_x\right).
\end{aligned} \tag{3.1.21}$$

We introduce the notation $(X_\mu \sigma_\mu)^T = X_\mu^\dagger \sigma_\mu$ and note that eq. (3.1.14) can conveniently be rewritten as

$$(\Delta^\dagger(x) C)^T = \Delta^\dagger(x) C = \left(C^\dagger \Delta(x)\right)^\dagger. \tag{3.1.22}$$

Inserting this at suitable places in eq. (3.1.21), e.g.

$$\left(\Delta^\dagger(x) C_s f_x C_r^\dagger \Delta(x)\right)^\dagger = \left(C_s^\dagger \Delta(x) f_x \Delta^\dagger(x) C_r\right)^T = \left(C_s^\dagger(1-P)C_r\right)^T, \tag{3.1.23}$$

after some algebra one obtains

$$\begin{aligned}
&\tfrac{1}{4}\partial_\mu^2 \mathrm{Tr}\,\mathrm{tr}_2\left(C_r^\dagger(P+1)C_s f_x + C_s^\dagger(P+1)C_r f_x\right) = \\
&\quad -2\mathrm{Tr}\,\mathrm{tr}_2(C_s^\dagger P C_r + C_r^\dagger P C_s) f_x b^\dagger \mathrm{tr}_2(P) b f_x \\
&\quad -2\mathrm{Tr}\,\mathrm{tr}_2(P C_r f_x b^\dagger (C_s^\dagger P)^T f_x b^\dagger + C_r^\dagger P b f_x (P C_s)^T b f_x).
\end{aligned} \tag{3.1.24}$$

This agrees with $\mathrm{Tr}(Z_\mu^r Z_\mu^s)$ obtained from eq. (3.1.12). As an aside, we realise that the zero-modes corresponding to the centre of mass of the instanton, for which

$$C^{(\nu)} = b\sigma_\nu, \tag{3.1.25}$$

are identical to curvature components,

$$Z_\mu(C^{(\nu)}) = Z_\mu(b\sigma_\nu) = F_{\mu\nu}. \tag{3.1.26}$$

Using this fact, one may derive an expression for the the action density in terms of the ADHM data:

$$\begin{aligned}
-\mathrm{Tr} F_{\mu\nu} F_{\mu\nu} &= -\tfrac{1}{2}\partial_\mu^2 \mathrm{Tr}\,\mathrm{tr}_2(b^\dagger \bar{\sigma}_\nu P b \sigma_\nu f_x + 4 f_x b^\dagger b) \\
&= -2\partial_\mu^2 \mathrm{Tr}\,\mathrm{tr}_2(2 f_x b^\dagger b - b^\dagger \Delta f_x \Delta^\dagger b f_x).
\end{aligned} \tag{3.1.27}$$

On the other hand,

$$\partial_\mu^2 \log\det f_x = -\mathrm{Tr}\left(f_x \partial_\mu^2 f_x^{-1} - (f_x \partial_\mu f_x^{-1})^2\right), \tag{3.1.28}$$

which equals the rhs. of eq. (3.1.27), resulting in the remarkable expression for the action density [80]

$$\mathrm{Tr} F_{\mu\nu} F_{\mu\nu} = -\partial_\mu^2 \partial_\nu^2 \log\det f_x. \tag{3.1.29}$$

Here f_x is considered a $k \times k$ complex hermitean (non-quaternionic) matrix.

Using the asymptotic behaviour

$$f_x = 1_k / x^2, \quad x^2 \to \infty, \tag{3.1.30}$$

one can readily evaluate the L_2 norm in eq. (1.5.8) as a boundary term at $x^2 \to \infty$. Using in addition that $Z(C)\bar{\sigma}_i = Z(C\bar{\sigma}_i)$ and identifying the tangent space to the ADHM data with the vector space itself, the (see [66]) hyperKähler isometric property of the ADHM construction is proven

$$\begin{aligned} g_{\mathcal{M}}(Z, Z') &= \tfrac{1}{2}\mathrm{Tr}\,\mathrm{tr}_2\left(Y^\dagger Y' + c^\dagger c' + c'^\dagger c\right), \\ \vec{\omega}_{\mathcal{M}}(Z, Z') \cdot \vec{\sigma} &= \tfrac{1}{2}\sigma_i \mathrm{Tr}\,\mathrm{tr}_2 \bar{\sigma}_i \left(Y^\dagger Y' + c^\dagger c' - c'^\dagger c\right). \end{aligned} \tag{3.1.31}$$

The right hand side of eq. (3.1.31) gives a natural metric and Kähler forms on the space $\hat{\mathcal{A}}$ of ADHM matrices Δ. By inserting eq. (3.1.26) in eq. (3.1.31), one readily retrieves the action of the instanton to be $8\pi^2 k$ and hence that an ADHM matrix satisfying the ADHM constraint corresponds to a charge k instanton.

We now consider the geometry of the space $\hat{\mathcal{A}}$ in its own right and notice that with metric and Kähler forms on $\hat{\mathcal{A}}$ defined from eq. (3.1.31) as

$$\begin{aligned} g &= \tfrac{1}{2}\mathrm{Tr}\,\mathrm{tr}_2\left(dB^\dagger dB + 2d\lambda^\dagger d\lambda\right), \\ \vec{\omega} \cdot \vec{\sigma} &= \tfrac{1}{2}\sigma_i \mathrm{Tr}\,\mathrm{tr}_2\,\bar{\sigma}_i\left(dB^\dagger \wedge dB + 2d\lambda^\dagger \wedge d\lambda\right), \end{aligned} \tag{3.1.32}$$

$\hat{\mathcal{A}}$ is hyperKähler. The $U(k)$ transformations of eq. (3.1.10) leave $(g, \vec{\omega})$ invariant and therefore form a group of triholomorphic isometries of $\hat{\mathcal{A}}$. The associated moment map reads

$$\vec{\mu} = \tfrac{1}{2}\mathrm{tr}_2\left[\left(B^\dagger B + \lambda^\dagger \lambda\right)\vec{\sigma}\right]. \tag{3.1.33}$$

Its zero set $\vec{\mu}^{-1}(0)$ is formed by the solutions to the ADHM constraint

$$\bar{\eta}_{\mu\nu} B_\mu B_\nu + \tfrac{1}{2}\tau_a \otimes \mathrm{tr}_2(\tau_a \lambda^\dagger \lambda) = 0. \tag{3.1.34}$$

Thus it is shown that an element $\Delta \in \vec{\mu}^{-1}(0)$ corresponds to a charge k instanton solution. The gauge connection (3.1.4) is not affected by the $U(k)$ transformations (3.1.10), which therefore have to be divided out to obtain the instanton moduli space $\vec{\mu}^{-1}(0)/U(k)$. The dimension of this quotient, the moduli space of ADHM data, is $4kn$. This follows from a counting procedure: Each matrix B_μ is described by k^2 parameters, resulting in $4k^2$ parameters in total for B; $4nk$ parameters are associated with λ, giving $4k^2 + 4nk$ parameters for Δ. The diagonal of $B^\dagger B$ commutes with $\mathbb{H}$, hence the diagonal elements of $\lambda^\dagger\lambda$, which are hermitean 2×2 matrices, should commute with $\mathbb{H}$. This gives $3k$ constraints. For the off-diagonal part of $B^\dagger B + \lambda^\dagger\lambda$ to satisfy the ADHM constraint, $3 \times 2 \times \frac{1}{2}k(k-1)$ conditions should be met. Finally, the T symmetry further reduces the number of parameters by k^2. One then quickly arrives at a dimensionality of $4kn$ for the instanton moduli space. As it is a hyperKähler quotient, the moduli space of ADHM data $\vec{\mu}^{-1}(0)/U(k)$ is hyperKähler [24, 26]. Global gauge transformations of the instanton, $\lambda \to g\lambda$, $g \in SU(n)$, which are included as moduli, form a triholomorphic isometry, as follows from eq. (3.1.32). Since $SU(n)$ acts on the left, it commutes with $U(k)$ acting on the right. Therefore, $SU(n)$ descends as a group of triholomorphic isometries to the moduli space of ADHM data, the hyperKähler quotient $\vec{\mu}^{-1}(0)/U(k)$, reflecting the gauge symmetry of the instanton solution. As the ADHM construction is an isometry and the moduli space of ADHM data $\vec{\mu}^{-1}(0)/U(k)$ is hyperKähler the same holds for the moduli space of instantons on $\mathbb{R}^4$. Note that for this to hold it was necessary to include the $SU(n)$ global gauge symmetry as moduli.

With the natural metric on the space of ADHM data, the conditions on C for it to be a tangent vector to the space of ADHM data, have a natural interpretation. The first equation is the deformation of the ADHM constraint whereas the second condition guarantees orthogonality of C to any infinitesimal T transformation with respect to the metric (3.1.32).

Many aspects featuring in the construction above have their counterparts in the Nahm transformation. This is not unexpected from the discussion in section 2.3.1. In fact, one can make a one to one identification of the objects in section 2.3.1 and those featuring here:

$$\begin{aligned}
\hat{D}_x^{\text{ext}} &= \Delta(x),\\
\hat{D}_x &= B(x),\\
2\pi\tilde{\alpha} &= \lambda,\\
\hat{\psi}^{\text{ext}}(x) &= v(x),\\
\hat{\psi}(x) &= u(x),\\
s_x &= 1_n.
\end{aligned} \tag{3.1.35}$$

The reality constraint is similar to the vanishing of the imaginary quaternions in the Weitzenböck formula, which leads to the selfduality of the Nahm connection. The symmetries in the ADHM construction can be traced back to the triviality of the gauge action in the Nahm transformation and the unitary symmetry of the fermionic zero modes. The matrix inverses are the analogues of the Green's functions. Finally, the possibility to compute the instanton moduli space metric in terms of the ADHM data reflects the hyperKähler isometry of the Nahm transformation.

3.2 Weyl zero-modes

Weyl zero-modes in the background of the instanton solution play a central role in the Nahm transformation. It is therefore useful to have expressions for them within the ADHM formalism.

To find the Weyl zero-mode Ψ solving

$$\bar{\sigma}_\mu D_\mu \Psi(x) = 0 \tag{3.2.1}$$

we note that (see eq. (3.1.20))

$$\begin{aligned} v^\dagger \partial_\mu (P b \sigma_\mu f_x) &= v^\dagger (\partial_\mu P) b \sigma_\mu f_x + v^\dagger P b \sigma_\mu f_x (\bar{\sigma}_\mu b^\dagger \Delta(x) + \Delta^\dagger(x) b \sigma_\mu) f_x \\ &= 2 v^\dagger b \sigma_\mu f_x \Delta^\dagger(x) b \sigma_\mu f_x + 4 v^\dagger b f_x b^\dagger \Delta(x) f_x \\ &= 0, \end{aligned} \tag{3.2.2}$$

where we used that $\sigma_\mu \Delta^\dagger(x) b \sigma_\mu = -2 b^\dagger \Delta(x)$. Using charge conjugation

$$\varepsilon \sigma_\mu^\dagger \varepsilon = -\tilde{\sigma}_\mu, \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.2.3}$$

where the tilde denotes transposing with respect to the spinor indices, we can rewrite eq. (3.2.2) as the Weyl equation

$$\begin{aligned} -v^\dagger \partial_\mu (v (v^\dagger b f_x) \sigma_\mu) &= v^\dagger \partial_\mu (v \varepsilon \bar{\sigma}_\mu \varepsilon \tilde{v}^\dagger b f_x) = (\partial_\mu + A_\mu) \varepsilon \bar{\sigma}_\mu \varepsilon \tilde{v}^\dagger b f_x \\ &= \varepsilon \bar{\sigma}_\mu D_\mu \varepsilon \tilde{v}^\dagger b f_x = 0. \end{aligned} \tag{3.2.4}$$

Therefore, zero-modes for the Weyl equation are given by $\varepsilon \tilde{v}^\dagger b f_x$. Viewed as a $2n \times k$ matrix, each column represents one of the k independent zero-modes. Using eq. (3.1.20), we find for the $k \times k$ normalisation matrix

$$(\varepsilon \tilde{v}^\dagger b f_x)^\dagger \varepsilon \tilde{v}^\dagger b f_x = \mathrm{tr}_2 \left((\varepsilon v^\dagger b f_x)^\dagger \varepsilon v^\dagger b f_x \right) = -\frac{1}{4} \partial_\mu^2 f_x. \tag{3.2.5}$$

This now results in the formulae for the normalised Weyl zero-modes

$$\Psi(x) = \frac{1}{\pi} \varepsilon \tilde{v}^\dagger b f_x, \quad \Psi^\dagger(x) \Psi(x) = -\frac{1}{4\pi^2} \partial_\mu^2 f_x, \tag{3.2.6}$$

as follows from integration over $\mathbb{R}^4$, using the asymptotic behaviour of f_x as given in eq. (3.1.30).

4 The $k = 1$ $SU(2)$ caloron

4.1 Introduction

Instantons and BPS monopoles possess remarkable properties. They exist as exact solutions with arbitrary charges and with an action or energy, proportional to their integer charges. Therefore the multi-charge solutions can be seen as built from constituents of unit charge. Indeed, for BPS monopoles the absence of an interaction energy can be understood - for large separation - as a cancellation between the electro-magnetic and scalar interactions [68, 72].

Instantons are selfdual solutions with finite action. For non-compact manifolds, the solutions must approach vacua in the non-compact directions. Due to the topology of the base manifold, these vacua can be non-trivial and can give rise to extra parameters for these selfdual solutions. For periodic instantons on $\mathbb{R}^3 \times S^1$ (calorons), the vacuum label is given by the eigenvalues of the holonomy around S^1 at spatial infinity, cf. section 2.2. Equivalently, one may consider this vacuum as the background field on which the solution is superposed. In this respect, they are very similar to monopole solutions in broken gauge theories, a non-trivial vacuum generally breaking the gauge symmetry.

Calorons can be seen to have as constituents BPS monopoles [32, 63] (for charge one, n for $SU(n)$), as follows from Nahm's work [77]. The constituents are such that the net magnetic and electric charge of the caloron vanishes. Unlike for the ordinary multi-monopoles, the BPS constituents are hence of opposite charge, and thus have an attractive electro-magnetic interaction. Nevertheless, also here exact solutions exist with an *action* that does not depend on the parameters, though the solutions become static only for large separations. In order to have this non-trivial situation the holonomy at spatial infinity has to be non-trivial, breaking the gauge invariance spontaneously. The eigenvalues of this holonomy uniquely fix the masses of the constituent monopoles. Their separation - not surprisingly - is related to the scale parameter of the caloron solution.

In this chapter we study periodic $SU(2)$ instantons with topological charge one and arbitrary holonomy. Central to the success in providing explicit and relatively simple new solutions is the construction of the relevant Green's function. In the context of the Nahm transformation, introduced here as the Fourier transform of the ADHM data, this can be reduced to a quantum mechanical problem on the circle with a piecewise constant potential and well-defined delta function singularities related to the holonomy. We will find compact expressions for the gauge field and action density of the solution and investigate the properties of the caloron. The moduli space is described in terms of the constituent monopoles, which in the approach taken here appear as explicit lumps in the action density. Furthermore, the constituent monopole nature of these instantons will be related to work by Taubes in which he showed how to make gauge configurations with non-trivial topological charge out of monopole fields [95].

Independently, the recent work in ref. [58, 59] has taken the constituent monopole description [32, 63, 77] as the starting point, suitably superposing two BPS monopole solutions to form a caloron solution.

Periodic instantons have been discussed first in the context of finite temperature field theory [42, 41], where the period ($\mathcal{T}$) is the inverse temperature in euclidean field theory, as discussed in chapter 1. A non-trivial value of the holonomy will modify the vacuum fluctuations and thereby leads to a non-zero vacuum energy density as compared to a trivial holonomy. It was on the basis of this observation that calorons with non-trivial holonomy were deemed irrelevant in the infinite volume limit [41]. It should be emphasised though, that the semi-classical one-instanton calculation is no longer considered a reliable approximation. At finite temperature A_0 can be seen to play the role of a Higgs field and in a strongly interacting environment one could envisage regions with this Higgs field pointing predominantly in a certain direction, and nevertheless having at infinity a trivial Higgs field. Given a finite density of periodic instantons, in an infinite volume solutions with non-trivial holonomy (in some average sense) may well have a role to play.

In the construction of the charge one caloron with non-trivial holonomy, we pick a particular gauge. In the periodic gauge, the spatial components of the vacuum connection at infinity can be gauged to zero. The A_0 component can only be gauged to a constant, e.g. $A_0 = 2\pi i\vec{\omega}\cdot\vec{\tau}$, when the total magnetic charge is vanishing. This connection has obviously a non-trivial holonomy at infinity

$$\mathcal{P}(\vec{x}) = P\,\exp(\int_0^{\mathcal{T}} A_0(\vec{x},x_0)dx_0) \to e^{2\pi i\vec{\omega}\cdot\vec{\tau}}.$$

Alternatively, connections on $\mathbb{R}^3\times S^1$ can be formulated by embedding them in $\mathbb{R}^4$ and demanding periodicity modulo gauge transformations. Gauging with a non-periodic gauge transformation $g(\vec{x},x_0) = e^{2\pi i x_0\vec{\omega}\cdot\vec{\tau}}$, starting from the periodic gauge with zero A_i and constant $A_0 = 2\pi i\vec{\omega}\cdot\vec{\tau}$ at infinity, sets *all* gauge fields to zero at infinity. In that case we have

$$A_\mu(\vec{x},x_0+\mathcal{T}) = e^{2\pi i\vec{\omega}\cdot\vec{\tau}}A_\mu(\vec{x},x_0)e^{-2\pi i\vec{\omega}\cdot\vec{\tau}}, \tag{4.1.1}$$

with $\mathcal{T}$ the period in the imaginary time direction, i.e. the inverse temperature. Clearly, $e^{2\pi i\vec{\omega}\cdot\vec{\tau}} \equiv g_0(\vec{x})$ is the transition function or cocycle. Using the proper expression for the holonomy along a path traversing the boundary between coordinate patches [5],

$$\mathcal{P}(\vec{x}) = P\,\exp(\int_0^{\mathcal{T}} A_0(\vec{x},x_0)dx_0)g_0(\vec{x}), \tag{4.1.2}$$

we find the same value for the holonomy in this gauge, the holonomy at infinity now solely being carried by the cocycle $g_0(\vec{x})$. It is in this so-called "algebraic" gauge that we will calculate the generalised caloron solutions.

The remainder of this chapter is organised as follows. Section 4.2 contains further details of the ADHM formalism special for gauge group $SU(2)$ and necessary for the construction of the caloron in section 4.3. It is made evident how the ADHM approach to calorons is related with the Nahm formalism discussed in chapter 2 by Fourier transformation. By relying on the ADHM construction we profit from the vast knowledge on multi-instanton calculus within this formalism. Section 4.3 forms the calculational core of this chapter, in which we derive the gauge potential and a particularly simple expression for the action density. The various properties and symmetries of the caloron are unravelled in section 4.4. In section 4.5 the moduli space of the caloron is described. Section 4.6 contains a discussion on the relation with Taubes's work, abelian projection [52] and possible applications to QCD.

4.2 The ADHM construction for $SU(2)$

There is a special version of the ADHM construction for $SU(2)$ instantons on $\mathbb{R}^4$ which reduces the number of initial parameters [1, 2] as compared to that in the exposition in section 3.1. The matrices B_μ can be taken real symmetric, and a similar constraint on the elements of the $2\times 2k$ complex matrix λ allow these to be combined into k elements of the quaternions tensored over the real numbers. The data are then referred to as *framed* data.

The framed ADHM data thus consist of a quaternionic row vector $\lambda=(\lambda_1,\cdots,\lambda_k)$ and a quaternionic, symmetric $k\times k$ matrix B ($\lambda_p\equiv\lambda_p^\mu\sigma_\mu$ and $B_{p,p'}\equiv B^\mu_{p,p'}\sigma_\mu$, with $\lambda_p^\mu\in\mathbb{R}$ and $B^\mu_{p,p'}=B^\mu_{p',p}\in\mathbb{R}$). The symmetries in the ADHM data are now also slightly different, the group of T symmetries is $O(k)$ rather than $U(k)$: The transformations

$$\begin{aligned}\lambda\to\lambda T^{-1},\ B\to TBT^{-1},\ T\in O(k),\\ \lambda\to g\lambda,\quad g\in SU(2),\end{aligned}\tag{4.2.1}$$

both leave the quadratic ADHM constraint untouched. The first does not change $A_\mu(x)$, whereas the second induces a global gauge transformation. One must divide out these symmetries in order to obtain all gauge inequivalent solutions, cf. section 3.1. This reduces the dimension of the space of gauge inequivalent solutions to $8k(-3)$, depending on whether or not the global gauge degrees of freedom are included as moduli. Considering the g as moduli, the moduli space is an $8k$ dimensional hyperKähler manifold.

The special form of the ADHM formalism for $SU(2)$ allows for a more convenient form of the fields. With $v(x)$ as in eq. (3.1.2), the gauge potential reads

$$A_\mu(x)=v^\dagger(x)\partial_\mu v(x),\tag{4.2.2}$$

where $v(x)$ is a $(k+1)$ dimensional quaternionic vector, the normalised solution to

$$\Delta^\dagger(x)v(x)=0.\tag{4.2.3}$$

To solve for $v(x)$ in eq. (4.2.3), we introduce (a matrix of spinors) $u(x)$, and obtain

$$v(x) = \phi^{-\frac{1}{2}}(x) \begin{pmatrix} -1 \\ u(x) \end{pmatrix}, \quad u(x) = (B^\dagger - x^\dagger)^{-1}\lambda^\dagger, \tag{4.2.4}$$

where

$$\phi(x) = 1 + \lambda G_x \lambda^\dagger = 1 + u^\dagger(x)u(x) \tag{4.2.5}$$

accounts for the normalisation of $v(x)$. In terms of these quantities, the gauge potential reads

$$A_\mu(x) = \tfrac{1}{2}\phi^{-1}(x)(u^\dagger(x)\partial_\mu u(x) - \partial_\mu u^\dagger(x)u(x)), \tag{4.2.6}$$

where it was used that $\phi(x)$ is a scalar function for $SU(2)$.

In the case at hand, eq. (4.2.6) is of little practical use as it stands. We therefore rearrange it such that we can express A_μ in terms of evaluations of the Green's function f_x. To simplify the manipulations, a second Green's function G_x is introduced. Together the Green's functions read

$$f_x = (\Delta(x)^\dagger \Delta(x))^{-1} \in \mathbb{R}^{k\times k}, \quad G_x = ((B-x)^\dagger(B-x))^{-1} \in \mathbb{H}^{k\times k}. \tag{4.2.7}$$

The Green's functions f_x and G_x are related, as can be seen from the expansion of f_x in terms of G_x,

$$f_x = (G_x^{-1} + \lambda^\dagger\lambda)^{-1} = G_x - G_x\lambda^\dagger \sum_{n=0}^{\infty}(-\lambda G_x \lambda^\dagger)^n \lambda G_x = G_x - \phi^{-1}(x) G_x \lambda^\dagger \lambda G_x. \tag{4.2.8}$$

Acting on eq. (4.2.8) with $\lambda^\dagger$ on the right and/or with λ on the left, yields

$$G_x\lambda^\dagger = \phi(x) f_x \lambda^\dagger, \quad \phi(x) = (1 - \lambda f_x \lambda^\dagger)^{-1}. \tag{4.2.9}$$

Using $\partial_\mu(B^\dagger - x^\dagger)^{-1} = (B^\dagger - x^\dagger)^{-1}\bar\sigma_\mu(B^\dagger - x^\dagger)^{-1} = (B-x)G_x\bar\sigma_\mu(B-x)G_x$, we get

$$A_\mu(x) = \phi^{-1}(x)\ \lambda G_x \bar\eta_{\mu\nu}(B-x)_\nu G_x \lambda^\dagger. \tag{4.2.10}$$

We substitute $G_x\lambda^\dagger = \phi(x) f_x \lambda^\dagger$, eq. (4.2.9), and noting that

$$\partial_\nu f_x^{-1} = \partial_\nu G_x^{-1} = -2(B-x)_\nu, \quad (\partial_\mu f_x^{-1}) f_x = -f_x^{-1}\partial_\mu f_x, \tag{4.2.11}$$

we arrive at the following compact result for the gauge potential (see also ref. [21])

$$A_\mu(x) = \tfrac{1}{2}\phi(x)\partial_\nu\left(\lambda \bar\eta_{\mu\nu} f_x \lambda^\dagger\right), \tag{4.2.12}$$

using once again that f_x commutes with the quaternions. When $\bar\eta_{\mu\nu}$ is moved through λ, one finds an expression for A_μ in terms of (derivatives of) "expectation values" of the Green's function f_x,

$$A_\mu(x) = \tfrac{1}{2}\phi(x)\sigma_\alpha \bar\eta_{\mu\nu}\bar\sigma_\beta \partial_\nu \phi_{\alpha\beta}, \tag{4.2.13}$$

where

$$\phi_{\alpha\beta}(x) = \phi_{\beta\alpha}(x) = (\lambda_\alpha f_x \lambda^t_\beta). \tag{4.2.14}$$

At this point we can make contact with the well-known 't Hooft Ansatz [51, 87], cf. section 1.3. This forms a subclass of the ADHM construction with λ real ($\lambda_p = \sigma_0 \rho_p$) and $B_{p,p'} = \delta_{p,p'} y_p$ diagonal, corresponding to k instantons with scales ρ_p at positions y_p. This $5k$ dimensional family trivially satisfies the ADHM constraints. In this simpler situation the gauge potential can be written even in terms of a *single* scalar potential $\phi(x) = 1 + \sum_p \rho_p^2/|x-y_p|^2$ as $A_\mu(x) = \frac{1}{2}\bar{\eta}_{\mu\nu}\partial_\nu \log\phi(x)$, since $\phi_{00}(x) = 1 - \phi^{-1}(x)$.

The expression for the action density,

$$\mathrm{Tr} F^2_{\mu\nu}(x) = -\partial^2_\mu \partial^2_\nu \log\det f_x,$$

is regular everywhere. It can be rewritten using eq. (4.2.8) and eq. (4.2.11) as

$$\mathrm{Tr} F^2_{\mu\nu}(x) = -\tfrac{1}{2}\partial^2_\mu \partial^2_\nu \log\det G_x + \partial^2_\mu \partial^2_\nu \log\phi(x). \tag{4.2.15}$$

The factor $\frac{1}{2}$ is due to G_x being considered as a quaternionic and f_x as a real $k \times k$ matrix. For the 't Hooft Ansatz, $\partial^2_\mu \log\det G_x$ vanishes, except for delta functions at $x = y_k$, and we retrieve the known result [51, 87], $\mathrm{Tr} F^2_{\mu\nu} = \partial^2_\mu \partial^2_\nu \log\phi(x)$, which is indeed singular at these points.

4.3 The construction of the caloron

In this section we describe the ADHM construction of caloron solutions with non-trivial holonomy. This will be a two-step process. Crucial will be the interpretation of the ADHM data as the Fourier coefficients of the Weyl operator in the Nahm transformation. In the strategy taken here, the caloron is built as an infinite, periodic (gauge-twisted) chain of instantons. It will be shown how we can realise this within the ADHM construction, by solving the quadratic constraint on the ADHM data. To find $A_\mu(x)$ we use again a Fourier transform to construct $\hat{f}_x(z, z')$, the Green's function of an ordinary second order differential equation, which allows for the determination of $\phi_{\alpha\beta}(x) = (\lambda_\alpha f_x \lambda^t_\beta)$, see eq. (4.2.13).

The boundary conditions $A_\mu(x + \mathcal{T}) = e^{2\pi i \vec{\omega}\cdot\vec{\tau}} A_\mu(x) e^{-2\pi i \vec{\omega}\cdot\vec{\tau}}$ are satisfied when

$$u_k(x + \mathcal{T}) = u_{k-1}(x)\exp(-2\pi i \vec{\omega}\cdot\vec{\tau}), \tag{4.3.1}$$

as is seen from eq. (4.2.6). This is implemented by the periodicity constraints

$$\lambda_{k+1} = e^{2\pi i \vec{\omega}\cdot\vec{\tau}}\lambda_k, \quad B(x+\mathcal{T})_{m,n} = B(x)_{m-1,n-1}, \tag{4.3.2}$$

where $B(x) = B - x$. It now follows that

$$B_{p+1,p'+1} = B_{p,p'} + \mathcal{T}\delta_{p,p'}, \tag{4.3.3}$$

the inhomogeneous part of which is solved by having $\ldots, -2\mathcal{T}, -\mathcal{T}, 0, \mathcal{T}, 2\mathcal{T}, \ldots$ on the diagonal of B. We still have to determine the remainder of B, called $\hat{A}$ (anticipating its interpretation as Nahm connection), that contains its off-diagonal entries. In order to satisfy eq. (4.3.2), $\hat{A}$ has to be of a convolutive type $\hat{A}_{p,p'} = \hat{A}_{p-p'}$, such that

$$\lambda_p = e^{2\pi i p\vec{\omega}\cdot\vec{\tau}}\zeta, \quad \zeta = \rho q, \quad B_{p,p'} = \mathcal{T}p\delta_{p,p'} + \hat{A}_{p-p'}. \tag{4.3.4}$$

Here ζ is an arbitrary quaternion. Its length $\rho = |\zeta|$ is the scale parameter of the caloron. The $SU(2)$ element $q = \zeta/\rho$ describes its combined spatial and gauge orientation. The diagonal of $\hat{A}_{p,p'}$ is necessarily constant, $\hat{A}^{\rm diag}_{p,p'} \equiv \xi$, and plays the role of the position of the caloron. The ADHM data can now be readily interpreted as describing a periodic array of instantons, with temporal spacing $\mathcal{T}$ and relative gauge orientation $e^{2\pi i\vec{\omega}\cdot\vec{\tau}}$, with off-diagonal terms to account for the non-linear constraints. To simplify notations, we use the scale invariance of the selfduality equations to set $\mathcal{T} = 1$. On dimensional grounds one can easily reinstate the proper $\mathcal{T}$ dependence when required.

When we perform the Fourier transformation, B will be transformed into a Weyl operator, λ and $\lambda^\dagger\lambda$ into delta function singularities and $u(x)$ into a spinor (to be more precise a 2×2 matrix with as columns $\hat{\psi}^i_x$, cf. section 2.1 and eq. (3.1.35)):

$$\begin{aligned}
&\sum_{p,p'} B_{p,p'}(x)e^{2\pi i(pz-p'z')} = \frac{\delta(z-z')}{2\pi i}\hat{D}_x(z'),\\
&\hat{D}_x(z) = \sigma_\mu D^\mu_x(z) = \frac{d}{dz} + \hat{A}(z) - 2\pi i x,\\
&\sum_p \lambda_p e^{-2\pi i p z} = \hat{\lambda}(z),\\
&\hat{\lambda}(z) = (P_+\delta(z-\omega) + P_-\delta(z+\omega))\zeta, \quad P_\pm = \tfrac{1}{2}(1 \pm \hat{\omega}\cdot\vec{\tau}),\\
&\sum_{p,p'} \lambda^\dagger_p\lambda_{p'} e^{2\pi i(pz'-p'z)} = \hat{\lambda}^\dagger(z')\hat{\lambda}(z) = \delta(z-z')\hat{\Lambda}(z),\\
&\hat{\Lambda}(z) = \bar{\zeta}\hat{\lambda}(z) = \hat{\lambda}^\dagger(z)\zeta,\\
&\sum_p u_p(x)e^{2\pi i p z} = \hat{\psi}_x(z).
\end{aligned} \tag{4.3.5}$$

All these objects are defined on S^1, or more appropriately from the Nahm perspective, on $\mathbb{R}^4/\hat{H} = \mathbb{R}^4/(\mathbb{R}^3\times\mathbb{Z})$. Note that $\hat{A}(z) = \sigma_\mu\hat{A}_\mu(z) = 2\pi i\sum_p \exp(2\pi i p z)\hat{A}_p$, such that from the symmetry of $\hat{A}_{p,p'}$ (implying $\hat{A}_p = \hat{A}_{-p}$) it follows that $\hat{A}_\mu(z)$ is imaginary such that the differential operator $\hat{D}_x(x)$ is exactly the dual Weyl operator $\hat{D}_x(\hat{A})$ introduced in section 2.1. Combining these features, we can interpret the Fourier transform of the ADHM construction as the inverse (or second) Nahm transformation.

The symmetries in the ADHM construction for periodic instantons lead to a $U(1)$ gauge symmetry for $\hat{A}_\mu(z)$. In order for eq. (4.2.1) to preserve the periodicity constraint eq. (4.3.2), T has to be of a convolutive type, $T_{p,p'} = T_{p-p'}$. Defining the periodic function $\hat{g}(z) = \sum_p \exp(2\pi i p z) T_p$ and using the fact that T is orthogonal ($T^{-1}_{p,p'} = T_{p'-p}$ and $\sum_{p''} T_{p'+p} T_{p'} = \delta_{p,0}$), one concludes that $\hat{g}(z) \in U(1)$. A gauge transformation with $\hat{g}(z)$ leaves $\hat{A}_i(z)$ invariant and transforms $\hat{A}_0(z)$ to $\hat{A}_0(z) - d\log\hat{g}(z)/dz$. Note that $\hat{A}_0(z)$ can be gauged away, apart from a constant (as the holonomy is gauge invariant). Hence we may choose $\hat{A}_0(z) = 2\pi i \xi_0$.

The quadratic ADHM constraint can be formulated as $\Im(\Delta^\dagger(x)\Delta(x)) = 0$, or

$$\Im(\hat{D}^\dagger_x(z)\hat{D}_x(z) + 4\pi^2\hat{\Lambda}(z)) = 0, \tag{4.3.6}$$

cf. section 3.1. From the Weitzenböck formula,

$$\hat{D}^\dagger_x(z)\hat{D}_x(z) = -(\hat{D}^\mu_x(z)\hat{D}^\mu_x(z) + \tfrac{1}{2}\bar{\eta}_{\mu\nu}\hat{F}_{\mu\nu}), \tag{4.3.7}$$

(cf. eq. (2.1.3)) we find

$$\begin{aligned} \frac{d}{dz}\hat{A}(z) &= \tfrac{1}{2}\bar{\eta}_{\mu\nu}\hat{F}_{\mu\nu}(z) = -\Im\hat{D}^\dagger_x(z)\hat{D}_x(z) = 4\pi^2\Im\hat{\Lambda}(z) \\ &= 2\pi^2(\bar{\zeta}\hat{\omega}\cdot\vec{\tau}\zeta)(\delta(z-\omega) - \delta(z+\omega)), \end{aligned} \tag{4.3.8}$$

using eq. (4.3.6) and $\hat{F}_{ij}(z) = 0$. This leads to

$$\hat{A}(z) = \sigma_\mu \hat{A}_\mu(z) = 2\pi i[\xi - \pi(\bar{\zeta}\hat{\omega}\cdot\vec{\sigma}\zeta)\Theta_\omega(z)], \tag{4.3.9}$$

where (see fig. 4-1)

$$\Theta_\omega(z) = (\chi_{[-\omega,\omega]}(z) - 2\omega). \tag{4.3.10}$$

Here $\chi_{[a,b]}(z) = 1$ if $z \in [a,b]\mathrm{mod}1$ and 0 elsewhere. We have arranged $2\pi i\xi = \int dz\ \hat{A}(z)$, such that $B(x)$ has a single zero mode for $x = \xi$, to agree with the interpretation of ξ as the position (centre of mass) of the caloron. Fourier transforming back, we retrieve the matrix representation of $B(x)$

$$B_{p,p'}(x) = (p + \xi - x)\delta_{p,p'} - \bar{\zeta}\hat{\omega}\cdot\vec{\sigma}\zeta\frac{\sin(2\pi(p-p')\omega)}{p-p'}(1-\delta_{p,p'}). \tag{4.3.11}$$

The moduli space is thus parametrised by the caloron position ξ and by its scale and orientation $\zeta = \rho q$, with $\xi_0 \in S^1$, $\vec{\xi} \in \mathbb{R}^3$, $\rho \in \mathbb{R}^+$ and $q \in SU(2)$.

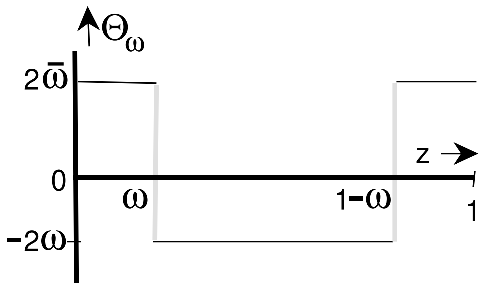


Figure 4-1. The function $\Theta_\omega(z)$.

We end this first step in the construction by noting that the delta function singularities arise precisely as predicted by the general properties of the Nahm transformation, discussed in section 2.2 and that $\hat{A}(z)$ constructed in this section solves the Nahm equations, whereby we will find all charge one selfdual solutions for gauge group $SU(2)$ on $\mathbb{R}^3\times S^1$.

For the second step we have to find the Green's function f_x. For further notational simplification we absorb ξ by a translation (we have already used the scale invariance to fix $\mathcal{T}=1$) such that after a Fourier transformation the definition of f_x, eq. (4.2.7), can be cast into a differential equation for $\hat{f}_x(z,z')\equiv\sum_{p,p'}f_x^{p,p'}e^{2\pi i(pz-p'z')}$

$$\left\{\left(\frac{1}{2\pi i}\frac{d}{dz}-x_0\right)^2+s^2\chi_{[-\omega,\omega]}(z)+r^2\chi_{[\omega,1-\omega]}(z)+\frac{\rho^2}{2}(\delta(z-\omega)+\delta(z+\omega))\right\}\hat{f}_x(z,z')=\delta(z-z'). \tag{4.3.12}$$

Here, the radii r and s are given by

$$r^2=\tfrac{1}{2}\mathrm{tr}_2(\vec{x}\cdot\vec{\tau}-2\pi\omega\rho^2\bar{q}\hat{\omega}\cdot\vec{\tau}q)^2,\quad s^2=\tfrac{1}{2}\mathrm{tr}_2(\vec{x}\cdot\vec{\tau}+2\pi\bar{\omega}\rho^2\bar{q}\hat{\omega}\cdot\vec{\tau}q)^2,\quad (\bar{\omega}=\tfrac{1}{2}-\omega), \tag{4.3.13}$$

and can be interpreted as the respective centre of mass radii of the two constituent monopoles of the caloron. Note that $\bar{q}\hat{\omega}\cdot\vec{\tau}q$ shows how global gauge rotations are to be correlated to spatial rotations so as to keep the holonomy unchanged. We will come back to this in the next section. The symmetries of $\Delta^\dagger(x)\Delta(x)$ imply for the Green's function $\hat{f}_x(z,z')$ the following relations,

$$\hat{f}_x(z,z')=\hat{f}_x(-z,-z')^*=\hat{f}_x(-z',-z)=\hat{f}_x(z',z)^*. \tag{4.3.14}$$

In particular we have $\hat{f}_x(\omega,\omega)=\hat{f}_x(-\omega,-\omega)\in\mathbb{R}$ and $\hat{f}_x(\omega,-\omega)=\hat{f}_x(-\omega,\omega)^*$. The Green's function is that of ordinary quantum mechanics on a circle with a piecewise constant potential and delta function singularities of strength $\frac{1}{2}\rho^2$ at the jumping points $z=\pm\omega$. It can thus be constructed in the usual straightforward (but tedious) method of matching the value of the Green's function and its derivative (up to the appropriate jumps) at $z=\pm\omega$ and $z=z'$. How this can be done in a systematic way is discussed in section 5.4. The result reads

$$\hat{f}_x(z,z')=\chi_{[-\omega,\omega]}(z')\left(\chi_{[-\omega,\omega]}(z)\hat{f}_x^d(z,z',r,s,\omega)+\chi_{[\omega,1-\omega]}(z)\hat{f}_x^o(z,z',r,s,\omega)\right)+$$
$$\chi_{[\omega,1-\omega]}(z')\left(\chi_{[\omega,1-\omega]}(z)\hat{f}_x^d(z-\tfrac{1}{2},z'-\tfrac{1}{2},s,r,\bar{\omega})+\chi_{[-\omega,\omega]}(z)\hat{f}_x^o(z',z,r,s,\omega)^*\right). \tag{4.3.15}$$

In the following the diagonal component $\hat{f}_x^d(z,z')$ is only defined strictly for $z,z'\in[-\omega,\omega]$ and the off-diagonal component $\hat{f}_x^o(z,z')$ only for $z\in[\omega,1-\omega]$ and $z'\in[-\omega,\omega]$. For z or z' outside of these intervals, one first has to map back to the interval $[-\omega,1-\omega]$, using periodicity.

$$\hat{f}_x^d(z,z',r,s,\omega)=e^{2\pi ix_0(z-z')}\pi(rs\psi)^{-1}\Big\{e^{-2\pi ix_0\mathrm{sign}(z-z')}r\sinh(2\pi s|z-z'|)$$

$$-s^{-1}\cosh(2\pi s(z+z'))\left[\pi\rho^2 r\cosh(4\pi r\bar{\omega})+\tfrac{1}{2}(r^2-s^2+\pi^2\rho^4)\sinh(4\pi r\bar{\omega})\right]$$
$$+s^{-1}\cosh(2\pi s(2\omega-|z-z'|))\left[\pi\rho^2 r\cosh(4\pi r\bar{\omega})+\tfrac{1}{2}(r^2+s^2+\pi^2\rho^4)\sinh(4\pi r\bar{\omega})\right]$$
$$+\sinh(2\pi s(2\omega-|z-z'|))\left[r\cosh(4\pi r\bar{\omega})+\pi\rho^2\sinh(4\pi r\bar{\omega})\right]\Big\},$$

$$\begin{aligned}\hat{f}^o_x(z,z',r,s,\omega)=e^{2\pi i x_0(z-z')}\pi(rs\psi)^{-1}\Big\{&\pi\rho^2\sinh(2\pi r(1-z-\omega))\sinh(2\pi s(z'+\omega))\\&+r\cosh(2\pi r(z-1+\omega))\sinh(2\pi s(z'+\omega))\\&-s\sinh(2\pi r(z-1+\omega))\cosh(2\pi s(z'+\omega))\\&+e^{-2\pi i x_0}\Big[s\sinh(2\pi r(z-\omega))\cosh(2\pi s(z'-\omega))\\&\quad-r\cosh(2\pi r(z-\omega))\sinh(2\pi s(z'-\omega))\\&\quad-\pi\rho^2\sinh(2\pi r(z-\omega))\sinh(2\pi s(z'-\omega))\Big]\Big\},\end{aligned} \tag{4.3.16}$$

where we have introduced the scalar function

$$\begin{aligned}\psi=&-\cos(2\pi x_0)+\cosh(4\pi r\bar{\omega})\cosh(4\pi s\omega)+\frac{(r^2+s^2+\pi^2\rho^4)}{2rs}\sinh(4\pi r\bar{\omega})\sinh(4\pi s\omega)\\&+\pi\rho^2\left(s^{-1}\sinh(4\pi s\omega)\cosh(4\pi r\bar{\omega})+r^{-1}\sinh(4\pi r\bar{\omega})\cosh(4\pi s\omega)\right).\end{aligned} \tag{4.3.17}$$

In particular,

$$\hat{f}_x(\omega,-\omega)=\pi(rs\psi)^{-1}e^{4\pi i x_0\omega}\left\{e^{-2\pi i x_0}r\sinh(4\pi s\omega)+s\sinh(4\pi r\bar{\omega})\right\}, \tag{4.3.18}$$

$$\begin{aligned}\hat{f}_x(\omega,\omega)=\pi(rs\psi)^{-1}\Big\{&s\sinh(4\pi r\bar{\omega})\cosh(4\pi s\omega)+r\sinh(4\pi s\omega)\cosh(4\pi r\bar{\omega})\\&+\pi\rho^2\sinh(4\pi r\bar{\omega})\sinh(4\pi s\omega)\Big\}\end{aligned}$$

and (see eq. (4.2.9))

$$\phi(x)=(1-\lambda f_x\lambda^\dagger)^{-1}=(1-\rho^2\hat{f}_x(\omega,\omega))^{-1}\equiv\psi/\hat{\psi}, \tag{4.3.19}$$

where

$$\hat{\psi}=-\cos(2\pi x_0)+\cosh(4\pi r\bar{\omega})\cosh(4\pi s\omega)+\frac{(r^2+s^2-\pi^2\rho^4)}{2rs}\sinh(4\pi r\bar{\omega})\sinh(4\pi s\omega). \tag{4.3.20}$$

We now use eqs. (4.2.13-4.2.14) to determine A_μ. The scalar functions $\phi_{\alpha\beta}$ are all defined in terms of $\hat{f}_x(\omega,\pm\omega)$. To get compact expressions, we introduce the complex function

$$\chi=\rho^2\hat{f}_x(\omega,-\omega)=e^{4\pi i x_0\omega}\frac{\pi\rho^2}{\psi}\left\{e^{-2\pi i x_0}s^{-1}\sinh(4\pi s\omega)+r^{-1}\sinh(4\pi r\bar{\omega})\right\}. \tag{4.3.21}$$

We choose $\vec{\omega}$ in the $z-$direction and $q=1$, which can always be achieved by performing a suitable gauge and spatial rotation. We find for those functions $\phi_{\alpha\beta}$ that are non-zero

$$\phi_{00}=\tfrac{1}{2}(1-\phi^{-1}+\mathrm{Re}\chi),\quad \phi_{33}=\tfrac{1}{2}(1-\phi^{-1}-\mathrm{Re}\chi),\quad \phi_{03}=\phi_{30}=\tfrac{1}{2}\mathrm{Im}\chi, \tag{4.3.22}$$

such that with the use of eq. (4.2.13) (for a matrix W, $\mathrm{Re}W \equiv \frac{1}{2}(W+W^\dagger)$)

$$A_\mu = -\frac{i}{2}\bar{\eta}^3_{\mu\nu}\tau_3\partial_\nu \log\phi - \frac{i}{2}\phi\mathrm{Re}\left((\bar{\eta}^1_{\mu\nu} - i\bar{\eta}^2_{\mu\nu})(\tau_1 + i\tau_2)\partial_\nu\chi\right). \tag{4.3.23}$$

For $\omega = 0$ this reduces to the Harrington-Shepard solution of the caloron with trivial holonomy, since in that case $\chi = 1 - \phi^{-1}$. Cf. eq. (1.3.13).

The selfduality of eq. (4.3.23) follows from eq. (3.1.8), but has also been checked numerically. In the asymptotic regime of large distances $|\vec{x}|$,

$$\hat{f}_x(z,z') \approx \frac{\pi}{|\vec{x}|}e^{-2\pi|\vec{x}||z-z'|+2\pi i x_0(z-z')}, \tag{4.3.24}$$

(with $|z-z'|$ the obvious distance function on the circle) from which it follows that A_μ tends to zero at spatial infinity. The holonomy at spatial infinity is then fully carried by the cocycle, and equals $e^{2\pi i\vec{\omega}\cdot\vec{\tau}}$ as required. The non-trivial holonomy becomes even more transparent in the periodic gauge by performing a gauge transformation $g(x) = e^{-2\pi i x_0\vec{\omega}\cdot\vec{\tau}}$. This yields

$$A^{\mathrm{per}}_\mu = -\frac{i}{2}\bar{\eta}^3_{\mu\nu}\tau_3\partial_\nu \log\phi - \frac{i}{2}\phi\mathrm{Re}\left((\bar{\eta}^1_{\mu\nu} - i\bar{\eta}^2_{\mu\nu})(\tau_1 + i\tau_2)(\partial_\nu + 4\pi i\omega\delta_{\nu,0})\tilde{\chi}\right) + \delta_{\mu,0}2\pi i\omega\tau_3, \tag{4.3.25}$$

where

$$\tilde{\chi} \equiv e^{-4\pi i x_0\omega}\chi = \frac{\pi\rho^2}{\psi}\left\{e^{-2\pi i x_0}s^{-1}\sinh(4\pi s\omega) + r^{-1}\sinh(4\pi r\bar{\omega})\right\}. \tag{4.3.26}$$

In this gauge we immediately read off the constant background field at spatial infinity, $A^{\mathrm{per}}_\mu = 2\pi i\vec{\omega}\cdot\vec{\tau}\delta_{\mu,0}$, responsible for the holonomy in the periodic gauge. This concludes the construction of the caloron solution.

To use eq. (3.1.29) for the action density we have to regularise the determinant, which for calorons diverges. However, $\partial_\mu \log\det f_x$ turns out to be finite. With the help of eq. (4.2.11) we find

$$\partial_\mu \log\det f_x = \partial_\mu \mathrm{Tr}\,\log f_x = \frac{1}{\pi i}\mathrm{Tr}\,\hat{D}^\mu_x f_x = \frac{1}{\pi i}\int_{S^1} dz\ \lim_{z'\to z}\hat{D}^\mu_x(z)\hat{f}_x(z,z'), \tag{4.3.27}$$

where Tr denotes the Hilbert space trace. We use point-splitting to define

$$\lim_{z'\to z}\frac{d}{dz}\hat{f}_x(z,z') \equiv \tfrac{1}{2}\left(\lim_{\epsilon\downarrow 0}\frac{d}{dz}\hat{f}_x(z+\epsilon,z') + \lim_{\epsilon\downarrow 0}\frac{d}{dz}\hat{f}_x(z-\epsilon,z')\right)\Bigg|_{z'=z}, \tag{4.3.28}$$

in accordance with the Fejér theorem for the convergence of Fourier series, see e.g. ref. [100]. In section 5.4 we will prove that

$$\mathrm{Tr}\,\hat{D}^\mu_x f_x = -\pi i\partial_\mu \log\psi, \tag{4.3.29}$$

leading to the following miraculously simple result

$$-\mathrm{Tr}F^2_{\mu\nu}(x) = \partial^2_\mu\partial^2_\nu \log\det f_x = -\partial^2_\mu\partial^2_\nu \log\psi. \tag{4.3.30}$$

Note that ψ is positive definite and smooth, despite its appearance. The same applies for eq. (4.3.30). The action density $-\frac{1}{2}\mathrm{Tr}F^2_{\mu\nu}(x)$ takes its maximal value at $x_0 = 0$. Eq. 4.3.30 was verified numerically, using eq. (4.3.23). Since the action density is a total derivative, one can express the total action in terms of a surface integral at spatial infinity. Using that $\partial^2_\mu \log\psi = 4\pi/|\vec{x}| + \mathcal{O}(|\vec{x}|^{-4})$, one easily verifies that for the topological charge

$$k = -\frac{1}{8\pi^2}\int \mathrm{tr}F\wedge F = -\frac{1}{16\pi^2}\int d_4x\ \mathrm{tr}F^2_{\mu\nu}(x) = 1. \tag{4.3.31}$$

In the appendix we give the expression for the Green's function G_x, from which it follows that

$$\partial^2_\mu\partial^2_\nu \log\det G_x = -\tfrac{1}{2}\partial^2_\mu\partial^2_\nu \log\hat{\psi}, \tag{4.3.32}$$

in accordance with eq. (4.2.15) and eq. (4.3.19).

4.4 Properties of the caloron solution

We first settle the issue of orientations in colour and real space. In general only the centre Z_2 of the group of global gauge transformations will leave the gauge potential invariant. The framing - embedding of the solution in colour space - is in general not invariant under global gauge rotations. For non-trivial holonomy ($\omega\bar{\omega} \neq 0$, or $\mathcal{P} \neq \pm 1$ at infinity), also the holonomy is not invariant under such global gauge rotations, except for a $U(1)$ subgroup generated by $\hat{\omega}\cdot\vec{\tau}$, which in the monopole terminology generates the unbroken gauge symmetry. For each choice of the holonomy - which can not change under continuous deformations - we have a separate caloron parameter space. It should be noted that the spatial orientation is given by the preferred axis that appears in the formula for the action density and in the definition of the two radii r and s, eq. (4.3.13). The action density has an axial symmetry around $\hat{a}$ defined through

$$\hat{a}\cdot\vec{\tau} = \bar{q}\hat{\omega}\cdot\vec{\tau}q = \bar{\zeta}\hat{\omega}\cdot\vec{\tau}\zeta/\rho^2. \tag{4.4.1}$$

When q is part of the $U(1)$ subgroup generated by $\hat{\omega}\cdot\vec{\tau}$, it does not affect the orientation of the solution, and indeed can be pulled through in eq. (4.3.4), to be identified with the global gauge invariance of eq. (4.2.1) associated to this residual $U(1)$. The dimension of the moduli space of *gauge inequivalent* solutions at fixed holonomy, including the position of the caloron described by ξ_μ, is thus 7 for non-trivial and 5 for trivial holonomy. A global residual $U(1)$ gauge transformation (or a global $SU(2)$ gauge transformation in case of trivial holonomy), however, does change the framing of the solutions and the moduli space of framed calorons is 8 dimensional. Including these global gauge degrees of freedom will reveal the hyperKähler structure of the moduli space, to be discussed in the next section.

Since a global gauge transformation leaves $\hat{a}$ invariant, we can best describe the parameters of the solutions for the choice where $\hat{\omega} = \hat{e}_3$, i.e. $\hat{\omega}$ is pointing in the positive x_3-direction. Due to the residual gauge group, to any point on the two sphere defined by the symmetry axes of the caloron solution, a full $U(1)$ can be associated. This gives the Hopf fibration of $S^3 = SU(2)$ over S^2, the fiber being $U(1)$. Using Euler angles we may choose the parametrisation

$$q = e^{i\Upsilon\frac{\tau_3}{2}} e^{i(\frac{\pi}{2}-\theta)\frac{\tau_2}{2}} e^{-i\varphi\frac{\tau_1}{2}}, \quad 0 \le \Upsilon \le 4\pi, \quad 0 \le \varphi \le 2\pi, \quad 0 \le \theta \le \pi, \tag{4.4.2}$$

which leads, for $\hat{\omega} = \hat{e}_3$, to the axis of axial symmetry $\hat{a} = (\cos\theta, \sin\theta\sin\varphi, \sin\theta\cos\varphi)$. The variable Υ describes the residual $U(1)$ gauge group generated by $\tau_3 (= \hat{\omega}\cdot\vec{\tau})$. With $q = q_\mu \sigma_\mu$ ($|q| = 1$) we can introduce the Maurer-Cartan one-forms

$$\Sigma_i = 2\eta^i_{\mu\nu} q_\mu dq_\nu, \quad d\Sigma_i = -\tfrac{1}{2}\varepsilon_{ijk}\Sigma_j \wedge \Sigma_k. \tag{4.4.3}$$

In terms of the Euler angles these read

$$\begin{aligned} \Sigma_1 &= -\cos\Upsilon \sin\theta d\varphi + \sin\Upsilon d\theta, \\ \Sigma_2 &= \sin\Upsilon \sin\theta d\varphi + \cos\Upsilon d\theta, \\ \Sigma_3 &= d\Upsilon + \cos\theta d\varphi. \end{aligned} \tag{4.4.4}$$

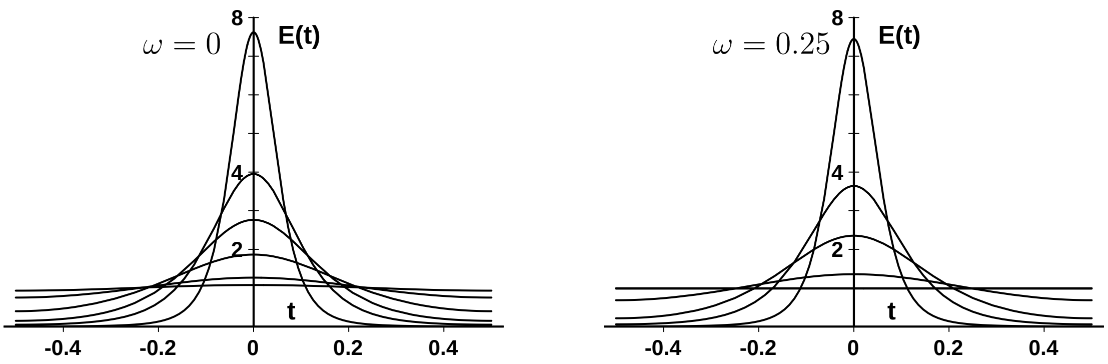


Figure 4-2. Time evolution of the caloron solution. During one period ($\mathcal{T} = 1$), we plot the "energy" as a function of time, $E(t) \equiv -\frac{1}{16\pi^2}\int_{\mathbb{R}^3} d_3x \, \mathrm{tr} F^2_{\mu\nu}$, for $\rho = 0.1, 0.2, 0.3, 0.5, 1.0, 2.0$. For small values of ρ, the caloron is short-lived and instanton-like, whereas for large values, $\rho > 1$, the profile flattens and the caloron becomes static and monopole-like.

In order to visualise the caloron solution, we can use eq. (4.3.30). A time-slice of the caloron shows that it generically consists of two lumps. In figs. 4-2 and 4-3 the time dependence is studied for various values of ρ. For small ρ the caloron approaches the ordinary single instanton solution, with no dependence on ω, as $\rho \to 0$ is equivalent to $\mathcal{T} \to \infty$. For $\rho = 0$ the action density is concentrated in one point and we are dealing with an ideal, delta function like instanton. Finite size effects set in when the size of the instanton becomes of the order of the compactification length $\mathcal{T}$, i.e. when the caloron bites in its own tail. This occurs at roughly $\rho = \frac{1}{2}\mathcal{T}$. At this point, for $\omega\bar{\omega} \ne 0$ (i.e. the holonomy $\mathcal{P} \ne \pm 1$), two lumps are formed, whose

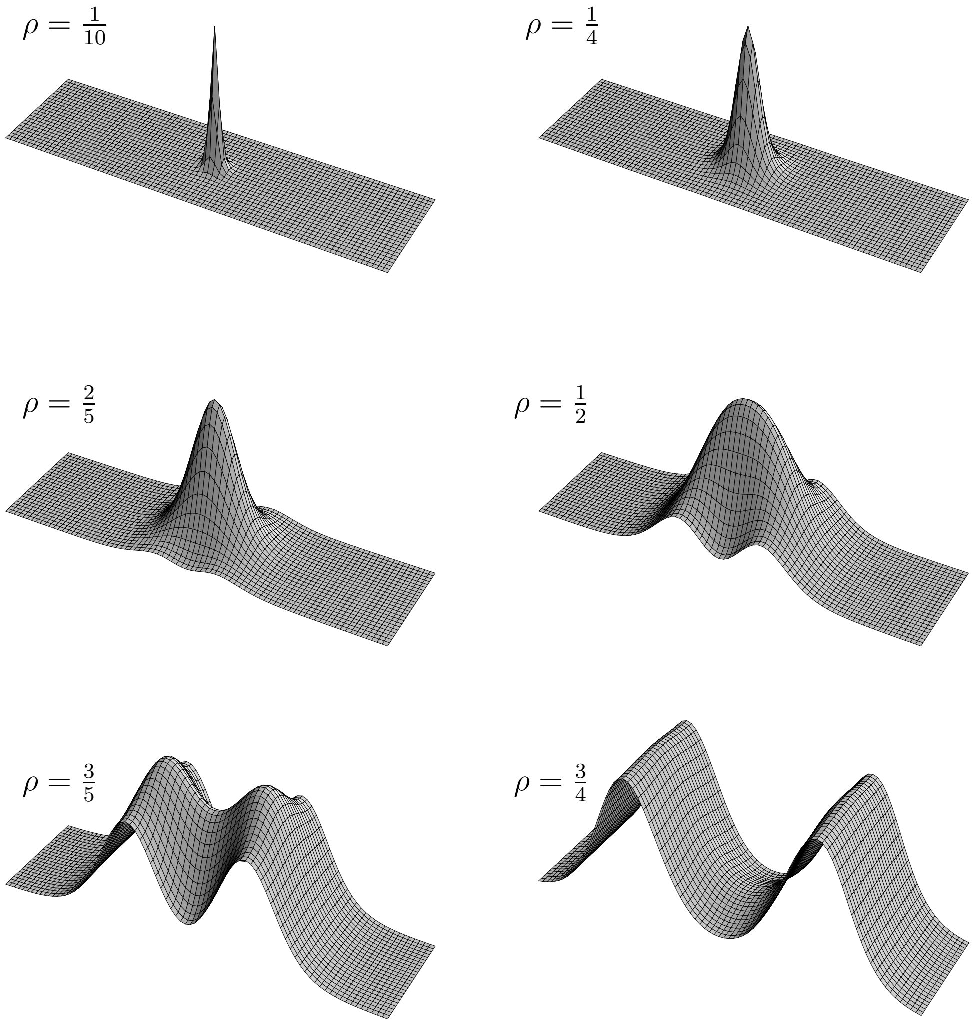


Figure 4-3. Finite size effects: from instanton to monopole. Action densities in the $z-t$-plane for $x=y=0$, $\omega=\frac{1}{4}$, $\rho=\frac{1}{10}, \frac{1}{4}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}$, and $\frac{3}{4}$ in units of $\mathcal{T}$ (left to right and top to bottom) for the $SU(2)$ $k=1$ caloron. The maxima correspond to $-\mathrm{Tr}F_{\mu\nu}^2 = 9.28\times10^5$, 1.98×10^4, 2.05×10^3, 560, 210 and 128.

separation grows as $\pi\rho^2/\mathcal{T}$ (cf. eq. (4.3.13)). These finite size effects are depicted in fig. 4-3.

At large ρ the solution spreads out over the entire circle in the euclidean time direction and becomes static in the limit $\rho\to\infty$. So for large ρ the lumps are well separated, see fig. 4-4. When far apart they become spherically symmetric. As they are static and selfdual they are necessarily BPS monopoles. One can show in this limit that they have unit, but opposite, magnetic charges and that the two lumps have spatial scales proportional to respectively $1/\bar{\omega}$ and $1/\omega$ (see sect. 4.6). Their action densities (or energy densities in this static limit) scale with $\bar{\omega}^4$ and ω^4. After integration, this results in monopole masses of respectively $16\pi^2\bar{\omega}/\mathcal{T}$ and $16\pi^2\omega/\mathcal{T}$ for the two lumps, their mass ratio is therefore $\bar{\omega}/\omega$. The total energy, simply obtained by addition, indeed conforms with the unit topological charge of the solution. For

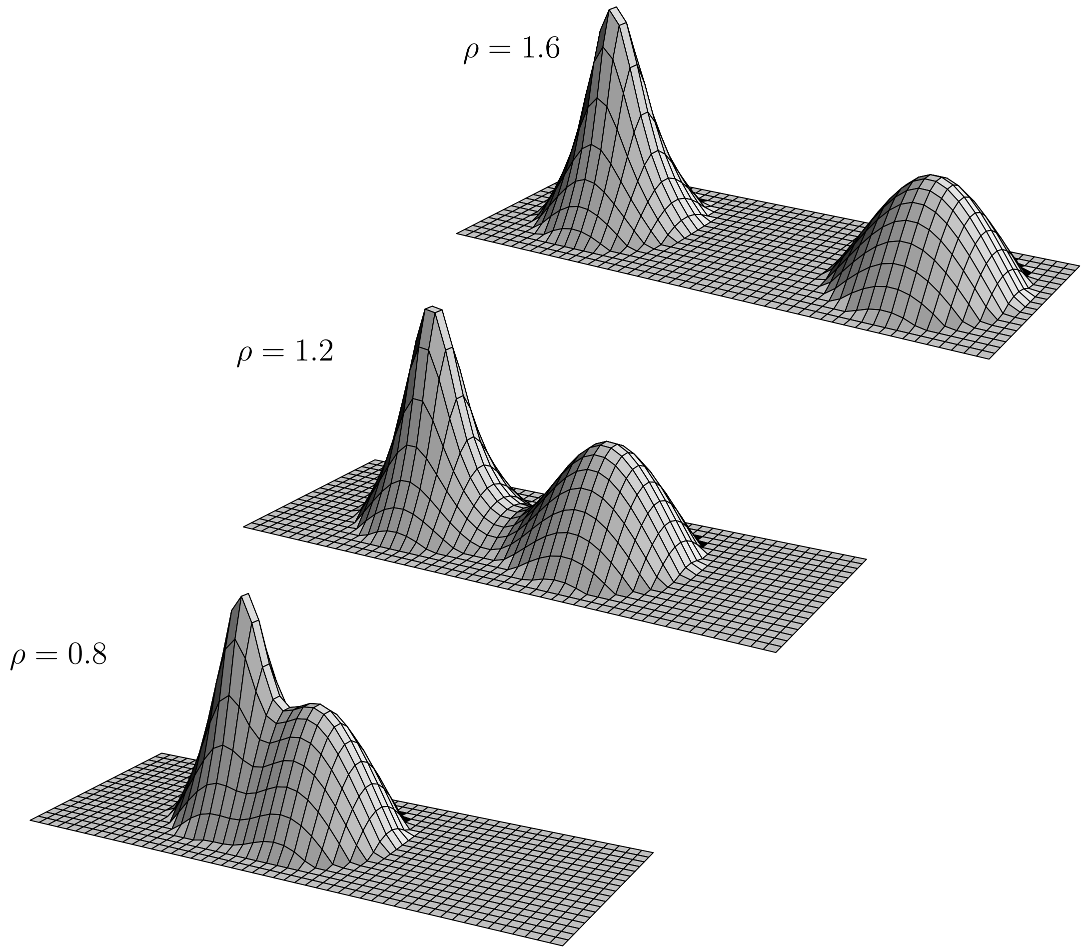


Figure 4-4. Shown are caloron profiles for $\omega = 0.125$ ($\mathcal{T} = 1$), with $\rho = 0.8, 1.2, 1.6$ (from bottom to top). This illustrates the growing separation of the two lumps with ρ. Once the constituents are separated, the lumps are spherically symmetric and do not change their shape upon further separation. Vertically is plotted the action density at $x_0 = 0$, on equal logarithmic scales for all profiles. They were cut off at an action density below $1/e^2$.

$\omega = 0$ or $\omega = \frac{1}{2}$, the second lump is absent and the solution is spherically symmetric. For generic ω, the solution has only an axial symmetry around the axis $\hat{a}$ connecting the two lumps. For $\omega = \frac{1}{4}$, the lumps are equally sized, and the solution has a mirror symmetry in the plane perpendicular to $\hat{a}$, see fig. 4-5.

These aspects can be readily retrieved by inspecting eq. (4.3.23), and in particular eq. (4.3.30), for the limit of large ρ and realising that r and s are the centre off mass radii of the constituent monopoles. If ρ is small, the caloron is best described in terms of the instanton picture, whereas for large ρ the two-monopole picture is more appropriate. For the constituent picture of oppositely charged BPS monopoles to be correct, the field has to behave like a magnetic (and electric) dipole at large distances. Indeed one easily finds that the field strength decays as $1/|\vec{x}|^3$ for distances much larger than $\pi\rho^2/\mathcal{T}$. Note that for $\omega = 0$ we have the standard Harrington-Shepard

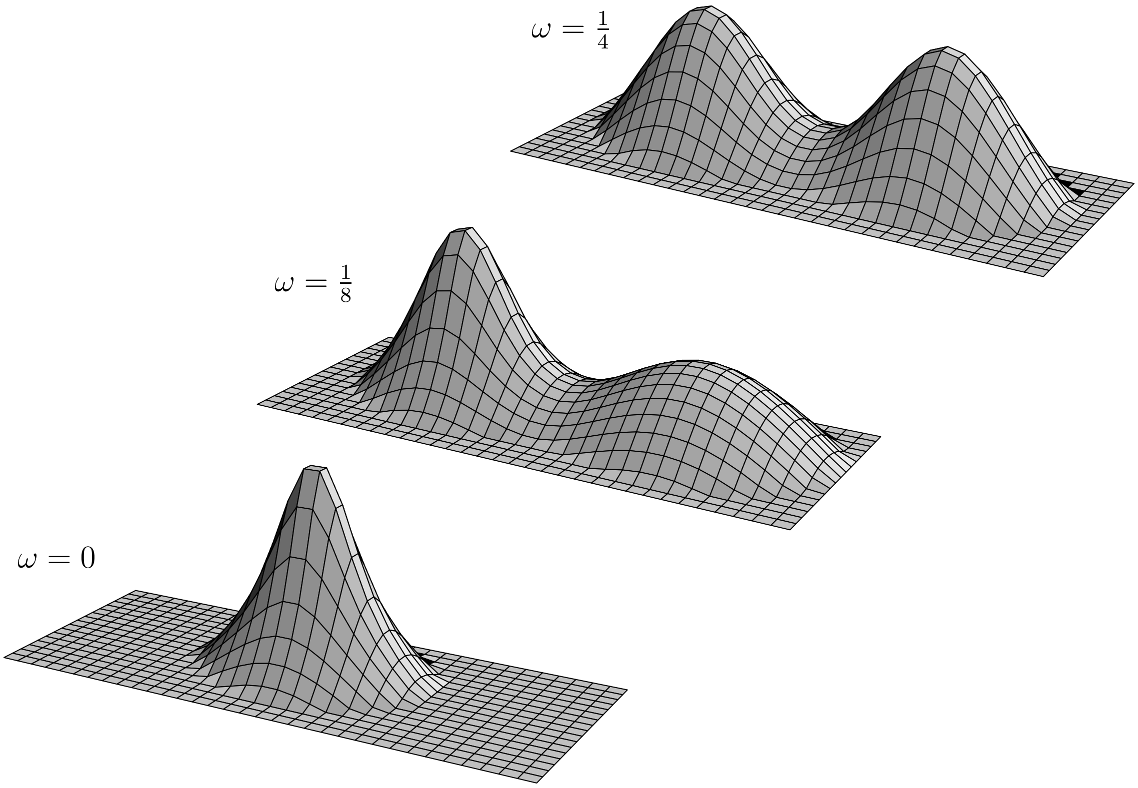


Figure 4-5. Profiles for calorons at $\omega = 0, \frac{1}{8}, \frac{1}{4}$(from left to right) with $\rho = \mathcal{T} = 1$. The axis connecting the lumps, separated by a distance π (for $\omega \neq 0$), corresponds to the direction of $\hat{a}$. The other direction indicates the distance to this axis, making use of the axial symmetry of the solutions. The mass ratio of the two lumps is approximately $\omega/\bar{\omega}$, i.e. zero (no second lump), a third and one (equal masses), for the respective values of ω. Vertically is plotted the action density at $x_0 = 0$, on equal logarithmic scales for all profiles. They were cut off at an action density below $1/e$.

caloron [42], which in the limit of large ρ was already shown by Rossi [88] to become the standard BPS monopole (after a singular gauge transformation), as discussed in the section 1.3.

Interpreting the Nahm data (eq. (4.3.9)) as the juxtaposition of two sub-intervals of lengths 2ω and $2\bar{\omega}$ respectively, with constant Nahm connections $\hat{A}(z)$, leads to a more indirect way of understanding the composite nature of the caloron. Indeed, $\hat{A} = 0$ gives the standard BPS-monopole, adding a constant merely translates the solution in space. This was described in detail in section 2.3.2. Thus, each interval gives rise to a BPS monopole on $\mathbb{R}^3 \times S^1$, and we can in a good approximation add the two connections corresponding to the two sub-intervals. The ρ^2 dependence of $\hat{A}(z)$ explains the large separation of the constituent lumps for large ρ. As the lengths of the intervals are given by the asymptotic Higgs vacuum expectation value of the corresponding monopoles, the mass ratio $\bar{\omega}/\omega$ of the lumps is easily explained by noting that the mass of a BPS monopole is proportional to the Higgs vacuum expectation value. The above interpretation underlies the approach taken in ref. [58, 59]. The expression for the gauge potential given there is precisely the sum alluded to above, plus gauge-like terms and gauge transformations required for glueing them together.

4.5 The moduli space

The moduli space $\mathcal{M}$ of the caloron solutions has as its coordinates ξ and ζ. The global gauge degrees of freedom ($SU(2)$ for trivial and $U(1)$ for non-trivial holonomies) should be included, so as to make the solution space hyperKähler, cf. section 3.1. The moduli space of these so-called framed calorons is a product of the base manifold $\mathbb{R}^3 \times S^1$, parametrised by ξ and the (Taub-NUT) space parametrised by ζ, forming the non-trivial part of the moduli space and describing the relative coordinates of the two constituent monopoles, quite similar to that for the two-monopole moduli space [3]. It should be noted that $\zeta \to -\zeta$, corresponding to the centre of the $SU(2)$, leaves $A_\mu(x)$ invariant, such that we have to divide out this symmetry to obtain the space of framed calorons.

As explained in section 1.5.2, the metric on this space is given by the Riemannian metric on the gauge theory configuration space, restricted to the space of solutions. The metric is then given by ($g_M^{\mu\nu} = \delta^{\mu\nu}$ being the flat metric on $M = \mathbb{R}^3 \times S^1$)

$$g_{\mathcal{M}}(Z, Z') = \int_M g_M^{\mu\nu} \mathrm{tr}\left(Z_\mu^\dagger(x) Z'_\nu(x)\right). \tag{4.5.1}$$

with Z_μ and Z'_μ two vectors tangent to the space of caloron solutions. The gauge structure requires them to be transverse to gauge deformations, thereby satisfying the background (or Coulomb) gauge condition

$$D_\mu^{\mathrm{ad}}(A) Z^\mu = 0. \tag{4.5.2}$$

The requirement that $A_\mu + Z_\mu$ is selfdual leads to the so-called deformation equations

$$D^{\mathrm{ad}}_{[\mu} Z_{\nu]} = \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta} D^{\mathrm{ad}}_{[\alpha} Z_{\beta]}, \tag{4.5.3}$$

and in the algebraic gauge we have to require in addition (see eq. (4.1.1))

$$Z_\mu(x+1) = e^{2\pi i \vec{\omega}\cdot\vec{\tau}} Z_\mu(x) e^{-2\pi i \vec{\omega}\cdot\vec{\tau}}. \tag{4.5.4}$$

The tangent vectors (zero modes) can be found by varying the caloron solution with respect to the coordinates ξ and ζ, which will automatically satisfy the deformation equation eq. (4.5.3) and periodicity eq. (4.5.4), but generally one has to apply an infinitesimal gauge transformation Φ^r, compatible with eq. (4.5.4), to transform to the Coulomb gauge, eq. (4.5.2). Hence,

$$Z_\mu^r = \delta_r A_\mu + D_\mu \Phi^r, \tag{4.5.5}$$

where the label r indicates the parameters (or coordinates) of the moduli space. For the metric (eq. (4.5.1)) to exist, zero modes should of course be normalisable.

For instantons on $\mathbb{R}^4$, the zero modes can be determined within the ADHM formalism [81], as was discussed in section 3.1. Thus one can calculate the metric in terms of the ADHM data. This reflects the fact that the Nahm transformation (and the ADHM construction) is a hyperKähler isometry [26, 14]. We have, also for the framed data,

$$\delta\Delta = \begin{pmatrix} \delta\lambda \\ \delta B \end{pmatrix} \equiv \begin{pmatrix} c \\ Y \end{pmatrix} = C, \tag{4.5.6}$$

and for the calorons, in addition periodicity (eq. (4.5.4)) requires

$$Y_{p,p'} = Y_{p-1,p'-1}, \quad c_{p+1} = e^{2\pi i \vec{\omega}\cdot\vec{\tau}} c_p. \tag{4.5.7}$$

In terms of the deformation C of the framed ADHM data, the zero mode reads [81]

$$Z_\mu = -v^\dagger(x) C \bar{\sigma}_\mu f_x u(x) \phi^{-\frac{1}{2}}(x) + \phi^{-\frac{1}{2}}(x) u^\dagger(x) f_x \sigma_\mu C^\dagger v(x). \tag{4.5.8}$$

and

$$D_\mu^{\rm ad} Z^\mu(x) = \phi^{-1} u^\dagger(x) f_x \sigma_\mu \left(C^\dagger \Delta(x) - \Delta^\dagger(x) C\right) \bar{\sigma}_\mu f_x u(x). \tag{4.5.9}$$

(Note that for $W = W_\mu\sigma_\mu$, $\sigma_\mu W \bar{\sigma}_\mu = 4W_0 = 2\mathrm{tr}_2 W$.) Combining the deformation of the quadratic ADHM constraint (as the $\Im$ part) with the Coulomb gauge condition (as the $\Re$ part), imposes eq. (3.1.14) which for framed $SU(2)$ ADHM data reduces to

$$(\Delta^\dagger(x) C) = (\Delta^\dagger(x) C)^t. \tag{4.5.10}$$

To satisfy the Coulomb gauge condition, the $\Re$ part of this equation being equivalent to

$$\mathrm{tr}_2(\Delta^\dagger(x) C - C^\dagger \Delta(x)) = 0, \tag{4.5.11}$$

one has only the T invariance of eq. (4.2.1) available (i.e. the $U(1)$ gauge invariance of $\hat{A}$), since a global gauge rotation would distort the framing. We write $T \in O(k)$ as $T = \exp(-\delta X)$, with $\delta X^t = -\delta X$. Like T, also δX has to satisfy $\delta X_{p,p'} = \delta X_{p-p'}$, and can be interpreted as the Fourier coefficients of an infinitesimal gauge function $\delta\hat{g}(z)$ on the circle. We now write the zero modes C in ADHM language as a variation of Δ plus a compensating gauge transformation,

$$C = \delta\Delta + \delta_X \Delta, \quad \delta_X \Delta = \begin{pmatrix} \lambda\delta X \\ [B, \delta X] \end{pmatrix}. \tag{4.5.12}$$

Inserting this in eq. (4.5.11) we find

$$\mathrm{tr}_2\left((\delta\Delta^\dagger)\Delta - \Delta^\dagger\delta\Delta - 2\delta X\Lambda + \left[B^\dagger, \delta X\right] B - B^\dagger \left[B, \delta X\right]\right) = 0, \tag{4.5.13}$$

cf. eq. (3.1.18). where we used that $[\Lambda, \delta X] = 0$, since the gauge symmetry (described by δX) is abelian. After Fourier transformation, with $\delta(z - z')\delta\hat{X}(z) = \sum_{p,p'} X_{p,p'}(x) e^{2\pi i (pz - p'z')}$, this equation reads

$$-\frac{1}{4\pi^2}\frac{d^2\delta\hat{X}(z)}{dz^2} + |\zeta|^2(\delta(z-\omega) + \delta(z+\omega))\delta\hat{X}(z) = -\frac{i}{2}\rho^2(\delta(z-\omega) - \delta(z+\omega))\hat{\omega}\cdot\vec{\Sigma}, \tag{4.5.14}$$

using that with the help of the Maurer-Cartan one-forms, eq. (4.4.3), we can write

$$\mathrm{tr}_2\left((\delta\zeta\bar\zeta-\zeta\delta\bar\zeta)\hat\omega\cdot\vec\sigma\right)=4\hat\omega^a\eta^a_{\mu\nu}\zeta_\mu\delta\zeta_\nu=2\rho^2\hat\omega\cdot\vec\Sigma. \tag{4.5.15}$$

The solution to the differential equation for $\delta\hat X(z)$ gives the infinitesimal gauge transformations needed to go to Coulomb gauge. One finds

$$\delta\hat X(z)=2\pi^2 i\rho^2\hat\omega\cdot\vec\Sigma(1+8\pi^2\omega\bar\omega\rho^2)^{-1}\int_0^z\Theta_\omega(z')dz' \tag{4.5.16}$$

which is a zig-zag wave (see fig. 4-6), vanishing at $2z\in\mathbb{Z}$ and taking its extremal values at $z=\pm\omega$,

$$\delta\hat X_\omega\equiv\pm\delta\hat X(\pm\omega)=4i\pi^2\omega\bar\omega\rho^2\hat\omega\cdot\vec\Sigma(1+8\pi^2\omega\bar\omega\rho^2)^{-1}. \tag{4.5.17}$$

Note that for the variations with respect to the caloron position, ξ, no compensating gauge transformation is needed.

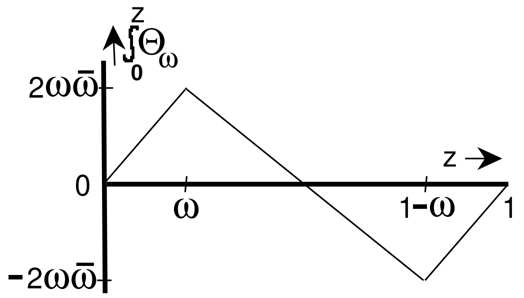


Figure 4-6. The function $\int_0^z\Theta_\omega(z')dz'$.

In order to evaluate the metric, eq. (4.5.1), it is sufficient to compute $g_{\mathcal M}(Z,Z)$ for Z related to an arbitrary deformation of the moduli parameters, as determined by C in eq. (4.5.12) and eq. (4.5.16). For this we employ the relation (3.1.19) due to Corrigan, adapted to the $SU(2)$ situation

$$\begin{aligned}\mathrm{Tr}(Z_\mu^\dagger(x)Z_\mu(x))&=-\tfrac12\partial^2\,\mathrm{tr}\,\mathrm{Tr}\,\left(C^\dagger(2-\Delta(x)f_x\Delta^\dagger(x))Cf_x\right)\\&=-\tfrac12\partial^2\,\mathrm{tr}\,\mathrm{Tr}\,\big(2(c^\dagger c+Y^\dagger Y)f_x\\&\qquad-(c^\dagger\lambda+Y^\dagger B(x))f_x(\lambda^\dagger c+B^\dagger(x)Y)f_x\big)\,,\end{aligned} \tag{4.5.18}$$

which is derived from eq. (4.5.8), making use of eq. (4.5.10). We introduce the Fourier transforms $\hat c(z)\equiv\sum_p\exp(2\pi ipz)c_p$ and $\hat Y(z)$, with

$$\delta(z-z')\hat Y(z)=\sum_{p,p'}e^{2\pi i(pz-p'z')}Y_{p,p'}, \tag{4.5.19}$$

such that

$$\begin{aligned}\hat c(z)&=\delta\hat\lambda(z)+\hat\lambda(z)\delta\hat X(z)=P_+\delta(z-\omega)(\delta\zeta+\zeta\delta\hat X_\omega)+P_-\delta(z+\omega)(\delta\zeta-\zeta\delta\hat X_\omega),\\\hat Y(z)&=\frac{1}{2\pi i}\left(\delta\hat A(z)+\frac{d}{dz}\delta\hat X(z)\right)\\&=\delta\xi+\pi\left(-\delta\bar\zeta\hat\omega\cdot\vec\sigma\zeta-\bar\zeta\hat\omega\cdot\vec\sigma\delta\zeta+\rho^2\hat\omega\cdot\vec\Sigma(1+8\pi^2\omega\bar\omega|\zeta|^2)^{-1}\right)\Theta_\omega(z).\end{aligned} \tag{4.5.20}$$

We may use the special structure of $\hat{c}(z)$ and $\hat{\lambda}(z)$ (eq. (4.3.5)) - formed out of the combinations $\delta(z\mp\omega)P_\pm$, with $P_\pm^2=P_\pm$ and $P_\pm P_\mp=0$ - to deduce

$$\hat{c}^\dagger(z)\hat{c}(z')=\delta(z-z')\hat{c}^\dagger(z)<\hat{c}>,\quad \hat{c}^\dagger(z)\hat{\lambda}(z')=\delta(z-z')\hat{c}^\dagger(z)<\hat{\lambda}>,\quad <h>\equiv\int_{S^1}h(z)dz. \tag{4.5.21}$$

For calorons, this allows us to turn Corrigan's identity into

$$\begin{aligned}\mathrm{tr}_2(Z_\mu^\dagger(x)Z_\mu(x)) = {}& -\partial^2\int_{S_1}dz\ \mathrm{tr}_2\left(\left[\hat{Y}^\dagger(z)\hat{Y}(z)+\hat{c}^\dagger(z)<\hat{c}>\right]\hat{f}_x(z,z)\right) \qquad (4.5.22)\\ &+\tfrac{1}{2}\partial^2\int_{S^1}dzdz'\ \mathrm{tr}\left([\hat{\mathcal{C}}(z)+\hat{\mathcal{Y}}_x(z)]\hat{f}_x(z,z')[\hat{\mathcal{Y}}_x^\dagger(z')+\hat{\mathcal{C}}^\dagger(z')]\hat{f}_x(z',z)\right),\end{aligned}$$

with

$$\hat{\mathcal{Y}}_x(z)=(2\pi i)^{-1}\hat{Y}^\dagger(z)\hat{D}_x(z),\quad \hat{\mathcal{C}}(z)=\hat{c}^\dagger(z)<\hat{\lambda}>. \tag{4.5.23}$$

In the integration over space-time, the ∂_0^2 part does not contribute due to periodicity. The integral is therefore reduced to a boundary term at spatial infinity, $|\vec{x}|\to\infty$. In this limit ρ^2 can be neglected and the two radii r and s in eq. (4.3.13) become equal. In particular in this limit the potential in eq. (4.3.12) equals $|\vec{x}|^2$, independent of z. From this one concludes that asymptotically $\hat{f}_x(z,z')$ becomes a function of $z-z'$ and therefore that $\hat{D}_x(z)\hat{f}_x(z,z')\hat{D}_x^\dagger(z')=4\pi^2\delta(z-z')+\mathcal{O}(r^{-1})$. This can also be deduced from the asymptotic form of $\hat{f}_x(z,z')$ in eq. (4.3.24), which implies $\hat{f}_x(z,z)=\pi/r+\mathcal{O}(r^{-2})$. Thus $\int_{S^1}dz'\ \mathrm{tr}_2[\hat{\mathcal{Y}}_x(z)\hat{f}_x(z,z')\hat{\mathcal{Y}}_x(z')\hat{f}_x(z,z')]=\mathrm{tr}_2[\hat{Y}^\dagger(z)\hat{Y}(z)\hat{f}_x(z,z)]$, to be combined with the first term in eq. (4.5.22). Using eq. (4.5.21), we also have $\int_{S^1}dz'\ \mathrm{tr}_2[\hat{\mathcal{C}}(z)\hat{f}_x(z,z')\hat{\mathcal{C}}^\dagger(z')\hat{f}_x(z,z')]=\mathrm{tr}_2[<\hat{c}^\dagger><\hat{\lambda}><\hat{\lambda}^\dagger>\hat{c}(z)\hat{f}_x^2(z,z)]=\mathcal{O}(r^{-2})$, and $2\pi i\int_{S^1}dz'\ \mathrm{tr}_2[\hat{\mathcal{Y}}(z)\hat{f}_x(z,z')\hat{\mathcal{C}}^\dagger(z')\hat{f}_x(z,z')]=\sum_{s=\pm}\mathrm{tr}_2[\hat{Y}^\dagger(z)D_x(z)f_x(z,s\omega)<\hat{\lambda}^\dagger><\hat{c}>P_sf_x(s\omega,z)]$, which after integration over z is $\mathcal{O}(r^{-2})$. Only those terms that are $\mathcal{O}(r^{-1})$ will contribute and we obtain the following remarkably simple result

$$g_{\mathcal{M}}(Z,Z)=2\pi^2\mathrm{tr}(<\hat{Y}^\dagger\hat{Y}>+2<\hat{c}^\dagger><\hat{c}>). \tag{4.5.24}$$

Inserting eq. (4.5.20) gives the metric (we put $\hat{\omega}=\hat{e}_3$)

$$ds^2=2\pi^2\left\{2d\xi_\mu d\xi^\mu+(1+8\pi^2\omega\bar{\omega}\rho^2)\left(4d\rho^2+\rho^2\left(\Sigma_1^2+\Sigma_2^2\right)\right)+\rho^2(1+8\pi^2\omega\bar{\omega}\rho^2)^{-1}\Sigma_3^2\right\}. \tag{4.5.25}$$

The first part describes the flat metric of the base manifold $\mathbb{R}^3\times S^1$, the remainder forms the non-trivial part of the metric. They separate because $\int\Theta_\omega(z)dz=0$, see eq. (4.3.10).

We introduce a "radial" coordinate X and "mass" parameter M

$$X^2=8\pi^2\rho^2,\quad M^{-2}=16\omega\bar{\omega}, \tag{4.5.26}$$

and rewrite the non-trivial part of the moduli space metric as

$$ds^2=\left(1+\frac{X^2}{16M^2}\right)\left(dX^2+\tfrac{1}{4}X^2(\Sigma_1^2+\Sigma_2^2)\right)+\tfrac{1}{4}X^2\left(1+\frac{X^2}{16M^2}\right)^{-1}\Sigma_3^2. \tag{4.5.27}$$

This metric is the Taub-NUT metric [79] as given in [45]. It is a selfdual Einstein manifold [44, 28, 38] and is hyperKähler [3, 47]. The latter property is inherited from the hyperKähler structure of the base manifold $\mathbb{R}^3 \times S^1$, preserved by the selfduality equations [26]. Therefore, the $SU(2)$ moduli space for calorons becomes

$$\mathcal{M}_{\text{framed}} = (\mathbb{R}^3 \times S^1) \times \text{Taub-NUT}/Z_2. \tag{4.5.28}$$

Note that Z_2 corresponds with $\zeta = q = \pm 1$, i.e. the centre of the SU(2) gauge group. With $\zeta \to -\zeta$ leaving the gauge fields unchanged, this gives rise to an orbifold singularity (at $\zeta\!=\!0$) and $(\mathbb{R}^3 \times S^1) \times \text{Taub-NUT}$ is a double cover of $\mathcal{M}_{\text{framed}}$.

For small ρ or ω, when $X^2/M^2 \to 0$, the metric becomes that of $\mathbb{R}^4$, since $\frac{1}{4}(\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2)$ (see eq. (4.4.3)) is the metric on the unit three-sphere. With $\rho \to 0$ corresponding to $\mathcal{T} \to \infty$, this describes the moduli space of a charge one instanton on $\mathbb{R}^4$, whereas for $\omega = 0$ we have the standard Harrington-Shepard caloron moduli space. In both cases this is parametrised by the scale and $SU(2)$ group orientation (to make the moduli space hyperKähler) and $\mathcal{M}_{\text{framed}} = \mathbb{R}^4 \times \mathbb{R}^4/Z_2$, see ref. [26, 34].

For large ρ (i.e. large X), or equivalently for $\mathcal{T} \to 0$, Taub-NUT space is a squashed S^3, that is $S^2 \times S^1$, with S^1 a non-trivial (Hopf) fibration over S^2. This is best studied by introducing a radial coordinate R through $X^2 = 8MR$, which brings the Taub-NUT metric to the form [44]

$$ds^2 = \left(1 + \frac{2M}{R}\right)(dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)) + 4M^2\left(1 + \frac{2M}{R}\right)^{-1}(d\Upsilon + \cos\theta d\phi)^2, \tag{4.5.29}$$

also familiar from the (spatial part of the) Kaluza-Klein monopole solution [90, 40]. Since $\Upsilon \in [0, 4\pi]$ we read-off from the asymptotic form of the metric that the compactification radius equals $4M$. At large ρ, $\mathcal{M}_{\text{framed}}$ is therefore of the form $(\mathbb{R}^3 \times S^1) \times (\mathbb{R}^3 \times' S^1)/Z_2$. It is natural to view this as the product space of two single BPS monopole moduli spaces. The first $\mathbb{R}^3$ represents the centre of mass and the second the relative coordinates. The first S^1 corresponds to x_0 and can be seen as a global $U(1)$. The other gives the relative $U(1)$ orientation on which the Z_2 acts. This Z_2 does *not* act on the positions, as the monopoles have in general different masses and are hence not identical objects. A similar description is valid for the $SU(2)$ two-monopole moduli space, see ref. [36, 3]. At *large* separations its metric is precisely of the Taub-NUT form, but with a *negative* mass parameter M, as was shown by Manton [71], using the asymptotic form of the interactions. The complete metric was constructed by Atiyah and Hitchin (see ref. [3]) in terms of elliptic integrals, using its symmetries and hyperKähler structure. As is clear from the expression of the metric, the point $\rho = 0$ gives no singularity in the metric, which behaves like the flat metric on $\mathbb{R}^4$. However, as a Z_2 has to be divided out there is a conic orbifold singularity, corresponding to the constituents sitting on top of each other when the action density is concentrated in one point. Then the size of the lump is no longer given by the inverse masses of the monopoles but by ρ itself, resulting in an ideal, delta function like instanton.

Finally it is interesting to note that the moduli space of an $SU(3)$ monopole with maximal symmetry breaking to $U(1)\times U(1)$ and charges $(1,1)$ (see ref. [19, 35, 60]) is Taub-NUT with a positive mass parameter, as for the caloron. This is not surprising, as its Nahm data, see the appendix of chapter 6, are similar to those of the caloron. As the metric can be formulated in terms of the Nahm data, we would indeed expect the metric to be similar. More remarks on this issue can be found in 6.4.

4.6 Discussion

The interpretation of the ADHM data for a periodic instanton as the Fourier coefficients of the Nahm transformed Weyl operator extends naturally to higher charges and other gauge groups [64] as will be exploited in the next chapters. For charge one calorons in $SU(N)$ the determination of the Green's function remains a problem of quantum mechanics on the circle with a piecewise constant potential (on n subintervals, separated by delta functions), see chapter 5. Also the formalism to compute the metric on the moduli space will be generalised, in chapter 6. Due to the relation with the ADHM approach, one may wonder [63] whether there is some advantage in obtaining monopole solutions from the calorons by sending certain scales to infinity - in the limit of which the solution becomes static and constituents separate. Indeed, such a limit will be taken in section 5.5.1. For higher charges the Nahm bundle is no longer abelian and the construction is more complicated. For generalisations to further compactifications, e.g. $\mathbb{R}^2 \times T^2$ and $\mathbb{R} \times T^3$ (see ref. [6]), note that the 't Hooft Ansatz [51, 87] diverges when summing over more than one direction. This will correspond to all holonomies trivial and one may well have no solutions in that case. A dramatic particular example of a non-existence proof for charge one instantons is T^4, see ref. [14], a situation where indeed an existence proof of Taubes [96] does not hold.

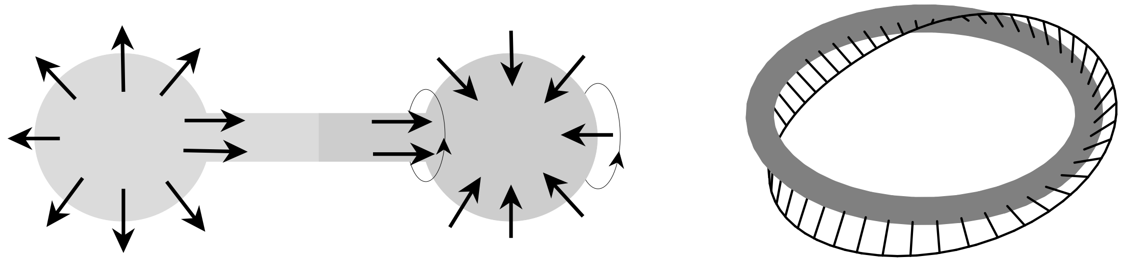


Figure 4-7. The non-contractible loop is constructed from two oppositely charged monopoles by rotating one of them, as indicated on the left. On the right is a closed monopole line, rotating its frame when completing the circle.

We now recall briefly Taubes's arguments for building gauge fields with topological charge one out of monopole fields [95, 94]. Although his construction was within the Standard Model with a genuine Higgs field, the same argument applies to the caloron case, using A_0 as the Higgs field. As we saw in section 2.2 (eqs. (2.2.2,2.2.4)), nontrivial $SU(2)$ monopole fields can be classified by the winding number $k_1 = -k_2$ of maps from S^2 to $SU(2)/U(1) \sim S^2$. We consider at this point configurations at a *fixed* time t, $\{A_\mu(\vec{x})\}$. In the sector where the net winding vanishes, we study a one-parameter family of configurations, $\{A_\mu(\vec{x},t)\}$ (the parameter can, but need not, be seen as the time t). When this configuration is made out of monopoles with opposite charges, in a suitable gauge the isospin orientations behave as shown in fig. 4-7, sufficiently far from the core of both monopoles. We note that the arrows match in the "throat" of the configuration. This remains true if we rotate *only one* of the monopoles around the axis of the throat. Clearly, the net magnetic winding remains zero, but the fields of two monopoles will no longer cancel when brought together, despite the fact that the long range abelian components do cancel. The non-contractible loop is now constructed by letting t affect a *full* rotation.

Taubes describes this by creating a monopole anti-monopole pair, bringing them far apart, rotating one of them over a full rotation and finally bringing them together to annihilate. The four dimensional configuration constructed this way is topologically non-trivial. Since an anti-monopole travelling forward in time is a monopole travelling backwards in time, we can describe the same as a closed monopole line (or loop). It represents a topologically non-trivial configuration when the monopole makes a full rotation while moving along the closed monopole line (see fig. 4-7). The non-trivial topology discussed by Taubes [95] ($\pi_1(M_0(S^2;S^2))=\mathbb{Z}$) is just the Hopf fibration, except that now it is more natural to see S^1 as the base manifold and S^2 as the fibre, which rotates (twists) while moving along the circle formed by the closed monopole line . The only topological invariant available to characterise this homotopy type is precisely the Pontryagin index. This link between instanton and monopole charges was generalised in [55] by Jahn.

It was mentioned that the short range components of the fields can not be fully cancelled due to the non-trivial "twist" along the monopole line, so they have to be responsible for the Pontryagin index. Indeed, in the computation of the total topological charge of the new configurations as the integral over $\partial_\mu^2\partial_\nu^2\log\psi$ (eq. (4.3.31)), the massive component of the field gives rise to

$$\psi = 2e^{4\pi(r\bar{\omega}+s\omega)}(1+\mathcal{O}(|\vec{x}|^{-1})) = 2e^{2\pi|\vec{x}|}(1+\mathcal{O}(|\vec{x}|^{-1})), \quad \partial_\mu^2\log\psi = 4\pi/|\vec{x}|+\mathcal{O}(|\vec{x}|^{-4}), \tag{4.6.1}$$

and thus yields the surviving boundary term, but at the same time does not contribute to the action density, since $\partial_\mu^2|\vec{x}|^{-1}=0$.

We now inspect more closely the monopole content of the new calorons. For this we choose $\rho/\mathcal{T}$ large, such that the monopoles are well separated and static. There are two world lines of monopoles running in opposite directions (due to their opposite charges), closed due to the periodic boundary conditions. At smaller separations the solutions are far from static, with the attractive force driving the constituents together, after which they annihilate. In that case the world lines form a single closed monopole line, as mentioned above. It should be noted though, that for small $\rho/\mathcal{T}$ the constituents become rather extended. Nevertheless, such closed monopole lines are characterised by rotation of the local monopole field over precisely one full rotation when completing the circle, since the caloron has unit topological charge. It prevents the field from decaying to the trivial configuration. It is this "twist" that provides the closed monopole line its stability.

Before continuing, we observe that the calorons are given in a singular gauge, as is usual for the ADHM construction. The function $\hat{\psi}$ (eq. (4.3.20)) has an isolated zero at $x = 0$. This can be traced to the zero-mode in $B - x$, responsible for the non-trivial topology of the solution. This singularity is easily seen to be removed by a gauge transformation, locally of the form $x/|x|$ (viewing x as a quaternion). We now assume that $\omega\bar{\omega} \neq 0$ and consider the region outside the core of both monopoles, i.e. $r\bar{\omega} > 1$ and $s\omega > 1$. In this region

$$\begin{aligned} \phi &= \frac{r+s+\pi\rho^2}{r+s-\pi\rho^2} + \mathcal{O}(e^{-8\pi\min(s\omega,r\bar{\omega})}), \\ \tilde{\chi} &= \frac{4\pi\rho^2}{(r+s+\pi\rho^2)^2}\left\{re^{-4\pi r\bar{\omega}}e^{-2\pi i x_0} + se^{-4\pi s\omega}\right\}\left(1+\mathcal{O}(e^{-8\pi\min(s\omega,r\bar{\omega})})\right). \end{aligned} \tag{4.6.2}$$

Substituting this in eq. (4.3.25) we find the solution to be time independent and abelian, up to exponential correction

$$\begin{aligned} A_0 &= -\frac{i}{2}\tau_3\partial_3\log\phi + 2\pi i\omega\tau_3, \quad A_k = -\frac{i}{2}\tau_3\varepsilon_{kj3}\partial_j\log\phi, \\ E_k &= F_{k0} = -\frac{i}{2}\tau_3\partial_k\partial_3\log\phi, \quad B_k = \varepsilon_{kij}\partial_i A_j = -\frac{i}{2}\tau_3\left(\partial_k\partial_3\log\phi - \delta_{k3}\partial_j^2\log\phi\right). \end{aligned} \tag{4.6.3}$$

For convenience we rotate $\hat{a}$ to $\hat{e}_3$. Self-duality, $\vec{E}=\vec{B}$, requires $\log\phi$ to be harmonic. We first note that when *neglecting* the exponential corrections, ϕ^{-1} vanishes on the interval $-2\pi\rho^2\omega \leq x_3 \leq 2\pi\rho^2\bar{\omega}$ at $x_1 = x_2 = 0$ (we denote the characteristic function on this interval by $\chi_\omega(x_3)$). A careful analysis reveals $\partial_j^2\log\phi = -4\pi\delta(x_1)\delta(x_2)\chi_\omega(x_3)$ (that it vanishes away from the zeros of ϕ^{-1} follows by a direct computation). The term $-\frac{i}{2}\tau_3\delta_{k3}\partial_j^2\log\phi$ in the expression for the magnetic field corresponds precisely to the Dirac string singularity, carrying the return flux. One finds that $\partial_k E_k = \partial_k B_k = \frac{i}{2}\tau_3\partial_3\partial_j^2\log\phi = 2\pi i\tau_3(\delta_3(\vec{s}) - \delta_3(\vec{r}))$, when ignoring this return flux, which in the full theory is absent [48, 85] (indeed as noted before ϕ^{-1} has only an isolated zero at $x = 0$, corresponding to a gauge singularity).

Finally, to confirm our expectations it remains to identify the rotation of one of the monopoles so as to guarantee the topologically non-trivial nature of the configuration. Inspecting the behaviour in the core region of the monopoles, described by $\tilde{\chi}$ in eq. (4.6.2), gives the following factorisation

$$\tilde{\chi} = e^{-2\pi i x_0}\chi^{(1)}(r) + \chi^{(2)}(s). \tag{4.6.4}$$

While one of the monopoles has a static core, the other has a time dependent phase rotation - equivalent to a (gauge) rotation - precisely of the type required to form a non-contractible loop, as the phase makes a full rotation when closing by the periodic boundary conditions in the time direction.

Although interpreting A_0 as the Higgs field allows one to introduce monopoles in pure gauge theory, there are some subtle differences. In the static limit the BPS equations imply $F_{i0} = D_i A_0$ and we would be tempted to call the solution a dyon. In the Higgs model dyons are constructed by taking A_0 proportional to the Higgs field Φ [11, 86, 56]. By a time dependent gauge transformation A_0 can be gauged to zero. This gauge transformation is generated by A_0, precisely the unbroken generator, as A_0 is proportional to the Higgs field. The resulting electric field is now given by $\partial_0 A_i$ and is *not* quantised. In the Higgs model $B_i = D_i\Phi$ and $E_i = -\partial_0 A_i$. In pure gauge theory it makes, however, no sense to separate $D_i\Phi = D_i A_0$ from $\partial_0 A_i$. Gauge invariance requires that they occur in the combination $F_{i0} = D_i A_0 - \partial_0 A_i$. The electric field is necessarily fixed and quantised as soon as we interpret A_0 as the Higgs field. Nevertheless, we *can* consider dyons also in pure gauge theories, but for this we have to add a term proportional to $\theta F^* F$ to the lagrangian [101]. The electric charge is now proportional to θ, so no longer quantised. But, unlike in the Higgs model, it is the same for all monopoles.

It should be noted that in the Higgs model the construction of the non-contractible loop generates an electric charge due to the (gauge) rotation along the closed monopole line, when interpreting the loop parameter as time. The electric charge is proportional to the rate of rotation and can vary along the monopole line. However, integrated along a closed monopole line the charge is fixed and proportional to the number of rotations, which hence plays the role of a winding number. In pure gauge theory this winding can not be read off from the long range field components, but for both cases the fields in the core are responsible for the Pontryagin number (an abelian field can not contribute to this topological charge).

There is a natural context in which the analogy with the Higgs model is more precise. For this we have to add time as a fifth dimension, such that four dimensional space is compactified on a circle. In the limit of zero compactification radius ($\mathcal{T} \to 0$) the new calorons become genuine monopoles and can obtain dyonic charges in the sense of Julia and Zee [56]. It is in this context that Taub-NUT metric describes the scattering of oppositely charged monopoles on $\mathbb{R}^3 \times S^1$, in exactly the same way as the Atiyah-Hitchin metric describes the scattering of like-charged monopoles on $\mathbb{R}^3$ [70, 3].

Monopoles appear also in the context of 't Hooft's abelian projection [52] as (gauge) singularities. The lesson learned from the above analysis is that in order to include the non-trivial topological charge, important for fermion zero modes, breaking of the axial $U(1)$ symmetry [50] and presumably for chiral symmetry breaking, one needs to keep some information on the behaviour near the core of these monopoles. This allows one to combine the attractiveness of the dual superconductor picture of confinement [49] in terms of monopole degrees of freedom, with the success of the instanton liquid model [89]. There have been many attempts to make an effective monopole model for the long range confining properties of QCD, see e.g. ref. [91]. Also there have been many studies in lattice gauge theory, using the idea of abelian projection implemented by the so-called maximal abelian gauge [57], in order to extract the monopole content of the theory. It was observed that the string tension is saturated by the monopole fields [93, 84]. More recently it was found that after abelian projection instantons contain closed monopole lines [18, 43, 12, 15]. As emphasised in ref. [7], in the light of Taubes's construction this was to be expected. Here we have shown in more detail how one can make fields with non-zero topological charge out of monopole degrees of freedom, with as example the well defined setting of calorons with non-trivial holonomy. What is *minimally* required is a frame associated to each monopole, whose rotation is a topological invariant for closed monopole lines. Such closed monopole lines can shrink, but one will be left over with what represents an instanton. It would be interesting to build a hybrid model based on the instanton liquid and monopoles, and see how successful it is in capturing the appropriate phenomenology.

To conclude, it is sensible to take the monopole content of instantons serious in the broader context sketched here. There is a somewhat destructive (but reversible) gauge invariant method of investigating the monopoles inside an instanton. First the instanton is heated just a little. Then a non-trivial value of the holonomy is added at infinity, without disturbing the instanton significantly (true for $\mathcal{T}$ sufficiently large). Now it has to be squeezed (or heated) hard. Out come the two constituent monopoles, in a direction determined by the choice we have made for the holonomy at infinity (which does not change under heating). The new caloron solutions were also found on the lattice [30] using improved cooling techniques [29] and twisted boundary conditions.

Appendix A: The Green's function G_x

In this appendix the solution for the Green's function $G_x = (B^\dagger(x)B(x))^{-1}$ is presented, which after Fourier transformation satisfies the equation

$$\left\{\left(\frac{1}{2\pi i}\frac{d}{dz}-x_0\right)^2+s^2\chi_{[-\omega,\omega]}(z)+r^2\chi_{[\omega,1-\omega]}(z)-\frac{\rho^2}{2}\hat{\omega}\cdot\vec{\tau}(\delta(z-\omega)-\delta(z+\omega))\right\}\hat{G}_x(z,z') = \delta(z-z'), \tag{4.A.1}$$

with r and s given as in eq. (4.3.13). Its solution is given by

$$\begin{aligned}\hat{G}_x(z,z') = \chi_{[-\omega,\omega]}(z')&\left(\chi_{[-\omega,\omega]}(z)\hat{G}^d_x(z,z',r,s,\omega)+\chi_{[\omega,1-\omega]}(z)\hat{G}^o_x(z,z',r,s,\omega)\right) \qquad (4.A.2)\\ +\chi_{[\omega,1-\omega]}(z'))&\left(\chi_{[\omega,1-\omega]}(z)\hat{G}^d_x(\tfrac{1}{2}-z,\tfrac{1}{2}-z',s,r,\bar{\omega})^*+\chi_{[-\omega,\omega]}(z)\hat{G}^o_x(z',z,r,s,\omega)^*\right).\end{aligned}$$

Like for $\hat{f}_x$ the diagonal component $\hat{G}^d_x(z,z')$ is only defined strictly for $z,z'\in[-\omega,\omega]$ and the off-diagonal component $\hat{G}^o_x(z,z')$ only for $z\in[\omega,1-\omega]$ and $z'\in[-\omega,\omega]$. For z or z' outside of these intervals, one first has to map back to the interval $[-\omega,1-\omega]$, using periodicity.

$$\hat{G}^d_x(z,z',r,s,\omega)=e^{2\pi i x_0(z-z')}\pi(rs\hat{\psi})^{-1}\Big\{e^{-2\pi i x_0 \mathrm{sign}(z-z')}r\sinh(2\pi s|z-z'|)$$

$$
\begin{aligned}
&+s^{-1}\sinh(4\pi r\bar\omega)\Big[\pi s\rho^2\hat\omega\cdot\vec\tau\sinh(2\pi s(z+z'))\\
&\qquad\qquad +\tfrac{1}{2}(s^2-r^2+\pi^2\rho^4)\cosh(2\pi s(z+z'))\\
&\qquad\qquad +\tfrac{1}{2}(s^2+r^2-\pi^2\rho^4)\cosh(2\pi s(2\omega-|z-z'|))\Big]\\
&+r\cosh(4\pi r\bar\omega)\sinh(2\pi s(2\omega-|z-z'|))\Big\},
\end{aligned}
\tag{4.A.3}
$$

$$
\begin{aligned}
\hat G_x^o(z,z',r,s,\omega)=e^{2\pi i x_0(z-z')}\pi(rs\hat\psi)^{-1}\Big\{&\pi\rho^2\hat\omega\cdot\vec\tau\sinh(2\pi r(1-z-\omega))\sinh(2\pi s(z'+\omega))\\
&+r\cosh(2\pi r(z-1+\omega))\sinh(2\pi s(z'+\omega))\\
&-s\sinh(2\pi r(z-1+\omega))\cosh(2\pi s(z'+\omega))\\
&e^{-2\pi i x_0}\Big[s\sinh(2\pi r(z-\omega))\cosh(2\pi s(z'-\omega))\\
&\qquad -r\cosh(2\pi r(z-\omega))\sinh(2\pi s(z'-\omega))\\
&\qquad +\pi\rho^2\hat\omega\cdot\vec\tau\sinh(2\pi r(z-\omega))\sinh(2\pi s(z'-\omega))\Big]\Big\}.
\end{aligned}
$$

In particular,

$$
\hat G_x(\omega,-\omega)=\pi(rs\hat\psi)^{-1}e^{4\pi i x_0\omega}\left\{e^{-2\pi i x_0}r\sinh(4\pi s\omega)+s\sinh(4\pi r\bar\omega)\right\},
\tag{4.A.4}
$$

$$
\begin{aligned}
\hat G_x(\pm\omega,\pm\omega)=\pi(rs\hat\psi)^{-1}\Big\{&s\sinh(4\pi r\bar\omega)\cosh(4\pi s\omega)+r\sinh(4\pi s\omega)\cosh(4\pi r\bar\omega)\\
&\pm\pi\rho^2\hat\omega\cdot\vec\tau\sinh(4\pi r\bar\omega)\sinh(4\pi s\omega)\Big\}.
\end{aligned}
$$

which can be used to verify eq. (4.3.19) as derived from eq. (4.2.5), and eq. (4.3.32).

5 Monopole constituents inside $SU(n)$ calorons

5.1 Introduction

In this chapter, we generalise the results of chapter 4 for $k = 1$ calorons for gauge group $SU(2)$ to gauge group $SU(n)$. The most important qualitative result was the identification of two oppositely charged elementary BPS monopoles as constituents of the $SU(2)$ caloron. This composite nature forms a refinement of a more transparent compositeness that follows from the action of instantons and BPS monopoles being proportional to the winding numbers or charges describing the topology of these solutions, suggesting that they are composed out of elements of unit charge. We will find by computing the action density that charge one calorons for the gauge group $SU(n)$ are composed out of n basic BPS monopoles [11, 86], whose magnetic charges cancel exactly, generalising the $SU(2)$ situation. These n BPS monopoles will appear as explicit lumps in the action density profiles.

For clarity of the presentation, the result, which is surprisingly simple, will be given first, in the next section 5.2. The construction is then exposed in two subsequent sections. Section 5.3 contains the Nahm formalism for the caloron, which is -as for the $SU(2)$ case in section 4.3- derived from the ADHM construction using Fourier transformation. The construction of the caloron is thus reduced to a Green's function problem, which is solved in section 5.4. The properties of the $k = 1$ $SU(n)$ caloron are discussed in section 5.5. In particular, it will be interesting to consider static limits of the caloron, obtaining multi-monopoles.

5.2 The result

The labels of the $SU(n)$ calorons are given by the topology of the vacuum at spatial infinity the solution necessarily approaches for the action to be finite. This classification was presented in section 2.2. We consider the calorons with no net magnetic charge, in which case the Polyakov loop (holonomy) (eq. (2.2.1)) at spatial infinity becomes constant. Its eigenvalues $e^{2\pi i\mu_m}$ are part of the topological labels and will play an important role in the construction,

$$\lim_{|\vec{x}|\to\infty} \mathcal{P}(\vec{x}) = \mathcal{P}_\infty = V\mathcal{P}_\infty^0 V^{-1}, \quad \mathcal{P}_\infty^0 = \exp[2\pi i \mathrm{diag}(\mu_1, \ldots, \mu_n)], \tag{5.2.1}$$

V denoting a constant gauge transformation, cf. eq. (2.2.2). We recall that, making use of the gauge symmetry, we can choose the eigenvalues such that

$$\mu_1 < \ldots < \mu_n < \mu_{n+1} \equiv \mu_1 + 1, \quad \sum_{m=1}^{n} \mu_m = 0, \tag{5.2.2}$$

assuming maximal symmetry breaking for the moment. We define $\nu_m = \mu_{m+1} - \mu_m$, related to the mass of the $m^{\rm th}$ constituent monopole. Standard arguments, also following from the construction below, gives $4n$ instanton parameters for fixed $\mathcal{P}_\infty$, including the global gauge transformations that do not change $\mathcal{P}_\infty$. We will see that $3n$ parameters can be interpreted as the positions $(\vec{y}_m)$ of the constituents. The remaining parameters in this interpretation are the $n-1$ phases related to the unbroken gauge group $U(1)^{n-1}$, on which the action density does not depend and the position of the caloron in time, which we fix to be 0 by translational invariance. Also we will use the scale invariance to set $\mathcal{T} = 1$. Where needed, the proper $\mathcal{T}$ dependence can be reinstated on dimensional grounds. We will find the following surprisingly simple formula

$$-\tfrac{1}{2}\mathrm{Tr}F_{\mu\nu}^2 = -\tfrac{1}{2}\partial_\mu^2\partial_\nu^2 \log\psi. \tag{5.2.3}$$

where the positive scalar potential ψ is defined as

$$\psi(x) = \tfrac{1}{2}\mathrm{tr}_2 \prod_{m=1}^{n} \left\{ \begin{pmatrix} r_m & |\vec{y}_m - \vec{y}_{m+1}| \\ 0 & r_{m+1} \end{pmatrix} \begin{pmatrix} \cosh(2\pi\nu_m r_m) & \sinh(2\pi\nu_m r_m) \\ \sinh(2\pi\nu_m r_m) & \cosh(2\pi\nu_m r_m) \end{pmatrix} \frac{1}{r_m} \right\} - \cos(2\pi x_0). \tag{5.2.4}$$

Here $r_m = |\vec{x} - \vec{y}_m|$ denotes the centre of mass radius of the $m^{\rm th}$ monopole. The order of matrix multiplication is crucial, $\prod_{m=1}^{n} A_m \equiv A_n \ldots A_1$.

5.3 The ADHM-Nahm formalism

We will construct the caloron in the so-called algebraic gauge, related to the periodic gauge by the non-periodic gauge transformation

$$g(\vec{x}, x_0) = V \exp[2\pi i x_0 \mathrm{diag}(\mu_1, \ldots, \mu_n)] V^{-1}. \tag{5.3.1}$$

In this gauge, the background field $2\pi i\,\mathrm{diag}(\mu_1, \cdots, \mu_n)$ in eq. (2.2.4) is removed and we have the alternative boundary condition,

$$A_\mu(\vec{x}, x_0 + \mathcal{T}) = \mathcal{P}_\infty A_\mu(\vec{x}, x_0) \mathcal{P}_\infty^{-1}. \tag{5.3.2}$$

As in the $SU(2)$ case, the non-trivial holonomy will fully be carried by the cocycle, cf. eq. (4.1.2). Since in the absence of magnetic windings, $\mathcal{P}_\infty$ can always be gauged to a constant diagonal form, we assume henceforth $\mathcal{P}_\infty = \mathcal{P}_\infty^0$ without loss of generality. The periodic instanton of charge one is obtained in the algebraic gauge (5.3.2) by taking an infinite array of elementary instantons, relatively gauge-rotated over $\mathcal{P}_\infty$.

To implement this in the ADHM formalism we take the specific solution in eq. (3.1.3) for the zero mode vector $v(x)$ in the ADHM construction,

$$v(x) = \begin{pmatrix} -1_n \\ u(x) \end{pmatrix} \phi^{-\frac{1}{2}}(x), \quad u(x) = (B^\dagger - x^\dagger 1_k)^{-1}\lambda^\dagger, \quad \phi(x) = 1_n + u^\dagger(x)u(x),$$

where ϕ is an $n \times n$ positive hermitean matrix. In terms of these, one obtains the gauge potential from eq. (3.1.4),

$$A_\mu(x) = \phi^{-\frac{1}{2}}(x)(u^\dagger(x)\partial_\mu u(x))\phi^{-\frac{1}{2}}(x) + \phi^{\frac{1}{2}}(x)\partial_\mu\phi^{-\frac{1}{2}}(x).$$

For eq. (5.3.2) to hold, it is then required that

$$u_{p+1}(x+1) = u_p(x)\mathcal{P}_\infty^{-1}, \quad p \in \mathbb{Z}. \tag{5.3.3}$$

This imposes periodicity constraints on the data

$$\lambda_{p+1} = \mathcal{P}_\infty\lambda_p, \quad B_{p,p'}(x+1) = B_{p-1,p'-1}(x), \tag{5.3.4}$$

with $B(x) = B - x1_k$, which imply

$$\lambda_p = \mathcal{P}_\infty^p\zeta, \quad B_{p,p'} = \sigma_0\delta_{p,p'} + \hat{A}_{p-p'}, \quad p, p' \in \mathbb{Z}. \tag{5.3.5}$$

The off-diagonal part $\hat{A}$ is still to be determined. Fourier transformation translates the ADHM formalism to the Nahm language. B is cast into a Weyl operator,

$$\begin{aligned}
&\sum_{p,p'\in\mathbb{Z}} B_{p,p'}(x)e^{2\pi i(pz-p'z')} = \frac{\delta(z-z')}{2\pi i}\hat{D}_x(z'),\\
&\hat{D}_x(z) = \sigma_\mu\hat{D}_x^\mu(z) = \frac{d}{dz} + \hat{A}(z) - 2\pi i x,\\
&\hat{A}(z) = \sigma_\mu\hat{A}_\mu(z), \quad \hat{A}_\mu(z) = 2\pi i\sum_{p\in\mathbb{Z}} e^{2\pi i p z}\hat{A}_{\mu\,p},
\end{aligned} \tag{5.3.6}$$

cf. eq. (3.1.35), and $\lambda^\dagger\lambda$ into a singularity structure describing the matching conditions for $\hat{A}(z)$,

$$\begin{aligned}
&\sum_{p\in\mathbb{Z}} e^{-2\pi p i z}\lambda_p = \sum_{p\in\mathbb{Z}} e^{2\pi i p(\mu_m - z)}P_m\zeta = \hat{\lambda}(z),\\
&\hat{\lambda}(z) = \sum_{m\in\mathbb{Z}/n\mathbb{Z}} \delta(z-\mu_m)P_m\zeta,\\
&\sum_{p,p'\in\mathbb{Z}} \lambda_p^\dagger e^{2\pi i(pz-p'z')}\lambda_{p'} = \delta(z-z')\hat{\Lambda}(z),\\
&\hat{\Lambda}(z) = \sum_{m\in\mathbb{Z}/n\mathbb{Z}} \delta(z-\mu_m)\zeta^\dagger P_m\zeta = \zeta^\dagger\hat{\lambda}(z).
\end{aligned} \tag{5.3.7}$$

Here we introduced the projection operators $P_m = e_m e_m^t$, where e_m is the $m^{\rm th}$ unit vector, in terms of which

$$\mathcal{P}_\infty = \sum_{m\in\mathbb{Z}/n\mathbb{Z}} \exp(2\pi i\mu_m) P_m$$

and

$$\lambda_p = \sum_{m\in\mathbb{Z}/n\mathbb{Z}} \exp(2\pi i p\mu_m) P_m\zeta.$$

The group index $m \in \mathbb{Z}/n\mathbb{Z}$ is a cyclic variable. We also used that for any two objects a, b of type $a_p = \mathcal{P}_\infty^p \alpha$, $p \in \mathbb{Z}$, the Fourier transforms defined as $\hat{a}(z) = \sum_{p\in\mathbb{Z}} \exp(-2\pi i p z) a_p$, have the property

$$\begin{aligned} \hat{a}^\dagger(z)\hat{b}(z') &= \delta(z-z')\, \hat{a}(z)^\dagger < \hat{b} > = \delta(z-z') < \hat{a}^\dagger > \hat{b}(z) \\ &= \delta(z-z') \sum_{m\in\mathbb{Z}/n\mathbb{Z}} \delta(z-\mu_m)\alpha^\dagger P_m \beta, \end{aligned} \tag{5.3.8}$$

where $< H > \equiv \int_{S^1} H(z) dz$. The quadratic ADHM constraint translates into

$$\tfrac{1}{2}[\hat{D}_\mu(z), \hat{D}_\nu(z)]\bar{\eta}_{\mu\nu} = 4\pi^2 \Im \hat{\Lambda}(z), \tag{5.3.9}$$

cf. eq. (4.3.6). We introduce a $U(1)$ fibration of $\mathbb{C}^2$ over $\mathbb{R}^3$ (cf. eq. (6.1.2)) to write

$$\zeta^\dagger P_m \zeta = \zeta^\dagger_{(m)}\zeta_{(m)} = \frac{1}{2\pi}(\rho_m + \vec{\rho}_m\cdot\vec{\tau}), \quad \rho_m = |\vec{\rho}_m|. \tag{5.3.10}$$

The $U(1)$ fibre arises due to the $U(1)$ phase ambiguity in defining $\zeta_{(m)}$ from $\vec{\rho}_m$. This phase ambiguity is resolved later. Eqs. (5.3.9, 5.3.10) now lead to the caloron Nahm equation

$$\frac{d}{dz}\hat{A}_j(z) = 2\pi i \sum_{m\in\mathbb{Z}/n\mathbb{Z}} \delta(z-\mu_m)\rho_m^j, \tag{5.3.11}$$

which is abelian in the $k=1$ situation at hand, see [77]. As integration of eq. (5.3.11) over S^1 gives a constraint on ζ,

$$\sum_{m\in\mathbb{Z}/n\mathbb{Z}} \vec{\rho}_m = \pi \mathrm{tr}_2(\vec{\tau}\zeta^\dagger\zeta) = \vec{0}, \tag{5.3.12}$$

we can introduce vectors $\vec{y}_m, m \in \mathbb{Z}/n\mathbb{Z}$, such that $\vec{\rho}_m = \vec{y}_m - \vec{y}_{m-1}$. The vectors $\vec{y}_m$ are to be interpreted as the constituent monopole positions. We now find for the spacelike components of $\hat{A}(z)$,

$$\hat{A}_j(z) = 2\pi i \sum_{m\in\mathbb{Z}/n\mathbb{Z}} \chi_{[\mu_m,\mu_{m+1}]}(z) y_m^j. \tag{5.3.13}$$

Note that the Nahm equations determine $\vec{y}_m$ up to the global $\mathbb{R}^3 \times S^1$ position variable

$$\xi = \frac{1}{2\pi i}\int_{S^1} \hat{A}(z)dz, \quad \vec{\xi} = \sum_{m\in\mathbb{Z}/n\mathbb{Z}} \nu_m \vec{y}_m. \tag{5.3.14}$$

We recall $\nu_m = \mu_{m+1} - \mu_m$ is related to the mass of the m^{th} constituent, so that $\vec{\xi}$ is the centre of mass of the caloron. The T symmetry, eq. (3.1.10), in the ADHM construction is mapped to a $U(1)$ gauge symmetry, with gauge group $\hat{\mathcal{G}} = \{g(z)|g : z \to e^{-ih(z)} \in U(1)\}$, acting as

$$\hat{A}(z) \to \hat{A}(z) + i\frac{d}{dz}h(z), \quad \zeta_m \to \zeta_m e^{ih(\mu_m)}. \tag{5.3.15}$$

For calorons, $g(z)$ is periodic and can be used to set $\hat{A}_0(z)$ to a constant. A piecewise linear $U(1)$ gauge function $h(z)$ shifts the $U(1)$ phase ambiguities in $\zeta_{(m)}$ to $\hat{A}_0(z)$, which thus becomes piecewise constant. Therefore, all $4n$ moduli are included in the following solution to the Nahm equations

$$\hat{A}(z) = 2\pi i \sum_{m\in\mathbb{Z}/n\mathbb{Z}} \chi_{[\mu_m,\mu_{m+1}]}(z)\left(\frac{\tau_m}{4\pi\nu_m}\sigma_0 + \vec{y}_m\cdot\vec{\sigma}\right), \tag{5.3.16}$$

where $\tau = (\tau_1, \ldots, \tau_n)^t$ takes values in $\mathbb{R}^n$. Using the gauge function

$$g(z) = \sum_{m\in\mathbb{Z}/n\mathbb{Z}} \chi_{[\mu_m,\mu_{m+1}]}(z)\exp(2\pi i(z-\mu_m)\frac{k_m}{\nu_m}), \quad k_m \in \mathbb{Z}, \tag{5.3.17}$$

which leaves the $U(1)$ phases of ζ unaffected, we can restrict τ to the torus $\mathbb{R}^n/(4\pi\mathbb{Z})^n$. In this gauge, the moduli describing the general caloron are the position vectors $\vec{y}_m$, comprised in $\vec{y} = (\vec{y}_1, \ldots, \vec{y}_n)$ and the torus coordinate τ describing the $U(1)^{n-1}$ residual gauge symmetry and the temporal position of the caloron. Strictly speaking, these variables are coordinates on the cover of the moduli space of framed calorons. The true moduli space is obtained by dividing out the centre of the gauge group. This leads to orbifold singularities.

Under Fourier transformation, the Green's function f_x (eq. (5.3.18)) for calorons becomes $\hat{f}_x(z,z') \equiv \sum_{p,p'\in\mathbb{Z}} f_x^{p,p'} e^{2\pi i(pz-p'z')}$ and is a solution of the differential equation

$$\begin{aligned}\left\{\left(\frac{1}{2\pi i}\frac{d}{dz} - x_0\right)^2 + \sum_{m=1}^{n}\chi_{[\mu_m,\mu_{m+1}]}(z)\, r_m^2 + \frac{1}{2\pi}\sum_{m=1}^{n}\delta(z-\mu_m)|\vec{y}_m - \vec{y}_{m-1}|\right\}\hat{f}_x(z,z') \\ = \delta(z-z').\end{aligned} \tag{5.3.18}$$

Here we took $\hat{A}_0(z) = 0$, obtained in the $U(1)$ gauge where $\hat{A}_0 = $ constant by absorbing ξ_0 in x_0. Here $r_m = |\vec{x} - \vec{r}_m|$ is the centre of mass radius of the m^{th} constituent. Expressions for $\hat{f}_x$ in other gauges are obtained by using that under the action of $\hat{\mathcal{G}}$, $\hat{f}_x$ transforms as

$$\hat{f}_x(z,z') \to g(z)\hat{f}_x(z,z')g(z')^*, \quad g(z) \in \hat{\mathcal{G}}. \tag{5.3.19}$$

The Green's function in eq. (5.3.18) is a solution of a quantum-mechanical problem on the circle with a piecewise constant potential and delta function impurities. This solution is obtained by solving it on each sub-interval, where $\hat{f}_x(z,z')$ is of simple exponential form. Starting at $z=z'$ and matching properly at $z=\mu_m$ so as to account for the scattering by the impurity, we can go full circle to return at $z=z'$ where one last matching accounts for the delta function at the rhs. of eq. (5.3.18). This programme will be carried out in the next section.

5.4 Green's function techniques

In this section we discuss the necessary Green's function techniques for the $k=1$, $SU(n)$ caloron. The relevant Green's function $\hat{f}_x(z,z')$ is the solution to the differential equation (5.3.18). The symmetries in this equation imply the relations

$$\hat{f}_x(z,z')=\hat{f}_x(-z,-z')^*=\hat{f}_x(-z',-z)=\hat{f}_x(z',z)^*. \tag{5.4.1}$$

The aim of this section is to calculate $\hat{f}_x(z,z)$ and $\det\hat{f}_x$. On the m^{th} subinterval $[\mu_m,\mu_{m+1}], m\in\mathbb{Z}/n\mathbb{Z}$, $\hat{f}_x(z,z')$ satisfies

$$\left\{\left(\frac{1}{2\pi i}\frac{d}{dz}-x_0\right)^2+r_m^2\right\}\hat{f}_x(z,z')=0. \tag{5.4.2}$$

For $(z,z')\in[\mu_m,\mu_{m+1}]\times[\mu_{m'},\mu_{m'+1}]$, the Green's function is the sum of two exponentials. Solving the total Green's function then amounts to patching the exponentials together. We write the Green's function and its derivative as a two-component vector

$$\begin{pmatrix}\hat{f}_x(z,z')\\ \left(\frac{d}{dz}-2\pi i x_0\right)\hat{f}_x(z,z')\end{pmatrix}=W(r_m,z)c_{mm'}(z'),\ z\in[\mu_m,\mu_{m+1}],\ z'\in[\mu_{m'},\mu_{m'+1}], \tag{5.4.3}$$

where W is a matrix containing the two independent solutions to eq. (5.4.2) and their derivatives

$$W(r,z)\equiv e^{2\pi i x_0 z}\begin{pmatrix}e^{2\pi r z} & e^{-2\pi r z}\\ 2\pi r e^{2\pi r z} & -2\pi r e^{-2\pi r z}\end{pmatrix} \tag{5.4.4}$$

and $c_{mm'}(z')$ is a two-dimensional column vector containing the z' dependent integration constants. For our purposes, we will only need $f_x(z,z')$ in the limit $z'\to z$, i.e. z and z' lie in the same interval. For $z,z'\in[\mu_m,\mu_{m+1}]$, c^a_{mm} and c^b_{mm} indicate $z<z'$ and $z>z'$. The matrices W enjoy the following properties

$$\begin{aligned}\det W &= -4\pi r e^{4\pi i x_0 z},\\ W^{-1}(r,z) &= -\frac{e^{-2\pi i x_0 z}}{4\pi r}\begin{pmatrix}-2\pi r e^{-2\pi r z} & -e^{-2\pi r z}\\ -2\pi r e^{2\pi r z} & e^{2\pi r z}\end{pmatrix}.\end{aligned} \tag{5.4.5}$$

The delta functions at $z=\mu_m$ appearing in eq. (5.3.18) give two matching conditions,

$$\lim_{z\downarrow\mu_m}\hat{f}_x(z,z')-\lim_{z\uparrow\mu_m}\hat{f}_x(z,z')=0,$$

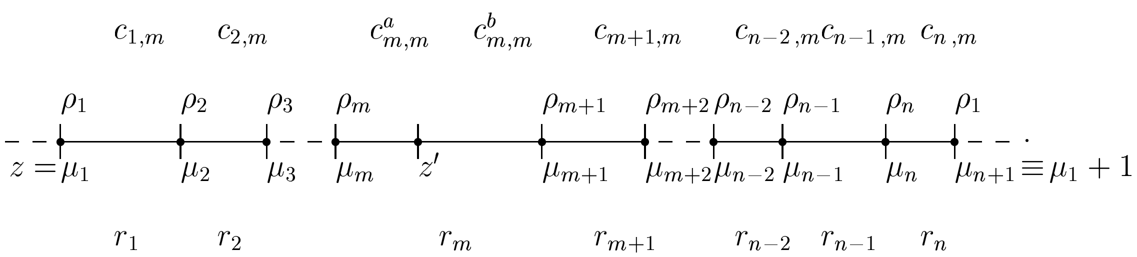
$$\lim_{z\downarrow\mu_m}\frac{d}{dz}\hat{f}_x(z,z')-\lim_{z\uparrow\mu_m}\frac{d}{dz}\hat{f}_x(z,z')=2\pi|\vec{y}_m-\vec{y}_{m-1}|\hat{f}(\mu_m,z'), \tag{5.4.6}$$

whereas at $z=z'$ we find

$$\lim_{z\downarrow z'}\hat{f}_x(z,z')-\lim_{z\uparrow z'}\hat{f}_x(z,z')=0,$$

$$\lim_{z\downarrow z'}\frac{d}{dz}\hat{f}_x(z,z')-\lim_{z\uparrow z'}\frac{d}{dz}\hat{f}_x(z,z')=-4\pi^2. \tag{5.4.7}$$

Let us assume $z'\in[\mu_m,\mu_{m+1}]$. The following partition of the circle on which z lies is then appropriate



The matching at $z=\mu_j$, eq. (5.4.6), in the matrix form of eq. (5.4.3) reads

$$W(r_j,\mu_j)c_{j,m}-W(r_{j-1},\mu_j)c_{j-1,m}=\begin{pmatrix}0&0\\2\pi\rho_j&0\end{pmatrix}W(r_{j-1},\mu_j)c_{j-1,m}, \tag{5.4.8}$$

and implies the recursion relation for $j\neq 1$

$$\begin{aligned}c_{j,m}&=W^{-1}(r_j,\mu_j)\begin{pmatrix}1&0\\2\pi\rho_j&1\end{pmatrix}W(r_{j-1},\mu_j)c_{j-1,m}\\&\equiv U_jc_{j-1,m},\end{aligned} \tag{5.4.9}$$

where we introduced the scattering matrices $U_j, j\neq 1$, which are time independent and have the property

$$\det U_j=\frac{r_{j-1}}{r_j}. \tag{5.4.10}$$

Special care should be taken at the matching where the circle is closed ($z=\mu_1,\mu_1+1$), which reads

$$\begin{aligned}c_{1,m}&=W^{-1}(r_1,\mu_1)\begin{pmatrix}1&0\\2\pi\rho_1&1\end{pmatrix}W(r_n,\mu_1+1)c_{n,m}\\&\equiv U_1c_{n,m}.\end{aligned} \tag{5.4.11}$$

Here U_1 satisfies

$$\begin{aligned}U_1&=W^{-1}(r_1,\mu_1)\begin{pmatrix}1&0\\2\pi\rho_l&1\end{pmatrix}W(r_n,\mu_1)\begin{pmatrix}e^{2\pi r_n}&0\\0&e^{-2\pi r_n}\end{pmatrix}e^{2\pi ix_0},\\ \det U_1&=\frac{r_n}{r_1}e^{4\pi ix_0}.\end{aligned} \tag{5.4.12}$$

We now solve $c^a_{m,m}$ and $c^b_{m,m}$. This is achieved by writing both in terms of $c_{m+1,m}$ using the matching condition at $z = \mu_{m+1}$,

$$c^b_{m,m} = U^{-1}_{m+1} c_{m+1,m}, \tag{5.4.13}$$

and the recursion relations, eqs. (5.4.9, 5.4.11),

$$c^a_{m,m} = \left\{ \prod_{l \overset{*}{=} m+2}^{m} U_l \right\} c_{m+1,m}. \tag{5.4.14}$$

The star denotes taking the product over representatives of $\mathbb{Z}/n\mathbb{Z}$, possibly extending over $l = n$, and the order in the product is crucial,

$$\prod_{l \overset{*}{=} l_1}^{l_2} A_l \equiv A_{l_2} \cdots A_{l_1} \text{ if } l_2 > l_1, \tag{5.4.15}$$

in which case we usually omit the star, and

$$\prod_{l \overset{*}{=} l_1}^{l_2} A_l \equiv A_{l_2} \cdots A_1 A_n \cdots A_{l_1} \text{ if } l_1 > l_2. \tag{5.4.16}$$

The matching condition in $z = z'$, eq. (5.4.7), implies

$$W(r_m, z') c^b_{mm} - W(r_m, z') c^a_{mm} = \begin{pmatrix} 0 \\ -4\pi^2 \end{pmatrix} \equiv V. \tag{5.4.17}$$

This implements the full circle scattering process and $c_{m+1,m}$ can now be solved in terms of V and W,

$$c_{m+1,m} = \left(1 - \prod_{l \overset{*}{=} m+2}^{m+1} U_l \right)^{-1} U_{m+1} W^{-1}(r_m, z') V. \tag{5.4.18}$$

The necessary coefficient vectors for the Green's function in eq. (5.4.3) near $z = z'$ now follow from inserting this in eqs. (5.4.13) and (5.4.14),

$$\begin{aligned} c^a_{mm} &= \left(\prod_{l \overset{*}{=} m+2}^{m} U_l \right) \left(1 - \prod_{l' \overset{*}{=} m+2}^{m+1} U_{l'} \right)^{-1} U_{m+1} W^{-1}(r_m, z') V, \\ c^b_{mm} &= U^{-1}_{m+1} \left(1 - \prod_{l \overset{*}{=} m+2}^{m+1} U_l \right)^{-1} U_{m+1} W^{-1}(r_m, z') V. \end{aligned} \tag{5.4.19}$$

To manipulate with $\left(1-\prod_{l\stackrel{*}{=}m+2}^{m+1} U_l\right)^{-1}$, we note

$$\left(1-\prod_{l\stackrel{*}{=}m+2}^{m+1} U_l\right)^{-1} = \det{}^{-1}\left(1-\prod_{l=1}^{n} U_l\right)\left(1-\mathrm{Minor}\left(\prod_{l'\stackrel{*}{=}m+2}^{m+1} U_{l'}\right)\right). \quad (5.4.20)$$

Here we used that for a 2×2 matrix $\det(1_2 - A) = 1 - \mathrm{tr}_2 A + \det A$, and hence

$$\det\left(1-\prod_{l\stackrel{*}{=}m+2}^{m+1} U_l\right) = 1 - \mathrm{tr}_2\prod_{l=1}^{n} U_l + \det\prod_{l=1}^{n} U_l = \det\left(1-\prod_{l=1}^{n} U_l\right). \quad (5.4.21)$$

It is convenient to introduce H_m and $\mathcal{A}_m$ as

$$\begin{aligned} H_m &= e^{-2\pi i x_0 \nu_m} W(r_m,\mu_{m+1}) W^{-1}(r_m,\mu_m) \\ &= \begin{pmatrix} \cosh 2\pi\nu_m r_m & \sinh 2\pi\nu_m r_m/2\pi r_m \\ 2\pi r_m \sinh 2\pi\nu_m r_m & \cosh 2\pi\nu_m r_m \end{pmatrix} \end{aligned} \quad (5.4.22)$$

and

$$\mathcal{A}_m \equiv \frac{1}{r_m}\begin{pmatrix} r_m & |\vec{y}_m - \vec{y}_{m+1}| \\ 0 & r_{m+1} \end{pmatrix}\begin{pmatrix} \cosh 2\pi\nu_m r_m & \sinh 2\pi\nu_m r_m \\ \sinh 2\pi\nu_m r_m & \cosh 2\pi\nu_m r_m \end{pmatrix}, \quad (5.4.23)$$

related to H_m via

$$\mathcal{A}_m = \begin{pmatrix} 0 & \frac{1}{2\pi r_m} \\ \frac{r_{m+1}}{r_m} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2\pi|\vec{y}_m - \vec{y}_{m+1}| & 1 \end{pmatrix} H_m \begin{pmatrix} 0 & 1 \\ 2\pi r_m & 0 \end{pmatrix}. \quad (5.4.24)$$

We then obtain

$$\left\{\begin{matrix} \det \\ \mathrm{tr}_2 \end{matrix}\right\}\prod_{l=1}^{n} U_l = \left\{\begin{matrix} e^{4\pi i x_0} \\ e^{2\pi i x_0}\mathrm{tr}_2\prod_{l=1}^{n} H_l \begin{pmatrix} 1 & 0 \\ 2\pi\rho_l & 1 \end{pmatrix} = e^{2\pi i x_0}\mathrm{tr}_2\prod_{l=1}^{n}\mathcal{A}_l \end{matrix}\right. . \quad (5.4.25)$$

For this, and later use, it is useful to note that

$$\begin{aligned} \prod_{l\stackrel{*}{=}m+1}^{m} U_l = e^{2\pi i x_0} W^{-1}(r_m,\mu_m) \begin{pmatrix} 1 & 0 \\ 2\pi\rho_m & 1 \end{pmatrix} H_{m-1} \begin{pmatrix} 1 & 0 \\ 2\pi\rho_{m-1} & 1 \end{pmatrix} H_{m-2} \cdots \\ \cdots \begin{pmatrix} 1 & 0 \\ 2\pi\rho_1 & 1 \end{pmatrix} H_n \begin{pmatrix} 1 & 0 \\ 2\pi\rho_n & 1 \end{pmatrix} \cdots H_{m+1} \begin{pmatrix} 1 & 0 \\ 2\pi\rho_{m+1} & 1 \end{pmatrix} H_m W(r_m,\mu_m). \end{aligned} \quad (5.4.26)$$

In terms of the scalar function ψ,

$$\psi = -\cos 2\pi x_0 + \tfrac{1}{2}\mathrm{tr}_2\prod_{l=1}^{n}\mathcal{A}_l, \quad (5.4.27)$$

we find

$$\det\left(1-\prod_{l=1}^{n}U_l\right)=-2e^{2\pi ix_0}\psi,\tag{5.4.28}$$

by which we completed the construction of the coefficient vectors for the Green's function in eq. (5.4.19). We can now evaluate the Green's function in the region $\mu_m\leq z'\leq z\leq\mu_{m+1}$,

$$\begin{aligned}\hat f_x(z,z') &= \begin{pmatrix}1 & 0\end{pmatrix}W(r_m,z)c^b_{mm}(z') \qquad (5.4.29)\\ &= \begin{pmatrix}1 & 0\end{pmatrix}W(r_m,z)U^{-1}_{m+1}\left(1-\prod_{l\stackrel{*}{=}m+2}^{m+1}U_l\right)^{-1}U_{m+1}W^{-1}(r_m,z')\begin{pmatrix}0\\ -4\pi^2\end{pmatrix}.\end{aligned}$$

Using the fact that for a 2×2 matrix A

$$\begin{pmatrix}a & b\end{pmatrix}\mathrm{Minor}(A)\begin{pmatrix}c\\ d\end{pmatrix}=\begin{pmatrix}d & -c\end{pmatrix}A\begin{pmatrix}b\\ -a\end{pmatrix},\tag{5.4.30}$$

we can rewrite this last expression as

$$\begin{aligned}\hat f_x(z,z') = \frac{\pi}{r_m\psi}e^{2\pi ix_0(z-z')}e^{-2\pi ix_0}\Bigg(&\sinh 2\pi r_m(z-z') \qquad (5.4.31)\\ &-\tfrac{1}{2}\begin{pmatrix}e^{2\pi r_m z'} & e^{-2\pi r_m z'}\end{pmatrix}\left\{\prod_{l\stackrel{*}{=}m+1}^{m}U_l\right\}\begin{pmatrix}-e^{-2\pi r_m z}\\ e^{2\pi r_m z}\end{pmatrix}\Bigg).\end{aligned}$$

In terms of the $\mathcal{A}_l$, using eq. (5.4.26), we obtain a relatively compact expression for the Green's function for $\mu_m\leq z'\leq z\leq\mu_{m+1}$,

$$\begin{aligned}\hat f_x(z,z') = \tfrac{\pi e^{2\pi ix_0(z-z')}}{r_m\psi}\{&e^{-2\pi ix_0}\sinh(2\pi(z-z')r_m)+\\ &\langle v_m(z')|\mathcal{A}_{m-1}\cdots\mathcal{A}_1\mathcal{A}_n\cdots\mathcal{A}_m|w_m(z)\rangle\}.\end{aligned}\tag{5.4.32}$$

Here the spinors v_m and w_m are defined by

$$\begin{aligned}v^1_m(z) &= -w^2_m(z)=\sinh\left(2\pi(z-\mu_m)r_m\right),\\ v^2_m(z) &= w^1_m(z)=\cosh\left(2\pi(z-\mu_m)r_m\right).\end{aligned}\tag{5.4.33}$$

Using $\hat f_x(z,z')=\hat f_x(z',z)^*$, eq. (5.4.1), we can derive an expression for the Green's function for unordered $z,z'\in[\mu_m,\mu_{m+1}]$,

$$\begin{aligned}\hat f_x(z,z') = \frac{\pi e^{2\pi ix_0(z-z')}}{\psi r_m}\Bigg[&e^{-2\pi ix_0\mathrm{sgn}(z-z')}\sinh(2\pi r_m|z-z'|)+ \qquad (5.4.34)\\ &\mathrm{tr}_2\left(B_m(z+z',|z-z'|)\prod_{l\stackrel{*}{=}m}^{m-1}\mathcal{A}_l\right)\Bigg].\end{aligned}$$

Here, $B_m(z+z', z-z') = |w_m(z)\rangle\langle v_m(z')|$. Equivalently,

$$B_m(u,u') = \tag{5.4.35}$$
$$\frac{1}{2}\begin{pmatrix} -\sinh(2\pi r_m u')+\sinh(2\pi r_m(u-2\mu_m)) & \cosh(2\pi r_m u')+\cosh(2\pi r_m(u-2\mu_m)) \\ \cosh(2\pi r_m u')-\cosh(2\pi r_m(u-2\mu_m)) & -\sinh(2\pi r_m u')-\sinh(2\pi r_m(u-2\mu_m)) \end{pmatrix}.$$

To calculate $\partial_\mu \log\det f_x = \partial_\mu \mathrm{Tr}\log f_x = -\mathrm{Tr}(\partial_\mu f_x^{-1})f_x = 2\mathrm{Tr}B_\mu(x)f_x$ we need to evaluate

$$\frac{1}{\pi i}\int_{\hat{S}^1} \hat{D}_\mu(z)\hat{f}_x(z,z)dz.$$

We first evaluate the time-component, $\mu = 0$. The action of the z derivative appearing in $\hat{D}_0$ is defined by point-splitting,

$$\mathrm{Tr}(\frac{d}{dz}\hat{f}_x) = \left(\lim_{\epsilon\to 0}\int_{\hat{S}^1} dz[\frac{d}{dz}\hat{f}_x(z,z'+\epsilon) + \frac{d}{dz}\hat{f}_x(z,z'-\epsilon)]/2\right)_{z'=z}. \tag{5.4.36}$$

As $\hat{f}_x(z,z') \equiv e^{2\pi i x_0(z-z')}F_x(z,z')$,

$$(\frac{d}{dz} - 2\pi i x_0)\hat{f}_x(z,z') = e^{2\pi i x_0(z-z')}\frac{d}{dz}F_x(z,z'). \tag{5.4.37}$$

The z derivative of a function of $|z-z'|$ is given by

$$\frac{d}{dz}g(|z-z'|) = \mathrm{sgn}(z-z')g'(|z-z'|), \tag{5.4.38}$$

which yields zero when evaluated at $z = z'$, provided that $g(z)$ is smooth. Therefore, the B_m part of the z derivative of $\hat{f}_x(z,z')$ in eq. (5.4.34) is given by

$$\frac{d}{dz}B_m(z+z',|z-z'|)\Big|_{z'=z} = \frac{d}{dz}B_m(2z,0)/2. \tag{5.4.39}$$

For the first part of the z derivative of $\hat{f}_x(z,z')$ in eq. (5.4.34), which is not smooth, we find

$$\begin{aligned}\frac{d}{dz}\left(e^{-2\pi i x_0 \mathrm{sgn}(z-z')}\sinh(2\pi r_m|z-z'|)\right) = \\ e^{-2\pi i x_0\mathrm{sgn}(z-z')}[-2\pi i x_0\delta(z-z')\sinh(2\pi r_m|z-z'|) + \\ 2\pi r_m \mathrm{sgn}(z-z')\cosh(2\pi r_m|z-z'|)].\end{aligned} \tag{5.4.40}$$

Integrating this expression using point splitting, we obtain

$$\mathrm{Tr}(\frac{d}{dz} - 2\pi i x_0)\hat{f}_x = -2\pi^2 i\frac{\sin 2\pi x_0}{\psi} + \int_{\hat{S}^1}\frac{d}{dz}\hat{f}_x(z,z)/2. \tag{5.4.41}$$

As $\hat{f}_x(z,z)$ is a smooth periodic function, the integral of $d\hat{f}_x(z,z)/dz$ over the circle is zero. Therefore

$$\mathrm{Tr}(\frac{1}{2\pi i}\frac{d}{dz} - x_0)\hat{f}_x = -\tfrac{1}{2}\partial_0\log\psi. \tag{5.4.42}$$

In the evaluation of the spatial components, $\mu = i$, it is worthwhile to rewrite the Green's function,

$$\begin{aligned}
\hat{f}_x(z,z) &= \frac{\pi}{\psi r_m}\mathrm{tr}_2 B_m(2z,0)\prod_{l\stackrel{*}{=}m}^{m-1}\mathcal{A}_l \\
&= \frac{\pi}{\psi r_m}\mathrm{tr}_2\begin{pmatrix} 0 & 1 \\ 2\pi r_m & 0\end{pmatrix}\mathcal{H}_m B_m(2z,0)\begin{pmatrix} 0 & \frac{1}{2\pi r_m} \\ 1 & 0\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2\pi\rho_m & 1\end{pmatrix}\times \\
&\quad\times\prod_{l\stackrel{*}{=}m+1}^{m-1}\left(H_l\begin{pmatrix} 1 & 0 \\ 2\pi\rho_l & 1\end{pmatrix}\right),
\end{aligned} \tag{5.4.43}$$

using the relation between $\mathcal{A}_m$ and H_m, eq. (5.4.24), and defining

$$\mathcal{H}_m = \begin{pmatrix} \cosh 2\pi\nu_m r_m & \sinh 2\pi\nu_m r_m \\ \sinh 2\pi\nu_m r_m & \cosh 2\pi\nu_m r_m\end{pmatrix}. \tag{5.4.44}$$

A short calculation reveals

$$\int_{\mu_m}^{\mu_{m+1}} dz\begin{pmatrix} 0 & 1 \\ 2\pi r_m & 0\end{pmatrix}\mathcal{H}_m B_m(2z,0)\begin{pmatrix} 0 & \frac{1}{2\pi r_m} \\ 1 & 0\end{pmatrix} = \frac{1}{4\pi}\frac{\partial}{\partial r_m}H_m. \tag{5.4.45}$$

As the integral can be "pulled through the product", it then follows immediately that

$$\begin{aligned}
\mathrm{Tr} f_x B_i(x) &= \sum_{m=1}^{n}\int_{\mu_m}^{\mu_{m+1}}\frac{1}{2\pi i}\hat{D}_i(z)\hat{f}_x(z,z)dz \\
&= \sum_{m=1}^{n}\int_{\mu_m}^{\mu_{m+1}}\left(\frac{1}{2\pi i}\hat{A}_i(z) - x_i\right)\hat{f}_x(z,z)dz \\
&= \sum_{m=1}^{n}\frac{y_m^i - x^i}{4\psi r_m}\mathrm{tr}_2\left(\left(\frac{\partial}{\partial r_m}H_m\right)\begin{pmatrix} 1 & 0 \\ 2\pi\rho_m & 1\end{pmatrix}\prod_{l\stackrel{*}{=}m+1}^{m-1}\left(H_l\begin{pmatrix} 1 & 0 \\ 2\pi\rho_l & 1\end{pmatrix}\right)\right) \\
&= -\tfrac{1}{2}\partial_i\log\psi
\end{aligned} \tag{5.4.46}$$

One thus concludes from eqs. (5.4.42, 5.4.46) that

$$\partial_\mu\log\det f_x = -\partial_\mu\mathrm{Tr}\log f_x^{-1} = -\frac{i}{\pi}\mathrm{Tr}\hat{D}_\mu\hat{f}_x = -\partial_\mu\log\psi(x) \tag{5.4.47}$$

and therefore

$$-\tfrac{1}{2}\mathrm{Tr}F_{\mu\nu}^2(x) = \tfrac{1}{2}\partial_\mu^2\partial_\nu^2\log\det f_x = -\tfrac{1}{2}\partial_\mu^2\partial_\nu^2\log\psi(x). \tag{5.4.48}$$

This completes the proof of eq. (5.2.3).

5.5 Discussion

We briefly consider the properties of the $SU(n)$ charge one caloron. From eq. (5.2.4) we see that the $m^{\rm th}$ constituent monopole can be located at arbitrary $\vec{y}_m$, with arbitrary mass $8\pi^2\nu_m/\mathcal{T}$, subject only to the constraint $\sum_m \nu_m = 1$, choosing $\mathcal{P}_\infty$, $\vec{\xi}$ and ζ appropriately. As the action density, eq. (5.2.3), is expressed as a total derivative, the action is easily found by partial integration, with the expected result of $8\pi^2$. The size of the instanton is related to the differences in position of the constituent monopoles. As we work in units of $\mathcal{T}$, the situation of a small scale (nearby constituents), corresponds to large $\mathcal{T}$, i.e. to an instanton on $\mathbb{R}^4$. At the other extreme one has well separated lumps for small $\mathcal{T}$, i.e. in the static limit. In figure 5-1 we present a typical $SU(3)$ caloron for decreasing values of $\mathcal{T}$ using eq. (5.2.3).

When the lumps are far apart, they do not deform each other and become spherically symmetric. Since the solution is selfdual, the constituents have to be basic BPS monopoles. This can be proven by carefully analysing eq. (5.2.3) for the limit where $r_m \ll r_l$ for all $l \neq m$, in which case the action density approaches $-\frac{1}{2}\partial_\mu^2\partial_\nu^2 \log[\sinh(2\pi\nu_m r_m)/r_m]$. This is precisely the behaviour of the BPS monopole [88]. The other constituents need not be well-separated from each other for the above argument to hold.

The $SU(2)$ results of chapter 4, eqs. (4.3.30, 4.3.17), are retrieved by putting $\mu_1 = -\omega$, $\mu_2 = \omega$ and $\mu_3 = 1-\omega$, such that $\nu_1 = \mu_2 - \mu_1 = 2\omega$ and $\nu_2 = \mu_3 - \mu_2 = 1 - 2\omega \equiv 2\bar{\omega}$. Furthermore one identifies $r_1 = s$, $r_2 = r$ and $|\vec{y}_1 - \vec{y}_0| = |\vec{y}_2 - \vec{y}_1| = \pi\rho^2$.

5.5.1 The $(1,1,\ldots,1)$ monopole

When sending the $m^{\rm th}$ constituent to infinity (i.e. $|\vec{y}_m| \to \infty$) the caloron becomes static. What remains are $n-1$ monopole constituents with a combined magnetic charge opposite to the magnetic charge of the $m^{\rm th}$ constituent monopole that has been removed. As the solution is static in this limit one is left with an $SU(n)$ BPS monopole. Indeed, for $|\vec{y}_m| \to \infty$ we see from the solution of the Nahm equation, eq. (5.3.11), that $\hat{A}_i(z)$ lives on an interval, rather than on the circle, as is appropriate for the $SU(n)$ monopole [76]. From these Nahm data it is read off that it is the $(1,1,\ldots,1)$ monopole. (See also the appendix to chapter 6). One readily obtains the energy density of this monopole by taking the limit $|\vec{y}_m| \to \infty$ in eq. (5.2.4), to arrive at

$$\mathcal{E}(\vec{x}) = -\tfrac{1}{2}\mathrm{Tr}F_{\mu\nu}^2(\vec{x}) = -\tfrac{1}{2}\Delta^2 \log\tilde{\psi}_{\rm m}(\vec{x}), \tag{5.5.1}$$

$$\tilde{\psi}_{\rm m}(\vec{x}) = \tfrac{1}{2}\mathrm{tr}_2\left\{\frac{1}{r_{n-1}}\begin{pmatrix} s_{n-1} & c_{n-1} \\ 0 & 0\end{pmatrix}\prod_{m=1}^{n-2} A_m\right\}. \tag{5.5.2}$$

(see [65] for some special cases). One easily verifies that it decays as $1/|\vec{x}|^4$ - which is the behaviour of a monopole- as opposed to $1/|\vec{x}|^6$ without removing the $m^{\rm th}$ constituent.

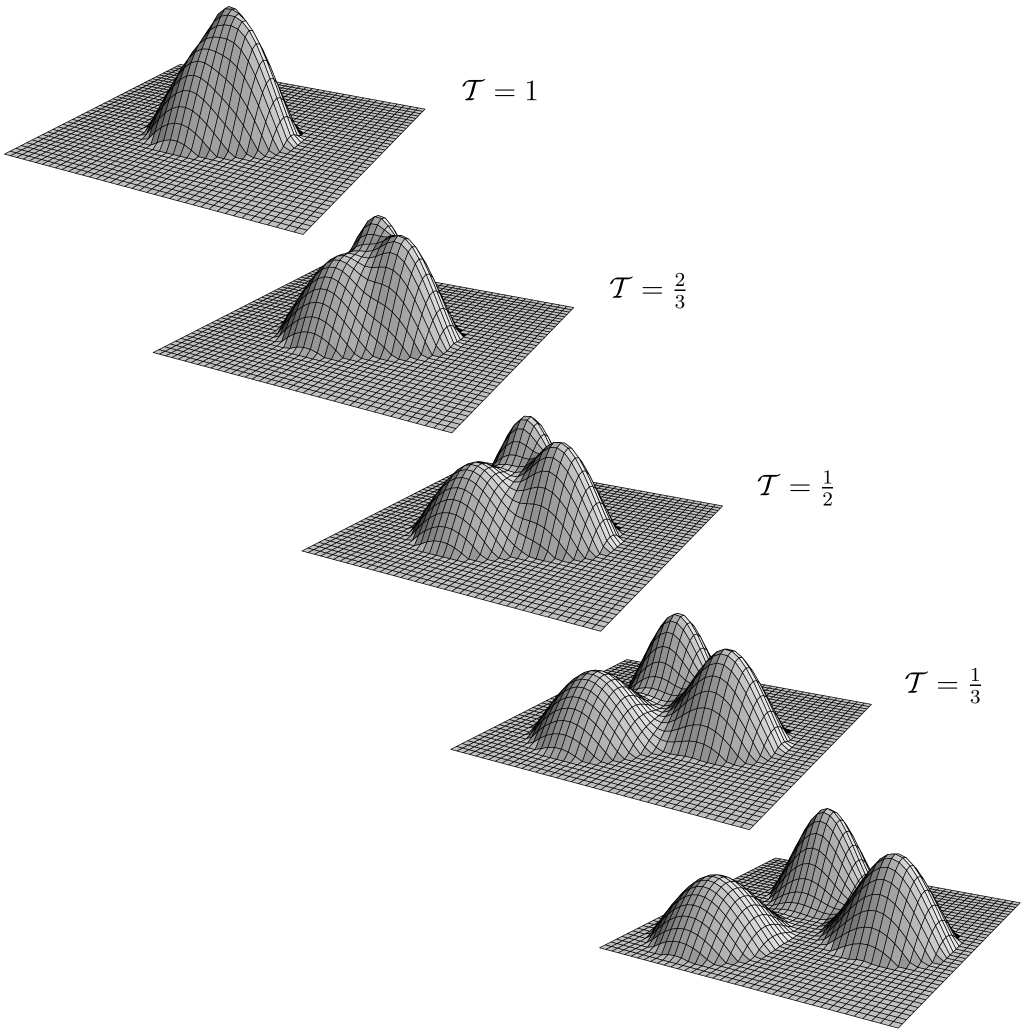


Figure 5-1. Action densities for the $SU(3)$ caloron on a logarithmic scale, cut off at $1/2e$, for $t = 0$ in the plane defined by $\vec{y}_1 = (-\frac{1}{2}, -\frac{1}{2}, 0)$, $\vec{y}_2 = (0, \frac{1}{2}, 0)$ and $\vec{y}_3 = (\frac{1}{2}, -\frac{1}{4}, 0)$, in units of $\mathcal{T}^{-1}$, for $1/\mathcal{T} = 1$, 3/2,2, 3 and 4 from top to bottom, $(\mu_1, \mu_2, \mu_3) = (-17, -2, 19)/60$. The constituent monopoles become manifest with decreasing compactification length and have masses $8\pi^2\nu_i/\mathcal{T}$, $(\nu_1, \nu_2, \nu_3) = (0.25, 0.35, 0.4)$.

Finally, we note that the $(1, 1, \ldots, 1)$ monopole has only one magnetic winding, as explained in section 2.2. It is opposite to the winding of the removed monopole, and hence, we can apply the reasoning in [95] explaining how the instanton charge arises also for $SU(n)$ from braiding two monopoles, cf. the $SU(2)$ situation discussed in section 4.6. Indeed, from the general formalism it follows that there is always a gauge in which the $(1, 1, \ldots, 1)$ monopole formed by $n - 1$ constituents is time-independent and where the last monopole carries the so-called Taubes-winding, even though its action density is time independent for well-separated constituents. This conclusion can also be drawn from the formalism developed in ref. [63, 58, 59], see also ref. [23].

5.5.2 Non-maximal symmetry breaking

The results have been derived for the case of maximal symmetry breaking, $\mu_m \neq \mu_{m+1}$. The situation of non-maximal symmetry breaking corresponds to a constituent obtaining zero mass, $\nu_m = 0$. In that case its centre of mass radius *drops out* of eq. (5.2.4), as follows from

$$\begin{pmatrix} r_m & |\vec{y}_m-\vec{y}_{m+1}| \\ 0 & r_{m+1} \end{pmatrix} \frac{1}{r_m} \begin{pmatrix} r_{m-1} & |\vec{y}_m-\vec{y}_{m-1}| \\ 0 & r_m \end{pmatrix} = \begin{pmatrix} r_{m-1} & |\vec{y}_m-\vec{y}_{m+1}|+|\vec{y}_m-\vec{y}_{m-1}| \\ 0 & r_{m+1} \end{pmatrix}. \tag{5.5.3}$$

This was also observed for $SU(2)$, in which case non-maximal symmetry breaking corresponds to a trivial Polyakov loop, $\mathcal{P}_\infty = \pm 1$, and the solution becomes that of Harrington and Shepard [42]. Hence the formula for the action density should also be valid for non-maximal symmetry breaking. More generally, we can consider, both for the caloron and for the $(1, 1, \ldots, 1)$ monopole, the situation of $N-1$ monopoles turning massless

$$\nu_K, \ldots, \nu_{K+N-2} = 0, \quad \mu_K = \ldots = \mu_{K+N-1}, \tag{5.5.4}$$

resulting in an enhanced residual symmetry to $SU(N) \times U(1)^{n-N}$. The corresponding centre of mass radii no longer appear in the expression for the action and energy densities, as follows from

$$\prod_{m=1}^{n} A_m \to \left\{ \prod_{m'=K+N-1}^{n} A_{m'} \right\} \begin{pmatrix} r_{K-1} & R_c \\ 0 & r_{K+N-1} \end{pmatrix} \begin{pmatrix} c_{K-1} & s_{K-1} \\ s_{K-1} & c_{K-1} \end{pmatrix} \frac{1}{r_{K-1}} \left\{ \prod_{m=1}^{K-2} A_m \right\}. \tag{5.5.5}$$

Here

$$R_c = |\vec{\rho}_K| + \ldots + |\vec{\rho}_{K+N-1}| = \pi \mathrm{tr}_2 \sum_{m=K}^{K+N-1} \zeta^\dagger_{(m)} \zeta_{(m)} \tag{5.5.6}$$

denotes what is known in the monopole literature as the "non-abelian cloud" parameter [62]. It is seen from the right hand side of eq. (5.5.6) that it is $SU(N)$ invariant. The $SU(N)$ transformations mix the positions of the massless monopoles, which therefore do not exist as individual particles. A way of seeing this physically is that the intrinsic length scales of the monopoles, proportional to their inverse masses, become infinitely large as their masses become small, so that they overlap and lose their identities, forming a cloud. This appearance of massless particles and infinite length scales illustrates a very general feature of systems near a transition to a more symmetric phase. In the situation of symmetry breaking to $SU(N_1) \times \cdots \times SU(N_s) \times U(1)^{n-N_1-\cdots-N_s}$, there will be s cloud parameters, $R_{c1}, \ldots, R_{cs}$ which are invariant under $SU(N_1), \ldots, SU(N_s)$ respectively.

5.5.3 Concluding remarks

Although the formalism can be extended easily to higher topological charges [64], the appropriate Nahm equation (i.e. solving the quadratic ADHM constraint) becomes a non-abelian problem, and finding solutions requires more powerful tools. Nevertheless, it is interesting to note that it is natural to conjecture that k instantons (i.e. an instanton of charge k) can be built from kn monopoles, since each instanton can be considered as being built from n BPS monopoles. The monopole constituents are only well separated when $\mathcal{T}$ is small, where the $4kn$ instanton parameters can be interpreted as $3kn$ positions and kn phases (including $\exp(2\pi i\xi_0/\mathcal{T})$). It should be emphasised that this monopole constituent picture has also some interesting phenomenological implications for the description of the long distance properties of QCD, discussed in more detail in section 4.6.

6 The metric of the $k = 1$ $SU(n)$ caloron

Moduli spaces of instantons and BPS monopoles have been subject to investigation for some time. The moduli space, quotient of the set of selfdual gauge connections by the group of gauge transformations, is a subset of the configuration space and its geometry therefore reflects physical properties of the system.

In this chapter the metric of the moduli space of $k = 1$ calorons with no magnetic windings is studied for the gauge group $SU(n)$, thus extending the analysis in section 4.5. Calorons are composed out of elementary BPS monopoles [63], as is seen from the action density, which was discussed in chapter 5. This becomes clear for small compactification lengths when the constituents are far apart. In particular, removing one of the monopoles to spatial infinity turns the $k = 1$ caloron into a BPS $SU(n)$ monopole, as described in section 5.5.1. In contrast, the situation of all monopoles nearly coalescing -in appropriate units corresponding to an infinite compactification length- gives back the ordinary instanton on $\mathbb{R}^4$. These various aspects are respected by the corresponding limits in the metric. The form of the metric was conjectured by Lee and Yi [63], using considerations of D-brane constructions and asymptotic monopole interactions. This chapter addresses the explicit calculation of the metric for the caloron moduli space and its limits.

Metric properties of moduli spaces of selfdual connections play an important role in the study of non-perturbative effects of gauge theories. For instantons the metric appears through the bosonic zero modes in the background of the charge one $SU(2)$ instanton in a calculation to study its physical effects [50]. The scattering of monopoles can be described as the geodesic motion on the moduli space [70], relating the metric to the Lagrangian of the interacting monopole system [71].

The metrics on these moduli spaces are hyperKähler [47]. This property derives formally from the nature of the selfduality equations themselves [3, 26], cf. section 1.5.2. It also appears in the ADHM construction of instantons of higher charge, as well as in the Nahm construction for monopoles, as a hyperKähler structure on the space of data [24, 25], cf. section 3.1.

The explicit computation of the metrics in this chapter is based on the isometric property of the Nahm transformation, known to hold for instantons on $\mathbb{R}^4$ and T^4, as well as for certain types of BPS monopoles [78]. It is believed to hold generally. For most situations considered in this chapter, an explicit proof seems not to be present in the literature, and will be given here. This allows for a determination of the metric on the moduli space of Nahm data. For monopoles, such a calculation was first done in [19] showing that the metric of the $(1, 1)$ data is a Taub-NUT space with positive mass parameter. Considerations based on asymptotic monopole interactions [37] reproduced this result [35, 60]. For the $(1, 1, \ldots, 1)$ monopole a similar equivalence was found [61, 74]. All these metrics are of so-called toric hyper-

Kähler type [33, 82, 83], and can be efficiently obtained as metrics on hyperKähler quotients [39]. An explicit calculation of the $k = 1$ $SU(2)$ caloron is extended here to $SU(n)$, generalising the techniques in section 4.5 . An alternative derivation using the hyperKähler quotient will also be given. There we will greatly benefit from the formalism in [74, 39], which is possible due to the similarity between the caloron and monopole Nahm data.

The outline of this chapter is as follows. Some preliminaries are discussed in section 6.1. The caloron metric is calculated in section 6.2. The instanton and monopole limits of the caloron are discussed in section 6.3. A unified description of instantons, calorons and monopoles is thus achieved. Other aspects of the caloron are commented on in the discussion. The appendix contains some technicalities on the $(1, 1, \ldots, 1)$ monopole.

6.1 Toric hyperKähler manifolds

The manifolds encountered in this chapter are all of toric hyperKähler type, which property will be explained in this section. The simplest example is $\mathbb{R}^4$, where there is one torus (circle) variable arising from a frequently used $U(1)$ fibration over $\mathbb{R}^3$, physically interpreted as a monopole phase and position. It is presented in terms of complex row 2-vectors that feature in the ADHM matrix λ and in the Nahm description of the $k = 1$ $SU(n)$ caloron, cf. section 5.3. Specifically, for a 2-dimensional complex row vector $\varsigma = (\varsigma_1, \varsigma_2)$, describing $\mathbb{R}^4$, the metric and Kähler forms read

$$g = \tfrac{1}{2}\mathrm{tr}_2(d\varsigma^\dagger d\varsigma), \quad \vec{\omega}\cdot\vec{\sigma} = \tfrac{1}{2}\sigma_i \mathrm{tr}_2 \bar{\sigma}_i d\varsigma^\dagger \wedge d\varsigma. \tag{6.1.1}$$

The complex structures act on ς by right multiplication with $-\sigma_i$. There is a triholomorphic $U(1)$ isometry with associated moment map

$$\varsigma \to e^{it}\varsigma, \quad \vec{\mu} = \tfrac{1}{2}\mathrm{tr}_2(-i\varsigma^\dagger\varsigma\vec{\bar{\sigma}}) = \tfrac{1}{2}\vec{r}. \tag{6.1.2}$$

The level sets are $U(1)$ fibres due to the phase ambiguity in defining ς from $\vec{r}$. This moment map and fibration also occurred in eq. (5.3.10). The $U(1)$ fibration becomes more manifest upon introducing new coordinates,

$$\varsigma = \varsigma^0 e^{i\frac{\psi}{2}}, \quad \psi \in \mathbb{R}/(4\pi\mathbb{Z}), \tag{6.1.3}$$

with for example $\varsigma_2^0(\vec{r})$ chosen real. A useful identity is

$$\tfrac{1}{2}\mathrm{tr}_2(\delta\varsigma_0^\dagger\varsigma_0 - \varsigma_0^\dagger\delta\varsigma_0) = -i|\vec{r}|\vec{\mathrm{w}}(\vec{r})\cdot d\vec{r}, \tag{6.1.4}$$

where $\vec{\mathrm{w}}(\vec{r})$ is the vector potential of the abelian Dirac monopole,

$$\vec{\nabla}_{\vec{r}} \times \vec{\mathrm{w}}(\vec{r}) = \vec{\nabla}_{\vec{r}}\frac{1}{|\vec{r}|}. \tag{6.1.5}$$

In the present form, the Dirac string lies along the positive z axis, other gauges are obtained by allowing for $\vec{r}$ dependent phase ambiguities. In terms of $(\vec{r}, \psi)$, the metric and Kähler forms on $\mathbb{R}^4$ read

$$\begin{aligned} ds^2 &= \frac{1}{4}\left(\frac{1}{|\vec{r}|}d\vec{r}^{\,2} + |\vec{r}|(d\psi + \vec{\mathrm{w}}(\vec{r})\cdot d\vec{r})^2\right), \\ \vec{\omega} &= (d\psi + \vec{\mathrm{w}}(\vec{r})\cdot d\vec{r}) \wedge d\vec{r} - \frac{1}{2r}d\vec{r}\wedge d\vec{r}. \end{aligned} \tag{6.1.6}$$

The $U(1)$ isometry is equivalent to a linear action

$$\psi \to \psi + 2t, \quad t \in \mathbb{R}/(2\pi\mathbb{Z}). \tag{6.1.7}$$

The general toric hyperKähler manifolds [83] have coordinates consisting of N three-vectors $\vec{x}_a \in \mathbb{R}^3, a = 1, \ldots, N$, and N torus variables ϕ_a, generalising the $U(1)$ in the previous example. Metric and Kähler forms read

$$\begin{aligned} g = d\vec{x}_a\Phi_{ab}\cdot d\vec{x}_b + \left(\frac{d\phi_a}{4\pi} + \vec{\Omega}_{ac}\cdot d\vec{x}_c\right)(\Phi^{-1})_{ab}\left(\frac{d\phi_b}{4\pi} + \vec{\Omega}_{bd}\cdot d\vec{x}_d\right), \\ \vec{\omega} = (\frac{d\phi_a}{4\pi} + \vec{\Omega}_{ab}\cdot d\vec{x}_b)\wedge d\vec{x}_a - \tfrac{1}{2}\Phi_{ab}d\vec{x}_b\wedge d\vec{x}_a. \end{aligned} \tag{6.1.8}$$

The potentials Φ and $\vec{\Omega}$ are ϕ_a independent, giving rise to N commuting triholomorphic isometries $\partial/\partial\phi_a$, corresponding to shifts on the torus. Closure of the Kähler forms is equivalent to

$$\frac{\partial}{\partial x_a^i}\Omega_{bc}^j - \frac{\partial}{\partial x_c^j}\Omega_{ba}^i = \varepsilon_{ijk}\frac{\partial}{\partial x_c^k}\Phi_{ab}, \quad \text{for all}\, a, b, c, i, j. \tag{6.1.9}$$

These equations are therefore called hyperKähler conditions [83, 33, 82], and generalise eq. (6.1.5). The metric in eq. (6.1.8) has an $SO(3)$ isometry, acting on the vectors $\vec{x}_a$, that rotates the complex structures. Toric hyperKähler manifolds are torus bundles over $(\mathbb{R}^3)^N$ [37]. Physically, the $\mathbb{R}^3$ vectors $\vec{x}_a$ are (relative) constituent monopole positions, whereas the torus describes the phases of the monopoles. In the Lagrangian interpretation of the metric, Φ and $\vec{\Omega}$ denote retarded interaction potentials for the constituents [37, 71] and it was considerations of this kind that led to the conjectures for the metric in [61, 63].

6.2 The caloron metric

For practical computations of metrics the formal reasoning in section 1.5.2 is of little use. Computing metrics on moduli spaces with the techniques presented there depends crucially on the construction of the Green's function of the covariant Laplacian and in the present situation, we do not even have an expression for A_μ readily available. We take a different route which uses multi-instanton calculus, suitably adapted to the caloron situation. This allows for calculating the metric in terms of the ADHMN data and makes it thus feasible to find a compensating gauge transformation or to perform the hyperKähler quotient.

6.2.1 Isometric property of the ADHM-Nahm construction.

We first recall that the computation of the metric on the moduli space of instantons on $\mathbb{R}^4$ can be entirely performed using ADHM techniques, as was explained in detail in section 3.1. Adapted to the $SU(n)$ caloron situation, this will translate into the formalism to calculate metrics in terms of Nahm data. In this way, we generalise the results in section 4.5.

In employing the metric properties of the ADHM construction in the caloron case, one has -in addition to the deformation equation and gauge orthogonality- the algebraic gauge condition eq. (5.3.2) to be satisfied

$$Z_\mu(x+1) = \mathcal{P}_\infty Z_\mu(x) \mathcal{P}_\infty^{-1}. \tag{6.2.1}$$

The periodicity constraint requires for the tangent vectors to the ADHM data in eq. (3.1.13) and the compensating gauge transformation in eq. (3.1.17)

$$Y_{p,p'} = Y_{p-1,p'-1}, \quad c_{p+1} = \mathcal{P}_\infty c_p, \quad \delta X_{p,p'} = \delta X_{p-p'}. \tag{6.2.2}$$

The compatibility of periodicity and nontrivial holonomy with the hyperKähler structure on the level of the ADHM-Nahm construction can be seen from the complex structures acting on Y and c as multiplication by $-\mathbf{i}, -\mathbf{j}, -\mathbf{k}$ on the right.

We define the Fourier transforms of the tangent vector

$$\begin{aligned} \hat{c}(z) &= \sum_{p\in\mathbb{Z}} \exp(-2\pi i p z) c_p = \sum_{m\in\mathbb{Z}/n\mathbb{Z}} \delta(z-\mu_m)\hat{c}_m, \\ \delta(z-z')\hat{Y}(z) &= \sum_{p,p'\in\mathbb{Z}} e^{2\pi i(pz-p'z')} Y_{p,p'}, \end{aligned} \tag{6.2.3}$$

and find after Fourier transformation of eqs. (3.1.14) the analogues of eq. (1.5.6) as the deformation of the Nahm equation and a gauge orthogonality condition

$$\begin{aligned} \frac{d}{dz}\hat{Y}_i(z) &= -i\pi \sum_{m\in\mathbb{Z}/n\mathbb{Z}} \mathrm{tr}_2 \bar{\sigma}^i (\zeta_m^\dagger \hat{c}_m + \hat{c}_m^\dagger \zeta_m)\delta(z-\mu_m), \\ \frac{d}{dz}\hat{Y}_0(z) &= -i\pi \sum_{m\in\mathbb{Z}/n\mathbb{Z}} \mathrm{tr}_2 (\zeta_m^\dagger \hat{c}_m - \hat{c}_m^\dagger \zeta_m)\delta(z-\mu_m). \end{aligned} \tag{6.2.4}$$

To evaluate the caloron metric we use eq. (3.1.19) and closely follow the reasoning in section 4.5. By Fourier transformation, Corrigan's formula is cast into

$$\begin{aligned} \mathrm{Tr} Z_\mu^\dagger(x) Z_\mu'(x) &= \\ & -\tfrac{1}{2}\partial^2 \mathrm{tr}_2 \int_{S^1} dz\, \Big([\hat{Y}^\dagger(z)\hat{Y}'(z) + \hat{Y}'^\dagger(z)\hat{Y}(z) + \\ & \qquad \hat{c}^\dagger(z) < \hat{c}' > + \hat{c}'^\dagger(z) < \hat{c} >] \hat{f}_x(z,z) \Big) \\ & +\tfrac{1}{2}\partial^2 \mathrm{tr}_2 \int_{S^1} dz dz'\, \Big([\hat{\mathcal{C}}(z) + \hat{\mathcal{Y}}_x(z)] \hat{f}_x(z,z') [\hat{\mathcal{Y}}_x'^\dagger(z') + \hat{\mathcal{C}}'^\dagger(z')] \hat{f}_x(z',z) \Big), \end{aligned} \tag{6.2.5}$$

where we introduced the shorthand notation

$$\hat{\mathcal{C}}(z) = \hat{c}^\dagger(z) < \hat{\lambda} >, \quad \hat{\mathcal{Y}}_x(z) = (2\pi i)^{-1}\hat{Y}^\dagger(z)\hat{D}_x(z). \tag{6.2.6}$$

In evaluating the integral over $\mathbb{R}^3 \times S^1$, the ∂_0^2 term gives no contribution because of periodicity. The term involving ∂_i^2 is evaluated by partial integration as a boundary term at spatial infinity, for which the asymptotic behaviour of the Green's function $\hat{f}_x(z,z')$ is needed. Since the asymptotic expression for the Green's function is independent of n,

$$\hat{f}_x(z,z') = \frac{\pi}{|\vec{x}|} e^{-2\pi|\vec{x}||z-z'|+2\pi i x_0(z-z')} + \mathcal{O}(|\vec{x}|^{-2}), \tag{6.2.7}$$

we can use, slightly adapted, the analysis for $SU(2)$ in section 4.5. Combining the first two lines in eq. (6.2.5) with the only surviving term of the third, we find the following gauge independent expression

$$\begin{aligned} g_{\mathcal{M}}(Z,Z') &= \tfrac{1}{2}\mathrm{tr}_2\left(< \hat{Y}^\dagger \hat{Y}' > + < \hat{c}^\dagger >< \hat{c}' > + < \hat{c}'^\dagger >< \hat{c} >\right), \qquad (6.2.8) \\ \omega^i_{\mathcal{M}}(Z,Z') &= \tfrac{1}{2}\mathrm{tr}_2\, \bar{\sigma}_i\left(< \hat{Y}^\dagger \hat{Y}' > + < \hat{c}^\dagger >< \hat{c}' > - < \hat{c}'^\dagger >< \hat{c} >\right). \end{aligned}$$

This proves that the metric and Kähler forms on the caloron moduli space can be computed as the metric on the Nahm data. In other words, for $k=1$ $SU(n)$ calorons, the Nahm construction is a hyperKähler isometry. A slightly modified proof shows this for monopoles of type $(1,1,\ldots,1)$ and can be found in the appendix.

The isometric property is essential for what follows. The metric on the caloron moduli spaces can now be calculated in terms of tangent vectors to the space of solutions to the Nahm equations, with infinitesimal gauge transformations performed where needed. This method, used in section 6.2.2, is called direct as it concentrates on the gauge orthogonal tangent vectors to the moduli space. An alternative method, given in section 6.2.3, uses the fact that the moduli space of data is an infinite dimensional hyperKähler quotient. It proceeds by using part of the $U(k)$ gauge symmetry to embed the moduli in a finite dimensional hyperKähler space. The metric on the moduli space is then found as the metric on a finite dimensional hyperKähler quotient, with the remaining gauge action to be divided out.

6.2.2 Direct computation

In the direct approach a compensating gauge function $\delta\hat{X}(z) = \sum_{p\in\mathbb{Z}} X_p \exp(2\pi i p z)$ has to be found to account for the tangent vectors

$$\hat{c}(z) = \sum_{m\in\mathbb{Z}/n\mathbb{Z}} \delta(z-\mu_m)\left(\delta\zeta_m + i\zeta_m\delta\hat{X}(\mu_m)\right), \quad \hat{Y}(z) = \frac{1}{2\pi i}\left(\delta\hat{A}(z) + i\frac{d}{dz}\delta\hat{X}(z)\right), \tag{6.2.9}$$

to be gauge orthogonal, eq. (6.2.4). The gauge orthogonality of $\hat{Y}(z)$ implies for the compensating gauge function $\delta\hat{X}(z)$

$$-\frac{1}{2\pi}\frac{d^2\delta\hat{X}(z)}{dz^2} + 2\delta\hat{X}(z)\sum_{m\in\mathbb{Z}/n\mathbb{Z}}\delta(z-\mu_m)|\vec{\rho}_m| = \sum_{m\in\mathbb{Z}/n\mathbb{Z}}\delta(z-\mu_m)\left[\frac{d\tau_m}{4\pi\nu_m} - \frac{d\tau_{m-1}}{4\pi\nu_{m-1}} - |\vec{\rho}_m|\vec{\mathrm{w}}_m(\vec{\rho}_m)\cdot d\vec{\rho}_m\right], \tag{6.2.10}$$

where we used eq. (6.1.4). This differential equation implies that $\delta\hat{X}(z)$ is continuous and piecewise linear. Therefore, $\delta\hat{X}(z)$ is fully determined by the values $\delta\hat{X}_m$ it takes at $z = \mu_m$, which are comprised in the vector $\delta\hat{X} = (\delta\hat{X}_1, \ldots, \delta\hat{X}_n) \in \mathbb{R}^n$. In the gauge chosen, all functions are either constants on the subintervals (μ_m, μ_{m+1}), or fixed by values at $z = \mu_m$. Therefore, the entire computation can conveniently be performed in terms of n dimensional vectors and $n \times n$ matrix operators acting thereon, at the cost of introducing some extra notation. For taking derivatives, we will use the $n \times n$ matrix

$$S = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & & \\ & & & 1 & -1 \\ -1 & & & & 1 \end{pmatrix}, \tag{6.2.11}$$

with unspecified entries zero. In addition we introduce the vector $\vec{\rho} = (\vec{\rho}_1, \ldots, \vec{\rho}_n) \in \mathbb{R}^{3n}$ (recall that $\vec{y} = (\vec{y}_1, \ldots, \vec{y}_n) \in \mathbb{R}^{3n}$) and diagonal matrices

$$N = \mathrm{diag}(\nu_1, \ldots, \nu_n),\ \vec{W} = \frac{1}{4\pi}\mathrm{diag}(\vec{\mathrm{w}}_1(\vec{\rho}_1), \ldots, \vec{\mathrm{w}}_n(\vec{\rho}_n)),\ V^{-1} = 4\pi\mathrm{diag}(\rho_1, \ldots, \rho_n). \tag{6.2.12}$$

Introducing the symbol V anticipates its later interpretation as potential. In the sequel, all matrix multiplications between n-dimensional objects are implicitly assumed. The transpose ^{t} acts only on the indices running from 1 to n.

The Nahm connection is now represented by the n dimensional vector

$$\hat{A} = (\hat{A}_1, \ldots, \hat{A}_n)^t = 2\pi(N^{-1}\frac{\tau}{4\pi} + \vec{y}\cdot\vec{\sigma}), \tag{6.2.13}$$

where $i\hat{A}_m$ is the value of $\hat{A}(z)$ on (μ_m, μ_{m+1}). The Nahm equation reduces to $\vec{\rho} = S^t\vec{y}$. Similarly $\hat{c}(z) = \sum_{m\in\mathbb{Z}/n\mathbb{Z}}\delta(z-\mu_m)\hat{c}_m$ and $\hat{Y}(z) = i\sum_{m\in\mathbb{Z}/n\mathbb{Z}}\chi_{[\mu_m,\mu_{m+1}]}(z)\hat{Y}_m$ are fixed by

$$\hat{c}_m = \delta\zeta_m + i\zeta_m\delta\hat{X}_m, \quad \hat{Y} = \frac{1}{2\pi}\delta\hat{A} - \frac{1}{2\pi}N^{-1}S\delta\hat{X}. \tag{6.2.14}$$

Integrating the differential equation (6.2.10) for $\delta\hat{X}(z)$ over small intervals $[\mu_m - \epsilon, \mu_m + \epsilon]$, $\epsilon \downarrow 0$, gives conditions on the values $\delta\hat{X}_m$. This yields

$$\frac{1}{2\pi}\left(S^tN^{-1}S + V^{-1}\right)\delta\hat{X} = (S^tN^{-1}\frac{d\tau}{4\pi} - V^{-1}\vec{W}S^t\cdot d\vec{y}), \tag{6.2.15}$$

where we used that $\int -d^2\delta\hat{X}(z)/dz^2 dz$ contributes

$$-\Big(\delta\hat{X}'(\mu_m+)-\delta\hat{X}'(\mu_m-)\Big)=-\left(\frac{1}{\nu_m}\left(\delta\hat{X}_{m+1}-\delta\hat{X}_m\right)-\frac{1}{\nu_{m-1}}\left(\delta\hat{X}_m-\delta\hat{X}_{m-1}\right)\right). \tag{6.2.16}$$

Eq. (6.2.15) is solved by

$$\frac{\delta\hat{X}}{2\pi}=VS^tG^{-1}\frac{d\tau}{4\pi}-\left(1-VS^tG^{-1}S\right)\vec{W}S^t\cdot d\vec{y}, \tag{6.2.17}$$

such that

$$\hat{Y}=d\vec{y}\cdot\vec{\sigma}+N^{-1}\frac{d\tau}{4\pi}-\frac{1}{2\pi}N^{-1}S\delta\hat{X}=d\vec{y}\cdot\vec{\sigma}+G^{-1}(\frac{d\tau}{4\pi}+S\vec{W}S^t\cdot d\vec{y}), \tag{6.2.18}$$

where we defined $G=N+SVS^t$. The integration over $\hat{S}^1$ to evaluate the metric on the Nahm data in eq. (6.2.8) is carried out as $<\hat{Y}^\dagger\hat{Y}>=\hat{Y}^\dagger N\hat{Y}$ using that each subinterval has length $\nu_m=\mu_{m+1}-\mu_m$. Thus we obtain

$$\begin{aligned}
\tfrac{1}{2}\mathrm{tr}_2<\hat{Y}^\dagger\hat{Y}> &= d\vec{y}^{\,t}\cdot Nd\vec{y}+(\frac{d\tau}{4\pi}+S\vec{W}S^t\cdot d\vec{y})^tG^{-1}NG^{-1}(\frac{d\tau}{4\pi}+S\vec{W}S^t\cdot d\vec{y}),\\
\tfrac{1}{4}\sigma_i\mathrm{tr}_2\bar{\sigma}_i<\hat{Y}^\dagger\wedge\hat{Y}> &= -\tfrac{1}{2}d\vec{y}^{\,t}N\wedge d\vec{y}\cdot\vec{\sigma}+\left(NG^{-1}(\frac{d\tau}{4\pi}+S\vec{W}S^t\cdot d\vec{y})\right)^t\wedge d\vec{y}\cdot\vec{\sigma}.
\end{aligned}$$

Using the properties (6.1.4,6.1.6) of ζ_m, the contribution to the metric of $\hat{c}_m$ defined in eq. (6.2.14) is found. One obtains

$$\begin{aligned}
&\mathrm{tr}_2<\hat{c}^\dagger><\hat{c}>= \\
&\quad d\vec{y}^{\,t}\cdot SVS^td\vec{y}+(\frac{d\tau}{4\pi}+S\vec{W}S^t\cdot d\vec{y})^tG^{-1}SVS^tG^{-1}(\frac{d\tau}{4\pi}+S\vec{W}S^t\cdot d\vec{y}),\\
&\tfrac{1}{2}\sigma_i\mathrm{tr}_2\bar{\sigma}_i<\hat{c}^\dagger>\wedge<\hat{c}>=\\
&\quad -\tfrac{1}{2}(SVS^td\vec{y})^t\wedge d\vec{y}\cdot\vec{\sigma}+\left(SVS^tG^{-1}(\frac{d\tau}{4\pi}+S\vec{W}S^t\cdot d\vec{y})\right)^t\wedge d\vec{y}\cdot\vec{\sigma},
\end{aligned} \tag{6.2.19}$$

where it is used that in the gauge chosen the phases of ζ are fixed. The metric and Kähler forms on moduli space of the uncentered caloron are now readily obtained

$$ds^2=\ d\vec{y}^{\,t}G\cdot d\vec{y}+(\frac{d\tau}{4\pi}+\vec{\mathcal{W}}\cdot d\vec{y})^tG^{-1}(\frac{d\tau}{4\pi}+\vec{\mathcal{W}}\cdot d\vec{y}), \tag{6.2.20}$$

$$\vec{\omega}=2(\frac{d\tau}{4\pi}+\vec{\mathcal{W}}\cdot d\vec{y})^t\wedge d\vec{y}-(Gd\vec{y})^t\wedge d\vec{y}, \tag{6.2.21}$$

$$G=N+SVS^t,\quad \vec{\mathcal{W}}=S\vec{W}S^t.$$

Equivalently writing

$$G_{m,m'}=\nu_m\delta_{mm'}-\frac{\delta_{m-1,m'}}{4\pi\rho_m}+\delta_{m,m'}\left(\frac{1}{4\pi\rho_m}+\frac{1}{4\pi\rho_{m+1}}\right)-\frac{\delta_{m+1,m'}}{4\pi\rho_{m+1}},\qquad m,m'\in\mathbb{Z}/n\mathbb{Z}, \tag{6.2.22}$$

reveals the form of G as given in [63]; thus we confirm the conjectured form for the metric in [63]. As is readily checked, from eqs. (6.1.5, 6.2.12) it follows that G and $\vec{\mathcal{W}}$ satisfy the hyperKähler conditions (6.1.9)

$$\vec{\nabla}_y G = \vec{\nabla}_y \times \vec{\mathcal{W}}, \quad \partial^i_m G_{m',m''} = \varepsilon_{ijk} \partial^j_m \left(\vec{\mathcal{W}}\right)^k_{m',m''}, \tag{6.2.23}$$

($\partial^i_m = \partial/\partial y^i_m$), which implies the Kähler forms in (6.2.21) to be closed and the caloron metric to be hyperKähler.

The metric has n commuting triholomorphic isometries,

$$\frac{\partial}{\partial \tau_m}, \quad m = 1, \ldots, n, \tag{6.2.24}$$

as G and $\vec{\mathcal{W}}$ are τ independent. The isometries correspond to shifts on the n-torus $\mathbb{R}^n/(4\pi\mathbb{Z})^n$ which describe the residual $U(1)^{n-1}$ gauge invariance and the temporal position

$$\xi_0 = \frac{1}{4\pi} \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \tau_m \in S^1, \tag{6.2.25}$$

of the caloron. Therefore, the caloron moduli space is a toric hyperKähler manifold, with dimension $4n$. $3n$ coordinates describe the monopole positions and n phase angles parameterise the temporal position and residual $U(1)^{n-1}$ gauge invariance in the case of maximal symmetry breaking. From the uncentered caloron metric in eq. (6.2.20), all other metrics discussed in this chapter can be obtained by taking suitable limits. In the next subsection the caloron metric will be obtained using the hyperKähler quotient.

The non-trivial part of the metric is obtained by splitting off the centre of mass coordinate ξ in eq. (5.3.14). To this aim, we express the metric in terms of ξ and $n-1$ relative monopole position vectors $\vec{\rho}_m$, using that $\vec{\rho}_n = -\sum_{m=1}^{n-1} \vec{\rho}_m$ because of eq. (5.3.12). The two sets of coordinates are related by the $n \times n$ dimensional centering matrix F_c,

$$F_\mathrm{c} = (S_c, N\mathrm{e}), \quad \begin{pmatrix} \tilde{\vec{\rho}} \\ \vec{\xi} \end{pmatrix} = F_\mathrm{c}^t \vec{y}. \tag{6.2.26}$$

Here, the $n \times (n-1)$ dimensional matrix S_c is obtained from S by omitting its last column, and we defined $\mathrm{e} = (1, \ldots, 1)^t \in \mathbb{R}^n$. A tilde denotes from now on the restriction to the first $n-1$ coordinates, e.g. $\tilde{\vec{\rho}} = (\vec{\rho}_1, \ldots, \vec{\rho}_{n-1})^t$. New torus coordinates $\tilde{\upsilon} = (\upsilon_1, \ldots, \upsilon_{n-1})^t$ are introduced as well

$$\tau = F_\mathrm{c} \begin{pmatrix} \tilde{\upsilon} \\ 4\pi\xi_0 \end{pmatrix}. \tag{6.2.27}$$

The centered metric will be again hyperKähler, as splitting off the centre off mass metric amounts to taking the hyperKähler quotient under the $U(1)$ action

$$\tau_m \to \tau_m + \nu_m t_c, \quad m = 1, \ldots, n, \quad t_c \in \mathbb{R}. \tag{6.2.28}$$

From eqs. (6.2.20, 6.2.21) it is seen that this action is a triholomorphic isometry whose moment map gives the centre of mass of the caloron

$$\vec{\mu} = \frac{1}{4\pi} \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \nu_m \vec{y}_m = \frac{\vec{\xi}}{4\pi}. \tag{6.2.29}$$

Indeed, the phase variables $\tilde{\upsilon}$ are invariant under the $U(1)$ action and can serve as coordinates on the quotient whereas the fibre coordinate ξ_0 changes as $\xi_0 \to \xi_0 + t_c$.

In the new basis the relative metric is expressed in terms of a relative mass matrix and relative interaction potentials

$$F_{\mathrm{c}}^{-1} G (F_{\mathrm{c}}^{-1})^t = \begin{pmatrix} \tilde{G}_{\mathrm{rel}} & \\ & 1 \end{pmatrix}, \quad \tilde{G}_{\mathrm{rel}} = \tilde{M} + \tilde{V}_{\mathrm{rel}},$$

$$(\tilde{V}_{\mathrm{rel}})_{mm'} = \tilde{V}_{mm'} + \frac{1}{4\pi|\vec{\rho}_n|}, \quad (\tilde{\vec{\mathcal{W}}}_{\mathrm{rel}})_{mm'} = \tilde{\vec{\mathcal{W}}}_{mm'} + \frac{\vec{\mathrm{w}}_n(\vec{\rho}_n)}{4\pi}, \tag{6.2.30}$$

where $m, m' = 1, \ldots, n-1$, $\vec{\rho}_n = -\sum_{m=1}^{n-1} \vec{\rho}_m$. The relative mass matrix $\tilde{M}$ is defined as

$$F_{\mathrm{c}}^t N^{-1} F_{\mathrm{c}} = \begin{pmatrix} \tilde{M}^{-1} & \\ & 1 \end{pmatrix}, \quad \tilde{M}^{-1} = \begin{pmatrix} \frac{1}{\nu_n} + \frac{1}{\nu_1} & -\frac{1}{\nu_1} & & \\ -\frac{1}{\nu_1} & \frac{1}{\nu_1} + \frac{1}{\nu_2} & -\frac{1}{\nu_2} & \\ \ddots & \ddots & \ddots & \\ & -\frac{1}{\nu_{n-3}} & \frac{1}{\nu_{n-3}} + \frac{1}{\nu_{n-2}} & -\frac{1}{\nu_{n-2}} \\ & & -\frac{1}{\nu_{n-2}} & \frac{1}{\nu_{n-2}} + \frac{1}{\nu_{n-1}} \end{pmatrix}, \tag{6.2.31}$$

its explicit form allowing one to take limits that correspond to massless monopoles

$$\begin{aligned} \tilde{M} = \tilde{M}^t, \quad \tilde{M}_{mm'} &= (\nu_m + \cdots + \nu_{n-1})(1 - \nu_{m'} \cdots - \nu_{n-1}) \\ \text{for}\, m \geq m', &\quad m, m' = 1, \ldots, n-1. \end{aligned} \tag{6.2.32}$$

The centered metric and Kähler forms now read

$$g = d\xi_\mu d\xi_\mu + d\tilde{\vec{\rho}}^{\,t} \tilde{G}_{\mathrm{rel}} \cdot d\tilde{\vec{\rho}} + \Big(\frac{d\tilde{\upsilon}}{4\pi} + \tilde{\vec{\mathcal{W}}}_{\mathrm{rel}} \cdot d\tilde{\vec{\rho}}\Big)^t \tilde{G}_{\mathrm{rel}}^{-1} \Big(\frac{d\tilde{\upsilon}}{4\pi} + \tilde{\vec{\mathcal{W}}}_{\mathrm{rel}} \cdot d\tilde{\vec{\rho}}\Big),$$

$$\vec{\omega} = 2 d\xi_0 \wedge d\vec{\xi} - d\vec{\xi} \wedge d\vec{\xi} + 2\Big(\frac{d\tilde{\upsilon}}{4\pi} + \tilde{\vec{\mathcal{W}}}_{\mathrm{rel}} \cdot d\tilde{\vec{\rho}}\Big)^t \wedge d\tilde{\vec{\rho}} - (\tilde{G}_{\mathrm{rel}} d\tilde{\vec{\rho}})^t \wedge d\tilde{\vec{\rho}}. \tag{6.2.33}$$

The first terms give the centre of mass metric on $\mathbb{R}^3 \times S^1$, the other terms represent the non-trivial part of the metric. Both are toric hyperKähler, and have an $SO(3)$ invariance corresponding to spatial rotations.

6.2.3 HyperKähler quotient construction

We follow the approach in [74] for BPS monopoles of type $(1, 1, \ldots, 1)$ and consider the right hand side of eq. (6.2.8) as the natural metric and Kähler forms on the space of caloron Nahm data $\hat{\mathcal{A}}$

$$
\begin{aligned}
g_{\hat{\mathcal{A}}} &= \tfrac{1}{2}\mathrm{tr}_2\left(< d\hat{A}^\dagger d\hat{A} > +2 < d\hat{\lambda}^\dagger >< d\hat{\lambda} >\right), \\
\omega^i_{\hat{\mathcal{A}}} &= \tfrac{1}{2}\mathrm{tr}_2\,\bar{\sigma}_i\left(< d\hat{A}^\dagger \wedge d\hat{A} > +2 < d\hat{\lambda}^\dagger > \wedge < d\hat{\lambda} >\right).
\end{aligned}
\tag{6.2.34}
$$

One then notes that the group $\hat{\mathcal{G}}$ of $U(1)$ gauge transformations on $\hat{S}^1$ acts triholomorphically on $\hat{\mathcal{A}}$. The zero set of the associated moment map is formed by the set $\mathcal{N}$ of solutions to the Nahm equations, which after quotienting by the $U(1)$ gauge action $\hat{\mathcal{G}}$ on the dual S^1 gives the moduli space of Nahm data. By virtue of eq. (6.2.8) this quotient is isometric to the caloron moduli space,

$$
\mathcal{M} = \mathcal{N}/\hat{\mathcal{G}}. \tag{6.2.35}
$$

As both $\mathcal{N}$ and $\hat{\mathcal{G}}$ are infinite dimensional, it is not obvious that this procedure is well-defined. However, using the gauge action we can restrict to those solutions $\mathcal{N}_0$ to the Nahm equations which have constant $\hat{A}_0(z)$ on the subintervals (μ_m, μ_{m+1}). As the Nahm equations force $\hat{A}_i(z)$ to be piecewise constant, there are n quaternions specifying the Nahm connection, denoted by $y \in \mathbb{H}^n$. The singularities (or matching data) are described by n complex two-component vectors ζ_m, denoted by $\zeta \in \mathbb{C}^{n,2}$. Hence, $\mathcal{N}_0$ is a subset of the space $\hat{\mathcal{A}}_0 = \mathbb{H}^n \times \mathbb{C}^{n,2}$ of possible piecewise constant data, which has metric

$$
\begin{aligned}
g &= \tfrac{1}{2}\mathrm{Tr}\,\mathrm{tr}_2\left(dy^\dagger N dy + 2 d\zeta^\dagger d\zeta\right), \\
\omega_i &= \tfrac{1}{2}\mathrm{Tr}\,\mathrm{tr}_2\,\bar{\sigma}_i\left(dy^\dagger \wedge N dy + 2 d\zeta^\dagger \wedge d\zeta\right),
\end{aligned}
\tag{6.2.36}
$$

as is natural from eq. (6.2.8). On $\hat{\mathcal{A}}_0$, the gauge action $\hat{\mathcal{G}}$ is restricted to the set $\hat{\mathcal{G}}_0$ of gauge functions with piecewise linear and continuous logarithm. These are determined by the values h assumes at $z = \mu_m$. Under these gauge transformations, $\hat{A}$ and ζ change according to

$$
\zeta_m \to e^{it_m}\zeta_m, \quad \psi \to \psi + 2t, \quad y \to y - \frac{1}{2\pi}N^{-1}St, \tag{6.2.37}
$$

where $t = (h(\mu_1), \ldots, h(\mu_n)) \in \mathbb{R}^n/(2\pi\mathbb{Z})^n$ and $\psi = (\psi_1, \ldots, \psi_n)/(4\pi\mathbb{Z})^n$ denotes the phases of ζ. The lattices correspond to gauge transformations of type (5.3.17). Therefore the action of the restriction $\hat{\mathcal{G}}_0$ of $\hat{\mathcal{G}}$ on $\hat{\mathcal{A}}_0$ is equivalent to an $\mathbb{R}^n/(2\pi\mathbb{Z})^n$ action on $\mathbb{H}^n \times \mathbb{C}^{n,2}$. Thus we reduced the infinite dimensional hyperKähler quotient to a finite dimensional. This technique was also used for the $(1, 1, \ldots, 1)$ monopole metric [74]. The metric on the moduli space of Nahm data can now be computed as a metric on a hyperKähler quotient of a finite dimensional euclidean space by a toric group action. To do this we follow [39]. From the metric and Kähler forms on $\hat{\mathcal{A}}_0$, determined by inserting eqs. (1.5.2, 6.1.6) in eq. (6.2.36),

$$ds^2 = dy^\dagger N dy + d\vec{\rho}^{\,t} V \cdot d\vec{\rho} + (\frac{d\psi}{4\pi} + \vec{W} \cdot d\vec{\rho})^t V^{-1} (\frac{d\psi}{4\pi} + \vec{W} \cdot d\vec{\rho}), \qquad (6.2.38)$$
$$\vec{\omega} = -(N d\vec{y})^t \wedge d\vec{y} + 2(N dy_0)^t \wedge d\vec{y} + 2(\frac{d\psi}{4\pi} + \vec{W} \cdot d\vec{\rho})^t \wedge d\vec{\rho} - (V d\vec{\rho})^t \wedge d\vec{\rho},$$

the action (6.2.37) is seen to be triholomorphic. The moment map for this $\mathbb{R}^n/(2\pi\mathbb{Z})^n$ action reads

$$\vec{\mu} \cdot \vec{\sigma} = -\frac{1}{4\pi} S^t (y - \bar{y}) - \Im i \zeta^\dagger P \zeta, \qquad (6.2.39)$$

where $P = (P_1, \ldots, P_n)^t$, and has a zero set $\vec{\mu}^{-1}(0)$ given by the solutions $\hat{A}$ corresponding to $\vec{\rho} = S^t \vec{y}$. Therefore, the space of piecewise constant solutions to the Nahm data is $(\hat{A}, \zeta) \in \mathcal{N}_0 = \vec{\mu}^{-1}(0) \subset \hat{\mathcal{A}}_0$. The moduli space of Nahm data is this set quotiented by the reduction of the gauge action in eq. (5.3.15), or equivalently $\mathbb{R}^n$. Hence

$$\mathcal{M} = \mathcal{N}/\hat{\mathcal{G}} = \mathcal{N}_0/\hat{\mathcal{G}}_0 = \vec{\mu}^{-1}(0)/(\mathbb{R}^n/(2\pi)^n). \qquad (6.2.40)$$

The metric on $\vec{\mu}^{-1}(0)$ reads

$$ds^2 = d\vec{y}^{\,t}(SVS^t + N) \cdot d\vec{y} + (\frac{d\psi}{4\pi} + \vec{W} \cdot S^t d\vec{y})^t V^{-1} (\frac{d\psi}{4\pi} + \vec{W} \cdot S^t d\vec{y}) + dy_0^t N dy_0,$$
$$\vec{\omega} = 2(\frac{d\tau}{4\pi} + \vec{\mathcal{W}} \cdot d\vec{y})^t \wedge d\vec{y} - (G d\vec{y})^t \wedge d\vec{y}. \qquad (6.2.41)$$

The n vector

$$\frac{\tau}{4\pi} = S \frac{\psi}{4\pi} + N y_0, \qquad (6.2.42)$$

is invariant under the $\mathbb{R}^n/(2\pi\mathbb{Z})^n$ action (6.2.37) and can therefore be used as coordinate on the quotient $\vec{\mu}^{-1}(0)/\mathbb{R}^n = \mathcal{M}$, together with $\vec{y}$. Cotangent vectors involving $d\psi$ have a vertical component, i.e. lie along the $\mathbb{R}^n/(2\pi\mathbb{Z})^n$ fibre. The horizontal and vertical part of the metric are separated by inserting $y_0 = \frac{1}{4\pi} N^{-1}(\tau - S\psi)$ and completing the squares to obtain

$$\begin{aligned} ds^2 \; = \; & d\vec{y}^{\,t} G \cdot d\vec{y} + \frac{d\tau^t}{4\pi} N^{-1} \frac{d\tau}{4\pi} + d\vec{y}^{\,t} S \cdot \vec{W} V^{-1} \vec{W} \cdot S^t d\vec{y} \\ & -(S^t N^{-1} \frac{d\tau}{4\pi} - V^{-1} \vec{W} S^t \cdot d\vec{y})^t (V^{-1} + S^t N^{-1} S)^{-1} (S^t N^{-1} \frac{d\tau}{4\pi} - V^{-1} \vec{W} S^t \cdot d\vec{y}) \\ & +\varphi^t (V^{-1} + S^t N^{-1} S) \varphi, \end{aligned} \qquad (6.2.43)$$

where the one-form φ denotes the component along the $\mathbb{R}^n/(2\pi\mathbb{Z})^n$ fibre

$$\varphi = \frac{d\psi}{4\pi} + (V^{-1} + S^t N^{-1} S)^{-1} V^{-1} \vec{W} S^t \cdot d\vec{y} - (V^{-1} + S^t N^{-1} S)^{-1} S^t N^{-1} \frac{d\tau}{4\pi}. \quad (6.2.44)$$

Horizontal projecting to the metric on $\vec{\mu}^{-1}(0)/(\mathbb{R}^n/(2\pi\mathbb{Z})^n)$ amounts to discarding the last term in eq. (6.2.43) and one obtains (after reorganising) the metric on the caloron moduli space $\mathcal{M}$ given in eq. (6.2.20). For the Kähler forms, this projection is generally not necessary: eq. (6.2.41) is precisely the Kähler form in eq. (6.2.21). This is a manifestation of the degeneracy of the Kähler forms along the gauge orbit, needed for the hyperKähler quotient to be well defined.

6.3 Instanton and monopole limits of the caloron

From the caloron metric, other toric hyperKähler manifolds can be obtained by taking suitable limits. For large $\mathcal{T}$ or equivalently all ρ_m small, one expects the metric to approach the moduli space for $k = 1$ $SU(n)$ instantons on $\mathbb{R}^4$. To study this limit, we consider the centered metric eq. (6.2.33). For small ρ_m, the elements of the relative mass matrix $\tilde{M}$ in eq. (6.2.30) are dominated by the ρ_m^{-1} terms in $\tilde{V}_{\rm rel}$,

$$F_{\rm c}^{-1}G(F_{\rm c}^{-1})^t = \begin{pmatrix} \tilde{G}_{\rm rel} & \\ & 1 \end{pmatrix} \to \begin{pmatrix} \tilde{V}_{\rm rel} & \\ & 1 \end{pmatrix}, \quad \rho_m \to 0, \quad m = 1, \ldots, n-1, \quad (6.3.1)$$

resulting in the asymptotic form for the non-trivial part of the metric and Kähler forms

$$g_{\rm limit} = d\tilde{\vec{\rho}}^{\,t}\tilde{V}_{\rm rel} \cdot d\tilde{\vec{\rho}} + (\frac{d\tilde{\upsilon}}{4\pi} + \tilde{\vec{\mathcal{W}}}_{\rm rel} \cdot d\tilde{\vec{\rho}})^t \tilde{V}_{\rm rel}^{-1} (\frac{d\tilde{\upsilon}}{4\pi} + \tilde{\vec{\mathcal{W}}}_{\rm rel} \cdot d\tilde{\vec{\rho}}),$$

$$\vec{\omega}_{\rm limit} = 2(\frac{d\tilde{\upsilon}}{4\pi} + \tilde{\vec{\mathcal{W}}}_{\rm rel} \cdot d\tilde{\vec{\rho}})^t \wedge d\tilde{\vec{\rho}} - (\tilde{V}_{\rm rel} d\tilde{\vec{\rho}})^t \wedge d\tilde{\vec{\rho}}. \quad (6.3.2)$$

The caloron with trivial gauge holonomy has the same limiting metric, as follows directly from taking the limit $\nu_1, \ldots, \nu_{n-1} \to 0, \nu_n \to 1$ of the caloron relative mass matrix in eq. (6.2.32). The phase variables are now given by $\upsilon_m = \tau_m + \ldots + \tau_{n-1} \in \mathbb{R}/(4\pi\mathbb{Z})$, cf. eq. (6.2.27). The Kähler forms $\vec{\omega}_{\rm limit}$ are closed, since the hyperKähler conditions (6.1.9) are satisfied

$$\vec{\nabla}_\rho \tilde{G}_{\rm rel} = \vec{\nabla}_\rho \times \tilde{\vec{\mathcal{W}}}_{\rm rel}, \quad (6.3.3)$$

hence the limiting metric for large $\mathcal{T}$ is hyperKähler. It is known as the Calabi metric.

This limit was discussed in [63] using indirect arguments. With the techniques presented in this chapter, it is easy to prove explicitly that the limiting metric is indeed the metric for both the ordinary $k = 1$ $SU(n)$ instantons on $\mathbb{R}^4$ and the calorons with trivial holonomy. It follows immediately when realising that the $4(n-1)$ dimensional Calabi space can be obtained as the hyperKähler quotient of $\mathbb{H}^n$ by a $U(1)$ action [39]. This quotient emerges naturally from both the construction of the charge one $SU(n)$ instanton and the trivial holonomy caloron. First note that there is a one to one correspondence between the ADHM data of the $k = 1$ $SU(n)$ instanton and the Nahm data of the trivial holonomy caloron in the $\hat{\mathcal{G}}$ gauge with constant $\hat{A}_0(z)$. The latter are given in terms of $(\xi, \zeta) \in \mathbb{H} \times \mathbb{C}^{n,2}$ as $\hat{A}(z) = 2\pi i\xi, \quad \hat{\lambda}(z) = \delta(z)\zeta$ and directly translate into ADHM data $\lambda = \zeta, \quad B = \xi$ for the instanton. With only one subinterval, the metric on the Nahm data now reduces to the expression for the instanton (3.1.32). Having restricted to constant $\hat{A}_0(z)$, the remaining transformations in $\hat{\mathcal{G}}_0$ leave ξ invariant, apart from confining ξ_0 to the circle through $g(z) = \exp(2\pi i l z), l \in \mathbb{Z}$. For their action on the matching data only the $U(1)$ formed by the values $g(0)$ is relevant. Therefore, in both cases the nontrivial part of the moduli space is the quotient of $\mathbb{C}^{n,2}$ (with $g = \frac{1}{2}\mathrm{tr}_2 d\zeta^\dagger d\zeta$, $\omega_i = \frac{1}{2}\mathrm{tr}_2\,(\bar{\sigma}_i d\zeta^\dagger \wedge d\zeta)$) by the $U(1)$ action

$$\zeta_m \to e^{it}\zeta_m, \quad \psi_m \to \psi_m + 2t, \qquad m = 1, \ldots n, \quad t \in \mathbb{R}/(2\pi\mathbb{Z}). \tag{6.3.4}$$

(Identifying $\mathbb{C}^2$ and $\mathbb{H}$, this quotient is readily seen to be equivalent to that discussed in eq. (36) of [39]). The corresponding moment map, zero set and invariants are given by

$$\vec{\mu} = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}/n\mathbb{Z}} \vec{\rho}_m, \qquad \sum_{m\in\mathbb{Z}/n\mathbb{Z}} \vec{\rho}_m = \vec{0}, \quad \tilde{\upsilon}_m = \psi_m - \psi_n, \quad m = 1, \ldots, n-1. \tag{6.3.5}$$

Expressing the metric on the zero set in terms of invariants and the terms involving $d\psi_n$ describing the fibre part, one obtains [39] the Calabi metric in eq. (6.3.2).

The Calabi metric has an $SU(n)$ triholomorphic isometry, reflecting the $SU(n)$ gauge symmetry of the $k = 1$ instanton and trivial holonomy caloron. As explained in section 6.1 for the instanton, it emerges as the $SU(n)$ acting on ζ in eq. (6.3.4) on the left, commuting with $U(1)$, and descending to the quotient. A direct calculation using a compensating gauge transformation gives the same result.

In sections 4.5 and 5.5 it was explicitly shown from the action density that removing one of the constituent monopoles of the caloron to spatial infinity, $|\vec{y}_n| \to \infty$ turns it into a static selfdual $SU(n)$ solution, i.e. a monopole in the BPS limit. Indeed, this limit corresponds to the compactification length going to zero. The Nahm data suggest that the remnant is the $(1, 1, \ldots, 1)$ monopole. We will show indeed that the metric in this limit has the required form.

Removing a constituent is described by a hyperKähler quotient. Consider the $U(1)$ action that changes the phase of the mth monopole in the uncentered caloron

$$\tau_m \to \tau_m + t, \quad t \in \mathbb{R}/(4\pi\mathbb{Z}). \tag{6.3.6}$$

It is a triholomorphic isometry as follows from eqs. (6.2.20, 6.2.21). Its moment map $\vec{\mu}_{\rm fix}$ is exactly the position of the mth monopole, $\vec{\mu}_{\rm fix} = \vec{y}_m/(4\pi)$. Therefore, the metric on the quotient, the caloron moduli space with the mth constituent fixed, is hyperKähler irrespective of its position. For finite $|\vec{y}_m|$, the resulting metric on the quotient $\vec{\mu}_{\rm fix}^{-1}(\vec{y}_m)/\mathbb{R}$ is complicated, and no longer $SO(3)$ symmetric. Removing the constituent, $|\vec{y}_m| \to \infty$, i.e. fixing it at spatial infinity, gives the hyperKähler metric of the remnant BPS monopole, with a simple form and $SO(3)$ symmetry restored.

The metric and Kähler forms with the nth monopole far away, in which case $\rho_1^{-1}, \rho_n^{-1} \to 0$, reads

$$(g, \vec{\omega}) = (g_n, \vec{\omega}_n) + (g_{\rm m}, \vec{\omega}_{\rm m}). \tag{6.3.7}$$

Here the removed monopole is described by the metric $g_n = \nu_n d\vec{y}_n^2 + {\nu_n}^{-1} d\tau_n^2/(4\pi^2)$ and Kähler forms $\vec{\omega}_n = d\tau/(2\pi) \wedge d\vec{y}_n - \nu_n d\vec{y}_n \wedge d\vec{y}_n$, and the remnant by

$$g_{\rm m} = d\vec{y}_{\rm m}{}^t G_{\rm m} \cdot d\vec{y}_{\rm m} + (\frac{d\tau_{\rm m}}{4\pi} + \vec{\mathcal{W}}_{\rm m} \cdot d\vec{y}_{\rm m})^t G_{\rm m}^{-1}(\frac{d\tau_{\rm m}}{4\pi} + \vec{\mathcal{W}}_{\rm m} \cdot d\vec{y}_{\rm m}),$$

$$\vec{\omega}_{\mathrm{m}} = -\tfrac{1}{2}(G_{\mathrm{m}} d\vec{y}_{\mathrm{m}})^t \wedge d\vec{y}_{\mathrm{m}} + (\frac{d\tau_{\mathrm{m}}}{4\pi} + \vec{\mathcal{W}}_{\mathrm{m}} \cdot d\vec{y}_{\mathrm{m}})^t \wedge d\vec{y}_{\mathrm{m}}, \tag{6.3.8}$$

where

$$G_{\mathrm{m}} = N_{\mathrm{m}} + S_{\mathrm{m}} V_{\mathrm{m}} S_{\mathrm{m}}^t, \quad \vec{\mathcal{W}}_{\mathrm{m}} = S_{\mathrm{m}} \vec{W}_{\mathrm{m}} S_{\mathrm{m}}^t,$$

$$V_{\mathrm{m}}^{-1} = 4\pi \mathrm{diag}(\rho_2, \ldots, \rho_{n-1}), \quad \vec{W}_{\mathrm{m}} = \mathrm{diag}(\vec{\mathrm{w}}_2(\vec{\rho}_2), \ldots, \vec{\mathrm{w}}_{n-1}(\vec{\rho}_{n-1}))/(4\pi),$$

$$N_{\mathrm{m}} = \mathrm{diag}(\nu_1, \ldots, \nu_{n-1}), \tag{6.3.9}$$

$$y_{\mathrm{m}} = (y_1, \ldots, y_{n-1})^t, \;\; \vec{\rho}_{\mathrm{m}} = (\vec{\rho}_2, \ldots, \vec{\rho}_{n-1})^t, \;\; \tau_{\mathrm{m}} = (\tau_1, \ldots, \tau_{n-1})^t,$$

$$S_{\mathrm{m}} = \begin{pmatrix} -1 & & & \\ 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \\ & & & 1 \end{pmatrix} \in \mathbb{R}^{n-1,n-2}. \tag{6.3.10}$$

More explicitly, the potential term in eq. (6.3.8) reads

$$4\pi S_{\mathrm{m}} V_{\mathrm{m}} S_{\mathrm{m}}^t = \begin{pmatrix} \frac{1}{\rho_2} & -\frac{1}{\rho_2} & & \\ -\frac{1}{\rho_2} & \frac{1}{\rho_2} + \frac{1}{\rho_3} & -\frac{1}{\rho_3} & \\ \ddots & \ddots & \ddots & \\ & -\frac{1}{\rho_{n-2}} & \frac{1}{\rho_{n-2}} + \frac{1}{\rho_{n-1}} & -\frac{1}{\rho_{n-1}} \\ & & -\frac{1}{\rho_{n-1}} & \frac{1}{\rho_{n-1}} \end{pmatrix} \in \mathbb{R}^{n-1,n-1}. \tag{6.3.11}$$

The vector potential $\vec{\mathcal{W}}_{\mathrm{m}}$ has a similar structure. The metric in eq. (6.3.8) is that of the uncentered $SU(n)$ monopole of type $(1,1,\ldots,1)$. The calculation of the metric on its space of Nahm data was performed in [39, 74]. Details on the Nahm construction of the $(1,1,\ldots,1)$ monopole and a proof of its isometric property as well as an outline of the calculation of the metric can be found in the appendix to this chapter.

To connect with [61], we have to centre the monopole. We introduce

$$F_{\mathrm{m}} = \begin{pmatrix} S_{\mathrm{m}}, & \frac{1}{\nu} N_{\mathrm{m}} \mathrm{e}_{\mathrm{m}} \end{pmatrix} \in \mathbb{R}^{n-1,n-1}, \tag{6.3.12}$$

where $\mathrm{e}_{\mathrm{m}} = (1, \ldots, 1)^t \in \mathbb{R}^{n-1}$ and $\nu = \sum_{m=1}^{n-1} \nu_m$ denotes the mass of the monopole. The relative position variables $\vec{\rho}_{\mathrm{m}}$ are reinstated and the centre of mass $\mathbb{R}^3$ position is separated off using

$$\vec{y}_{\mathrm{m}} = (F_{\mathrm{m}}^t)^{-1} \begin{pmatrix} \vec{\rho}_{\mathrm{m}} \\ \vec{\xi}_{\mathrm{m}} \end{pmatrix}, \quad \vec{\xi}_{\mathrm{m}} = \frac{1}{\nu} \sum_{m=1}^{n-1} \nu_m \vec{y}_m. \tag{6.3.13}$$

The mass matrix in this basis is given by

$$F_{\mathrm{m}}^t N_{\mathrm{m}}^{-1} F_{\mathrm{m}} = \begin{pmatrix} M_{\mathrm{m}}^{-1} & \\ & \nu^{-1} \end{pmatrix},$$

$$M_{\rm m}^{-1} = \begin{pmatrix} \frac{1}{\nu_1}+\frac{1}{\nu_2} & -\frac{1}{\nu_2} & & & \\ -\frac{1}{\nu_2} & \frac{1}{\nu_2}+\frac{1}{\nu_3} & -\frac{1}{\nu_3} & & \\ \ddots & \ddots & \ddots & & \\ & -\frac{1}{\nu_{n-3}} & \frac{1}{\nu_{n-1}}+\frac{1}{\nu_{n-2}} & -\frac{1}{\nu_{n-2}} \\ & & -\frac{1}{\nu_{n-2}} & \frac{1}{\nu_{n-2}}+\frac{1}{\nu_{n-1}} \end{pmatrix},$$

$$M_{\rm m} = M_{\rm m}^t, \quad (M_{\rm m})_{m,m'} = \nu^{-1}(\nu_1+\cdots+\nu_m)(\nu_{m'+1}+\cdots+\nu_{n-1}), \quad \text{for}\, m' \geq m. \tag{6.3.14}$$

Furthermore, alternative torus coordinates $\chi_{\rm m} = (\chi_1, \ldots, \chi_{n-2})$ are introduced, as well as a global $U(1)$ phase $\xi_{0,\rm m}$,

$$\tau_{\rm m} = F_{\rm m}\begin{pmatrix} \chi_{\rm m} \\ \xi_0 \end{pmatrix}, \quad \xi_{0,\rm m} = \sum_{m=1}^{n-1}\tau_m. \tag{6.3.15}$$

In the new coordinates, the uncentered metric is the sum of the centre of mass and relative metric

$$g_{\rm m} = \nu d\vec{\xi}_{\rm m}\cdot d\vec{\xi}_{\rm m} + \nu^{-1}d\xi_{0,\rm m}^2 + g_{\rm m}^c, \tag{6.3.16}$$

where the nontrivial part

$$g_{\rm m}^c = d\vec{\rho}_{\rm m}^{\,t}(M_{\rm m}+V_{\rm m})\cdot d\vec{\rho}_{\rm m} + (\frac{d\chi_{\rm m}}{4\pi}+\vec{W}_{\rm m}\cdot d\vec{\rho}_{\rm m})^t(M_{\rm m}+V_{\rm m})^{-1}(\frac{d\chi_{\rm m}}{4\pi}+\vec{W}_{\rm m}\cdot d\vec{\rho}_{\rm m}) \tag{6.3.17}$$

is the Lee-Weinberg-Yi metric [61]. It is of toric hyperKähler form. Thus we proved that the $(1,1,\ldots,1)$ monopole is a limit of the caloron, identifying the static remnant in section 5.5.

6.4 Discussion

Since the metric describes the Lagrangian for adiabatic motion on the moduli space [70], it reflects the interactions of the monopole constituents. The constituent nature of the caloron solution, easily extracted from the action density, eq. (5.2.3), should therefore also be reflected in the metric.

This density allow for an unambiguous identification of elementary BPS monopoles as constituents of calorons, and $(1,1,\ldots,1)$ monopoles, as in the limit where $r_m \ll r_l$ for all $l \neq m$ the action density approaches that of the single BPS monopole, see section 5.5. The corresponding limit in the uncentered metrics reveals

$$ds^2|_m = \nu_m d\vec{y}_m\cdot d\vec{y}_m + \frac{1}{\nu_m}d\tau_m^2 \tag{6.4.1}$$

for the part describing the $m^{\rm th}$ constituent, as all interaction potentials approach zero with the other constituents far away. Eq. (6.4.1) is the flat metric on $\mathbb{R}^3\times S^1$, the twofold cover of the moduli space for the elementary BPS monopole. Therefore the limit of the (cover of the) caloron moduli space -corresponding to all monopoles well separated- can be seen as a product of elementary BPS monopole moduli spaces.

We obtained the metric for the $k = 1$ $SU(n)$ caloron assuming symmetry breaking to the maximal torus $U(1)^{n-1}$ with arbitrarily chosen holonomy eigenvalues μ_m. In the situation of non-maximal breaking, some of the eigenvalues of the holonomy become equal, resulting in some monopoles acquiring zero mass. The form of the relative mass matrices defined as inverses suggests that dramatic things happen when one or more of the constituents acquire zero mass. However, as is clear from the explicit forms of $M, M_{\rm m}$ in equations (6.2.12, 6.2.32, 6.3.9, 6.3.14), all limits can be taken smoothly. This assertion was explicitly checked for the trivial holonomy caloron, with all but one monopoles having zero mass. Therefore one can study most efficiently all symmetry breaking patterns, both for $k = 1$ calorons and for monopoles of type $(1, 1, \ldots 1)$, just by inserting the proper values for μ_m, rather than having to calculate the metric for each case separately. From the ADHM-Nahm construction (3.1.3, 3.1.4), this $SU(N)$ symmetry is seen to leave the holonomy invariant. It will descend to the quotient in the hyperKähler quotient construction of the metric, and therefore, the metric will be $SU(N)$ invariant as well, just like in the case of the trivial holonomy caloron. As the explicit form of the metric can readily be found by inserting eq. (5.5.4) in the mass matrices (6.2.12, 6.3.9), it will not be given here.

The fact that the $SU(n+1)$ $(1, 1, \ldots, 1)$ monopole and the $k = 1$ $SU(n)$ caloron both consist out of n constituent BPS monopoles in combination with the fact that the former can be obtained out of an $SU(n+1)$ caloron, suggests a great similarity between their metrics. We consider the relevant situation for quantum chromodynamics, the $SU(3)$ caloron. Removing one monopole to infinity gives the $SU(3)$ monopole of type (1,1). There remain two constituents, of masses proportional to ν_1, ν_2. The relative metric of the $(1, 1)$ monopole is Taub-NUT [19, 35, 60] with positive mass parameter, as follows from eq. (6.3.17)

$$g_{TN} = U(\vec{\rho})d\vec{\rho}^2 + U(\vec{\rho})^{-1}(\frac{d\psi}{4\pi} + \frac{Q\vec{\rm w}(\vec{\rho})}{4\pi} \cdot d\vec{\rho})^2, \quad U(\vec{\rho}) = \frac{\nu_1\nu_2}{\nu_1 + \nu_2} + \frac{Q}{4\pi|\vec{\rho}|}, \qquad (6.4.2)$$

$\vec{\rho}$ denoting the separation of the constituents, $Q = 1$. The relative metric for the $SU(2)$ caloron is also a Taub-NUT space, as proven in section 4.5. (The metric obtained there in eq. (4.5.29) checks with eq. (6.4.2) apart from the normalisation $4\pi^2$, as $\pi\rho^2, \Upsilon$ in section 4.5 is related to $|\vec{\rho}|, \psi$ in eq. (6.4.2)). However, the interaction strength, depending on the distance between the monopoles, for the caloron is $Q = 2$, twice that of the $SU(3)$ monopole. Both solitons can be considered as built out of two interacting constituent BPS monopoles, and have a four-dimensional relative moduli space. As each matching point in the Nahm construction gives rise to an interaction between monopoles of distinct type, this is to be expected. In [58, 59] this was attributed to the fact that the constituent monopoles in the $SU(3)$ (1,1) case are charged with respect to different $U(1)$, whereas for the caloron, they are oppositely charged with respect to the *same* $U(1)$, generated by $\hat{\omega} \cdot \vec{\tau}$.

However, the true moduli spaces of the $k = 1$ $SU(2)$ caloron and the $SU(3)$ $(1, 1)$ monopole are different. In the relative moduli space of the $k = 1$ $SU(2)$ caloron,

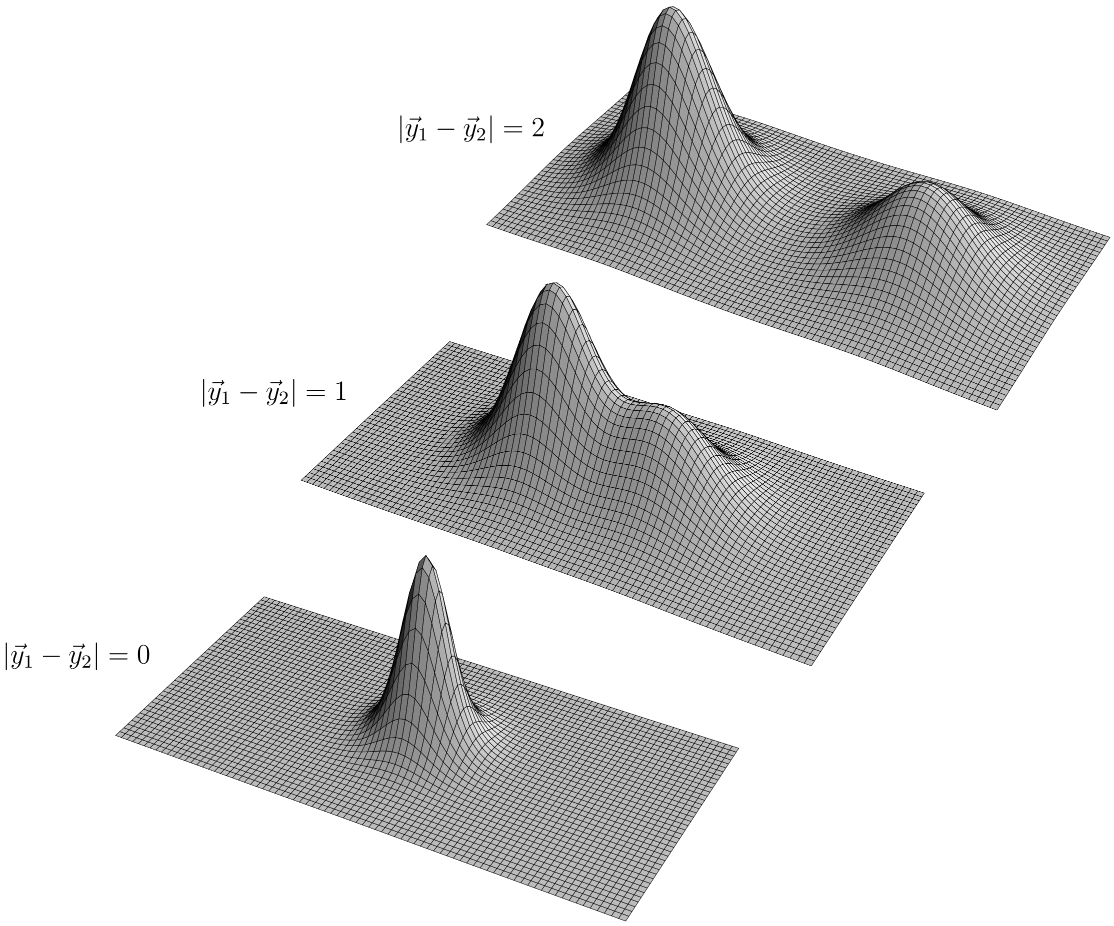


Figure 6-1. Energy density profiles for the $SU(3)$ $(1,1)$ monopole with various intermonopole distances $|\vec{y}_1 - \vec{y}_2| = 2, 1, 0$ on equal horizontal and different vertical scales. The asymptotic value of the Higgs field corresponds to $(\mu_1, \mu_2, \mu_3) = (-\frac{31}{60}, \frac{2}{60}, \frac{29}{60})$. The maxima correspond to $\mathcal{E} = 125., 169.$ and $1039.$ respectively.

there is a Z_2 to be divided out to obtain the Z_2 orbifold of Taub-NUT space,

$$\mathcal{M}^{\rm cal}_{SU(2)} = \mathbb{R}^3 \times S^1 \times \frac{\text{Taub-NUT}}{Z_2}. \tag{6.4.3}$$

The singular point, the apex of the cone, corresponds to $\rho = 0$ when the constituents are on top of each other and one is dealing with an ideal instanton (as depicted in figs. 4-3, 4-4). Thus the orbifold singularity corresponds to the delta-function action density. For the $SU(3)$ $(1,1)$ monopole, the moduli space is [35, 60]

$$\mathcal{M}^{(1,1)\rm mon}_{SU(3)} = \mathbb{R}^3 \times \frac{\mathbb{R} \times \text{ Taub-NUT}}{\mathbb{Z}} \tag{6.4.4}$$

where $\mathbb{R}/\mathbb{Z}$ replaces the S^1 in eq. (6.4.3) and there is no such finite discrete symmetry to be divided out. We therefore can go smoothly to the origin of Taub-NUT space.

Indeed, an energy density profile of the $(1,1)$ monopole reveals that the two constituents can be placed on top of each other while the energy stays finitely distributed. This is depicted in figure 6-1, which was constructed using eq. (5.5.1).

Appendix A: The $(1,1,\ldots,1)$ monopole

The Nahm construction of the $(1,1,\ldots,1)$ monopole is similar to that of the $k=1$ $SU(n)$ caloron. The main difference is that the circle is replaced by the interval $[\mu_1,\mu_n]$. For the $(1,1,\ldots,1)$ monopole, the singularities reside at $z=\mu_2,\ldots,\mu_{n-1}$ [76, 54, 99]. Like for the caloron we introduce $\Delta^\dagger=(\lambda^\dagger(z),\frac{1}{2\pi i}\hat{D}_x^\dagger(z))$,

$$\lambda(z)=\sum_{m=2}^{n-1}\delta(z-\mu_m)\zeta_m,\quad \hat{D}_x(z)=\sigma_\mu\hat{D}_x^\mu(z)=\frac{d}{dz}+\hat{A}(z)-2\pi i x, \tag{6.A.1}$$

where $\hat{A}(z)$ is now defined on $[\mu_1,\mu_n]$. The Nahm construction is performed in terms of the normalised zero modes $v(x)$ of $\Delta(x)$

$$v(x)=\left(\begin{array}{c} s_x \\ \hat{\psi}_x(z)\end{array}\right),\quad \frac{1}{2\pi i}\hat{D}_x^\dagger(z)\hat{\psi}_x^m(z)+\sum_{m'=2}^{n-1}\delta(z-\mu_{m'})\zeta_{m'}^\dagger s_{xm'}^m=0, \tag{6.A.2}$$

$$v^\dagger(x)v(x)=s_x^\dagger s_x+\int_{\mu_1}^{\mu_n}dz\hat{\psi}_x^\dagger(z)\hat{\psi}_x(z)=1_n, \tag{6.A.3}$$

where $\hat{\psi}_x(z)=(\hat{\psi}_x^1(z),\ldots,\hat{\psi}_x^n(z))$ contains the n two-spinors defined on the interval $[\mu_1,\mu_n]$, and $s\in\mathbb{C}^{n-2,n}$. (The equation for $\hat{\psi}_x^m(z)$ is readily seen to have n solutions for fixed s_x [54]). Though the monopole is a static solution, it is preferable to have x_0 included as a dummy variable, the x_0 dependence trivially being implemented by $v(x)=e^{2\pi i x_0 z}v(\vec{x})$, so as to write concisely

$$A_\mu(\vec{x})=(\Phi(\vec{x}),\vec{A}(\vec{x}))=v^\dagger(x)\partial_\mu v(x), \tag{6.A.4}$$

with the inner product defined as in eq. (6.A.3). Performing all monopole calculations in terms of $\Delta(x)$ and $v(x)$, we can copy the caloron formalism. In particular, it follows that for eq. (6.A.4) to be selfdual, $\Delta^\dagger(x)\Delta(x)$ should commute with the quaternions. This is equivalent to the monopole Nahm equation

$$\frac{d}{dz}\hat{A}_j(z)=2\pi i\sum_{m=2}^{n-1}\delta(z-\mu_m)\rho_m^j. \tag{6.A.5}$$

Its solution $\hat{A}_j(z)$ can be written in terms of $n-1$ position vectors $\vec{y}_m$, $\vec{\rho}_m=\vec{y}_m-\vec{y}_{m-1}$, comprised in $\vec{y}_{\rm m}=(\vec{y}_1,\ldots,\vec{y}_n)^t$,

$$\vec{\rho}_{\rm m}=S_{\rm m}^t\vec{y}_{\rm m}, \tag{6.A.6}$$

implying

$$\hat{A}_j = 2\pi i \sum_{m=1}^{n-1} \chi_{[\mu_m,\mu_{m+1}]}(z)\vec{y}_m. \tag{6.A.7}$$

Like for the caloron, there is a gauge action on the Nahm data

$$\hat{A}(z) \to \hat{A}(z) + i\frac{d}{dz}h(z), \quad \zeta_m \to \zeta_m e^{ih(\mu_m)}, \quad m = 2,\ldots,n-1, \tag{6.A.8}$$

with gauge group $\hat{\mathcal{G}}_{\mathrm{m}} = \{g(z)|g : z \to e^{-ih(z)} \in U(1), g(\mu_1) = g(\mu_n) = 1\}$. The condition at the endpoints is required for id/dz to be hermitean on the space of gauge functions. Hence, for the monopole $\hat{\mathcal{G}}_{\mathrm{m}} = \{g(z)|g : z \to e^{-ih(z)} \in U(1), g(\mu_1) = g(\mu_n) = 1\}$. The $\mathcal{G}_{\mathrm{m}}$ action can be used to set $\hat{A}_0(z)$ constant, and to undo the $U(1)$ phase ambiguities in relating ζ_m to $\vec{\rho}_m, m = 2,\ldots,n-1$, hence ζ_m can be considered to have fixed phase. The monopole Nahm data can then be expressed in terms of $n-1$ quaternions

$$\hat{A}_{\mathrm{m}} = (\hat{A}_1,\ldots,\hat{A}_{n-1})^t = 2\pi(N_{\mathrm{m}}^{-1}\frac{\tau_{\mathrm{m}}}{4\pi} + \vec{y}_{\mathrm{m}}\cdot\vec{\sigma}), \tag{6.A.9}$$

$i\hat{A}_{\mathrm{m},m}$ denoting the value $\hat{A}(z)$ takes on (μ_m,μ_{m+1}).

In the gauge with constant $\hat{A}_0(z)$, the Green's function f_x in the monopole Nahm construction is the solution to the differential equation

$$\left\{\left(\frac{1}{2\pi i}\frac{d}{dz} - x_0\right)^2 + \sum_{m=1}^{n-1}\chi_{[\mu_m,\mu_{m+1}]}(z)\, r_m^2 + \frac{1}{2\pi}\sum_{m=2}^{n-1}\delta(z-\mu_m)|\vec{y}_m - \vec{y}_{m-1}|\right\}\hat{f}_x(z,z') \\ = \delta(z-z'), \tag{6.A.10}$$

whereas transformations to other gauges are realised by

$$\hat{f}_x(z,z') \to g(z)\hat{f}_x(z,z')g(z')^*, \quad g(z) \in \hat{\mathcal{G}}_{\mathrm{m}}. \tag{6.A.11}$$

The boundary condition for the monopole Green's function is determined by the requirement that $i\frac{d}{dz}$ be a hermitean operator, therefore the eigenfunctions of the left-hand side of eq. (6.A.10) vanish in the endpoints. This imposes by standard Sturm-Liouville theory

$$\hat{f}_x(\mu_1,z') = \hat{f}_x(\mu_n,z') = 0 \tag{6.A.12}$$

for the Green's function. This boundary condition is automatically satisfied when we obtain the monopole Green's function from the caloron Green's function, taking the limit $|\vec{y}_n| \to \infty$. One obtains

$$\hat{f}_x(z,z') = \frac{\pi e^{2\pi i x_0(z-z')}}{r_m\tilde{\psi}}\langle v_m(z')|\mathcal{A}_{m-1}\cdots\mathcal{A}_1\begin{pmatrix} s_{n-1} & c_{n-1} \\ 0 & 0\end{pmatrix}\frac{1}{r_{n-1}}\mathcal{A}_{n-2}\cdots\mathcal{A}_m|w_m(z)\rangle, \tag{6.A.13}$$

for $\mu_m \le z' \le z \le \mu_{m+1}$, $m = 1,\ldots,n-1$. Here $\mathcal{A}_m$ is defined in eq. (5.4.23), $v_m(z)$ and $w_m(z)$ in eq. (5.4.33) and $\tilde{\psi}$ in eq. (5.5.2). Indeed, the monopole Green's function in eq. (6.A.13) satisfies the boundary conditions eqs. (6.A.12).

The metric on the monopole moduli space is determined in terms of the L_2 norm of gauge orthogonal solutions $Z_{\rm m}$ to the linearised Bogomol'nyi equations. With A_0 identified as the Higgs field, and assuming all fields and zero modes being static, the conditions for a tangent vector to the monopole moduli space are identical to those for the tangent vector to an instanton moduli space, hence $Z_{\rm m}$ satisfies

$$D^{\mathrm{ad}^\dagger}(A)Z_{\rm m} = 0, \tag{6.A.14}$$

where ∂_0 acts trivially, but is kept to make later derivations more transparent. Metric and Kähler forms read

$$(g,\vec{\omega})(Z_{\rm m},Z'_{\rm m}) = \frac{1}{4\pi^2}\int_{\mathbb{R}^3} d^3x \mathrm{Tr} Z^\dagger_{\rm m}(\vec{x})Z'_{\rm m}(\vec{x}). \tag{6.A.15}$$

The formalism to compute the metric is copied from the caloron case. A tangent vector to the monopole moduli space is given by

$$-Z_{{\rm m}\mu}(\vec{x}) = \int_{[\mu_1,\mu_n]^2} dzdz'\left(\sum_{m'=2}^{n-1} s_x^\dagger \hat{c}_{m'}\delta(z-\mu_{m'}) + \hat{\psi}_x^\dagger(z)\hat{Y}(z)\right)\hat{f}_x(z,z')\bar{\sigma}_\mu\hat{\psi}_x(z') - h.c. \tag{6.A.16}$$

in terms of a tangent vector to the moduli space of monopole Nahm data

$$C = \begin{pmatrix}\hat{c}(z)\\ \hat{Y}(z)\end{pmatrix}, \quad \hat{c}(z) = \sum_{m=2}^{n-1}\hat{c}_m\delta(z-\mu_m), \tag{6.A.17}$$

satisfying the deformation and gauge orthogonality equations

$$\begin{aligned}\frac{d}{dz}\hat{Y}_i(z) &= -i\pi\sum_{m=2}^{n-1}\mathrm{tr}_2\bar{\sigma}^i(\zeta_m^\dagger\hat{c}_m + \hat{c}_m^\dagger\zeta_m)\delta(z-\mu_m),\\ \frac{d}{dz}\hat{Y}_0(z) &= -i\pi\sum_{m=2}^{n-1}\mathrm{tr}_2(\zeta_m^\dagger\hat{c}_m - \hat{c}_m^\dagger\zeta_m)\delta(z-\mu_m).\end{aligned} \tag{6.A.18}$$

To derive the analogue for monopoles of Corrigan's formula we trade each matrix multiplication in eq. (3.1.19) for an integration over $[\mu_1,\mu_n]$ or an inner product of type (6.A.3) and use the trivial x_0 dependence of $v(x)$ and $f_x(z,z')$ for the monopole to obtain

$$\begin{aligned}\mathrm{Tr}Z^\dagger_{\rm m}(x)Z'_{\rm m}(x) &= \\ &- \tfrac{1}{2}\nabla^2\mathrm{tr}_2\int_{[\mu_1,\mu_n]} dz\Big([\hat{Y}^\dagger(z)\hat{Y}'(z) + \hat{Y}'^\dagger(z)\hat{Y}(z)\\ &\qquad\qquad + \hat{c}^\dagger(z) < \hat{c}' > + \hat{c}'^\dagger(z) < \hat{c} >]\hat{f}_x(z,z)\Big)\\ &+ \tfrac{1}{2}\nabla^2\mathrm{tr}_2\int_{[\mu_1,\mu_n]^2} dzdz'\Big([\hat{\mathcal{C}}(z) + \hat{\mathcal{Y}}(z)]\hat{f}_x(z,z')[\hat{\mathcal{Y}}_x'^\dagger(z') + \hat{\mathcal{C}}'^\dagger(z')]\hat{f}_x(z',z)\Big),\end{aligned} \tag{6.A.19}$$

where now $\hat{\mathcal{C}}(z) = \sum_{m=2}^{n-1} \hat{c}_m^\dagger \zeta_m \delta(z - \mu_m)$, $\hat{\mathcal{Y}}_x(z) = (2\pi i)^{-1} \hat{Y}^\dagger(z) \hat{D}_x(z)$ and $< H > \equiv \int_{[\mu_1, \mu_n]} H(z) dz$. Compare eq. (6.2.5). The monopole metric is evaluated from eqs. (6.A.15,6.A.19) by partial integration, along the lines of the derivation in section 6.2.1. The monopole Green's function $f_x(z, z')$ behaves as in eq. (6.2.7). Thus we arrive at the isometric property of the Nahm construction for $(1, 1, \ldots, 1)$ monopoles,

$$\begin{aligned} g_{\mathcal{M}}(Z_{\mathrm{m}}, Z'_{\mathrm{m}}) &= \tfrac{1}{2} \mathrm{tr}_2 \left(< \hat{Y}^\dagger \hat{Y}' > + < \hat{c}^\dagger >< \hat{c}' > + < \hat{c}'^\dagger >< \hat{c} > \right), \\ \omega^i_{\mathcal{M}}(Z_{\mathrm{m}}, Z'_{\mathrm{m}}) &= \tfrac{1}{2} \mathrm{tr}_2 \bar{\sigma}_i \left(< \hat{Y}^\dagger \hat{Y}' > + < \hat{c}^\dagger >< \hat{c}' > - < \hat{c}'^\dagger >< \hat{c} > \right) \quad (6.A.20) \end{aligned}$$

An infinitesimal gauge transformation $\delta \hat{X}(z)$ is applied to obtain gauge orthogonality of the tangent vector C

$$\hat{c}(z) = \sum_{m=2}^{n-1} \delta(z - \mu_m) \hat{c} = \sum_{m=2}^{n-1} \delta(z - \mu_m) \left(\delta\zeta_m + i\zeta_m \delta\hat{X}(\mu_m) \right), \quad (6.A.21)$$

$$\hat{Y}(z) = i \sum_{m=1}^{n-1} \chi_{[\mu_m, \mu_{m+1}]} \hat{Y}_m = \frac{1}{2\pi i} \left(\delta\hat{A}(z) + i \frac{d}{dz} \delta\hat{X}(z) \right).$$

It vanishes in the endpoints $z = \mu_1, z = \mu_n$ and satisfies

$$\begin{aligned} -\frac{1}{2\pi} \frac{d^2 \delta\hat{X}(z)}{dz^2} &+ 2\delta\hat{X}(z) \sum_{m=2}^{n-1} \delta(z - \mu_m) |\vec{\rho}_m| \quad (6.A.22) \\ &= \sum_{m=2}^{n-1} \delta(z - \mu_m) \left[\frac{d\tau_m}{4\pi\nu_m} - \frac{d\tau_{m-1}}{4\pi\nu_{m-1}} - |\vec{\rho}_m| \vec{\mathrm{w}}_m(\vec{\rho}_m) \cdot d\vec{\rho}_m \right]. \end{aligned}$$

Therefore, it is piecewise linear and fixed by $\delta\hat{X} = (\delta\hat{X}_2, \ldots, \delta\hat{X}_{n-1})^t$, $\delta\hat{X}_m = \delta\hat{X}(\mu_m)$, $m = 2, \ldots, n-1$ where

$$\frac{1}{2\pi} (S^t_{\mathrm{m}} N^{-1}_{\mathrm{m}} S_{\mathrm{m}} + V^{-1}_{\mathrm{m}}) \delta\hat{X} = (S^t_{\mathrm{m}} N^{-1}_{\mathrm{m}} \frac{d\tau_{\mathrm{m}}}{4\pi} - V^{-1}_{\mathrm{m}} \vec{W}_{\mathrm{m}} S^t_{\mathrm{m}} \cdot d\vec{y}_{\mathrm{m}}), \quad (6.A.23)$$

(see eqs. (6.3.9, 6.3.10) for definitions). With the compensating gauge function found, the remaining manipulations to retrieve the uncentered monopole metric in eq. (6.3.8) from eqs. (6.A.20, 6.A.21) differ only in the m label and the dimensions of the matrices from those in section 6.2.2 and are therefore not repeated here.

To compute the metric using the hyperKähler quotient construction we follow and summarise the reasoning in [74, 39] and section 6.2.3. We have to find the metric on $\mathcal{N}_{\mathrm{m}}/\hat{\mathcal{G}}_{\mathrm{m}}$, where $\mathcal{N}_{\mathrm{m}}$ is the subset of the space $\hat{\mathcal{A}}_{\mathrm{m}}$ of monopole Nahm data containing the solutions to the Nahm equations. Making use of the $U(1)$ gauge symmetry for the monopole in eq. (6.A.8), we can restrict ourselves to piecewise constant $\hat{A}(z)$, characterised by $n-1$ quaternions corresponding to its values on the subintervals. Together with the $n-2$ complex two-vectors giving the matching data, these form the space $\hat{\mathcal{A}}_{0\mathrm{m}} = \mathbb{H}^{n-1} \times \mathbb{C}^{n-2,2} \ni (y_{\mathrm{m}}, \zeta_{\mathrm{m}})$. This space has natural metric

$$g = \tfrac{1}{2} \mathrm{Tr}\, \mathrm{tr}_2 \left(dy^\dagger_{\mathrm{m}} N_{\mathrm{m}} dy_{\mathrm{m}} + 2 d\zeta^\dagger_{\mathrm{m}} d\zeta_{\mathrm{m}} \right). \quad (6.A.24)$$

The set of piecewise constant solutions to the Nahm equations form $\mathcal{N}_{0,\mathrm{m}}$, which is a subset of $\hat{\mathcal{A}}_{0\mathrm{m}}$. The vector part of a piecewise constant solution to the monopole Nahm equation (i.e.$\mathcal{N}_{\mathrm{m},0}$) is fixed by eq. (6.A.6). We introduce the phases of ζ_{m} as $\psi_{\mathrm{m}} = (\psi_2, \ldots, \psi_{n-1})^t$. Having gauge fixed to constant $\hat{A}(z)$, the residual $U(1)$ gauge symmetry consists of gauge functions having piecewise linear and continuous logarithms, which vanish in the endpoints $z = \mu_1$ and $z = \mu_m$. This results in an $\mathbb{R}^{n-2}$ action on $\hat{\mathcal{A}}_{0\mathrm{m}}$, characterised by

$$y_{\mathrm{m}} \to y_{\mathrm{m}} - \frac{1}{2\pi} N_{\mathrm{m}}^{-1} S_{\mathrm{m}} t_{\mathrm{m}}, \quad \psi_{\mathrm{m}} \to \psi_{\mathrm{m}} + 2t_{\mathrm{m}}, \quad t_{\mathrm{m}} \in \mathbb{R}^{n-2}, \tag{6.A.25}$$

with moment map, zero set and invariants given by

$$\vec{\mu}_{\mathrm{m}} = -\frac{1}{2\pi} S_{\mathrm{m}}^t \vec{y}_{\mathrm{m}} + \frac{\vec{\rho}_{\mathrm{m}}}{2\pi}, \quad \vec{\rho}_{\mathrm{m}} = S_{\mathrm{m}}^t \vec{y}_{\mathrm{m}}, \quad \tau_{\mathrm{m}} = 4\pi N_{\mathrm{m}} y_{0\mathrm{m}} + S_{\mathrm{m}} \psi_{\mathrm{m}}. \tag{6.A.26}$$

A suitable notation being established, the algebra to obtain the metric and Kähler forms for the uncentered monopole in eq. (6.3.8) is now nearly identical to the hyper-Kähler quotient construction of the uncentered caloron metric, and one readily retrieves eq. (6.3.8). Actually, one only has to insert the m labels at appropriate places, just realising that the dimensionalities of the objects are slightly different.

7 Fermion zero-modes and reciprocity

The Weyl fermionic zero-modes in the background of the selfdual connection form the starting point in the Nahm transformation. The Nahm transformed connection is then calculated using eq. (2.1.2). In this chapter we will study these fermionic zero-modes for the $k = 1$ $SU(n)$ caloron and explicitly perform the Nahm transformation. We will thus retrieve $\hat{A}$ in the gauge with $\hat{A}_0 = 0$, using the Green's function in chapter 5. As the Nahm data are used as an ingredient in this calculation, we thus prove the involutive property of the Nahm transformation.

7.1 Fermion zero-modes

For the charge one caloron the zero-mode $\Psi_z(x)$ is the solution of

$$D_z^\dagger(A)\Psi_z(x) = -\bar{\sigma}_\mu(\partial_\mu + A_\mu - 2\pi i z_\mu)\Psi_z(x) = 0, \tag{7.1.1}$$

subject to the periodicity constraint

$$\Psi_z(x+1) = \mathcal{P}_\infty \Psi_z(x). \tag{7.1.2}$$

This is the relevant boundary condition for the Nahm transformation and compactifications. Here A_μ is in the algebraic gauge, eq. (5.3.2). The $\vec{z}$ dependence can be trivially found as a plane wave factor,

$$\Psi_z(x) = e^{2\pi i \vec{x}\cdot\vec{z}}\Psi_{z_0}(x). \tag{7.1.3}$$

In the rest of this section we can ignore this $\vec{z}$ dependence and z denotes z_0.

The caloron, considered as an instanton on $\mathbb{R}^4$ with infinite charge, has an infinite number of zero-modes, which can be computed within the ADHM formalism. They are given by eq. (3.2.6)

$$\Psi_p(x) = \frac{1}{\pi}\phi^{-\frac{1}{2}}(x)u^\dagger(x)_{p'I}f_x^{p',p}\varepsilon_{IJ}, \quad \Psi_p(x+1) = \mathcal{P}_\infty\Psi_{p-1}(x), \quad p \in \mathbb{Z}, \tag{7.1.4}$$

differing from each other via multiples of the holonomy. The linear combination

$$\Psi_z^{\mathrm{qp}}(x) = \sum_{p\in\mathbb{Z}} e^{-2\pi i p z}\Psi_p(x), \quad \Psi_z^{\mathrm{qp}}(x+1) = e^{-2\pi i z}\mathcal{P}_\infty\Psi_z^{\mathrm{qp}}(x), \tag{7.1.5}$$

is periodic up to a phase factor which is removed by a $U(1)$ gauge transformation:

$$\Psi_z(x) = e^{2\pi i z x_0}\Psi_z^{\mathrm{qp}}(x) \tag{7.1.6}$$

is the solution to eq. (7.1.1) with the proper boundary condition. For finite temperature applications one needs the anti-periodic combination

$$\Psi_z^{(-)}(x) = e^{2\pi i z x_0} \Psi^{\rm qp}_{z+\frac{1}{2}}(x), \quad \Psi_z^{(-)}(x+1) = -\mathcal{P}_\infty \Psi_z^{(-)}(x). \tag{7.1.7}$$

Combining eqs. (3.2.6, 7.1.6) results in

$$\begin{aligned} \Psi^\dagger_{z'}(x)\Psi_z(x) &= e^{2\pi i x_0(z'-z)} \sum_{p,p'} e^{2\pi i(pz-p'z')} \Psi^\dagger_p(x)\Psi_{p'}(x) \\ &= -\frac{1}{4\pi^2} e^{2\pi i x_0(z'-z)} \sum_{p,p'} e^{2\pi i(pz-p'z')} \partial^2_\mu f^{p,p'}_x \\ &= -\frac{1}{4\pi^2} e^{2\pi i x_0(z'-z)} \partial^2_\mu \hat{f}_x(z,z'), \end{aligned} \tag{7.1.8}$$

from which one reads off the normalisation of the zero-mode

$$|\Psi_z(x)|^2 = -\partial^2_\mu \frac{1}{4\pi r_m \psi} \langle v_m(z)|\mathcal{A}_{m-1}\cdots\mathcal{A}_1\mathcal{A}_n\cdots\mathcal{A}_m|w_m(z)\rangle, \quad \mu_m \le z \le \mu_{m+1}, \tag{7.1.9}$$

using that $\hat{f}_x(z,z)$ is known from eq. (5.4.32). It should be noticed that the norms do not depend on the gauge, hence the results also hold for the purely periodic gauge, related to the algebraic gauge through a gauge transformation of type eq. (5.3.1). As a result one obtains the norms of the purely periodic $\Psi^{(+)}(x)$ and antiperiodic $\Psi^{(-)}(x)$ zero-modes

$$|\Psi^{(+)}(x)|^2 = -\frac{1}{4\pi^2}\partial^2_\mu \hat{f}_x(0,0), \quad |\Psi^{(-)}(x)|^2 = -\frac{1}{4\pi^2}\partial^2_\mu \hat{f}_x(\tfrac{1}{2},\tfrac{1}{2}). \tag{7.1.10}$$

In figure 7-1 the norm squared of the fermion zero-modes in the background of an $SU(3)$ caloron is plotted for various values of z, comprising the purely periodic ($z=0$) and antiperiodic ($z=\frac{1}{2}$) cases. For comparison also the action density is given for the same configuration. Clearly, the zero-modes are localised around the monopole constituents that build up the caloron (cf. section 5.5), $|\Psi_z(x)|^2$ has a maximum near $\vec{y}_m$ for $\mu_m \le z \le \mu_{m+1}$. This generalises the $SU(2)$ result of [31], where the zero-modes were localised around the two constituents. For z near the eigenvalues μ_m, the zero-mode density becomes small and spreads out over the region connecting $\vec{y}_{m-1}$ and $\vec{y}_m$. This is shown in figure 7-2.

The localisation can be easily established analytically in the limit of large $|\vec{y}_i-\vec{y}_{i+1}|$ for all i, in which case one finds, when $z\in[\mu_m,\mu_{m+1}]$,

$$\hat{f}_x(z,z) = \frac{2\pi\sinh[2\pi(z-\mu_m)r_m]\sinh[2\pi(\mu_{m+1}-z)r_m]}{r_m\sinh[2\pi\nu_m r_m]}, \tag{7.1.11}$$

making explicit that the location of the zero-mode is determined by the interval that contains the appropriate value of z. From the ordering in eq. (5.2.2) it follows that $\mu_1 \le 0 \le \mu_n$, such that the periodic zero-mode is associated to the *static* constituent

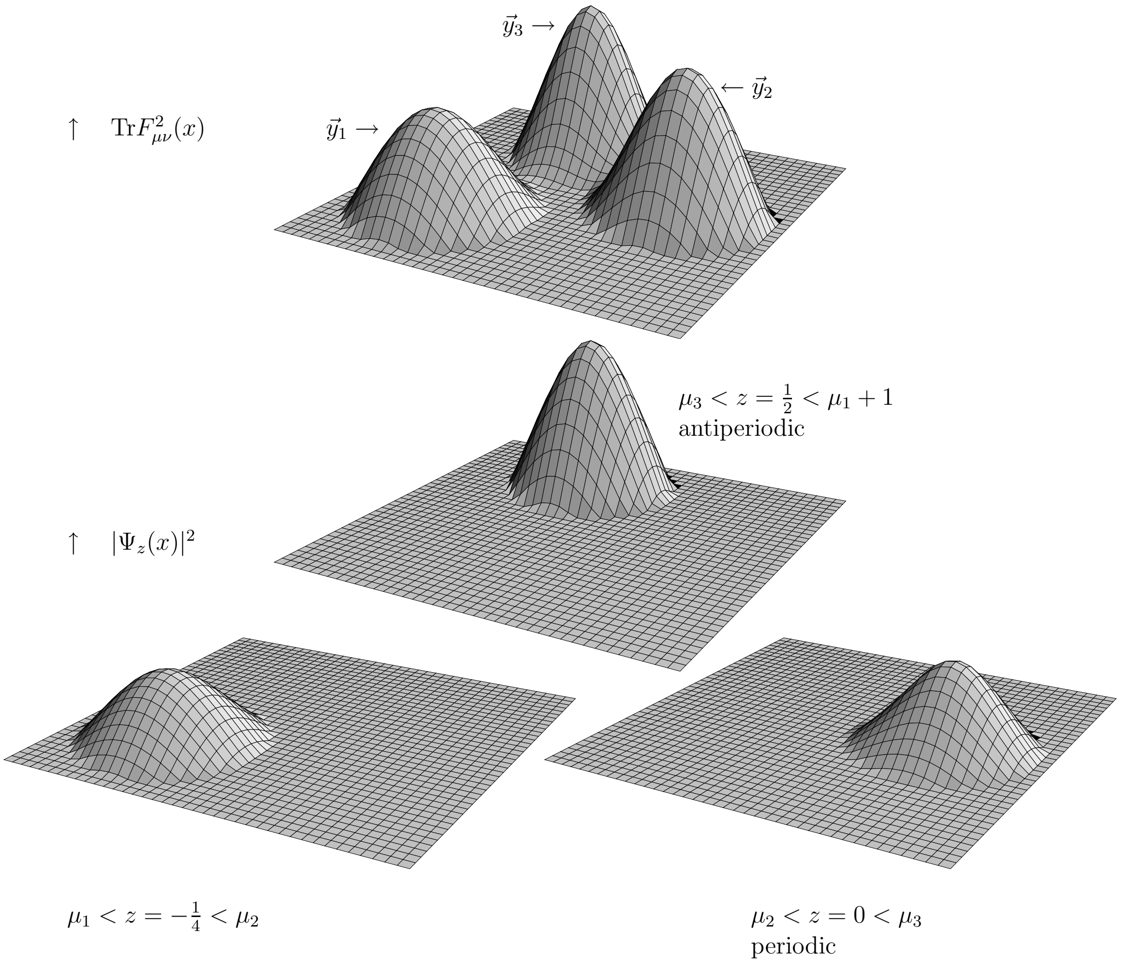


Figure 7-1. Weyl zero-mode density: The action density (top) for the $SU(3)$ caloron, cut off at $1/(2e)$, on a logarithmic scale, with $(\mu_1, \mu_2, \mu_3) = (-17, -2, 19)/60$ for $t = 0$ in the plane defined by $\vec{y}_1 = (-2, -2, 0)$, $\vec{y}_2 = (0, 2, 0)$ and $\vec{y}_3 = (2, -1, 0)$. Below the zero-mode density in the background of the $SU(3)$ caloron, for $z = \frac{1}{2}, 0, -\frac{1}{4}$ (clockwise), on equal logarithmic scales, cut off below $1/e^5$. The zero-modes are localised around the constituent monopoles.

(in some gauge) at $\vec{y}_m$, with $\mu_m \leq 0 \leq \mu_{m+1}$. This is precisely the condition for the existence of a zero-mode given by the Callias index theorem [16, 13] (see also the appendix of ref. [10]). Due to the static background (for well-separated constituents), time dependence of the zero-mode would be of the form $\exp(2\pi i k t)$ for k integer, shifting $z = 0$ by k, out of the interval that allows for a zero-mode.

Allowing for $k = \pm\frac{1}{2}$, for which $\exp(2\pi i k t)$ turns the periodic zero-mode anti-periodic, we can have situations where this anti-periodic zero-mode is associated to one of the static monopole constituents. A specific example for $SU(3)$ where this occurs is $(\mu_1, \mu_2, \mu_3) = (-0.48, -0.03, 0.51)$, yielding $(\nu_1, \nu_2, \nu_3) = (0.45, 0.54, 0.01)$. *Both* the periodic and the anti-periodic zero-mode are associated to the 2^{nd} constituent. We note that, apart from the fact that the 3^{rd} constituent is nearly massless,

$\uparrow \quad |\Psi_z(x)|^2$

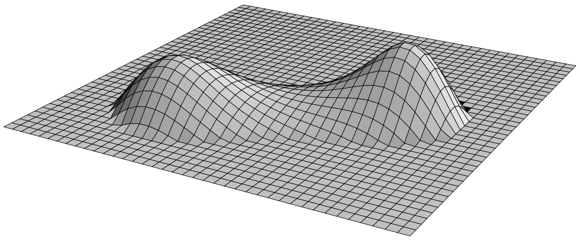


Figure 7-2. Weyl zero-mode density for $z = -\frac{2}{60} \pm \epsilon$ ($\epsilon = 3.33333 \times 10^{-7}$) at a logarithmic scale, cut off below $1/e^6$. The zero-mode density is considerably smaller than in figure 7-1 and is spread out over the region connecting monopoles 1 and 2.

both zero-modes are very broad since $\min(z - \mu_2, \mu_3 - z) = 0.03$ for $z = 0$ and 0.01 for $z = \frac{1}{2}$. For $SU(2)$ $z = 0$ is always midway between μ_1 and μ_2 and $z = \frac{1}{2}$ midway between μ_2 and $\mu_3 = 1 + \mu_1$). When z coincides with μ_i, the zero-mode is no longer normalisable, which is the origin of the delta function singularities in the Nahm transformation.

By removing one of the constituent monopoles to spatial infinity, the $k = 1$ $SU(n)$ caloron is changed into the $(1, 1, \ldots, 1)$ $SU(n)$ monopole, cf. section 5.5.1. Taking the corresponding limit in the Green's function gives the Green's function for the Nahm construction of the $(1, 1, \ldots, 1)$ monopole as was shown in the appendix to chapter 6), eq. (6.A.13). It then follows immediately that the density of the Weyl fermion zero-mode in the background of the $(1, 1, \ldots, 1)$ monopole is given by

$$|\Psi_{\mathrm{m}z}(\vec{x})|^2 = -\Delta \frac{1}{4\pi\tilde{\psi}} \langle v_m(z)| \mathcal{A}_{m-1} \cdots \mathcal{A}_1 \begin{pmatrix} s_{n-1} & c_{n-1} \\ 0 & 0 \end{pmatrix} \frac{1}{r_{n-1}} \mathcal{A}_{n-2} \cdots \mathcal{A}_m |w_m(z)\rangle, \tag{7.1.12}$$

for $\mu_m \leq z \leq \mu_{m+1}$, $m = 1, \ldots, n-1$. Here $\mathcal{A}_m$ is defined in eq. (5.4.23), $v_m(z)$ and $w_m(z)$ in eq. (5.4.33) and $\tilde{\psi}$ in eq. (5.5.2). As for the caloron, the zero-modes are localised around the constituents, according to the Callias index theorem. The analytic proof for localisation in eq. (7.1.11) is valid for the $(1, 1, \ldots, 1)$ monopole as well.

The adjoint fermion zero-modes $\Psi^{\mathrm{ad}}(x)$ in the background of the caloron and $(1, 1, \ldots, 1)$ monopole, $\bar{\sigma}_\mu D_\mu^{\mathrm{ad}} \Psi^{\mathrm{ad}}(x) = 0$ are given by the gauge zero-modes which were used in the computation of the metric in chapter 6, $\Psi^{\mathrm{ad}}(x) = \sigma_\mu Z_\mu(x)$, cf. eq. (1.5.7). Corrigan's formula in the form in eqs. (6.2.5, 6.A.19) gives an expression for the norm of these modes.

7.2 Nahm reciprocity

Performing the Nahm transformation amounts to calculating the integrals

$$\hat{A}_0(z) = \int_{\mathbb{R}^3\times S^1} d_4x\, \Psi_z^\dagger(x)\frac{d}{dz}\Psi_z(x), \quad \hat{A}_i(z) = 2\pi i\int_{\mathbb{R}^3\times S^1} d_4x\, \Psi_z^\dagger(x)x_i\Psi_z(x), \quad (7.2.1)$$

as follows from eqs. (2.1.2, 7.1.3). The spatial components of $\hat{A}$ are evaluated by partial integration as a boundary term,

$$\begin{aligned}\hat{A}_i(z) &= -\frac{i}{2\pi}\int_{\mathbb{R}^3\times S^1} d_4x\, x_i\partial_\mu^2\hat{f}_x(z,z)\\ &= \lim_{|\vec{x}|\to\infty}\frac{i}{2\pi}\int_{S^2\times S^1} d\Omega dx_0|\vec{x}|^2\left(\hat{x}_i\hat{f}_x(z,z)-\hat{x}_jx_i\partial_j\hat{f}_x(z,z)\right), \quad (7.2.2)\end{aligned}$$

using that the ∂_0^2 term does not contribute to the integral over the circle. To evaluate the boundary term, we need the asymptotic behaviour of $\hat{f}_x(z,z)$ in the limit of $|\vec{x}|\to\infty$. This requires a careful analysis.

The scalar potential ψ has the asymptotic behaviour

$$\psi(x) = \tfrac{1}{2}\mathrm{tr}_2\mathcal{A}_n\cdots\mathcal{A}_1 \to \tfrac{1}{2}e^{2\pi\sum_m \nu_m r_m}, \quad (7.2.3)$$

up to terms that are exponentially suppressed or of higher power in $|\vec{x}|^{-1}$. This follows from a multipole expansion of the product of $\mathcal{A}_m$ matrices,

$$\mathcal{A}_{m-1}\cdots\mathcal{A}_1\mathcal{A}_n\cdots\mathcal{A}_m \to e^{2\pi\sum_m \nu_m r_m}\tfrac{1}{2}\begin{pmatrix} 1+\frac{1}{2}\vec{\rho}_{m-1}\cdot\frac{\hat{x}}{|\vec{x}|} & 1+\frac{1}{2}\vec{\rho}_{m-1}\cdot\frac{\hat{x}}{|\vec{x}|}\\ 1-\frac{1}{2}\vec{\rho}_{m-1}\cdot\frac{\hat{x}}{|\vec{x}|} & 1-\frac{1}{2}\vec{\rho}_{m-1}\cdot\frac{\hat{x}}{|\vec{x}|}\end{pmatrix}, \quad (7.2.4)$$

which also implies

$$\langle v_m(z)|\mathcal{A}_{m-1}\cdots\mathcal{A}_1\mathcal{A}_n\cdots\mathcal{A}_m|w_m(z)\rangle \to \tfrac{1}{2}e^{2\pi\sum_m \nu_m r_m}, \quad (7.2.5)$$

up to terms that are exponentially suppressed or of higher order in $|\vec{x}|^{-1}$. From the asymptotics of the various factors in eq. (5.4.32) now follows the asymptotics of the Green's function,

$$\hat{f}_x(z,z) \to \frac{\pi}{|\vec{x}|}(1+\frac{\hat{x}}{|\vec{x}|}\cdot\vec{y}_m)\left(1+\mathcal{O}(|\vec{x}|^{-2},e^{-|\beta|\vec{x}||})\right) \quad \text{for} \quad |\vec{x}|\to\infty. \quad (7.2.6)$$

The Nahm transformed connection is then readily obtained from eq. (7.2.2),

$$\hat{A}^j(z) = 2\pi i y_m^j, \quad (7.2.7)$$

for $\mu_m \le z \le \mu_{m+1}$.

For the evaluation of $\hat{A}_0$ we use eq. (7.1.8)

$$\begin{aligned}\Psi_z^\dagger \frac{d}{dz}\Psi_z &= \lim_{z'\to z}\frac{d}{dz'}\Psi_z^\dagger(x)\Psi_{z'}(x)\\ &= -\frac{1}{4\pi^2}\lim_{z'\to z}\frac{d}{dz'}\{\partial_\mu^2 e^{2\pi i x_0(z'-z)}\hat{f}_x(z,z')\\ &\qquad -4\pi i(z'-z)e^{2\pi i x_0(z'-z)}\partial_0\hat{f}_x(z,z')\\ &\qquad -(2\pi i)^2(z'-z)^2 e^{2\pi i x_0(z'-z)}\hat{f}_x(z,z')\}. \qquad (7.2.8)\end{aligned}$$

The contribution of the first term on the rhs., which is evaluated by partial integration can be shown to vanish. One just uses the asymptotic behaviour of $d\hat{f}_x(z,z')/dz'|_{z=z'}$ which is similar to that of $\hat{f}_x(z,z')$. Also the other terms have zero contribution, as follows after integration over the circle or in the limit $(z'-z)\to 0$. Therefore, the time-component of the Nahm transformed connection is zero,

$$\hat{A}_0(z) = 0. \qquad (7.2.9)$$

We see that we have reconstructed the caloron Nahm data in eq. (5.3.16) for $\hat{A}_0 = 0$, when the caloron is localised at the origin (see the discussion just below eq. (5.3.18)). In this way we proved the full circle reciprocity: from the caloron Nahm data we reconstructed the Weyl zero-modes in the background of the caloron, which when used in the first Nahm transformation give back the original Nahm connection.

8 Conclusions

In this thesis periodic Yang-Mills instantons (or calorons) on $\mathbb{R}^3 \times S^1$ were studied. In the introduction some elementary properties of instantons and BPS monopoles, classical solutions in gauge theories, were discussed. Instantons are selfdual Yang-Mills connections. BPS monopoles are selfdual static solitons in spontaneously broken gauge theories. When the circumference of the S^1 is varied, the caloron interpolates between instantons (large circle) and monopoles (circle shrunk to a point) and it is therefore that calorons are objects worthwhile to investigate. Other motivations for their study presented in the introduction were their relevance for finite-temperature field theory and their serving as a toy-model for instantons on compactified spaces.

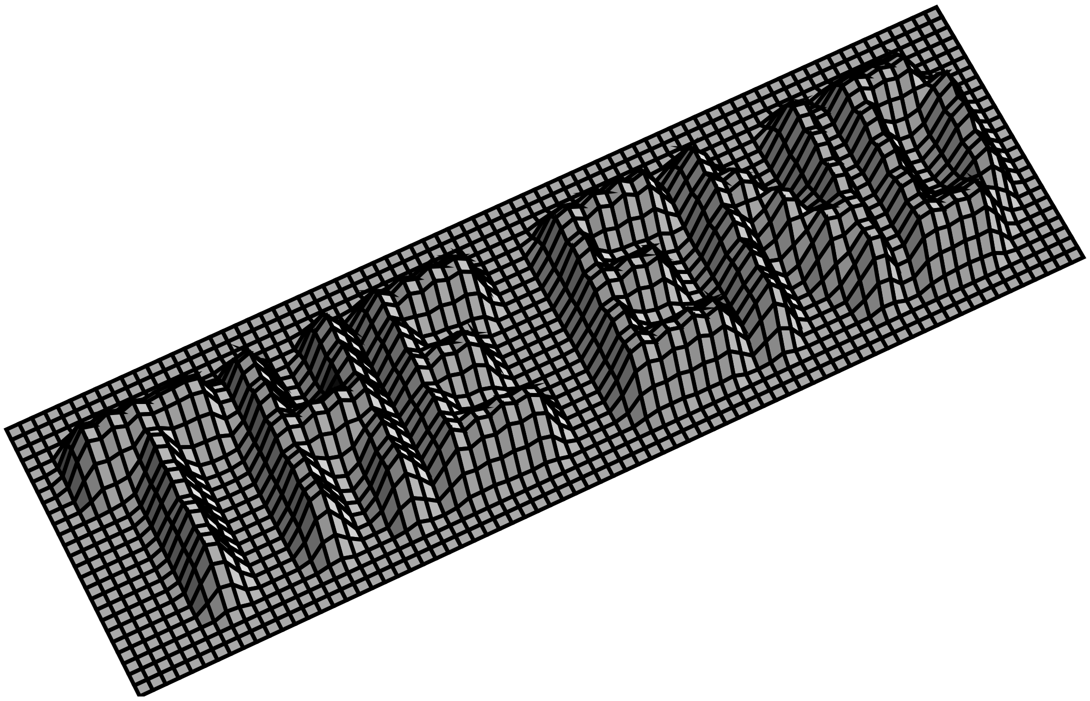
The calorons were studied using a suitable combination of the Nahm transformation and the ADHM formalism for multi-instantons. The Nahm transformation, which maps a selfdual connection onto a selfdual connection over a dual space, was treated in chapter 2 where it was explained how it maps calorons, selfdual connections on $\mathbb{R}^3 \times S^1$, to gauge fields living on a circle. The charge one calorons studied in this thesis are mapped to *abelian* gauge fields, which greatly simplifies their analysis. The Nahm transformation preserves the metric on the moduli space. The ADHM construction translates the study of multi-instanton solutions and their physical properties into a calculus of matrices satisfying a quadratic constraint that takes care of the selfduality. It was discussed at length in chapter 3.

In chapter 4 the detailed derivation was presented of the charge one periodic calorons with non-trivial holonomy for gauge group $SU(2)$. A suitable combination of the Nahm transformation and ADHM construction was used, in conjunction with the multi-instanton calculus in the ADHM formalism. This led to a quantum-mechanical scattering problem defined on the circle with a piecewise constant potential and delta-function impurities. The results rely on the feasibility to compute explicitly the relevant Green's function for this problem in terms of which the solution can be conveniently expressed. Using the explicit form of the Green's function, an expression for the gauge potential and action density could be derived for the $SU(2)$ $k = 1$ caloron. Action density profiles reveal two lumps, corresponding with elementary $SU(2)$ BPS monopoles. The two constituents can be placed at arbitrary positions. Their masses are related to the eigenvalues of the holonomy. The magnetic charges of the two constituents are opposite, and therefore the caloron has no net magnetic charge. The instanton charge can be seen as arising from a subtle braiding of the two monopole worldlines. This revives an old argument due to Taubes. Also discussed were the properties of the moduli space, $\mathbb{R}^3 \times S^1\times$ Taub-NUT$/Z_2$, and its metric. The Taub-NUT mass parameter is determined by the eigenvalues of the holonomy. Further issues dealt with were how to retrieve topological charge in the context of abelian projection and possible applications to QCD.

The techniques developed for $SU(2)$ could be generalised to the gauge group $SU(n)$. It was profitable to keep to the calculations formal. Thus the corresponding Green's function problem could be tackled and a compact expression could be derived for the action density of the $SU(n)$ charge one caloron with arbitrary non-trivial holonomy at spatial infinity. This was presented in chapter 5. It was shown explicitly that there are n lumps inside the caloron, each of which represents a BPS monopole. The masses of these constituents are related to the eigenvalues of the holonomy. The constituents can be positioned at arbitrary locations and can have arbitrary phase. Removing one of the constituents, i.e. putting it at spatial infinity, corresponds to shrinking the circle to a point: the static monopole limit. What remains is the $(1, 1, \ldots, 1)$ $SU(n)$ monopole, whose energy density was determined by taking the corresponding limit in the expression for the caloron action density. This forms an extension of the Rossi limit [88] to the gauge group $SU(n)$. The magnetic charge of the remnant is opposite to the magnetic charge of the removed monopole. The case of non-maximal symmetry breaking was considered, arising when two or more of the eigenvalues of the holonomy coincide. The main effect is the emergence of massless monopoles, which make up the so-called non-abelian cloud [62]. The non-abelian cloud parameter, featuring in the action and energy density and invariant under the enchanced unbroken symmetry group, is related to the relative positions of the massless constituents which themselves loose their identities.

Chapter 6 was devoted to the calculation of the metric of the charge one $SU(n)$ caloron, discussed in chapter 5. The calculation was based on the isometric property of the ADHMN construction. This allowed the calculation to be performed on the space of Nahm data, rather than using the gauge connection. This is a considerable simplification. The result was a metric of toric hyperKähler type, in accord with general principles and a conjecture by Lee and Yi [63]. An alternative approach mapped the calculation of the caloron metric to that of a metric on a finite dimensional hyperKähler quotient, giving the same result. The fact that the caloron consists of monopoles could be read off from the metric and by taking suitable limits, various toric hyperKähler metrics could be obtained. In particular the Lee-Weinberg-Yi metric [61] for the $(1, 1, \ldots, 1)$ monopole could be retrieved, for which the isometric property of the Nahm construction was proven as well. The instanton and monopole limits of the $k = 1$ $SU(n)$ caloron were considered as an example of the interpolation by calorons between instantons and monopoles.

In chapter 7 the fermionic zero-modes in the background of the charge one $SU(n)$ caloron were considered, which play a central role in the Nahm transformation. Using the ADHM calculus of chapter 3, the zero-mode density was given in terms of the Green's function derived in chapter 5. The main result is that for well-separated constituents the fermion zero-mode is localised around a single constituent. For $SU(2)$ the anti-periodic zero-mode is always associated to the constituent that carries Taubes-winding. For $SU(n > 2)$ this is also typically true in particular when the zero-mode is well localised. However, exceptions exist where both the periodic and anti-periodic zero-mode are associated to (possibly the same) static constituent(s), although this tends to be accompanied by nearly massless constituents, and rather delocalised zero-modes. By taking the monopole limit, the zero-mode density for the $(1, 1, \ldots, 1)$ monopole was derived. Using the zero-mode density, the Nahm transformation discussed in chapter 2 could be performed for the charge one $SU(n)$ caloron. Thus the Nahm connection, input in the calculation of the zero-mode density, was retrieved. This demonstrates the notion of reciprocity and showed the Nahm transformation to be an involution: when applied twice, it gives the identity.



An atypical $SU(73)$ $(1, 1, \ldots, 1)$ monopole

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